

Birkhoff's theorem for three-dimensional AdS gravityEloy Ayón-Beato,^{1,2*} Cristián Martínez,^{1†} and Jorge Zanelli^{1‡}¹*Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile*²*Departamento de Física, CINVESTAV-IPN, Apdo. Postal 14-740, 07000, México D.F., Mexico*

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All three-dimensional matter-free space-times with negative cosmological constant, compatible with cyclic symmetry, are identified. The only cyclic solutions are the 2+1 (BTZ) black hole with $SO(2) \times \mathbb{R}$ isometry, and the self-dual Coussaert-Henneaux space-times, with isometry groups $SO(2) \times SO(2,1)$ or $SO(2) \times SO(2)$.

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I. INTRODUCTION

Three-dimensional space-times satisfying the vacuum Einstein equations have constant curvature—positive, negative or zero, depending on the value of the cosmological constant. In view of this, it might seem surprising to find a number of nontrivial 2+1 geometries, analogous to four-dimensional space-times [1,2]. The key to understand this is the number of identifications that can be made on the maximal covering space. This is most dramatically observed in the case of 2+1 black hole, which can be obtained via identifications on AdS_3 [3,4].

A similar discussion also applies to higher dimensional space-times. In [5] and [6], it was shown that, under a suitable identification, the analogue of the nonrotating 2+1 black hole can be obtained in 3+1 dimensions. Similar solutions were constructed through identifications in higher dimensional AdS spacetimes [7,8]. Very recently, the general problem of identifications in AdS_d has been discussed in [9,10], and the conclusion is that the only possible black holes that can be obtained as quotients are the higher-dimensional generalizations of the nonrotating black hole in 2+1 dimensions [11]. Other physically acceptable spacetimes have also been obtained through identifications in Minkowski space as Kaluza-Klein reductions of supersymmetric vacua (see [12] for a classification).

Although the three dimensional black hole has been extensively studied over the past decade, the issue of its uniqueness has not been completely exhausted. One may ask, for instance, what family of geometries is determined by a given set of symmetries, analogous to Birkhoff's theorem, which states that any spherically symmetric solution of Einstein's equations in empty space in four dimensions is diffeomorphic to the maximally extended Schwarzschild solution in an open set [13].

Recent generalizations of Birkhoff's theorem to higher dimensions and to include matter sources as well as an extensive list of references can be found in [14]. In this reference—as in many others, like [15]—spherical symmetry [invariance under $SO(D-1)$] has been extensively dis-

cussed for arbitrary D . However, those general discussions leave out the three-dimensional case. This case is exceptional and should be treated separately because for $D=3$ the group of spatial rotations is Abelian. This means, in particular, that only for 2+1 dimensions “spherical symmetry” is compatible with nonvanishing angular momentum and therefore off-diagonal components in the metric must be allowed.

In this paper it is shown that, apart from the black hole geometry with two Killing vectors, cyclic symmetry [invariance under the action of $SO(2)$] in 2+1 dimensions also allows for the self-dual Coussaert-Henneaux (CH) spacetimes with four Killing vectors [16], and also for a different self-dual geometry with only two Killing vectors. These non-black-hole geometries are analogous to the Nariai solution, which exists in four dimensions with positive cosmological constant [17] (see [18] and references therein for the higher dimensional generalizations). The CH spacetimes are obtained as identifications of AdS_3 by self-dual generators of the two copies of $SO(2,1)$ of the isometry group of anti-de Sitter space, and have been recently shown to be relevant in the context of AdS/CFT correspondence [19]. The CH spacetimes were also independently obtained within the families of solutions derived in Ref. [20], but their properties were not explicitly discussed there.

The paper is organized as follows: In Sec. II the Einstein equations with a negative cosmological constant for 2+1 cyclic spacetimes are integrated. Three cases are identified depending on whether the norm (ν) of a certain gradient is positive, negative or null. In Sec. III the isometries for each of these cases are studied, concluding that the cases $\nu^2 > 0$ and $\nu^2 < 0$ correspond to different patches of the BTZ black hole with isometry $SO(2) \times \mathbb{R}$. The case $\nu^2 = 0$ corresponds to the self-dual CH spacetimes having $SO(2) \times SO(2,1)$ isometry. The last derivation involves an analysis of the Killing equations, which for this case cannot be solved in closed form in general. In Sec. IV it is shown that a further identification can be performed to produce a new self-dual time-dependent spacetime with $SO(2) \times SO(2)$ isometry group and without closed causal curves. Finally, Sec. V contains the conclusions and discussion. Some detailed calculations are included as Appendices.

II. CYCLIC VACUUM SOLUTIONS

Matter-free 2+1 gravity in the presence of a negative cosmological constant is described by the action

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$$S = \frac{1}{2\kappa} \int d^3x \sqrt{-g} (R + 2l^{-2}), \quad (1)$$

where $\Lambda = -l^{-2}$ and κ are the cosmological and gravitational constants, respectively. The corresponding Einstein field equations are

$$G_\mu^\nu = l^{-2} \delta_\mu^\nu. \quad (2)$$

In this section cyclic symmetric configurations satisfying Eqs. (2) will be discussed. A spacetime is called *cyclic symmetric* if it is globally invariant under the action of the one-parameter group $SO(2)$ [21]. The corresponding Killing vector field, $\mathbf{m} = \partial_\phi$, has norm $g_{\phi\phi} > 0$ and the most general metric with this symmetry can be written, in appropriate coordinates, as

$$\begin{aligned} \mathbf{g} = & -N(t,r)^2 F(t,r) dt^2 \\ & + \frac{dr^2}{F(t,r)} + Y(t,r)^2 (d\phi + W(t,r) dt)^2, \end{aligned} \quad (3)$$

where part of the freedom under coordinate transformations has been used to eliminate g_{tr} and $g_{\phi r}$. With this choice, the vacuum Einstein equations (2) can be readily integrated (see Appendix A). The solutions fall into different cases depending on the relative signs of F and the norm of the gradient $\nabla_\mu Y$,

$$\nu^2 \equiv \nabla_\mu Y \nabla^\mu Y = F \left((\partial_r Y)^2 - \frac{(\partial_t Y)^2}{N^2 F^2} \right). \quad (4)$$

Assuming $F > 0$, three cases can be distinguished, and in each case a different choice of coordinates can be made to render the metric in a more conventional form. Changing the sign of F , correspond to reversing the sign of ν^2 , so it is sufficient to analyze the case of positive F only.

A. Case $\nu^2 > 0$: black hole regions $r < r_-$ or $r > r_+$

If $\nabla_\mu Y \nabla^\mu Y > 0$, the radial coordinate can be chosen as $Y(t,r) = r$. This coordinate measures the perimeter, $2\pi r$, of the closed integral curves of the cyclic Killing field $\mathbf{m} = \partial_\phi$. With this choice for Y , the Einstein equations (see Appendix A) are easily integrated yielding

$$W(t,r) = -\frac{JN(t)}{2r^2} + W_0(t), \quad (5)$$

$$F(t,r) = F(r) = \frac{r^2}{l^2} - M + \frac{J^2}{4r^2}, \quad (6)$$

where $W_0(t)$ is an arbitrary function, $N(t,r) = N(t)$, and M and J are integration constants which are assumed to satisfy $|J| \leq Ml$ in order to avoid naked singularities. The function $F(r)$ is positive for $r < r_-$ or $r > r_+$, where r_\pm are the positive roots of the equation $F(r) = 0$. In this way, the metric takes the general form

$$\begin{aligned} \mathbf{g} = & - \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right) N(t)^2 dt^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right)^{-1} dr^2 \\ & + r^2 \left(d\phi + W_0(t) dt - \frac{J}{2r^2} N(t) dt \right)^2, \end{aligned} \quad (7)$$

where the radial coordinate lies in the region $\{r < r_-\} \cup \{r > r_+\}$. The spacetime described by Eq. (7) is locally equivalent to the regions outside the outer horizon ($r > r_+$) or inside the inner horizon ($r < r_-$) of the 2+1 black hole [3,4]. This can be made explicit performing the following coordinate transformation:

$$(t, r, \phi) \mapsto \left(\int N(t) dt, r, \phi + \int W_0(t) dt \right), \quad (8)$$

that respects the gauge choice $g_{tr} = 0 = g_{\phi r}$, $g_{\phi\phi} = r^2$.

B. Case $\nu^2 < 0$: black hole regions $r_- < r < r_+$

In the case $\nabla_\mu Y \nabla^\mu Y < 0$ the time coordinate can be identified with $Y(t,r) = -t$. Then, as can be seen from Appendix A, this implies $N(t,r) = N(r)$ and

$$W(t,r) = -\frac{J \int N(r) dr}{t^3} + W_1(t), \quad (9)$$

$$F(t,r) = \frac{1}{N(r)^2 f(t)}, \quad (10)$$

where $W_1(t)$ is an arbitrary function, and

$$f(t) = -\frac{t^2}{l^2} + M - \frac{J^2}{4t^2}, \quad (11)$$

where M and J are integration constants. In order to preserve the condition $F(t,r) > 0$, $f(t)$ must be positive as well. Hence, the above solution is valid in $t_- < t < t_+$, where t_\pm are the positive roots of the equation $f(t) = 0$. Thus, the metric takes the form

$$\begin{aligned} \mathbf{g} = & - \left(-\frac{t^2}{l^2} + M - \frac{J^2}{4t^2} \right)^{-1} dt^2 + \left(-\frac{t^2}{l^2} + M - \frac{J^2}{4t^2} \right) N(r)^2 dr^2 \\ & + t^2 \left(d\phi + W_1(t) dt - \frac{J \int N(r) dr}{t^3} dt \right)^2, \end{aligned} \quad (12)$$

with $t_- < t < t_+$. Finally, performing the coordinate transformation

$$\begin{aligned} (t, r, \phi) \mapsto & \left(r, t, \phi + \int W_1(t) dt - \int W_0(r) dr \right. \\ & \left. + \frac{J}{2t^2} \int N(r) dr \right), \end{aligned} \quad (13)$$

the above metric takes the same form (7), but with $r^2/l^2 - M + J^2/4r^2 < 0$, or equivalently, for $r_- < r < r_+$. Hence, the spacetime satisfying $F(t,r) > 0$ and $\nabla_\mu Y \nabla^\mu Y < 0$ is locally equivalent to the patch of the 2+1 black hole between the inner and outer horizons.

C. Case $\nu^2=0$: self-dual Coussaert-Henneaux space-times

The condition $\nabla_\mu Y \nabla^\mu Y = 0$ implies

$$\partial_t Y = FN \partial_r Y. \quad (14)$$

Combining the Einstein equations (A1a), (A1b), and (A1d) with this condition implies

$$G_r{}^r + FNG_r{}^t = \frac{J^2}{4Y^4} = \frac{1}{l^2}, \quad (15)$$

which means that $Y(t,r)^2 = l|J|/2 \equiv a^2$, and the angular momentum is completely determined by the constant norm of the cyclic Killing field. Hence, the 2+1 geometry (3) has the form

$$g = g^{(2)} + a^2(d\phi + Wdt)^2. \quad (16)$$

Furthermore, in this case the only nontrivial Einstein equation reads

$$\frac{1}{N} \partial_t \left(\frac{1}{N} \partial_t F^{-1} \right) - \frac{1}{N} \partial_r \left(\frac{1}{N} \partial_r (N^2 F) \right) = -\frac{8}{l^2}, \quad (17)$$

which just states that the metric $g^{(2)}$ describes a two-dimensional spacetime of constant negative curvature, $R^{(2)} = -8/l^2$.

Choosing the simple gauge $F(t,r) = 1$, Eq. (17) can be integrated at once for the function N ,

$$N(t,r) = N_0(t) \cosh[2r/l + H(t)], \quad (18)$$

where $N_0(t)$ and $H(t)$ are integration functions. Hence, Eq. (A2) can also be integrated giving

$$W(t,r) = \frac{N_0(t)}{a} \sinh[2r/l + H(t)] + W_0(t). \quad (19)$$

Then, making the coordinate transformation

$$(t,r,\phi) \mapsto \left(\int N_0(t) dt, r, \phi + \int W_0(t) dt \right), \quad (20)$$

and the rescaling $(t,r,\phi) \mapsto (2t/l, 2r/l, 2a\phi/l)$ the metric becomes

$$g = \frac{l^2}{4} (-dt^2 + dr^2 + 2 \sinh(r+H) dt d\phi + d\phi^2). \quad (21)$$

This metric describes a class of space-times of constant negative curvature with a cyclic Killing field of constant norm. The stationary case, $H = \text{const}$, corresponds to the self-dual space-times constructed by Coussaert and Henneaux [16]. For nonconstant $H(t)$, the geometry can be seen to be

diffeomorphic to the CH solution, but the coordinate transformation that relates the two metrics is far from obvious (see Appendix D 3).

III. GLOBAL STRUCTURE

The scope of Birkhoff's theorem in 3+1 dimensions and above is to identify the local geometries of the space-times compatible with some starting symmetry. In this same spirit, the previous analysis yields the local geometric features of cyclically symmetric space-times in 2+1 dimensions. However, since in 2+1 dimensions all solutions of the matter-free Einstein equations are locally diffeomorphic, this is insufficient to determine the space-time geometry at large. In this section, the global structure of the physical space-times¹ consistent with the conditions of Birkhoff's theorem is analyzed. The problem is to identify all the *global* isometries compatible with the cyclic symmetry, that is, to find all globally defined Killing vector fields \mathbf{K} , which in the (t,r,ϕ) coordinate basis read

$$\mathbf{K} = K^t(t,r,\phi) \partial_t + K^r(t,r,\phi) \partial_r + K^\phi(t,r,\phi) \partial_\phi, \quad (22)$$

satisfying the Killing equation,

$$(g_{\nu\alpha} \nabla_\mu + g_{\mu\alpha} \nabla_\nu) K^\alpha = 0. \quad (23)$$

As shown above, the 2+1 geometries compatible with cyclic symmetry are either a portion of the 2+1 black hole [Eq. (7), for $\nu^2 \neq 0$], or of the CH self-dual space-time [Eq. (21), for $\nu^2 = 0$]. The question is whether those solutions can be *globally* identified with those space-times or they are just *locally* diffeomorphic but globally inequivalent. The point is that coordinate transformations such as Eqs. (8), (13), and (20) in general change the identification in the covering AdS space.

In order to address the question, the global isometries of the solutions will be identified, which amounts to finding the Killing fields of the geometry explicitly. Assuming the metric (7), the Killing equation (23) can be fully integrated giving two independent, globally² defined, mutually commuting Killing vector fields. These fields span the isometry algebra $so(2) \oplus \mathbb{R}$. In the case of the metric (21) the Killing equations (23) cannot be reduced to quadratures in general due to the presence of the arbitrary function $H(t)$. Although this obscures the problem, it is still possible to identify the symmetry generated by the Killing algebra as $so(2) \oplus so(2,1)$, or upon one further identification, as $so(2) \oplus so(2)$ (see Sec. IV).

¹We restrict our attention to space-times without naked singularities or closed timelike curves.

²The term "global" is redundant, but is used here to emphasize that these Killing fields are defined throughout space-time and are not just solutions of Eq. (23) in an open neighborhood.

A. 2+1 black hole: $SO(2)\times\mathbb{R}$ isometry

The isometries of the metric (7) are found by directly solving the Killing equations (23) (for a detailed discussion, see Appendix B). The general conclusion of this analysis is that, apart from the cyclic Killing vector,

$$\mathbf{m} = \partial_\phi, \quad (24)$$

the geometry possesses another, globally defined, independent commuting Killing field,

$$\mathbf{k} = \frac{1}{N(t)}(\partial_t - W_0(t)\partial_\phi). \quad (25)$$

In adapted coordinates, given by

$$\tilde{t}(t, \phi) = \int N(t)dt, \quad \tilde{\phi}(t, \phi) = \phi + \int W_0(t)dt, \quad (26)$$

the Killing fields (24) and (25) can be written in the form

$$\mathbf{m} = \partial_{\tilde{\phi}}, \quad \mathbf{k} = \partial_{\tilde{t}}. \quad (27)$$

The fields \mathbf{m} and \mathbf{k} obviously generate the $SO(2)\times\mathbb{R}$ isometry algebra as in the BTZ geometry. The coordinate transformation $(t, \phi) \mapsto (\tilde{t}(t, \phi), \tilde{\phi}(t, \phi))$ is well defined since $N(t)$ is assumed to be nonvanishing, and this diffeomorphism is precisely the change of coordinates which turns the metric (7) into the 2+1 black hole metric.

B. Coussaert-Henneaux self-dual space-time: $SO(2)\times SO(2,1)$ isometry

The metric (21) has an $SO(2)$ isometry generated by the Killing vector $\mathbf{m} = \partial_\phi$. Additionally, it admits a family of Killing fields of the form (see Appendix C for details)

$$\mathbf{K}_{F,T} \equiv (F + \tanh(r+H)\dot{T})\partial_t + T\partial_r + \frac{\dot{T}}{\cosh(r+H)}\partial_\phi, \quad (28)$$

which commute with \mathbf{m} . The functions $F(t)$ and $T(t)$ satisfy the equations

$$\dot{F} + \dot{H}\dot{T} = 0, \quad (29a)$$

$$\ddot{T} + T + \dot{H}F = 0, \quad (29b)$$

where the dot denotes time derivative.

As shown in Appendix D, the Killing vectors $\mathbf{K}_{F,T}$ generate the $so(2,1)$ algebra, and additionally the geometry described by Eq. (21) is globally identical to the self-dual CH space-time. The proof is as follows: Since the system (29) has a three-dimensional space of solutions, the Killing vectors (28) span a three-dimensional family of globally defined fields. Moreover, the norm of these vector fields and their scalar products are constants throughout space-time. Now, if $\{F_1, T_1\}$ and $\{F_2, T_2\}$ are two linearly independent solutions

of the system (29), and \mathbf{K}_{F_1, T_1} and \mathbf{K}_{F_2, T_2} are the corresponding Killing fields, the following commutator algebra is found,

$$[\mathbf{K}_{F_1, T_1}, \mathbf{K}_{F_2, T_2}] = \mathbf{K}_{F_3, T_3}, \quad (30a)$$

$$[\mathbf{K}_{F_3, T_3}, \mathbf{K}_{F_1, T_1}] = c_{12}\mathbf{K}_{F_1, T_1} - c_{11}\mathbf{K}_{F_2, T_2}, \quad (30b)$$

$$[\mathbf{K}_{F_3, T_3}, \mathbf{K}_{F_2, T_2}] = c_{22}\mathbf{K}_{F_1, T_1} - c_{12}\mathbf{K}_{F_2, T_2}, \quad (30c)$$

where the structure functions are given by $c_{11} = 4l^{-2}\mathbf{g}(\mathbf{K}_{F_1, T_1}, \mathbf{K}_{F_1, T_1})$, $c_{22} = 4l^{-2}\mathbf{g}(\mathbf{K}_{F_2, T_2}, \mathbf{K}_{F_2, T_2})$, and $c_{12} = 4l^{-2}\mathbf{g}(\mathbf{K}_{F_1, T_1}, \mathbf{K}_{F_2, T_2})$. Since these scalars are constants and in particular, independent of $H(t)$, the Lie algebra (30) is the same as for $H(t)=0$, which is the $so(2,1)$ isometry subalgebra of the CH space-time. This is a strong indication that the metric (21) must be diffeomorphic to the CH metric,

$$\mathbf{g} = \frac{l^2}{4}(-d\hat{t}^2 + d\hat{r}^2 + 2\sinh\hat{r}d\hat{t}d\hat{\phi} + d\hat{\phi}^2). \quad (31)$$

The explicit form of the coordinate transformation, $(t, r, \phi) \mapsto (\hat{t}, \hat{r}, \hat{\phi})$, relating these two metrics as well as the details of the above proof are exhibited in Appendix D.

IV. FURTHER IDENTIFICATIONS

The uniqueness of the spacetimes of constant curvature hinges on the possibility of generating new geometries by means of identifications. In principle, any identification that does not introduce closed causal curves could be acceptable and this restricts identifications to be along spacelike Killing directions only. This condition, for instance, prevents further identifications on the BTZ geometry to obtain new spacetimes, since in that case the isometries $so(2)\oplus\mathbb{R}$ only admit an identification along the time direction \mathbb{R} , producing closed timelike curves.

The CH self-dual space-times (31) are obtained by identification of AdS_3 along one of the spacelike self-dual generators of the isometry algebra of anti-de Sitter space, $so(2,2) = so(2,1)\oplus so(2,1)$ [16]. The resulting isometry algebra $so(2)\oplus so(2,1)$ [see Eqs. (D25) for definitions] can be further reduced by an identification along one spacelike Killing vector in the unbroken $so(2,1)$ subalgebra. The resulting space-time is also a self-dual geometry but with only two Killing vectors corresponding to the isometry algebra $so(2)\oplus so(2)$.³ Indeed, this can be accomplished performing the following coordinate transformation to the CH space-time $(\hat{t}, \hat{r}, \hat{\phi}) \mapsto (\tau, \phi_1, \phi_2)$, where

$$\tau(\hat{t}, \hat{r}, \hat{\phi}) = \arcsin(\sin\hat{t}\cosh\hat{r}), \quad (32a)$$

$$\phi_1(\hat{t}, \hat{r}, \hat{\phi}) = \text{arctanh}\left(\frac{\tanh\hat{r}}{\cos\hat{t}}\right), \quad (32b)$$

³We thank R. Troncoso for pointing out this possibility to us.

TABLE I. The 2+1 geometries of constant negative curvature consistent with cyclic symmetry.

Case	Geometry	Killing fields	Isometry
$\nu^2 > 0$	BTZ ($r < r_-, r > r_+$)	2	$SO(2) \times \mathbb{R}$
$\nu^2 < 0$	BTZ ($r_- < r < r_+$)	2	$SO(2) \times \mathbb{R}$
$\nu^2 = 0$	CH	4	$SO(2) \times SO(2,1)$
		2	$SO(2) \times SO(2)$

$$\phi_2(\hat{t}, \hat{r}, \hat{\phi}) = \hat{\phi} + \operatorname{arctanh}(\tan \hat{t} \sinh \hat{r}). \quad (32c)$$

In these new coordinates, the spacelike Killing fields η_2 and m [see Eqs. (D25)] read

$$\eta_2 = \partial_{\phi_1}, \quad m = \partial_{\phi_2}, \quad (33)$$

where ϕ_2 is a new coordinate along the $SO(2)$ isometry which is identified as $\phi_2 = \phi_2 + 4\pi a/l$. The metric (31) is transformed into⁴

$$g = \frac{l^2}{4} (-d\tau^2 + d\phi_1^2 - 2 \sin \tau d\phi_1 d\phi_2 + d\phi_2^2). \quad (34)$$

Under the additional identification $\phi_1 = \phi_1 + 2\pi$, along $\eta_2 = \partial_{\phi_1}$, the isometry subalgebra $so(2,1)$ has been reduced to $so(2)$. Since the other Killing fields η_0 and η_1 do not commute with η_2 , they are not Killing fields of the resulting quotient space. Thus, the metric (34) with $0 \leq \phi_1 \leq 2\pi$ and $0 \leq \phi_2 \leq 4\pi a/l$ corresponds to a different time-dependent self-dual space-time with isometry $SO(2) \times SO(2)$ and without closed causal curves. This space-time is geodesically incomplete and the singularity is not hidden by a horizon as it occurs at $r=0$ in the massless BTZ geometry.⁵

V. DISCUSSION AND CONCLUSIONS

The 2+1 geometries of constant negative curvature consistent with cyclic symmetry are given in Table I.

This table exhausts all possible 2+1 geometries and no further identifications can be made on them, lest naked singularities or closed causal curves are introduced.

Unlike in higher dimensions, where Birkhoff's theorem assumes spherical symmetry, the solutions in 2+1 dimension are not restricted to have zero angular momentum, as is exhibited by the 2+1 black hole ($\nu^2 \neq 0$). This explains why previous results on Birkhoff's theorem does not apply to this case.

The self-dual space-times of Coussaert and Henneaux ($\nu^2 = 0$) arise from the accident in 2+1 dimensions that al-

lows to factor the AdS space isometry group $SO(2,2)$ as $SO(2,1) \times SO(2,1)$. This accident cannot be generalized for arbitrary dimensions. Furthermore, these self-dual CH solutions have a completely different topology from the 2+1 black hole; the product of two constant curvature spaces, $\text{AdS}_2 \times S^1$. In this sense, the CH solutions bear a resemblance with the Nariai space [17], which exists in four dimensions with positive cosmological constant. It would be interesting to investigate to what extent the Nariai solution is compatible with Birkhoff's theorem in presence of a positive cosmological constant.

The analysis of the $\nu^2 = 0$ case also illustrates some features of the problem that may be of use in other cases. The fact that the isometry algebra can be determined without knowing the explicit form of the solutions of the Killing equations is generic. This is a consequence of two facts: (i) The commutator of two Killing vectors is necessarily a Killing vector and (ii) only a linear combination of Killing vectors with constant coefficients is also a Killing vector. As a consequence, the structure constants of the isometry algebra are necessarily integration constants of the Killing equations. This explains the "remarkable" feature that the right hand side of Eqs. (30) contains only integration constants of the system (29), as shown in Appendix D.

The other interesting feature is related with the old problem of determining if two apparently different space-times having the same invariant quantities, including their isometry algebras, are the same space-time in different coordinates or not. For example, metrics (21) and (31) both represent spaces of constant negative curvature and isometry group $SO(2) \times SO(2,1)$. The approach we follow here rests on the fact that the coordinate transformation relating the two metrics, if it exists, must also relate the isometry algebras. Hence, the identification of the two families of Killing vectors leads to a class of transformations including the relevant one. The above process involves the integration of a linear PDE system. If the number of Killing vectors is sufficient, all the arbitrary functions that arise in the integration process are determined and the coordinate transformation is uniquely fixed, as in the present case (see Appendix D 3). On the contrary, the nonexistence of solutions of the PDE system would imply that the space-times under study must be different.

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⁴This metric can also be obtained from the self-dual CH spacetime (31) through the double Wick rotation $\hat{t} \rightarrow i\phi_1$, $\hat{r} \rightarrow i\tau$.

⁵We thank S. Ross for pointing out this to us.

**APPENDIX A: EINSTEIN EQUATIONS
FOR CYCLIC SYMMETRY**

For a metric of the form (3), the vacuum Einstein equations for 2+1 gravity (2) take the following form:

$$2Y(G_t{}^t - WG_\phi{}^\phi) = 2FY'' + \frac{Y^3(W')^2}{2N^2} + \frac{\dot{Y}\dot{F}}{N^2F^2} + Y'F' = \frac{2Y}{l^2}, \quad (\text{A1a})$$

$$2YG_r{}^r = \frac{(FY')}{N^2F^2} + \left(\frac{\dot{Y}}{N^2F}\right)' = 0, \quad (\text{A1b})$$

$$2YNG_\phi{}^\phi = \left(\frac{Y^3W'}{N}\right)' = 0, \quad (\text{A1c})$$

$$YNF(G_t{}^t - WG_\phi{}^\phi - G_r{}^r) = \left(\frac{\dot{Y}}{N}\right)' + N^2F^2\left(\frac{Y'}{N}\right)' = 0, \quad (\text{A1d})$$

$$-2YNG_\phi{}^r = \left(\frac{Y^3W'}{N}\right)' = 0, \quad (\text{A1e})$$

$$2N(G_\phi{}^\phi + WG_\phi{}^\phi) = -\left(\frac{(F^{-1})}{N}\right)' + \left(\frac{(N^2F)'}{N}\right)' - \frac{3Y^2(W')^2}{2N} = \frac{2N}{l^2}, \quad (\text{A1f})$$

where (\dots) and $(\dots)'$ denote time and radial derivatives, respectively. From Eqs. (A1c) and (A1e) it is clear that the quantity

$$J = \frac{Y^3}{N}W', \quad (\text{A2})$$

is an integration constant (angular momentum). The remaining equations determine the form of $W(t,r)$, $F(t,r)$, and $N(t,r)$, while $Y(t,r)$ is fixed by appropriate coordinate choices.

APPENDIX B: KILLING FIELDS FOR THE 2+1 BLACK HOLE
1. Generic case $r_+ \neq r_- \neq 0$

The isometries of metric (7) are found by directly solving the Killing equations (23). Redefining the mass and angular momentum in terms of the (positive) zeros r_\pm of the function (6), $M = (r_+^2 + r_-^2)/l^2$ and $J = 2r_+r_-/l$, the Killing equations for the radial component of the Killing vector becomes

$$\frac{\partial_r K^r}{K^r} = \frac{1}{2} \left(\frac{1}{r+r_+} + \frac{1}{r-r_+} + \frac{1}{r+r_-} + \frac{1}{r-r_-} \right) - \frac{1}{r}, \quad (\text{B1})$$

which can be integrated as

$$K^r(t,r,\phi) = \frac{[(r^2 - r_+^2)(r^2 - r_-^2)]^{1/2}}{r} F^r(t,\phi), \quad (\text{B2})$$

where $F^r = F^r(t,\phi)$ is an integration function. Similarly, the Killing equations for K^t and K^ϕ imply

$$\partial_r K^a = \frac{l^2 r [A^a(t,\phi)r^2 + B^a(t,\phi)]}{[(r^2 - r_+^2)(r^2 - r_-^2)]^{3/2}}, \quad a = t, \phi \quad (\text{B3})$$

where the functions A^a and B^a are defined as

$$A^t(t,\phi) \equiv \frac{l^2(\partial_t F^r - W_0 \partial_\phi F^r)}{N^2}, \quad (\text{B4a})$$

$$B^t(t,\phi) \equiv \frac{l r_+ r_- \partial_\phi F^r}{N}, \quad (\text{B4b})$$

$$A^\phi(t,\phi) \equiv \frac{l^2 W_0 (W_0 \partial_\phi F^r - \partial_t F^r)}{N^2} - \partial_\phi F^r, \quad (\text{B4c})$$

$$B^\phi(t,\phi) \equiv \frac{l r_+ r_- (\partial_t F^r - 2W_0 \partial_\phi F^r)}{N} + (r_+^2 + r_-^2) \partial_\phi F^r. \quad (\text{B4d})$$

From Eq. (B3) the r -dependence of K^t and K^ϕ can be explicitly found,

$$K^a(t,r,\phi) = F^a(t,\phi) - \{[(r_+^2 + r_-^2)A^a + 2B^a]r^2 - 2r_+^2 r_-^2 A^a - (r_+^2 + r_-^2)B^a\} \times \frac{l^2}{(r_+^2 - r_-^2)^2} \frac{1}{[(r^2 - r_+^2)(r^2 - r_-^2)]^{1/2}}, \quad a = t, \phi \quad (\text{B5})$$

where $F^t = F^t(t,\phi)$ and $F^\phi = F^\phi(t,\phi)$ are integration functions. The explicit dependence on r in the remaining Killing equations allows us to finally conclude that

$$F^t(t,\phi) = \frac{C_1}{N(t)}, \quad F^\phi(t,\phi) = -\frac{C_1 W_0(t)}{N(t)} + C_2, \quad (\text{B6})$$

where C_1 and C_2 are integration constant. Finally, the general form of expression of F^r is

$$F^r(u, v) = k_1 \exp\left(\frac{r_+ - r_-}{l} u\right) + k_2 \exp\left(\frac{r_+ + r_-}{l} v\right) + k_3 \exp\left(-\frac{r_+ - r_-}{l} u\right) + k_4 \exp\left(-\frac{r_+ + r_-}{l} v\right), \quad (\text{B7})$$

where k_1 , k_2 , k_3 , and k_4 are integration constants and the coordinates u and v are given by

$$u = \int [l^{-1}N(t) + W_0(t)] dt + \phi, \\ v = \int [l^{-1}N(t) - W_0(t)] dt - \phi. \quad (\text{B8})$$

The identification $\phi = \phi + 2\pi$ following from cyclic symmetry is respected by the Killing field only if $k_1 = k_2 = k_3 = k_4 = 0$. This means that the metric does not admit Killing vector fields with radial components, and the general form of \mathbf{K} for spacetimes (7) is

$$\mathbf{K} = \frac{C_1}{N(t)} \partial_t + \left(-\frac{C_1 W_0(t)}{N(t)} + C_2 \right) \partial_\phi. \quad (\text{B9})$$

2. Extreme case $r_+ = r_-$

The integration that yields Eq. (B5) cannot be done for the extreme case $r_+ = r_- \equiv r_e$, or $|J| = Ml$, and the treatment for this case is different from the previous one. However, a similar analysis leads to the following expressions:

$$F^r = k_1 u + k_2 + k_3 \exp(2vr_e/l) + k_4 \exp(-2vr_e/l), \quad (\text{B10})$$

$$F^t N = \frac{lk_1}{2} u^2 + lk_2 u + \frac{l^2 k_3}{2r_e} \exp(2vr_e/l) - \frac{l^2 k_4}{2r_e} \exp(-2vr_e/l) + C_1, \quad (\text{B11})$$

$$F^\phi + F^t W_0 = \frac{k_1}{2} u^2 + k_2 u - \frac{lk_3}{2r_e} \exp(2vr_e/l) + \frac{lk_4}{2r_e} \exp(-2vr_e/l) + C_2, \quad (\text{B12})$$

where k_1 , k_2 , k_3 , k_4 , C_1 , and C_2 are integration constants. As in the generic case, periodicity in ϕ implies $K^r = 0$, and the functions F^t and F^ϕ are given by Eq. (B6) as in the generic case. This allows to write the same general form (B9) for the Killing fields in the extreme case.

3. Zero angular momentum case $r_- = 0$

In this case, the integration yields

$$F^r(t, \phi) = F_1(t) \exp\left(\frac{r_+}{l} \phi\right) + F_2(t) \exp\left(-\frac{r_+}{l} \phi\right), \quad (\text{B13})$$

where F_1 and F_2 are integration constants. Again, the global identification $\phi = \phi + 2\pi$ required by cyclic symmetry implies $K^r = 0$.

4. Zero mass case $r_+ = r_- = 0$

In this case, direct integration yields

$$F^r = k_1 \tilde{t} + k_2 \tilde{\phi} + k_3, \quad (\text{B14})$$

$$F^t N = -\frac{k_1}{2} (\tilde{t}^2 + l^2 \tilde{\phi}^2) - (k_2 \tilde{t} - l^2 k_4) \tilde{\phi} - k_3 \tilde{t} + C_1, \quad (\text{B15})$$

$$F^\phi + F^t W_0 = -\frac{k_2}{2l^2} (\tilde{t}^2 + l^2 \tilde{\phi}^2) - (k_1 \tilde{t} + k_3) \tilde{\phi} + k_4 \tilde{t} + C_2, \quad (\text{B16})$$

where k_1 , k_2 , k_3 , k_4 , C_1 , and C_2 are integration constants, and the coordinates \tilde{t} and $\tilde{\phi}$ are defined in Eq. (26). As in the previous cases, cyclic symmetry implies $k_1 = k_2 = k_3 = k_4 = 0$. Thus, in all BTZ geometries $K^r = 0$ and the general form of the Killing fields is given by Eq. (B9).

APPENDIX C: KILLING FIELDS FOR THE SELF-DUAL SPACE-TIMES

The Killing equations (23) for the metric (21) read

$$\partial_r K^r = 0, \quad (\text{C1a})$$

$$\partial_r K^t - \frac{\partial_t K^r - \sinh u \partial_\phi K^r}{\cosh^2 u} = 0, \quad (\text{C1b})$$

$$\partial_r K^\phi + \frac{\partial_\phi K^r + \sinh u \partial_t K^r}{\cosh^2 u} = 0, \quad (\text{C1c})$$

$$\partial_\phi K^\phi + \sinh u \partial_\phi K^t = 0, \quad (\text{C1d})$$

$$\partial_t K^t - \sinh u \partial_t K^\phi = 0, \quad (\text{C1e})$$

$$\partial_t K^\phi - \partial_\phi K^t + \sinh u (\partial_t K^t + \partial_\phi K^\phi) + \cosh u (K^r + \dot{H} K^t) = 0, \quad (\text{C1f})$$

where $u = r + H$. Equation (C1a) implies $K^r(t, r, \phi) = K^r(t, \phi)$, and integration of Eqs. (C1b) and (C1c) directly yields

$$K^t(t, r, \phi) = F^t(t, \phi) + \frac{\partial_\phi K^r + \sinh u \partial_t K^r}{\cosh u}, \quad (\text{C2})$$

$$K^\phi(t, r, \phi) = F^\phi(t, \phi) + \frac{\partial_t K^r - \sinh u \partial_\phi K^r}{\cosh u}, \quad (\text{C3})$$

where F^t and F^ϕ are integration functions. Substituting these expressions, Eqs. (C1d)–(C1f) take the form

$$\alpha(t, \phi) + \beta(t, \phi) \sinh u + \gamma(t, \phi) \cosh u = 0. \quad (\text{C4})$$

Since these equations must be satisfied for any r , the (t, ϕ) -dependent coefficients must vanish independently, which implies the following system of equations:

$$\partial_\phi F^\phi = \partial_t F^\phi = 0, \quad (\text{C5a})$$

$$\partial_\phi F^t = 0, \quad (\text{C5b})$$

$$\partial_t^2 K^r = 0, \quad (\text{C5c})$$

$$\partial_t F^t + \dot{H} \partial_t K^r = 0, \quad (\text{C5d})$$

$$\partial_t^2 K^r - \partial_\phi^2 K^r + K^r + \dot{H} F^t = 0. \quad (\text{C5e})$$

From these equations it follows that $F^\phi(t, \phi) = C_4$, $F^t(t, \phi) = F(t)$, $K^r(t, \phi) = T(t) + \Phi(\phi)$, and

$$\ddot{T} + T + \dot{H} F = \frac{d^2 \Phi}{d\phi^2} - \Phi. \quad (\text{C6})$$

This equation fixes the angular dependence as

$$\Phi(\phi) = k_1 \exp(\phi) + k_2 \exp(-\phi) + k_3, \quad (\text{C7})$$

which is consistent with the identification $\phi = \phi + 4\pi a/l$ only if $k_1 = k_2 = 0$, and Φ is an irrelevant constant. Combining this with Eqs. (C5d) and (C6) yields

$$\dot{F} + \dot{H} \dot{T} = 0, \quad (\text{C8a})$$

$$\ddot{T} + T + \dot{H} F = 0. \quad (\text{C8b})$$

Thus, the general form of a Killing field for the metric (21) is

$$\mathbf{K} = (F + \tanh(r+H)\dot{T})\partial_t + T\partial_r + \left(C_4 + \frac{\dot{T}}{\cosh(r+H)} \right) \partial_\phi, \quad (\text{C9})$$

where F and T are solutions of Eqs. (C8a), (C8b) for a given function $H(t)$, as stated in Eq. (28).

APPENDIX D: THE $so(2,1)$ ISOMETRY SUBALGEBRA GENERATED BY $\mathbf{K}_{F,T}$

The general solution of the system (29) can be formally written as

$$\begin{pmatrix} F(t) \\ T(t) \\ \dot{T}(t) \end{pmatrix} = \mathcal{P} \left[\exp \left(\int_0^t \mathbf{M}(t') dt' \right) \right] \begin{pmatrix} F_0 \\ T_0 \\ \dot{T}_0 \end{pmatrix}, \quad (\text{D1})$$

where \mathcal{P} stands for the path-ordered product, and

$$\mathbf{M}(t) = \begin{pmatrix} 0 & 0 & -\dot{H}(t) \\ 0 & 0 & -1 \\ -\dot{H}(t) & 1 & 0 \end{pmatrix}. \quad (\text{D2})$$

The operator $\mathbf{M}(t)$ is a linear combination of $SO(2,1)$ generators, $\mathbf{M}(t) = \sigma_0 - \dot{H} \sigma_1$. The Killing fields (28) can be expressed as

$$\mathbf{K}_{F,T} = F\mathbf{e}_0 + T\mathbf{e}_1 + \dot{T}\mathbf{e}_2, \quad (\text{D3})$$

where the components are given by Eq. (D1) and

$$\mathbf{e}_0 = \partial_t, \quad \mathbf{e}_1 = \partial_r, \quad \mathbf{e}_2 = \tanh u \partial_t + \frac{\partial_\phi}{\cosh u}, \quad (\text{D4})$$

form an orthonormal frame for the spacetime (21), i.e., $\mathbf{g}(\mathbf{e}_a, \mathbf{e}_b) = (l^2/4) \eta_{ab}$, $0 \leq a \leq 2$. Hence, the formal solution (D1) can be interpreted as the evolution of the vector $\mathbf{K}_{F,T}$ in the orthonormal basis (D4) under a time-dependent Lorentz rotation acting on the vector of initial values $\mathbf{K}_0 \equiv F_0 \mathbf{e}_0 + T_0 \mathbf{e}_1 + \dot{T}_0 \mathbf{e}_2$. The norm of the Killing vectors is

$$\mathbf{g}(\mathbf{K}_{F,T}, \mathbf{K}_{F,T}) = \frac{l^2}{4} (-F^2 + T^2 + \dot{T}^2). \quad (\text{D5})$$

This expression is independent of the function $H(t)$, which reflects the fact that H can be gauged away by a change of coordinates, as will be shown shortly. Since the basis (D4) is orthonormal, the above norm is preserved under time-dependent Lorentz rotations. Hence, the right hand side of Eq. (D5) is constant in time, as can be directly checked from Eqs. (29). Thus, the norm of the Killing vector, $\mathbf{g}(\mathbf{K}_{F,T}, \mathbf{K}_{F,T})$, is equal to the norm of the corresponding vector of initial values, $\mathbf{g}(\mathbf{K}_0, \mathbf{K}_0)$. This also shows explicitly that the space of Killing vectors in the family (D3) is three-dimensional and in one to one correspondence with the vectors of initial values \mathbf{K}_0 . Consequently, given two Killing vectors, \mathbf{K}_{F_1, T_1} and \mathbf{K}_{F_2, T_2} , their scalar product,

$$\mathbf{g}(\mathbf{K}_{F_1, T_1}, \mathbf{K}_{F_2, T_2}) = \frac{l^2}{4} (-F_1 F_2 + T_1 T_2 + \dot{T}_1 \dot{T}_2), \quad (\text{D6})$$

is also time independent, as can also be directly verified from Eq. (29). Thus, given a set of Killing fields, the norm of each vector and their scalar products are fixed everywhere by their values at one point. In particular, the Killing fields are linearly independent everywhere if and only if the corresponding initial value vectors are linearly independent as well.

Although the Killing fields $\mathbf{K}_{F,T}$ cannot be written in closed form for a generic $H(t)$, the isometry algebra they generate can be identified from the properties of Eqs. (29). Let $\{F_1, T_1\}$ and $\{F_2, T_2\}$ be two linearly independent solutions of the system (29). Then, the corresponding Killing fields \mathbf{K}_{F_1, T_1} and \mathbf{K}_{F_2, T_2} are also linearly independent, and their norms and scalar product are the constants

$$\mathbf{g}(\mathbf{K}_{F_1, T_1}, \mathbf{K}_{F_1, T_1}) \equiv \frac{l^2}{4} c_{11}, \quad (\text{D7a})$$

$$\mathbf{g}(\mathbf{K}_{F_2, T_2}, \mathbf{K}_{F_2, T_2}) \equiv \frac{l^2}{4} c_{22}, \quad (\text{D7b})$$

$$\mathbf{g}(\mathbf{K}_{F_1, T_1}, \mathbf{K}_{F_2, T_2}) \equiv \frac{l^2}{4} c_{12}. \quad (\text{D7c})$$

Since Killing vectors form a Lie algebra under commutation, their commutator is also a solution of Eq. (23),

$$[\mathbf{K}_{F_1, T_1}, \mathbf{K}_{F_2, T_2}] = \mathbf{K}_{F_3, T_3}, \quad (\text{D8})$$

where the functions $\{F_3, T_3\}$ are also solutions of Eq. (29), given by

$$F_3 = T_1 \dot{T}_2 - T_2 \dot{T}_1, \quad (\text{D9a})$$

$$T_3 = F_1 \dot{T}_2 - F_2 \dot{T}_1, \quad (\text{D9b})$$

$$\dot{T}_3 = F_2 T_1 - F_1 T_2. \quad (\text{D9c})$$

The norm of the new Killing vector is

$$\mathbf{g}(\mathbf{K}_{F_3, T_3}, \mathbf{K}_{F_3, T_3}) \equiv \frac{l^2}{4} c_{33}, \quad (\text{D10})$$

which is also a constant of motion related to the other constants by

$$c_{33} = c_{12}^2 - c_{11} c_{22}. \quad (\text{D11})$$

The Killing fields \mathbf{K}_{F_1, T_1} , \mathbf{K}_{F_2, T_2} , and \mathbf{K}_{F_3, T_3} are linearly independent if and only if the determinant of their components $[\mathbf{K}_{F_i, T_i}^a]$, $0 \leq a \leq 2$, $1 \leq i \leq 3$,

$$\det[\mathbf{K}_{F_i, T_i}^a] = -c_{33}, \quad (\text{D12})$$

is nonvanishing. Starting with two linearly independent Killing fields \mathbf{K}_{F_1, T_1} and \mathbf{K}_{F_2, T_2} , three situations can be distinguished according to whether the plane spanned by their tangents is timelike, spacelike, or null:

(I) A timelike plane is spanned by one timelike and the other spacelike, or by two null vectors. In both cases (D11) implies $c_{33} > 0$. Hence, one timelike and two spacelike vectors, or two null and one spacelike vector.

(II) A spacelike plane requires both fields to be spacelike. Then, Schwarz's inequality implies $c_{33} < 0$. That is, one timelike and two spacelike vectors.

(III) A null plane is spanned by a null and a spacelike vector. Since without loss of generality they can be chosen

orthogonal then $c_{33} = 0$ and \mathbf{K}_{F_3, T_3} cannot be linearly independent from the other two.

1. Simple case: $c_{33} \neq 0$

If $c_{33} \neq 0$, using the system (29) it can be proved that the vectors \mathbf{K}_{F_1, T_1} , \mathbf{K}_{F_2, T_2} , and \mathbf{K}_{F_3, T_3} satisfy the following commutator algebra⁶

$$[\mathbf{K}_{F_3, T_3}, \mathbf{K}_{F_1, T_1}] = c_{12} \mathbf{K}_{F_1, T_1} - c_{11} \mathbf{K}_{F_2, T_2}, \quad (\text{D13a})$$

$$[\mathbf{K}_{F_3, T_3}, \mathbf{K}_{F_2, T_2}] = c_{22} \mathbf{K}_{F_1, T_1} - c_{12} \mathbf{K}_{F_2, T_2}. \quad (\text{D13b})$$

This applies to the two possibilities included in cases (I) and (II) above: two spacelike and one timelike vector, or two null and one spacelike vector.

Since the structure constants in the right hand side are seen from Eq. (D7) to be independent of $H(t)$, this algebra must be the same as that for $H=0$, which is the $so(2,1)$ isometry subalgebra of the self-dual CH spacetime. This can be made more explicit if the Killing fields are properly orthonormalized as

$$\boldsymbol{\eta}_0 = \frac{\mathbf{K}_{F_1, T_1}}{\sqrt{-c_{11}}}, \quad (\text{D14a})$$

$$\boldsymbol{\eta}_1 = \frac{c_{12} \mathbf{K}_{F_1, T_1} - c_{11} \mathbf{K}_{F_2, T_2}}{\sqrt{-c_{11}} \sqrt{c_{33}}}, \quad (\text{D14b})$$

$$\boldsymbol{\eta}_2 = \frac{\mathbf{K}_{F_3, T_3}}{\sqrt{c_{33}}}. \quad (\text{D14c})$$

(Here \mathbf{K}_{F_1, T_1} has been assumed to be timelike.) Then, from Eqs. (D8) and (D13) the commutation relations of the self-dual generators of $so(2,1)$ are recovered,

$$\begin{aligned} [\boldsymbol{\eta}_0, \boldsymbol{\eta}_1] &= \boldsymbol{\eta}_2, \\ [\boldsymbol{\eta}_1, \boldsymbol{\eta}_2] &= -\boldsymbol{\eta}_0, \\ [\boldsymbol{\eta}_2, \boldsymbol{\eta}_0] &= \boldsymbol{\eta}_1. \end{aligned} \quad (\text{D15})$$

Alternatively, if $c_{11} = 0 = c_{22}$ (then necessarily $c_{12} \neq 0$) and the algebra (D13) reduces to

$$[\mathbf{K}_{F_3, T_3}, \mathbf{K}_{F_1, T_1}] = c_{12} \mathbf{K}_{F_1, T_1}, \quad (\text{D16a})$$

$$[\mathbf{K}_{F_3, T_3}, \mathbf{K}_{F_2, T_2}] = -c_{12} \mathbf{K}_{F_2, T_2}, \quad (\text{D16b})$$

which is the same $so(2,1)$ algebra (D15) in a different basis. The corresponding orthonormalization is

⁶We thank M. Bustamante for helping us to elucidate this point.

$$\boldsymbol{\eta}_0 = \frac{1}{\sqrt{-2c_{12}}}(\mathbf{K}_{F_1, T_1} + \mathbf{K}_{F_2, T_2}), \quad (\text{D17a})$$

$$\boldsymbol{\eta}_1 = -\frac{\mathbf{K}_{F_3, T_3}}{c_{12}}, \quad (\text{D17b})$$

$$\boldsymbol{\eta}_2 = \frac{1}{\sqrt{-2c_{12}}}(\mathbf{K}_{F_1, T_1} - \mathbf{K}_{F_2, T_2}), \quad (\text{D17c})$$

where it has been assumed that the null fields are both future directed or past directed ($c_{12} < 0$). If these field were to point in opposite directions ($c_{12} > 0$), the sign inside the square root must be reversed, and exchange the definitions of $\boldsymbol{\eta}_0$ and $\boldsymbol{\eta}_2$.

2. Degenerate case: $c_{33}=0$

If the fields \mathbf{K}_{F_1, T_1} and \mathbf{K}_{F_2, T_2} are null ($c_{11}=0$) and spacelike ($c_{22}>0$) respectively, they span a null plane [case (III) above]. In this case $c_{33}=0$, and therefore \mathbf{K}_{F_1, T_1} , \mathbf{K}_{F_2, T_2} and $[\mathbf{K}_{F_1, T_1}, \mathbf{K}_{F_2, T_2}]$ are not linearly independent. However, it is possible to find another independent null Killing vector which, together with \mathbf{K}_{F_1, T_1} and \mathbf{K}_{F_2, T_2} , generate the same $so(2,1)$ algebra.

Without loss of generality they \mathbf{K}_{F_1, T_1} and \mathbf{K}_{F_2, T_2} can be taken to be orthogonal ($c_{12}=0$), and their commutator is not linearly independent, but is given by

$$[\mathbf{K}_{F_1, T_1}, \mathbf{K}_{F_2, T_2}] = \sqrt{c_{22}}\mathbf{K}_{F_1, T_1}. \quad (\text{D18})$$

Let \mathbf{K}_3 be another linearly independent null field not contained in the plane generated by \mathbf{K}_{F_1, T_1} and \mathbf{K}_{F_2, T_2} and orthogonal to \mathbf{K}_{F_2, T_2} . It can be then shown that \mathbf{K}_3 is also a Killing field. The scalar products between this null field and the two Killing fields are

$$\mathbf{K}_{F_1, T_1} \cdot \mathbf{K}_3 \equiv \frac{l^2}{4}k_{13}, \quad (\text{D19})$$

$$\mathbf{K}_{F_2, T_2} \cdot \mathbf{K}_3 \equiv \frac{l^2}{4}k_{23} = 0. \quad (\text{D20})$$

Since the fields \mathbf{K}_{F_1, T_1} , \mathbf{K}_{F_2, T_2} , and \mathbf{K}_3 are a local basis in the tangent space the metric can be written in this basis as

$$\mathbf{g} = \frac{4}{l^2} \left(2 \frac{\mathbf{K}_{F_1, T_1} \otimes_s \mathbf{K}_3}{k_{13}} + \frac{\mathbf{K}_{F_2, T_2} \otimes_s \mathbf{K}_{F_2, T_2}}{c_{22}} \right), \quad (\text{D21})$$

where \otimes_s stands for the symmetrized tensor product. Since \mathbf{K}_{F_1, T_1} and \mathbf{K}_{F_2, T_2} are Killing fields they must obey

$$\begin{aligned} 0 &= \frac{l^2 k_{13}}{8} \boldsymbol{\xi}_{\mathbf{K}_{F_1, T_1}} \mathbf{g} \\ &= \mathbf{K}_{F_1, T_1} \otimes_s \left([\mathbf{K}_{F_1, T_1}, \mathbf{K}_3] + \frac{k_{13}}{\sqrt{c_{22}}} \mathbf{K}_{F_2, T_2} \right), \end{aligned} \quad (\text{D22a})$$

$$\begin{aligned} 0 &= \frac{l^2 k_{13}}{8} \boldsymbol{\xi}_{\mathbf{K}_{F_2, T_2}} \mathbf{g} \\ &= \mathbf{K}_{F_1, T_1} \otimes_s ([\mathbf{K}_{F_2, T_2}, \mathbf{K}_3] - \sqrt{c_{22}}\mathbf{K}_3), \end{aligned} \quad (\text{D22b})$$

where the commutation relation (D18) has been used. The resulting conditions are both of the form $\mathbf{K}_{F_1, T_1} \otimes_s \mathbf{X} = 0$, and using the orthogonality properties of the basis they are equivalent to have $\mathbf{X} = 0$. Hence, the fact that \mathbf{K}_{F_1, T_1} and \mathbf{K}_{F_2, T_2} are Killing fields together with their commutation relation (D18) imply additional commutation relations. In order to show that \mathbf{K}_3 is also a Killing field we calculate the Lie derivative of the metric along this field

$$\boldsymbol{\xi}_{\mathbf{K}_3} \mathbf{g} = \frac{8}{l^2} \left(\frac{[\mathbf{K}_3, \mathbf{K}_{F_1, T_1}] \otimes_s \mathbf{K}_3}{k_{13}} + \frac{[\mathbf{K}_3, \mathbf{K}_{F_2, T_2}] \otimes_s \mathbf{K}_{F_2, T_2}}{c_{22}} \right) = 0, \quad (\text{D23})$$

where the last equality follows from the commutation relations implied by Eq. (D22). The commutation relations (D15) are recovered changing the basis to

$$\boldsymbol{\eta}_0 = \frac{1}{\sqrt{-2k_{13}}}(\mathbf{K}_{F_1, T_1} + \mathbf{K}_3), \quad (\text{D24a})$$

$$\boldsymbol{\eta}_1 = \frac{\mathbf{K}_{F_2, T_2}}{\sqrt{c_{22}}}, \quad (\text{D24b})$$

$$\boldsymbol{\eta}_2 = \frac{1}{\sqrt{-2k_{13}}}(\mathbf{K}_{F_1, T_1} - \mathbf{K}_3). \quad (\text{D24c})$$

3. Coordinate transformation

Since the solution (21) and the self-dual CH space-times possess the same isometries $so(2) \oplus so(2,1)$, this is a strong indication that these metrics should only differ in the choice of coordinates. For the self-dual CH spacetime (31) its isometry is spanned by the Killing fields $\mathbf{m} = \boldsymbol{\partial}_{\hat{\phi}}$ and

$$\boldsymbol{\eta}_0 = \boldsymbol{\partial}_{\hat{t}}, \quad (\text{D25a})$$

$$\boldsymbol{\eta}_1 = \tanh \hat{r} \cos \hat{t} \boldsymbol{\partial}_{\hat{t}} + \sin \hat{t} \boldsymbol{\partial}_{\hat{r}} + \frac{\cos \hat{t}}{\cosh \hat{r}} \boldsymbol{\partial}_{\hat{\phi}}, \quad (\text{D25b})$$

$$\boldsymbol{\eta}_2 = -\tanh \hat{r} \sin \hat{t} \boldsymbol{\partial}_{\hat{t}} + \cos \hat{t} \boldsymbol{\partial}_{\hat{r}} - \frac{\sin \hat{t}}{\cosh \hat{r}} \boldsymbol{\partial}_{\hat{\phi}}. \quad (\text{D25c})$$

In particular, the coordinate transformation that relates the metrics should be the same that relates the Killing vectors fields characterizing the same global isometries in the different coordinate bases. Using the above hint, it can be seen that the coordinate transformation $(t, r, \phi) \mapsto (\hat{t}, \hat{r}, \hat{\phi})$,

$$\hat{t}(t, r, \phi) = \arctan\left(\frac{\sqrt{-c}(F \cosh u + \dot{T} \sinh u)}{T(F \sinh u + \dot{T} \cosh u)}\right) - \int \frac{\sqrt{-c}F}{c - T^2} dt,$$

$$\hat{r}(t, r, \phi) = \operatorname{arcsinh}\left(\frac{F \sinh u + \dot{T} \cosh u}{\sqrt{-c}}\right),$$

$$\hat{\phi}(t, r, \phi) = \phi - \operatorname{arctanh}\left(\frac{T}{F \cosh u + \dot{T} \sinh u}\right),$$

where $u = r + H$ and the pair $\{F, T\}$ is any solution to equations (29) with $c \equiv -F^2 + T^2 + \dot{T}^2 < 0$, maps Eq. (21) into the self-dual CH metric (31).

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