Conformal quantum gravity with the Gauss-Bonnet term

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Conformal gravity is one of the most important models of quantum gravity with higher derivatives. We investigate the role of the Gauss-Bonnet term in this theory. The coincidence limit of the second coefficient of the Schwinger-DeWitt expansion is evaluated in an arbitrary dimension n. In the limit $n=4$ the Gauss-Bonnet term is topological and its contribution cancels. This cancellation provides an efficient test for the correctness of calculation and, simultaneously, clarifies the long-standing general problem concerning the role of the topological term in quantum gravity. For $n\neq 4$ the Gauss-Bonnet term becomes dynamical in the classical theory and relevant at the quantum level. In particular, the renormalization group equations in dimension $n=4-\epsilon$ manifest new fixed points due to quantum effects of this term.

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I. INTRODUCTION

At both classical and quantum levels, local conformal symmetry plays a special role in theories of gravity and their applications [1–3]. One of the most interesting issues is the violation of this symmetry at the quantum level. For the quantum theory of matter fields in curved space-time the violation of conformal symmetry is related to the well-known trace anomaly (see, e.g., Refs. [1,3] for the review). The important feature of the conformal anomaly is its universality for the matter (scalar, spinor, and vector) fields that contribute with the same sign to two of three terms of the anomaly. The opposite sign of the contributions takes place for the unphysical higher-derivative scalars and fermions [4,5]. In principle, one can choose the number of these higher derivative fields in such a way that they cancel the contributions of the matter fields. In this case the conformal symmetry holds at the one-loop level. The cancellation of the anomaly cannot be achieved in the known versions of conformal supergravity [6], and therefore the relation between the cancellation of conformal anomaly and what is supposed to be the fundamental theory [e.g., supergravity, which may be a low-energy limit of the (super)string/M theory] remains unclear within the semiclassical approach.

The violation of the conformal symmetry in quantum gravity is much less studied. One of the simpler theories of gravity that possesses local conformal symmetry is based on the Weyl action $\int d^4x\sqrt{-g}C^2$, where

$$C^2=C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}=R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-2R_{\alpha\beta}R^{\alpha\beta}+\frac{1}{3}R^2 \quad (1)$$

is the square of the Weyl tensor in $n=4$ dimensions. In order to provide renormalizability, one has to include topological and surface terms. In this way we arrive at the action

$$S_W=-\int d^4x\sqrt{-g}\left\{\frac{1}{2\lambda}C^2+\eta E+\tau\Box R\right\}. \quad (2)$$

Here

$$E=R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-4R_{\alpha\beta}R^{\alpha\beta}+R^2 \quad (3)$$

is the integrand of the Gauss-Bonnet topological term. The action (2) is conformal invariant, for it satisfies the conformal Noether identity

$$-\frac{2}{\sqrt{-g}}g_{\mu\nu}\frac{\delta S_W}{\delta g_{\mu\nu}}=0. \quad (4)$$

By dimensional reasons one can introduce into action (2) an extra term $\theta\cdot\int\sqrt{-g}R^2$. However, this expression possesses only global and non local conformal symmetry and hence it will not be included in the action. In order to complete the picture, let us note that a finite $\theta\cdot\int\sqrt{-g}R^2$ term may be generated as a quantum anomaly-induced correction, e.g., due to the renormalization of the $\int\sqrt{-g}\Box R$ term in (2). The anomalous violation of local conformal symmetry in the finite part of the one-loop effective action may produce the nonconformal divergences beyond the one-loop level. This effect has been investigated in Ref. [7] for the conformal scalar field and there are no reasons to expect that the situation for the conformal quantum gravity will be different.

The main purpose of the present paper is the one-loop renormalization and renormalization group in conformal quantum gravity. The renormalization structure depends on the form of divergences and corresponding counterterms. According to the standard expectations, despite the anomaly results from the one-loop renormalization, one-loop divergences in conformal quantum gravity must be conformally invariant. This property holds in all known examples of conformal matter fields, and one can expect that the same should be true for the Weyl quantum gravity based on action (2). The most natural result would be to meet the renormalization of the coefficients η,λ,τ but not the $\int\sqrt{-g}R^2$-type counterterm.

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At first sight there is not much difference whether the $\int\sqrt{-g}R^2$ term shows up in one-loop divergences or at higher loops. However, this may be relevant for some applications of quantum gravity. If nonconformal divergences appear only at higher loops, the conformal symmetry may be considered as a good approximation. For example, the one-loop renormalizability of conformal gravity provides the possibility of the successful realization of the anomaly-induced inflation scheme (see the discussion in Ref. [8]) in the presence of quantum gravity. At the same time, if the nonconformal divergence emerges at one-loop order, the conformal symmetry cannot be considered a reasonable approximation, because the running of the coefficient θ will be much stronger in this case. In this situation a quantized conformal matter on a curved classical background also cannot be considered as an approximation to the full theory involving quantum gravity (see, e.g., the discussion in Refs. [1,9]). Finally, we need to be sure whether the nonconformal divergence is present at the one-loop level in the Weyl quantum gravity.

The first explicit derivation of the one-loop divergences in the Weyl quantum gravity has been performed by Fradkin and Tseytlin [10] in the framework of background field method, properly modified for the higher-derivative theories [11]. The $\int\sqrt{-g}R^2$-type divergence has been encountered and the conformal invariance of the counterterms has been achieved through the use of the special procedure of conformal regularization. This regularization is nothing but the specific reparametrization of the background metric invented earlier in Ref. [12] (see also Refs. [13] and [14]). According to this procedure metric $g_{\mu\nu}$ has to be replaced by conformal metric $\tilde{g}_{\mu\nu}=g_{\mu\nu}P^2[g_{\mu\nu}]$, where scalar metric-dependent quantity $P[g_{\mu\nu}]$ is defined as a solution of the equation

$$\Box P=\frac{1}{6}RP. \tag{5}$$

When performing a local conformal transformation of the original metric

$$g_{\mu\nu}\to g'_{\mu\nu}=g_{\mu\nu}e^{2\sigma(x)},$$

the quantity $P[g_{\mu\nu}]$ transforms as $P\to P'=Pe^{-\sigma(x)}$, such that the metric $\tilde{g}_{\mu\nu}$ remains invariant. Another important property of the metric $\tilde{g}_{\mu\nu}$ is that the corresponding scalar curvature is zero $\tilde{R}=R(\tilde{g}_{\mu\nu})=0$. Therefore, after the original metric $g_{\mu\nu}$ is replaced by $\tilde{g}_{\mu\nu}$, the divergent $\int\sqrt{-g}R^2$ counterterm disappears and the expected invariant form of divergences gets restored. The procedure of conformal ‘‘regularization’’ has been generalized for the conformal quantum gravity coupled to conformal quantum matter fields in Ref. [14].

Is it correct to consider the replacement $g_{\mu\nu}\to\tilde{g}_{\mu\nu}$ as a kind of conformal regularization for the divergent part of the effective action of quantum gravity? It is easy to see that this procedure eliminates also the anomaly in the finite part of the effective action [15]. Therefore, this choice of background metric does not fit with numerous applications of conformal anomaly which we know. Furthermore, despite the choice of $\tilde{g}_{\mu\nu}$ as a background metric being mathematically consistent, it is not very appealing because, in particular, it eliminates the Einstein-Hilbert action. Consequently, the theory based on this metric may not have a consistent nonrelativistic limit. In general, the whole procedure appears to be an artificial addition to the background field method. If we really want to learn the role of conformal symmetry in quantum gravity, it is important to know if the appearance of the $\int\sqrt{-g}R^2$ counterterm is a calculational error or it is caused by inconsistency of the background field method applied to Weyl quantum gravity. The last option has been partially explored in Ref. [16], where the possible conflict between diffeomorphism and conformal gauge-fixing conditions has been discussed. It turned out that the counterterm $\int\sqrt{-g}R^2$ is gauge-fixing independent, exactly due to the renormalization of the terms $\int\sqrt{-g}C^2$ and $\int\sqrt{-g}E$. On the other hand, the counterterm $\int\sqrt{-g}\Box R$ is not protected from the gauge-fixing dependence and can be modified or even eliminated by the appropriate choice of the parameters of gauge fixing.

The second explicit derivation of the one-loop divergences in Weyl quantum gravity has been performed by Antoniadis, Mazur, and Mottola [17] using methods developed in Ref. [10]. The correctness of the β functions for the coefficients λ and η calculated in Ref. [10] has been confirmed. At the same time, the paper in Ref. [17] did not come across the suspicious $\int\sqrt{-g}R^2$ counterterm. Indeed, this result coincides with our general expectations discussed above, but the situation with the two conflicting results does not look acceptable. In what follows we shall perform a more general quantum calculation using dimensional regularization, starting from the action (2). In this way we will be able, in particular, to check the previous calculations [10] and [17]. Even more important, we may achieve a better understanding of the role of the Gauss-Bonnet term in quantum gravity in $n=4$ and $n=4-\epsilon$ dimensions.

As one of our objectives is to perform a very complicated calculation in conformal quantum gravity, we have to provide maximal safety with respect to possible calculational errors. Thus we shall use a new way of organizing calculations, which guarantees an efficient automatic verification of our result. Simultaneously, we shall resolve another long-standing problem of quantum gravity. In the well-known paper [18], Capper and Kimber noticed that the Gauss-Bonnet term may, in principle, play a significant role in quantum gravity. Usually this term is disregarded because it is topological and does not affect the classical equations of motion. However, this conclusion is true only if the Bianchi identity is satisfied. This implies the diffeomorphism invariance of the theory. However, when the theory is quantized through the Faddeev-Popov procedure, the diffeomorphism invariance is broken and the vector space extends beyond the physical degrees of freedom. In other words, after quantization not only the spin-2 but also the spin-1 and spin-0 components of the quantum metric become relevant, and the topological term may produce new vertices of interaction between these components. As a result, the quantum-gravitational loops may be, in principle, affected by the presence of the topological term. Of course, this output does not look probable, because if we include the topological term into the classical action, the gauge-fixing condition should modify and eventually compensate the new vertices. But this is a belief that is always good to verify. Such a verification is one of the purposes of the present paper.

We shall perform the one-loop calculation starting from the full action (2), taking the topological term into account. As it was predicted in Ref. [18], the contributions of this term penetrate all vertices or, in other words, all elements of the background field method technique. However, despite many intermediate formulas that strongly depend on the coefficient η, this dependence completely disappears in the final expression for the divergent part of the effective action in the $n \to 4$ limit. This cancellation provides a negative answer to the hypothesis raised by Capper and Kimber [18]. Moreover, it provides a very strong test for the correctness of the calculation. On the other hand, the η dependence is present in the $n \neq 4$ expression. Therefore, derivation of the relevant part of the effective action in an arbitrary dimension $n \neq 4$ opens the way for constructing the complete $4-\epsilon$ renormalization group equations in the conformal quantum gravity theory (2). As it will be shown below, the quantum effect of the Gauss-Bonnet term leads to new fixed points that have no analogs in the $n=4$ case.

The paper is organized as follows: In the next section we shall briefly describe the Lagrange quantization of theory (2). One can find a detailed description of this subject in Refs. [2,10,19,20]. In Sec. III the details of the bilinear expansion of higher-derivative gravity are presented. Some of the bulky expressions corresponding to this section are collected in Appendix A. Our expansions are more general than those that were known before [2,20,21], because they are performed for all higher-derivative terms including $\sqrt{-g}E$ and without taking into account the conformal gauge-fixing condition. This enables one, in principle, to derive divergences not only for the conformal case, but also for the general higher-derivative quantum gravity [10,21,22]. In the present paper we perform only the calculation for the Weyl theory and expect to report the results for the general case later on. In Sec. IV we derive the coincidence limit of the $a_2(x,x')$ coefficient of the Schwinger-DeWitt expansion. The expression is obtained for the general n-dimensional space-time, in order to see the effect of the Gauss-Bonnet term more explicitly. After that, we derive the divergences of the Weyl gravity at $n \to 4$ and establish their independence on the parameter η. In Sec. V the renormalization group in the $4-\epsilon$ dimensions is considered, and a number of new UV-stable and UV-unstable fixed points (due to the quantum effects of the topological term) are described. In the course of the calculations in Secs. IV and V we use the computer algebra program MAPLE (see, e.g., Ref. [23]). Finally, in the last section we draw our conclusions and discuss the possible form of the nonconformal finite contributions to the one-loop effective action.

II. QUANTIZATION AND GAUGE-FIXING DEPENDENCE

The quantum gravity calculation in the background field method (see, e.g., Ref. [2] for the introduction) implies the special parametrization of the metric

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}. \tag{6}$$

In the right-hand side (*r.h.s.*) of the last formula $g_{\mu\nu}$ is the background metric and $h_{\mu\nu}$ is the quantum field (integration variable in the path integral). The one-loop contribution $\bar{\Gamma}^{(1)}$ to the effective action of quantum gravity is defined as follows [10]:

$$\bar{\Gamma}^{(1)}[g_{\mu\nu}] = \frac{i}{2} \ln \mathrm{Det}\, \hat{\mathcal{H}} - \frac{i}{2} \ln \mathrm{Det}\, Y^{\alpha\beta} - i \ln \mathrm{Det}\, \hat{\mathcal{H}}_{gh}, \tag{7}$$

where $\hat{\mathcal{H}}$ is the bilinear (in quantum fields) form of the action (2) together with the gauge-fixing term

$$S_{GF} = \mu^{n-4} \int d^n x \sqrt{-g}\, \chi_\alpha Y^{\alpha\beta} \chi_\beta. \tag{8}$$

The operator $\hat{\mathcal{H}}_{gh}$ is a bilinear form of the action of the Faddeev-Popov ghosts and μ is the dimensional constant (renormalization parameter in the dimensional regularization). Expression (7) includes also $\ln \mathrm{Det}\, Y^{\alpha\beta}$, where $Y^{\alpha\beta}$ is the weight function. In the case of higher-derivative gravity theory, this term gives a relevant contribution to the effective action, because $Y^{\alpha\beta}$ is a second order differential operator [10].

Introducing the gauge-fixing term (8), one is fixing the diffeomorphism invariance. However, in the theory under consideration this is not sufficient, because there is another classical symmetry—local conformal invariance, which leads to a degeneracy even after the term (8) is introduced. Hence one has to choose the second gauge fixing condition. Following Fradkin and Tseytlin [10], we fix the conformal symmetry by imposing the constraint $h = h^\mu_\mu = 0$. The interference between the two gauge-fixing conditions may take place because the term (8) breaks the conformal symmetry in the background fields sector [16]. However, this breaking cannot lead to the nonconformal counterterms, because the latter can be shown to be insensitive to the choice of the gauge-fixing condition. The general gauge-fixing condition (here we restrict our attention to the linear background gauges) has the form

$$\chi^\mu = \nabla_\lambda h^{\lambda\mu} + \beta \nabla^\mu h,$$

$$Y_{\mu\nu} = \frac{1}{\alpha}(g_{\mu\nu}\Box + \gamma \nabla_\mu \nabla_\nu - \delta \nabla_\nu \nabla_\mu + p_1 R_{\mu\nu} + p_2 R g_{\mu\nu}), \tag{9}$$

where $\alpha, \beta, \gamma, \delta, p_1, p_2$ are arbitrary parameters. The action of the Faddeev-Popov ghosts has the form

$$S_{gh} = \int d^4 x \sqrt{-g}\, \bar{C}^\mu (\mathcal{H}_{gh})^\nu_\mu C_\nu, \tag{10}$$

where

$$\hat{\mathcal{H}}_{gh}=(\mathcal{H}_{gh})^{\nu}_{\mu}=-\delta^{\nu}_{\mu}\Box-\nabla^{\nu}\nabla_{\mu}-2\beta\nabla_{\mu}\nabla^{\nu}. \tag{11}$$

The parameter β is fixed in the conformal case due to the conformal symmetry condition $h^{\mu}_{\mu}=0$; hence $\beta=-1/n$. Other parameters may take different values and their choice may influence, in principle, the one-loop divergences. The general analysis [16] shows that the C^2, E, and R^2 counterterms cannot depend on these parameters while the $\Box R$-type counterterm may have such a dependence. In what follows we shall use these data extensively, namely, we will not pay attention to the irrelevant $\int\sqrt{-g}\Box R$ counterterm and, on the other hand, we shall choose the gauge-fixing parameters $\alpha,\gamma,\delta,p_1,p_2$ such that the calculation of other counterterms becomes simpler. Let us note that the dependence on the parameters p_1,p_2 has been explored and found irrelevant in Ref. [22] for the nonconformal version of the higher-derivative quantum gravity.

III. BILINEAR EXPANSION

The action (2) includes only higher-derivative conformal invariant and surface terms. There are no Einstein-Hilbert, cosmological, and $\int\sqrt{-g}R^2$ terms in the action, because none of them possesses local conformal symmetry. But for the sake of completeness, the bilinear expansions for all these terms will be given too. The parametrization of the quantum metric $h_{\mu\nu}$ will be chosen according to (6). Let us note that the relevant divergences in the theory (2) are independent of the choice of parametrization for the quantum metric [16]. When making the expansions of the elements of the gravitational action, we keep in mind that the relevant terms are of second order in $h_{\mu\nu}$. Hence we shall pay most attention to this order of the expansion. In what follows we indicate all quantities constructed from the total metric $g'_{\mu\nu}$ by primes (e.g., $g'^{\mu\nu}$, $\sqrt{-g'}$, $\Gamma'^{\gamma}_{\mu\nu}$, $R'^{\alpha}{}_{\mu\beta\nu}$ etc.), and reserve simpler notations (e.g., $g^{\mu\nu}$, $\sqrt{-g}$, $\Gamma^{\gamma}_{\mu\nu}$, $R^{\alpha}{}_{\mu\beta\nu}$ etc.) for the quantities constructed from the background metric $g_{\mu\nu}$.

For $g'^{\mu\nu}$ and $\sqrt{-g'}$ the expansions can be presented as

$$\begin{aligned} g'^{\mu\nu}&=g^{\mu\nu}-h^{\mu\nu}+h^{\mu}_{\lambda}h^{\lambda\nu}-h^{\mu}_{\lambda}h^{\lambda}_{\tau}h^{\tau\nu}+\cdots\\ &=g^{\mu\nu}_{(0)}+g^{\mu\nu}_{(1)}+g^{\mu\nu}_{(2)}+g^{\mu\nu}_{(3)}+\cdots \end{aligned} \tag{12}$$

and

$$\sqrt{-g'}=\sqrt{-g}\left(1+\frac{1}{2}h+\frac{1}{8}h^2-\frac{1}{4}h_{\mu\nu}h^{\mu\nu}+\cdots\right). \tag{13}$$

For the coefficients of the affine connection, using (12), we arrive at the expansion

$$\Gamma'^{\alpha}_{\mu\nu}=\Gamma^{\alpha}_{\mu\nu}+\delta\Gamma^{\alpha}_{\mu\nu},\qquad \delta\Gamma^{\alpha}_{\mu\nu}=\sum_{n=1}^{\infty}\delta\Gamma^{(n)}{}^{\alpha}_{\mu\nu}. \tag{14}$$

Here the tensors $\delta\Gamma^{(n)}{}^{\alpha}_{\mu\nu}$ are given by the expressions

$$\delta\Gamma^{(n)}{}^{\alpha}_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}_{(n-1)}(\nabla_{\mu}h_{\beta\nu}+\nabla_{\nu}h_{\beta\mu}-\nabla_{\beta}h_{\mu\nu}). \tag{15}$$

For the curvature tensor one can establish the following general expression:

$$\begin{aligned} R'^{\alpha}{}_{\beta\mu\nu}&=R^{\alpha}_{\beta\mu\nu}+\nabla_{\nu}\delta\Gamma^{\alpha}_{\beta\mu}-\nabla_{\mu}\delta\Gamma^{\alpha}_{\beta\nu}+\delta\Gamma^{\lambda}_{\beta\nu}\delta\Gamma^{\alpha}_{\lambda\mu}-\delta\Gamma^{\lambda}_{\beta\mu}\delta\Gamma^{\alpha}_{\lambda\nu}\\ &=R^{\alpha}_{\beta\mu\nu}+\sum_{n=1}^{\infty}R^{(n)}{}^{\alpha}_{\beta\mu\nu}. \end{aligned} \tag{16}$$

In the first and second orders in the quantum metric, $h_{\mu\nu}$, we obtain the following expressions for the Riemann tensor:

$$\begin{aligned} R^{(1)}{}^{\alpha}{}_{\beta\mu\nu}&=\frac{1}{2}(\nabla_{\mu}\nabla_{\beta}h^{\alpha}_{\nu}-\nabla_{\nu}\nabla_{\beta}h^{\alpha}_{\mu}+\nabla_{\nu}\nabla^{\alpha}h_{\beta\mu}-\nabla_{\mu}\nabla^{\alpha}h_{\beta\nu}+R^{\alpha}_{\lambda\mu\nu}h^{\lambda}_{\beta}-R^{\lambda}_{\beta\mu\nu}h^{\alpha}_{\lambda}),\\ R^{(2)}{}^{\alpha}{}_{\beta\mu\nu}&=\frac{1}{2}h^{\alpha\lambda}(\nabla_{\mu}\nabla_{\lambda}h_{\nu\beta}+\nabla_{\nu}\nabla_{\beta}h_{\mu\lambda}+\nabla_{\nu}\nabla_{\mu}h_{\lambda\beta})+\frac{1}{4}[\nabla_{\mu}h^{\alpha\lambda}(\nabla_{\lambda}h^{\nu\beta}-\nabla_{\beta}h^{\nu\lambda}-\nabla_{\nu}h^{\beta\lambda})\\ &\quad+\nabla_{\beta}h^{\lambda}_{\nu}(\nabla_{\lambda}h^{\alpha}_{\mu}-\nabla^{\alpha}h_{\lambda\mu})+\nabla_{\nu}h^{\lambda}_{\beta}(\nabla_{\lambda}h^{\alpha}_{\mu}-\nabla^{\alpha}h_{\lambda\mu})+\nabla^{\lambda}h_{\mu\beta}(\nabla_{\lambda}h^{\alpha}_{\nu}-\nabla^{\alpha}h_{\lambda\nu})]-(\mu\leftrightarrow\nu). \end{aligned} \tag{17}$$

For the Ricci tensor, similar expansions have the form

$$\begin{aligned} R^{(1)}_{\mu\nu}&=\frac{1}{2}(\nabla_{\lambda}\nabla_{\mu}h^{\lambda}_{\nu}+\nabla_{\lambda}\nabla_{\nu}h^{\lambda}_{\mu}-\nabla_{\mu}\nabla_{\nu}h-\Box h_{\mu\nu}),\\ R^{(2)}_{\mu\nu}&=\frac{1}{2}h^{\alpha\beta}(\nabla_{\alpha}\nabla_{\beta}h_{\mu\nu}+\nabla_{\mu}\nabla_{\nu}h_{\alpha\beta}-\nabla_{\alpha}\nabla_{\mu}h_{\nu\beta}-\nabla_{\alpha}\nabla_{\nu}h_{\mu\beta})+\frac{1}{2}\nabla_{\alpha}h^{\alpha\beta}(\nabla_{\beta}h_{\mu\nu}-\nabla_{\mu}h_{\nu\beta}-\nabla_{\nu}h_{\mu\beta})\\ &\quad+\frac{1}{2}\nabla_{\alpha}h_{\mu\beta}(\nabla^{\alpha}h^{\beta}_{\nu}-\nabla^{\beta}h^{\alpha}_{\nu})+\frac{1}{4}\nabla_{\mu}h^{\alpha\beta}\nabla_{\nu}h_{\alpha\beta}+\frac{1}{4}\nabla^{\beta}h(\nabla_{\mu}h_{\nu\beta}+\nabla_{\nu}h_{\mu\beta}-\nabla_{\beta}h_{\mu\nu}). \end{aligned} \tag{18}$$

For the scalar curvature we have the following expansions:

$$R^{(1)}=\nabla_\mu\nabla_\nu h^{\mu\nu}-\Box h-h_{\mu\nu}R^{\mu\nu},$$

$$R^{(2)}=h^{\alpha\beta}(\nabla_\alpha\nabla_\beta h+\Box h_{\alpha\beta}-\nabla_\alpha\nabla_\lambda h^\lambda_\beta-\nabla_\lambda\nabla_\alpha h^\lambda_\beta)+\nabla_\alpha h^{\alpha\lambda}(\nabla_\lambda h-\nabla_\beta h^\beta_\lambda)$$
$$-\frac{1}{4}\nabla_\lambda h\nabla^\lambda h+h_{\alpha\lambda}h^\lambda_\beta R^{\alpha\beta}+\frac{3}{4}\nabla_\lambda h_{\alpha\beta}\nabla^\lambda h^{\alpha\beta}-\frac{1}{2}\nabla_\alpha h_{\lambda\beta}\nabla^\beta h^{\alpha\lambda}. \tag{19}$$

With these expansions, we can derive part of the action that is quadratic in the quantum fields. It proves useful to consider an alternative version of the action (2),

$$S_W(n)=-\mu^{(n-4)}\int d^nx\sqrt{-g}\{xR^2_{\mu\nu\alpha\beta}+yR^2_{\mu\nu}+zR^2+\tau\Box R\}, \tag{20}$$

where the new parameters x, y, and z are related to η and λ as follows:

$$x=\frac{1}{2\lambda}+\eta,\quad y=-\frac{2}{(n-2)\lambda}-4\eta,$$
$$z=\eta+\frac{1}{\lambda(n-1)(n-2)}. \tag{21}$$

After some algebra we arrive at the formula

$$S^{(2)}=-\mu^{(n-4)}\int d^nx\{x(\sqrt{-g}R^2_{\mu\nu\alpha\beta})^{(2)}+y(\sqrt{-g}R^2_{\mu\nu})^{(2)}+z(\sqrt{-g}R^2)^{(2)}\}, \tag{22}$$

where the complicated expressions for the bilinear forms are collected in Appendix A.

Starting from the expression (22), and using (A4), (A5), and (A6), one can easily find the bilinear form of the action (20). The operator $\hat{\mathcal{H}}$ depends on the gauge fixing term (8). The gauge-fixing parameters $\alpha,\beta,\gamma,\delta,p_1,p_2$ in (9) will be chosen in such a way that the operator takes the simplest minimal form

$$\hat{\mathcal{H}}=\hat{K}\Box^2+\mathcal{O}(\nabla^2), \tag{23}$$

where $\hat{K}$ is a nondegenerate c-number operator. Two of the possible nonminimal fourth derivative terms $g_{\mu\nu}\nabla_\alpha\Box\nabla_\beta$ and $g_{\alpha\beta}\nabla_\mu\Box\nabla_\nu$ vanish due to the conformal gauge-fixing condition $h^\mu_\mu=0$. The simplest choice of the parameters providing the cancellation of the remaining nonminimal fourth-derivative structures $\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta$ and $g_{\nu\beta}\nabla_\mu\Box\nabla_\alpha$ is the following:

$$\alpha=\frac{2}{y+4x},\quad \gamma=\frac{2x-2z}{y+4x},$$
$$\delta=1,\quad p_1=p_2=0, \tag{24}$$

with $\beta=-1/n$ defined by the conformal gauge fixing (as already noted). Let us remark that this "minimal" choice of the gauge fixing is sensitive to the introduction of the Gauss-Bonnet term, as we expected. On the other hand, if we fix the value of η such that the sum of the Weyl term and the topological term gives

$$C^2-E=2W=2\left(R^2_{\mu\nu}-\frac{1}{3}R^2\right), \tag{25}$$

the gauge-fixing condition (24) coincides with that of Refs. [10,17].

After some algebraic calculations and using the commutators (A3), we find

$$[S+S_{gf}]^{(2)}=h^{\mu\nu}\hat{\mathcal{H}}h^{\alpha\beta},$$
$$\hat{\mathcal{H}}=\hat{K}\Box^2+\hat{D}^{\rho\lambda}\nabla_\rho\nabla_\lambda+\hat{N}^\mu\nabla_\mu-(\nabla_\mu\hat{Z}^\mu)+\hat{W}, \tag{26}$$

and $\hat{K}$, $\hat{D}^{\rho\lambda}$, $\hat{N}^\mu\nabla_\mu$, $\nabla_\mu\hat{Z}^\mu$ and $\hat{W}$ are matrices in the $h^{\mu\nu}$-space. In order to derive an explicit form of $\hat{N}^\mu$ and $\hat{Z}^\mu$ one has to extend the derivation of bilinear expressions of the Appendix A. However, the derivation of these quantities does not have much meaning, because the terms $\hat{N}^\mu\nabla_\mu$ and $(\nabla_\mu\hat{Z}^\mu)$ may be safely disregarded. The reason is that both expressions $\hat{N}^\mu$ and $\hat{Z}^\mu$ are covariant derivatives of curvatures. Therefore they may contribute only to the irrelevant gauge-fixing-dependent $\int\sqrt{-g}\Box R$-type counterterm, which we are not calculating here. Below we shall simply set both terms to zero.

Let us introduce the useful notation

$$\bar{\delta}_{\mu\nu,\alpha\beta}=\delta_{\mu\nu,\alpha\beta}-\frac{1}{n}g_{\mu\nu}g_{\alpha\beta}$$

for the projection operator into the traceless sector of the $h^{\mu\nu}$ space. Here, as usual,

$$\delta_{\mu\nu,\alpha\beta}=\frac{1}{2}(g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha}).$$

Since we assume the conformal gauge-fixing condition $h=0$, the tensor $\bar{\delta}_{\mu\nu,\alpha\beta}$ plays the role of the identity matrix. Without this condition the identity matrix is $\delta_{\mu\nu,\alpha\beta}$. Taking the conformal gauge into account, we find

$$(\hat{K})_{\mu\nu,\alpha\beta}=\left(\frac{y}{4}+x\right)\overline{\delta}_{\mu\nu,\alpha\beta}, \tag{27}$$

$$(\hat{D}^{\rho\lambda})_{\mu\nu,\alpha\beta}=-2xg_{\nu\beta}R_{\alpha}{}^{\rho\lambda}{}_{\mu}+4x\delta^{\rho}_{\nu}R^{\lambda}{}_{\alpha\mu\beta}+(3x+y)g^{\rho\lambda}R_{\mu\alpha\nu\beta}+2x\overline{\delta}_{\nu\beta,}{}^{\rho\lambda}R_{\mu\alpha}-(4x+2y)\delta^{\rho}_{\alpha}R^{\lambda}{}_{\mu}g_{\nu\beta}$$
$$-2xg_{\nu\beta}R_{\mu\alpha}g^{\rho\lambda}+\frac{y+2x}{2}\overline{\delta}_{\mu\nu,\alpha\beta}R^{\rho\lambda}-zg_{\nu\beta}\overline{\delta}_{\mu\alpha}{}^{\rho\lambda}R+\frac{1}{2}z\overline{\delta}_{\mu\nu,\alpha\beta}g^{\rho\lambda}R-2z\overline{\delta}_{\alpha\beta}{}^{\rho\lambda}R_{\mu\nu}, \tag{28}$$

and

$$(\hat{W})_{\mu\nu,\alpha\beta}=\frac{3x}{2}g_{\nu\beta}R_{\mu\rho\lambda\sigma}R_{\alpha}{}^{\rho\lambda\sigma}+\frac{x-y}{2}R^{\rho}{}_{\alpha\mu}{}^{\lambda}R_{\nu\beta\rho\lambda}+\frac{5x+y}{2}R^{\lambda}{}_{\alpha\mu}{}^{\rho}R_{\lambda\nu\beta\rho}+\frac{3x+y}{2}R^{\lambda}{}_{\mu}{}^{\rho}{}_{\nu}R_{\rho\alpha\lambda\beta}+\frac{y-5x}{2}R_{\mu\rho}R^{\rho}{}_{\alpha\nu\beta}$$
$$+\frac{y+2x}{2}R_{\mu\alpha}R_{\nu\beta}+\frac{3y}{2}g_{\nu\beta}R_{\mu\rho}R^{\rho}{}_{\alpha}+\frac{3z}{2}g_{\nu\beta}RR_{\alpha\mu}-\frac{z}{2}RR_{\nu\beta\alpha\mu}+zR_{\mu\nu}R_{\alpha\beta}-\frac{1}{4}(xR^{2}_{\rho\lambda\sigma\tau}+yR^{2}_{\rho\lambda}+zR^{2})(\overline{\delta}_{\mu\nu,\alpha\beta}). \tag{29}$$

In the above formulas we used special condensed notation that enables one to present the expressions in a relatively compact way. The idea of this condensed notation is that all the algebraic symmetries are implicit, including the symmetrizations in the couples of indices $(\alpha\beta)\leftrightarrow(\mu\nu)$, $(\alpha\leftrightarrow\beta)$ and $(\mu\leftrightarrow\nu)$, and also in the couple $(\rho\leftrightarrow\lambda)$ in the operator $\hat{D}^{\rho\lambda}$. In order to obtain the complete formula explicitly, one has to restore all the symmetries. For example,

$$R_{\mu\rho}R^{\rho}{}_{\alpha\nu\beta}\rightarrow\frac{1}{2}(R_{\mu\rho}R^{\rho}{}_{\alpha\nu\beta}+R_{\alpha\rho}R^{\rho}{}_{\mu\beta\nu})$$

restores the $(\alpha\beta)\leftrightarrow(\mu\nu)$ symmetry. The same procedure has to be applied also for the other symmetries $(\rho\leftrightarrow\lambda)$, $(\alpha\leftrightarrow\beta)$, and $(\mu\leftrightarrow\nu)$.

In order to use the Schwinger-DeWitt method for the fourth-derivative operator [10], we need to reduce it to the minimal form (23). To this end one has to multiply the operator (26) by the inverse matrix $\hat{K}^{-1}$, given by

$$(\hat{K}^{-1})^{\mu\nu,\alpha\beta}=\frac{4}{y+4x}\overline{\delta}^{\mu\nu,\alpha\beta}.$$

Let us notice that the matrix $\hat{K}^{-1}$ is a c-number operator and hence this multiplication does not affect the divergences. By straightforward algebra, one can find the minimal operator

$$\hat{H}=\hat{K}^{-1}\hat{\mathcal{H}}=\hat{1}\Box^{2}+\hat{V}^{\rho\lambda}\nabla_{\rho}\nabla_{\lambda}+\hat{U}, \tag{30}$$

where the new expressions

$$\hat{V}^{\rho\lambda}=\hat{K}^{-1}\hat{D}^{\rho\lambda},\qquad \hat{U}=\hat{K}^{-1}\hat{W} \tag{31}$$

already do not possess the symmetry in $(\alpha\beta)\leftrightarrow(\mu\nu)$. The expressions for these two matrices are the following:

$$(\hat{U})_{\mu\nu,\alpha\beta}=\frac{4}{y+4x}\left\{\frac{3x}{2}g_{\nu\beta}R_{\mu\rho\lambda\sigma}R_{\alpha}{}^{\rho\lambda\sigma}+\frac{5x+y}{2}R^{\lambda}{}_{\alpha\mu}{}^{\rho}R_{\lambda\nu\beta\rho}+\frac{3x+y}{2}R^{\lambda}{}_{\mu}{}^{\rho}{}_{\nu}R_{\rho\alpha\lambda\beta}+\frac{y-5x}{2}R_{\mu\rho}R^{\rho}{}_{\alpha\nu\beta}+\frac{y+2x}{2}R_{\mu\alpha}R_{\nu\beta}\right.$$
$$+\frac{3y}{2}g_{\nu\beta}R^{\lambda}{}_{\mu}R_{\alpha\lambda}-\frac{1}{4}(xR^{2}_{\rho\lambda\sigma\tau}+yR^{2}_{\rho\sigma}+zR^{2})(\overline{\delta}_{\mu\nu,\alpha\beta})+\frac{3z}{2}g_{\nu\beta}RR_{\alpha\mu}+\frac{x-y}{2}R^{\rho}{}_{\alpha\mu}{}^{\lambda}R_{\nu\beta\rho\lambda}+zR_{\mu\nu}R_{\alpha\beta}$$
$$\left.-\frac{z}{2}RR_{\nu\beta\alpha\mu}\right\}, \tag{32}$$

$$\hat{V}^{\rho\lambda}=\frac{4}{y+4x}\sum_{i=1}^{10}b_{i}\mathbf{k}_{i}, \tag{33}$$

where

$$\mathbf{k}_1 = g_{\nu\beta} g^{\rho\lambda} R_{\mu\alpha}, \quad \mathbf{k}_2 = \overline{\delta}_{\mu\nu,\alpha\beta} g^{\rho\lambda} R,$$

$$\mathbf{k}_3 = g^{\rho\lambda} R_{\mu\alpha\nu\beta}, \quad \mathbf{k}_4 = \delta_{\nu\beta,}{}^{\rho\lambda} R_{\mu\alpha},$$

$$\mathbf{k}_5 = \delta_{\nu\beta,}{}^{\rho\lambda} R g_{\mu\alpha}, \quad \mathbf{k}_6 = \overline{\delta}_{\mu\nu,\alpha\beta} R^{\rho\lambda},$$

$$\mathbf{k}_7 = \frac{1}{2}(\delta_\nu^{(\rho} R^{\lambda)}{}_{\alpha\beta\mu} + \delta_\beta^{(\rho} R^{\lambda)}{}_{\mu\nu\alpha}),$$

$$\mathbf{k}_8 = g_{\nu\beta} \delta_{(\mu}^{(\rho} R^{\lambda)}{}_{\alpha)}, \quad \mathbf{k}_9 = g_{\nu\beta} R_{(\alpha}{}^{\rho\lambda}{}_{\mu)},$$

$$\mathbf{k}_{10} = \frac{1}{2}(\overline{\delta}_{\alpha\beta,}{}^{\rho\lambda} R_{\mu\nu} + \overline{\delta}_{\mu\nu,}{}^{\rho\lambda} R_{\alpha\beta}), \tag{34}$$

and

$$b_1 = -2x, \quad b_2 = z/2, \quad b_3 = 3x + y,$$

$$b_4 = 2x, \quad b_5 = -z, \quad b_6 = x + y/2,$$

$$b_7 = -4x, \quad b_8 = -4x - 2y, \quad b_9 = -2x,$$

$$b_{10} = -2z. \tag{35}$$

The above form of $\hat{V}^{\rho\lambda}$ is helpful in organizing the cumbersome calculations of divergences, which will be described in the next section.

IV. DERIVATION OF DIVERGENCES

The algorithm for the one-loop divergences corresponding to the minimal fourth-order operator can be written as [10] (here we use the Euclidean signature of the metric, in order to be consistent with [10])

$$\frac{1}{2} \ln \mathrm{Det}\, \hat{H}|_{\mathrm{div}} = -\frac{\mu^{n-4}}{(4\pi)^2 (n-4)} \int d^n x \sqrt{g}\, \mathrm{tr} \lim_{x' \to x} a_2(x',x), \tag{36}$$

where

$$\lim_{x'\to x} a_2(x',x) = \frac{\hat{1}}{90} R^2_{\mu\nu\alpha\beta} - \frac{\hat{1}}{90} R^2_{\mu\nu} + \frac{\hat{1}}{36} R^2 + \frac{1}{6} \hat{\mathcal{R}}_{\mu\nu} \hat{\mathcal{R}}^{\mu\nu} - \hat{U} + \frac{1}{12} R \hat{V}^\rho{}_\rho - \frac{1}{6} R_{\rho\lambda} \hat{V}^{\rho\lambda} + \frac{1}{48} \hat{V}^\rho{}_\rho \hat{V}^\lambda{}_\lambda + \frac{1}{24} \hat{V}_{\rho\lambda} \hat{V}^{\rho\lambda}. \tag{37}$$

Here $\hat{\mathcal{R}}_{\mu\nu}$ is the commutator of the covariant derivatives acting in the tensor $h^{\alpha\beta}$ space,

$$\hat{\mathcal{R}}_{\mu\nu} = [\nabla_\mu, \nabla_\nu]. \tag{38}$$

The full collection of the traces of the expressions (37) is presented in Appendix B.

It is straightforward to find the contributions of the weight and ghost operators (the algorithm for the nonminimal vector operator can be found in Refs. [10,22])

$$-\frac{i}{2} \ln \det \hat{Y}|_{\mathrm{div}} = -\frac{1}{(n-4)(4\pi)^2} \int d^4 x \sqrt{-g} \left\{ \frac{11}{180} R^2_{\mu\nu\alpha\beta} - \frac{43}{90} R^2_{\mu\nu} + \frac{1}{9} R^2 \right\}, \tag{39}$$

$$-i \ln \det \hat{\mathcal{H}}_{gh}|_{\mathrm{div}} = -\frac{1}{(n-4)(4\pi)^2} \int d^4 x \sqrt{-g} \left\{ \frac{11}{90} E - \left(\frac{1}{3}\xi^2 - \frac{4}{3}\xi + \frac{7}{15} \right) R^2_{\mu\nu} - \left(\frac{1}{6}\xi^2 - \frac{1}{3}\xi + \frac{17}{30} \right) R^2 \right\}, \tag{40}$$

where the parameter ξ is given by

$$\xi = \frac{n-2}{2(n-1)}. \tag{41}$$

Collecting all the results from (37), (39), and (40) according to (7) and using the formulas from Appendix B, we arrive at the functional trace of the overall coincidence limit of the $a_2(x',x)$ coefficient

$$A_2^t = \lim_{x'\to x} \mathrm{sTr}\, a_2^t(x',x) = \lim_{x'\to x} [\mathrm{Tr}\, a_2(x',x)(\hat{\mathcal{H}}) - \mathrm{Tr}\, a_2(x',x)(\hat{Y}) - 2\, \mathrm{Tr}\, a_2(x',x)(\hat{\mathcal{H}}_{gh})]. \tag{42}$$

The last expression can be regarded as a functional supertrace of the coincidence limit of the $a_2(x',x)$ coefficient of the differential operator acting in the direct product of the tensor $h_{\mu\nu}$, vector (third ghost) and vector ghost spaces. The sign difference between the different terms in (42) is due to the different Grassmann parity of the fields, and the operator Tr includes integration, as usual.

Let us present the result in terms of the parameters η and λ:

$$A_2^t = -\mu^{n-4} \int d^n x \sqrt{-g} \{ \beta_1(n) E + \beta_2(n) C^2 + \beta_3(n) R^2 \}, \tag{43}$$

where the coefficients (β functions) $\beta_1(n)$, $\beta_2(n)$, and $\beta_3(n)$ are given by the expressions

$$\beta_i(n) = \delta_i^{(0)} + \delta_i^{(1)} \eta + \delta_i^{(2)} \eta^2, \quad i = (1,2,3). \tag{44}$$

The coefficients $\delta_j^{(i)}$are the following functions:

$$\delta_1^{(0)} = -\frac{15n^6+86n^5+201n^4-4842n^3+8104n^2+6624n-9648}{2880(n-1)(n-3)^2},$$

$$\delta_1^{(1)} = -\frac{(n-4)(n-2)(n^5-8n^4+39n^3-40n^2-196n+192)\lambda}{48n(n-1)(n-3)^2},$$

$$\delta_1^{(2)} = -\frac{(n-4)^2(n^3+9n^2+14n+12)(n-2)^2\lambda^2}{48(n-3)^2n^2},$$

$$\delta_2^{(0)} = \frac{(n-2)(5n^6+299n^5-1162n^4-2570n^3+15056n^2-18528n+6720)}{960n(n-1)(n-3)^2},$$

$$\delta_2^{(1)} = \frac{(n-4)(n^4-3n^3+50n-36)(n-2)^2\lambda}{48n(n-1)(n-3)^2},$$

$$\delta_2^{(2)} = \frac{(n-4)(n^3+10n^2+10n+24)(n-2)^3\lambda^2}{48(n-3)^2n^2},$$

$$\delta_3^{(0)} = \frac{(n-4)(5n^5+22n^4+179n^3-930n^2-112n+816)}{960(n-1)^2(n-3)},$$

$$\delta_3^{(1)} = \frac{(n-4)(n^4-4n^3-n^2+10n-12)(n-2)^2\lambda}{24n(n-1)^2(n-3)},$$

$$\delta_3^{(2)} = \frac{(n-4)^2(n+1)(n^2+2n+12)(n-2)^3\lambda^2}{96(n-1)(n-3)n^2}. \tag{45}$$

The above coefficients, despite their chaotic appearance, provide a lot of important information. First, they show that for $n\neq 4$ the Gauss-Bonnet topological term contributes to the effective action in a nontrivial way and in particular produces the $\int\sqrt{-g}R^2$-type term. On the other hand, it is remarkable that all $\delta_i^{(1)}$ and $\delta_i^{(2)}$ coefficients are proportional to $n-4$. Hence, for $n=4$ one can see that the η dependence completely disappears, and the final result, Eq. (43), becomes very simple. Let us write down the expression for the one-loop divergences

$$\Gamma_{\rm div}^{(1)} = -\frac{\mu^{n-4}}{(4\pi)^2(n-4)}\int d^nx\sqrt{-g}\left\{\frac{137}{60}E+\frac{199}{15}W\right\}$$
$$= \frac{\mu^{n-4}}{(4\pi)^2(n-4)}\int d^nx\sqrt{-g}\left\{\frac{87}{20}E-\frac{199}{30}C^2\right\}, \tag{46}$$

where we used Eq. (25) and the pseudo-Euclidean signature. The expression (46) coincides with that derived by Antoniadis, Mazur, and Mottola in Ref. [17]. Both coefficients in (46) also coincide with those derived by Fradkin and Tseytlin in Ref. [10]. However, we do not meet the pathological $\int\sqrt{-g}R^2$-type divergence [10] and hence there is no need to apply the conformal regularization [12,13] discussed in the Introduction. As long as our calculation is seriously tested by the cancellation of the numerous η-dependent terms in the $n\to 4$ limit, we strongly believe in its correctness. Thus, the $\int\sqrt{-g}R^2$-type one-loop divergence does not show up in the one-loop effective action of Weyl quantum gravity.

The expression (46) does not contain the divergences in the cosmological constant and of the Einstein-Hilbert type. This fact is due to our choice of the regularization procedure. As long as we start from the conformally invariant action, the theory does not possess dimensional parameters and therefore the divergences of these two sorts may be only quartic and quadratic divergences. Of course, in the dimensional regularization that we are using here, the quartic and quadratic divergences do not appear. However, one can easily see the cosmological divergences and those linear in R in other regularization schemes, for example in the covariant cutoff [10] or in the covariant Pauli-Villars [8] regularizations. Of course, the logarithmic divergences (46) that we have calculated do not depend on the choice of the regularization scheme.

Including the matter fields one meets additional contributions to the divergent coefficients in (46). As it was already noticed in the Introduction, the conventional scalars, fermions, and vectors give contributions of the same sign to both

β_1 and β_2, while the contributions of higher-derivative scalar and fermion have opposite sign [4–6]. The sign of the coefficients in (46) coincides with that of the conventional fields. Hence, since the $\int R^2$-type divergence is absent, one can use the method of Ref. [5] and adjust the number of higher-derivative scalars and fermions such that the one-loop divergences completely cancel. In a more complicated situation, when the matter coupled to quantum gravity possesses self-interaction, the quantum gravitational effects modify the divergences and the renormalization group trajectories also in the matter sector of the theory. This issue has been studied in detail [14,24,25] for the case of the higher-derivative gravity [26]. In both conformal [14] and general [24,25] cases the effect of quantum gravity is rather smooth and always favors the asymptotic freedom in the matter fields sector. Due to the absence of the $\int R^2$-type divergence the investigation for the conformal case can indeed be performed without the special conformal regularization (which we discussed in the Introduction), while the quantum-gravitational corrections to the β functions in the matter sector are exactly those derived in Ref. [14] (see also Ref. [2]). The reason is that these β functions do not depend on the scalar curvature and hence are not affected by the conformal regularization.

V. RENORMALIZATION GROUP EQUATIONS

The renormalization group (RG) equations for the theory (2) may be considered in two different ways [27]. The first possibility is to take usual $n=4$ β functions; in this case we meet exactly the same RG equations as in Ref. [10]. It proves useful to introduce a new parameter $\rho=-1/\eta$. Let us remark that the choice of λ as a coupling constant in the action (2) is fixed, because (i) λ is a parameter of the loop expansion in this theory and (ii) one cannot change the sign of λ without changing the positivity of the graviton energy. At the same time there are no similar constraints for the coefficient of the Gauss-Bonnet term and therefore the choice can be made according to convenience. The usual $n=4$ renormalization group equations for λ and ρ have the form

$$\frac{d\lambda}{dt}=\mu\frac{d\lambda}{d\mu}\bigg|_{n=4}=\beta_\lambda(4)=-a^2\lambda^2,\quad \lambda(0)=\lambda_0,$$
$$\frac{d\rho}{dt}=\mu\frac{d\rho}{d\mu}\bigg|_{n=4}=\beta_\rho(4)=-b^2\rho^2,\quad \rho(0)=\rho_0, \tag{47}$$

where

$$a^2=\frac{199}{15(4\pi)^2},\quad b^2=\frac{261}{60(4\pi)^2}. \tag{48}$$

The above equations indicate the UV asymptotic freedom in both parameters. In other words, there is a single fixed point $\lambda=\rho=0$ and it is stable in the high energy limit $t\to\infty$.

Let us now consider a more complicated version of the renormalization group equations, taking the dimension $n=4-\epsilon$ for $-1\leqslant\epsilon<1$. Mathematically this means that we do not take the limit $n\to 4$ in Eqs. (47). The renormalization group equations that emerge as a result of this procedure will be different from (47) and one can expect to see qualitative effects of the Gauss-Bonnet term in this framework.

A similar approach to the renormalization group proved fruitful in the two-dimensional quantum gravity [28], due to its relation to the concept of asymptotic safety [29] and to the discussion of the universality classes of quantum gravity theories [30]. The main idea of $2-\epsilon$ quantum gravity is the following. In the precisely $n=2$ dimensions, quantum gravity is a topological theory similar to that which we meet in $n=4$ starting from the pure Gauss-Bonnet term. However, if we generalize the theory for $n=2-\epsilon$, there is a dynamics (different from the Gauss-Bonnet theory, where the propagator does not appear even for $n\neq 4$) and at the quantum level one meets a nontrivial UV fixed point of the renormalization group [28–30]. Keeping this example in mind, it is natural to expect that the effect of the Gauss-Bonnet term on the renormalization group equations in $n=4-\epsilon$ may be nontrivial and in particular may produce new fixed points.

Consider the renormalization group equations for λ and ρ in $n=4-\epsilon$ dimension. The naive form of the β functions would be based on the "standard" expressions (47),

$$\beta_\lambda=-\epsilon\lambda+\beta_\lambda(4),\quad \beta_\rho=-\epsilon\rho+\beta_\rho(4), \tag{49}$$

indicating the one extra nonzero fixed point for each of the effective charges $\lambda(t)$ and $\rho(t)$. Indeed, the fixed point $\lambda=\rho=0$ remains stable in UV for $\epsilon>0$. However, this naive consideration is incorrect because the Gauss-Bonnet term gets dynamical for $n\neq 4$, affecting the renormalization group equations in a nontrivial way. Using the expressions (45), we arrive at the following correct form of the renormalization group equations, quite different from (49):

$$\frac{d\rho}{dt}=-\epsilon\rho+\frac{1}{(4\pi)^2}(f_1\rho^2-f_2\lambda\rho+f_3\lambda^2),$$
$$\frac{d\lambda}{dt}=-\epsilon\lambda-\frac{2\lambda^2}{(4\pi)^2}\left(g_1-g_2\frac{\lambda}{\rho}+g_3\frac{\lambda^2}{\rho^2}\right). \tag{50}$$

The coefficients $f_{1,2,3}$ and $g_{1,2,3}$ may be expressed via the coefficients $\delta_j^{(i)}$ from (45) as

$$f_1=\delta_1^{(0)},\quad f_2=\delta_1^{(1)}/\lambda,\quad f_3=\delta_1^{(2)}/\lambda^2,$$
$$g_1=\delta_2^{(0)},\quad g_2=\delta_2^{(1)}/\lambda,\quad g_3=\delta_2^{(2)}/\lambda^2. \tag{51}$$

One can note that $f_{1,2,3}$ and $g_{1,2,3}$ depend only on the parameter ϵ and not on the couplings. In the limit $\epsilon=0$ we come back to Eqs. (47). The renormalization group equations (50) are nonlinear and do not admit a simple analytic solution. For this reason we shall start from the search of the fixed points that are the values of λ and ρ for which both β functions vanish. Consequently, we explore the stability of these fixed points and establish the renormalization group flows for some particular values of ϵ.

In order to find fixed points, we consider the particular values of the parameter, $\epsilon=0.9$, $\epsilon=0.1$, $\epsilon=0.01$, ϵ

TABLE I. Numerical values of the coefficients for the particular values of ϵ.

ϵ	f_1	f_2	f_3	g_1	g_2	g_3
0.9	−16.77	−28.73	−36.48	2.359	−42.51	−46.98
0.1	−4.301	0.08	−0.016	6.385	−0.174	−0.318
0.01	−4.344	0.008	−0.0001	6.608	−0.016	−0.03
−0.01	−4.356	−0.008	−0.0001	6.659	0.016	0.03
−0.1	−4.416	−0.086	−0.013	6.902	0.146	0.286
−1	−5.416	−0.947	−0.81	9.98	1.087	2.526

$=-0.01$, $\epsilon=-0.1$, and $\epsilon=-1$. The numerical values of the coefficients for these cases are presented in Table I. The point $\epsilon=0.9$ is numerically close to $\epsilon=1$ $(n=3)$, where the expressions for the β functions become singular.

The numerical analysis shows that for each of the choices $\epsilon=0.9$, $\epsilon=0.1$, and $\epsilon=0.01$ there are four fixed points that are new compared to the $\epsilon=0$ case; while for the values $\epsilon=-0.01,-0.1,-1$, there are two new fixed points. The values of the parameters corresponding to these fixed points are shown in Table II.

The stability properties of the fixed points can be easily investigated in the linear approximation. The result is that, in the cases $\epsilon=0.9$, $\epsilon=0.1$, and $\epsilon=0.01$, the fixed points (λ_1,ρ_1) and (λ_2,ρ_2) are saddle points while the fixed points (λ_3,ρ_3) and (λ_4,ρ_4) are absolutely unstable in the UV limit $t\to\infty$. It is worth noticing the contrast with the $\epsilon=0$ renormalization group equations (47) with a single UV stable fixed point $\lambda=\rho=0$. In the $n>4$ cases $\epsilon=-0.01$, $\epsilon=-0.1$, and $\epsilon=-1$ there are only two extra fixed points, one of them UV stable and IR unstable and another one a saddle point (unstable in both UV and IR regimes). As shown in the Figs. 1 and 2, there are no additional (compared to the standard $\epsilon=0$ case) UV-stable fixed points for positive ϵ, and at the same time, for negative ϵ there is always one additional fixed point with stability in the UV domain.

An interesting observation concerning the renormalization group trajectories is that none of them crosses the line $\lambda=0$. In other words, the renormalization group flow in this theory is divided into two separate parts: one with $\lambda>0$ corresponds to the positively defined energy of the gravitons

TABLE II. Numerical values of the parameters corresponding to the new fixed points. None of them has an analog in the $\epsilon=0$ case.

Fixed point for $\epsilon=0.9$	1	2	3	4
λ_i	0	−6.817	−5.945	1.807
ρ_i	−8.475	−10.75	−12.52	−3.05
Fixed point for $\epsilon=0.1$	1	2	3	4
λ_i	0	−14.421	−1.232	16.236
ρ_i	−3.671	−3.159	−3.647	−3.706
Fixed point for $\epsilon=0.01$	1	2	3	4
λ_i	0	−5.228	−0.119	5.457
ρ_i	−0.364	−0.351	−0.363	−0.371
Fixed point for $\epsilon=-0.01$	1	2	3	4
λ_i	0	—	—	0.1186
ρ_i	0.3625	—	—	0.3628
Fixed point for $\epsilon=-0.1$	1	2	3	4
λ_i	0	—	—	1.147
ρ_i	3.576	—	—	3.597
Fixed point for $\epsilon=-1$	1	2	3	4
λ_i	0	—	—	8.001
ρ_i	29.157	—	—	30.239

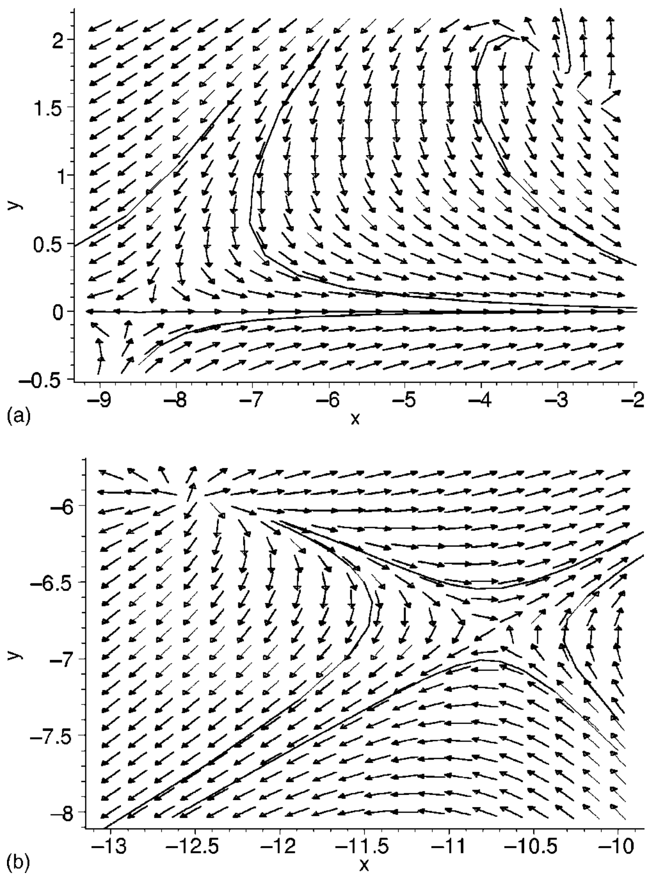

FIG. 1. Diagrams for $\epsilon=0.9$. The y axis represents the coupling λ and the x axis, the coupling ρ, as well as in all subsequent plottings. The left diagram shows the fixed points 1 and 4, and the other shows the points 2 and 3. The labels of the fixed points correspond to the numeration in Table II. The arrows indicate the direction of the renormalization group trajectory at the given point (we have also drawn some trajectories for illustrative purposes). One can distinguish stable, completely unstable, and saddle fixed points at these and further diagrams.

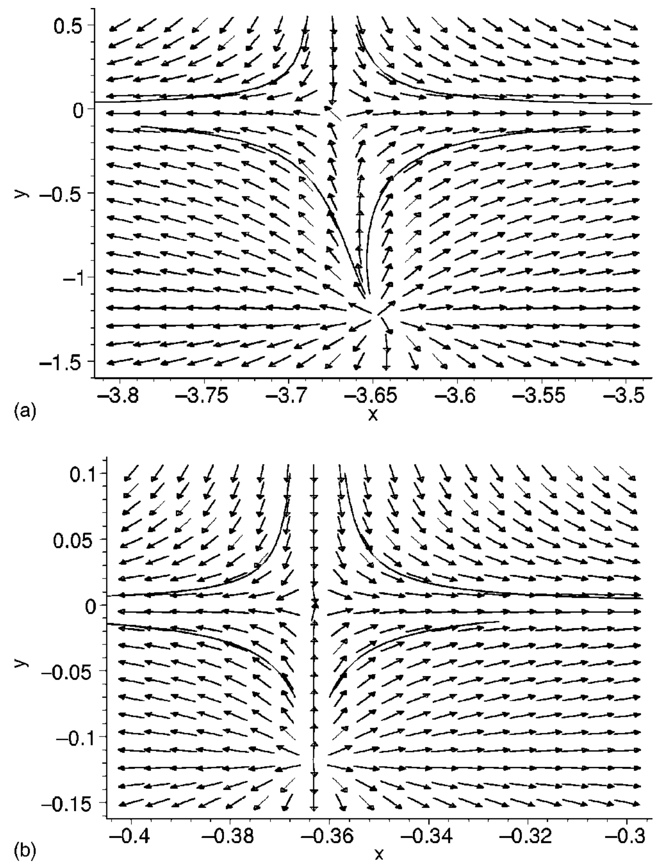

FIG. 2. The fixed points 1 and 3 are shown for the cases $\epsilon=0.1$ (left) and $\epsilon=0.01$ (right). Clearly, point 1 is a saddle point (unstable) and point 3 is UV unstable. Points 2 and 4 are similar (saddle and UV unstable, respectively) and are not plotted.

(massless spin-2 mode) and another one to the unphysical graviton sector with $\lambda<0$. There are examples of the qualitatively new UV-stable fixed points with $\lambda>0$ (see Figs. 3 and 4). At the same time there are no such examples for the case $\lambda<0$. One can suppose that this property of the fixed points is related to the limit $\epsilon\to 0$, where all new UV-stable fixed points presumably should tend to $\lambda=0$.

It is obvious that none of the fixed points that we have found so far coincides with the standard one $\lambda=\rho=0$ of the $n=4$ renormalization group. The natural question is whether it is true that the effect of the Gauss-Bonnet term is to eliminate the asymptotic freedom in $n=4-\epsilon$ dimensions. The answer to this question is definitely not. The source of our failure to see the standard fixed point is that we have used only the algebraic equations $\beta_\lambda=\beta_\rho=0$ and due to the nonpolynomial form of the β_λ function (50) one cannot see the fixed point with $\rho=0$ in this way. So, in order to complete our study we have to consider, especially, the possibility of simultaneous $\lambda\to 0$ and $\rho\to 0$. Using elementary transformations, one can check that the regimes $\lambda\ll\rho$ and $\rho\ll\lambda$ lead to contradictions. Therefore, we consider, additionally, the possibility of the special solution $\rho=k\lambda$, where k is a constant. Under this assumption, Eqs. (50) are consistent if

$$f_1k^3+(2g_1-f_2)k^2+(f_3-2g_2)k+2g_3=0, \tag{52}$$

with an additional restriction

$$b=\frac{2}{(4\pi k)^2}(g_1k^2-g_2k+g_3)>0, \tag{53}$$

dictated by the asymptotic freedom, in the UV regime. The origin of this condition is the following. After the relation $\rho=k\cdot\lambda$ is imposed, the equation for λ becomes

$$\frac{d\lambda}{dt}=-\epsilon\lambda-b\lambda^2. \tag{54}$$

The general solution of this equation has the form

$$\lambda(t)=\frac{\epsilon}{b(e^{\epsilon t}-1)}, \quad \lambda(t_0)>0. \tag{55}$$

It is easy to see that the asymptotic freedom in the UV limit $t\to+\infty$ holds for $\epsilon>0$ and (53) is satisfied. For $\epsilon<0$ and condition (53) satisfied, the situation is more complicated,

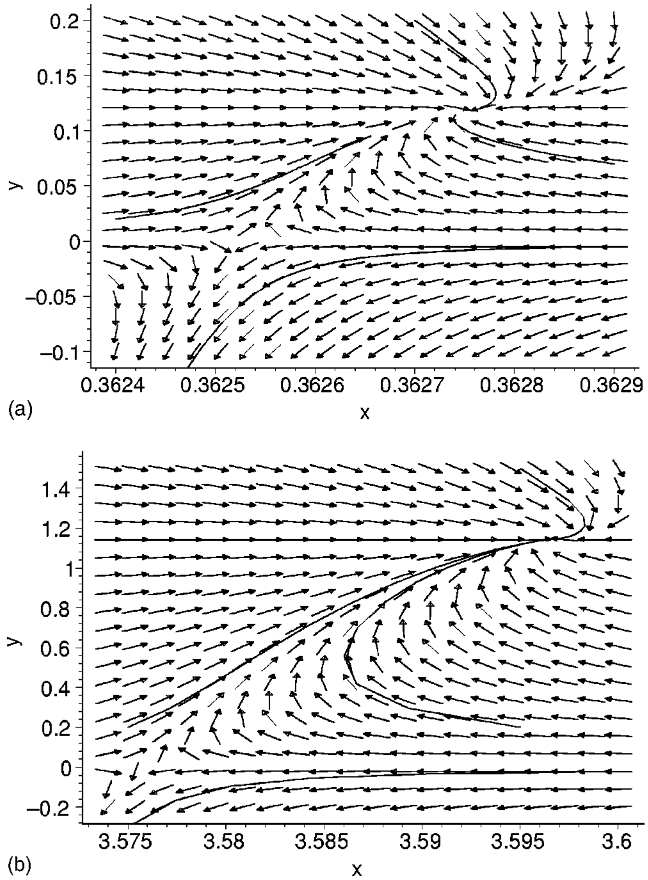

FIG. 3. Fixed points 1 and 4 for the cases $\epsilon = -0.01$ (left) and $\epsilon = -0.1$ (right). Point 1 is a saddle point (unstable) and point 4 is UV stable, contrary to the analogous points with positive ϵ.

because the UV-stable fixed point is nonzero $\lambda(t \to \infty) = -\epsilon/b > 0$. However, this fixed point tends to zero when $\epsilon \to 0$, and we can consider that the theory is asymptotically free in this sense. At the same time, independent on the sign of ϵ, the theory with $b < 0$ does not manifest the asymptotically free behavior in the UV limit.

In fact, there is no guarantee that the condition (53) is satisfied for every choice of ϵ and all real roots of (52). The

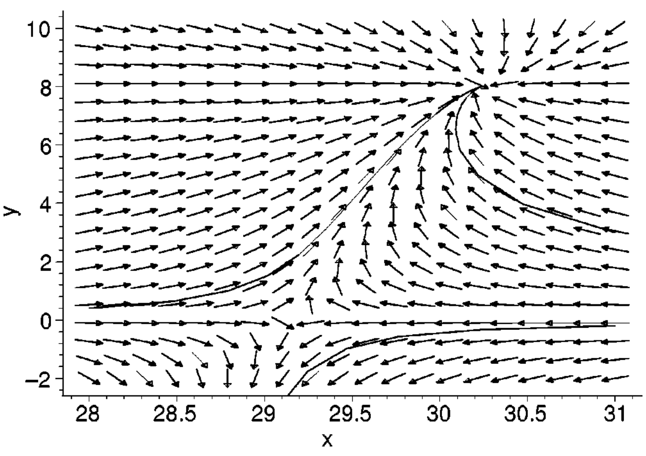

FIG. 4. The case $\epsilon = -1$, with a saddle point (1) and a UV-stable one (4).

numerical computations show that for $\epsilon = -0.01, -0.1$, and -1, Eq. (52) has only one real root and that this root satisfies the condition (53). On the other side, for $\epsilon = 0.9, 0.1$, and 0.01, Eq. (52) has three distinct real roots, one of them violating condition (53) for each case. Thus, there are solutions of the equations $\rho = k\lambda$ and (52) that do not satisfy Eq. (53). Let us note that, in all cases we examined, there are also solutions with the UV-stable fixed point (0,0). However, the asymptotic freedom depends on the choice of the initial condition on the λ-ρ plane. In some cases, when Eq. (53) does not hold on the special solution of Eqs. (50), the (0,0) point is not stable in the UV region.

Looking at Figs. 1–4, one can observe the renormalization group trajectories (for the $\epsilon > 0$ case) linking the IR-stable point (λ_3, ρ_3) to the UV-stable point (0,0), or alternatively the IR-stable point (λ_4, ρ_4) to (0,0). The situation is similar for $\epsilon < 0$, but here the renormalization group flow is inverted, linking the IR-stable point (0,0) to the UV-stable point (λ_4, ρ_4).

VI. CONCLUSIONS AND DISCUSSIONS

We have calculated the one-loop effective action for the Weyl gravity with the Gauss-Bonnet term. In the $n \to 4$ limit the quantum effects of the Gauss-Bonnet term cancel. This cancellation may be seen as a negative answer to the problem raised in Ref. [18]. This result is valid, at least, in the framework of the conformal quantum gravity. Another remarkable fact is that, in agreement with Ref. [17], there is no infinite $\int \sqrt{-g} R^2$ counterterm. Other sectors of the divergent part of the effective action are in perfect agreement with both earlier calculations [10,17].

Despite the one-loop divergences being conformal invariant, this symmetry may be broken at the one-loop level in the finite part of the effective action. The divergences of the $\int \sqrt{-g} C^2$- and $\int \sqrt{-g} E$-type produce the anomalous violation of the Noether identity (4), and as a result the finite part of the one-loop effective action contains usual nonlocal [31] anomaly-induced terms [4]. There may also be a local $\int \sqrt{-g} R^2$-type contribution that deserves special discussion. It is easy to see that there are two different possible sources of this term in the Weyl quantum gravity:

(i) If the calculation is performed in a dimensional regularization, the $\delta_3^{(0)}$ and $\delta_3^{(1)}$ terms in (45) are proportional to $n-4$ and therefore they produce the finite $\int \sqrt{-g} R^2$ term directly from A_2^t. It is remarkable that this contribution depends on the coefficient η of the Gauss-Bonnet term. According to Ref. [8], this contribution is a subject of strong ambiguity typical for the dimensional regularization. In general, the dimensional regularization is unable to predict any definite value for the coefficient of the finite $\int \sqrt{-g} R^2$ term.

(ii) The infinite $\int \sqrt{-g} \Box R$-type counterterm, which we did not calculate here, may produce a contribution to the conformal anomaly and eventually to the finite $\int \sqrt{-g} R^2$ term. However, this contribution is plagued by double ambiguities. First, the $\int \sqrt{-g} \Box R$-type counterterm itself is gauge-fixing dependent [16]. As already explained above, this is the reason why we did not calculate this counterterm. The second source of ambiguity is the derivation of anomaly and of the anomaly-induced effective action. In relation to the $\int\sqrt{-g}R^2$ term these procedures may be ambiguous. Detailed discussions of this issue have recently been given in Ref. [8], where the ambiguity has been confirmed not only for the traditional version of the dimensional regularization (where it is completely out of control) but also in a more physically covariant Pauli-Villars regularization with nonminimal scalar massive regulators. It is worth noticing that the status of this last ambiguity in the Weyl quantum gravity is quite different from that one in the semiclassical approach. In the last case the ambiguity is always reduced to the freedom of adding the $\int\sqrt{-g}R^2$ term to the classical action of vacuum, while in the former case this operation would increase the number of physical degrees of freedom (see, e.g., Ref. [2] and references therein) and hence cannot be seen as the legal operation for the theory (2).

In any case the local conformal invariance in Weyl gravity is violated at the one-loop level by quantum corrections. Hence, despite the fact that the general higher-derivative quantum gravity is indeed renormalizable [19,32], the particular conformal version is multiplicatively nonrenormalizable at higher loops. Our results show, however, that the conformal quantum gravity can be regarded as a good approximation. The corresponding procedure means that one can start from the theory with a very small coefficient of the $\int\sqrt{-g}R^2$ term. Due to the one-loop renormalizability of the conformal theory this coefficient will remain very small at the quantum level. If we consider the conformal quantum gravity in this framework, the problem of ambiguity of the anomalous $\int\sqrt{-g}R^2$ term is irrelevant and we can regard this theory as a useful particular example of the higher-derivative quantum gravity models.

One of the outcomes of our investigation is new fixed points of the renormalization group flows that appear due to quantum effects of the topological Gauss-Bonnet term in $4-\epsilon$ dimensions [33]. One can expect even a greater number of nontrivial fixed points for a general higher-derivative quantum gravity, with the Einstein-Hilbert, cosmological, and $\int R^2$ terms included.

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APPENDIX A: BILINEAR EXPANSIONS QUADRATIC IN CURVATURE TERMS

In this appendix we collect the cumbersome expressions with bilinear expansions of the relevant terms of the second order in curvature. Furthermore, we present the transformation of these terms to the form that is useful for the derivation of the effective action. The initial set of the bilinear expansions has the following form:

$$
\begin{aligned}
(\sqrt{-g}R^2_{\mu\nu\alpha\beta})^{(2)} = \sqrt{-g}h^{\mu\nu}\Big\{ & \delta_{\mu\nu,\alpha\beta}\Box^2 - g_{\mu\alpha}\nabla_\beta\Box\nabla_\nu + \nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu - g_{\mu\alpha}\nabla_\rho\nabla_\beta\nabla_\nu\nabla^\rho + 4R^\rho{}_{\alpha\nu\beta}\nabla_\mu\nabla_\rho + \delta_{\mu\nu,\alpha\beta}R_{\rho\lambda}\nabla^\rho\nabla^\lambda + R_{\mu\alpha\nu\beta}\Box \\
& - 2R^\rho{}_{\mu\alpha\nu}\nabla_\beta\nabla_\rho - 2g_{\mu\nu}R_{\rho\alpha\lambda\beta}\nabla^\lambda\nabla^\rho - 4g_{\nu\beta}R_{\alpha\rho\lambda\mu}\nabla^\lambda\nabla^\rho + \frac{7}{2}g_{\nu\alpha}R_{\mu\rho\lambda\tau}R_\beta{}^{\rho\lambda\tau} + g_{\mu\nu}R_{\rho\lambda\tau\alpha}R^{\rho\lambda\tau}{}_\beta \\
& - \frac{1}{4}\left(\delta_{\mu\nu,\alpha\beta} - \frac{1}{2}g_{\mu\nu}g_{\alpha\beta}\right)R^2_{\rho\lambda\tau\theta} - \frac{1}{2}R_{\mu\alpha\rho\lambda}R_{\nu\beta}{}^{\rho\lambda}\Big\}h^{\alpha\beta},
\end{aligned}
$$

$$
\begin{aligned}
(\sqrt{-g}R^2_{\mu\nu})^{(2)} = \sqrt{-g}h^{\mu\nu}\Big\{ & \frac{1}{2}\nabla_\alpha\nabla_\mu\nabla_\beta\nabla_\nu + \frac{1}{4}\delta_{\mu\nu,\alpha\beta}\Box^2 + \frac{1}{2}g_{\nu\alpha}\nabla_\rho\nabla_\mu\nabla_\beta\nabla^\rho - \frac{1}{2}g_{\alpha\beta}\nabla_\rho\nabla_\mu\nabla_\nu\nabla^\rho - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\rho\nabla_\beta\nabla^\rho \\
& + \frac{1}{4}g_{\mu\nu}g_{\alpha\beta}\nabla_\rho\Box\nabla^\rho - \frac{1}{2}g_{\alpha\nu}\nabla_\beta\nabla_\mu\Box + \frac{1}{2}g_{\mu\nu}\nabla_\alpha\nabla_\beta\Box - \frac{1}{2}g_{\nu\alpha}\Box\nabla_\beta\nabla_\mu - 2g_{\nu\alpha}R^\rho{}_\beta\nabla_\mu\nabla_\rho + \frac{1}{2}\delta_{\mu\nu,\alpha\beta}R^{\rho\lambda}\nabla_\rho\nabla_\lambda \\
& + g_{\alpha\beta}R^\rho{}_\nu\nabla_\rho\nabla_\mu - R_{\mu\beta}\nabla_\alpha\nabla_\nu + g_{\nu\alpha}R_{\mu\beta}\Box + 2g_{\nu\alpha}R^\rho{}_\mu\nabla_\rho\nabla_\beta + g_{\mu\nu}R^\rho{}_\beta\nabla_\alpha\nabla_\rho - \frac{1}{2}g_{\mu\nu}g_{\alpha\beta}R^{\rho\lambda}\nabla_\rho\nabla_\lambda - 2g_{\mu\beta}R^\rho{}_\nu\nabla_\alpha\nabla_\rho \\
& + \frac{1}{8}(g_{\mu\nu}g_{\alpha\beta} - 2\delta_{\mu\nu,\alpha\beta})R^2_{\rho\lambda} + R_{\mu\alpha}R_{\nu\beta} + 2g_{\nu\alpha}R_{\mu\rho}R^\rho{}_\beta - g_{\alpha\beta}R_{\mu\rho}R^\rho{}_\nu\Big\}h^{\alpha\beta},
\end{aligned}
\tag{A1}
$$

and

$$(\sqrt{-g}R^2)^{(2)}=\sqrt{-g}h^{\mu\nu}\Big\{\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta-g_{\alpha\beta}\nabla_\mu\nabla_\nu\Box-g_{\mu\nu}\Box\nabla_\alpha\nabla_\beta+g_{\mu\nu}g_{\alpha\beta}\Box^2-g_{\nu\alpha}R\nabla_\beta\nabla_\mu-2R_{\mu\nu}\nabla_\alpha\nabla_\beta+g_{\mu\nu}R\nabla_\alpha\nabla_\beta$$
$$+2g_{\alpha\beta}R_{\mu\nu}\Box+\frac{1}{2}(\delta_{\mu\nu,\alpha\beta}-g_{\mu\nu}g_{\alpha\beta})R\Box+2Rg_{\nu\beta}R_{\mu\alpha}-g_{\mu\nu}RR_{\alpha\beta}+\frac{1}{8}(g_{\mu\nu}g_{\alpha\beta}-2\delta_{\mu\nu,\alpha\beta})R^2+R_{\mu\nu}R_{\alpha\beta}\Big\}h^{\alpha\beta}. \tag{A2}$$

It proves necessary to establish some commutation relations between covariant derivatives. In the expressions below we have omitted those terms that may contribute only to the total derivatives in the effective action. Also, for the sake of brevity we broke the symmetries in the pairs of indices $(\alpha\beta)$ and $(\mu\nu)$. These symmetries must be restored for practical calculations. We can write

$$g_{\nu\beta}\nabla^\lambda\nabla_\mu\nabla_\alpha\nabla_\lambda h^{\alpha\beta}=(g_{\nu\beta}R_{\rho\mu}\nabla^\rho\nabla_\alpha+R_{\nu\beta\lambda\mu}\nabla^\lambda\nabla_\alpha+g_{\nu\beta}\nabla_\mu\Box\nabla_\alpha+g_{\nu\beta}R^\lambda{}_\alpha\nabla_\lambda\nabla_\mu+R_{\nu\beta\alpha\lambda}\nabla^\lambda\nabla_\mu)h^{\alpha\beta},$$
$$g_{\nu\beta}\Box\nabla_\alpha\nabla_\mu h^{\alpha\beta}=(g_{\nu\beta}R_{\mu\alpha}\Box+R_{\nu\beta\alpha\mu}\Box+g_{\nu\beta}\nabla_\mu\Box\nabla_\alpha+g_{\nu\beta}R_{\rho\mu}\nabla^\rho\nabla_\alpha+2R_{\nu\beta\lambda\mu}\nabla^\lambda\nabla_\alpha)h^{\alpha\beta},$$
$$g_{\nu\beta}\nabla^\lambda\nabla_\alpha\nabla_\mu\nabla_\lambda h^{\alpha\beta}=\Big(g_{\nu\beta}R_{\rho\mu}\nabla^\rho\nabla_\alpha+R_{\nu\beta\lambda\mu}\nabla^\lambda\nabla_\alpha+g_{\nu\beta}\nabla_\mu\Box\nabla_\alpha+g_{\nu\beta}R^\lambda{}_\alpha\nabla_\lambda\nabla_\mu+R_{\nu\beta\alpha\lambda}\nabla^\lambda\nabla_\mu+g_{\nu\beta}R_{\mu\alpha}\Box$$
$$+R_{\nu\beta\alpha\mu}\Box+\frac{1}{2}g_{\nu\beta}R_{\mu\tau\rho\lambda}R_\alpha{}^{\tau\rho\lambda}+\frac{1}{2}R_{\alpha\mu}{}^{\rho\lambda}R_{\nu\beta\rho\lambda}\Big)h^{\alpha\beta},$$
$$g_{\nu\beta}\nabla_\alpha\Box\nabla_\mu h^{\alpha\beta}=(2g_{\nu\beta}R^\rho{}_{\alpha\lambda\mu}\nabla_\rho\nabla^\lambda+2R_{\nu\beta\lambda\mu}\nabla_\alpha\nabla^\lambda+R_{\mu\alpha\beta\nu}\Box+g_{\nu\beta}(2R_{\rho\alpha}\nabla_\mu\nabla^\rho+R_{\alpha\mu}\Box+\nabla_\mu\Box\nabla_\alpha$$
$$+2R_{\beta\nu\lambda\alpha}\nabla_\mu\nabla^\lambda)h^{\alpha\beta},$$
$$g_{\alpha\beta}\nabla_\rho\nabla_\lambda\nabla^\rho\nabla^\lambda h^{\alpha\beta}=(R_{\rho\lambda}\nabla^\rho\nabla^\lambda+\Box^2)h, \tag{A3}$$
$$\nabla_\alpha\nabla_\mu\nabla_\beta\nabla_\nu h^{\alpha\beta}=(R^\rho{}_{\alpha\beta\nu}\nabla_\rho\nabla_\mu+R_{\nu\beta}\nabla_\alpha\nabla_\mu+2R_{\mu\alpha}\nabla_\nu\nabla_\beta-R^\rho{}_{\nu\alpha\mu}\nabla_\rho\nabla_\beta+\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta)h^{\alpha\beta},$$
$$\nabla_\alpha\nabla_\beta\nabla_\mu\nabla_\nu h^{\alpha\beta}=[\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta+4R_{\mu\alpha}\nabla_\nu\nabla_\beta-R^\rho{}_{\nu\alpha\mu}(\nabla_\rho\nabla_\beta+\nabla_\beta\nabla_\rho)+2R^\rho{}_{\alpha\beta\mu}\nabla_\rho\nabla_\nu]h^{\alpha\beta},$$
$$\nabla^\lambda\nabla_\mu\nabla_\lambda\nabla_\nu h=(R_{\mu\rho}\nabla^\rho\nabla_\nu-R_{\rho\nu\lambda\mu}\nabla^\lambda\nabla^\rho+\nabla_\mu\Box\nabla_\nu)h,$$
$$\Box\nabla_\mu\nabla_\nu h=(2R^\rho{}_{\mu\nu}{}^\lambda\nabla_\rho\nabla_\lambda+R_{\rho\mu}\nabla^\rho\nabla_\nu+\nabla_\mu\Box\nabla_\nu)h,$$
$$\nabla_\alpha\nabla^\lambda\nabla_\beta\nabla_\lambda h^{\alpha\beta}=(R_{\rho\beta}\nabla_\alpha\nabla^\rho+R^\rho{}_{\alpha\beta}{}^\lambda\nabla_\rho\nabla_\lambda+\nabla_\alpha\Box\nabla_\beta)h^{\alpha\beta}.$$

Using these relations, we can rewrite the bilinear expansions in a more useful form,

$$(\sqrt{-g}R^2_{\mu\nu\alpha\beta})^{(2)}=\sqrt{-g}h^{\mu\nu}\Big\{\delta_{\mu\nu,\alpha\beta}\Box^2+g_{\nu\beta}R_\alpha{}^{\rho\lambda}{}_\mu(2\nabla_\rho\nabla_\lambda-4\nabla_\lambda\nabla_\rho)-R_{\nu\beta\lambda\mu}(\nabla^\lambda\nabla_\alpha-2\nabla_\alpha\nabla^\lambda)+3R_{\mu\alpha\nu\beta}\Box+5R_{\rho\alpha\nu\beta}\nabla_\mu\nabla^\rho$$
$$-g_{\nu\beta}R_{\rho\alpha}(\nabla_\mu\nabla^\rho+\nabla^\rho\nabla_\mu)-g_{\nu\beta}R_{\rho\mu}(\nabla_\alpha\nabla^\rho+\nabla^\rho\nabla_\alpha)-2g_{\nu\beta}\nabla_\mu\Box\nabla_\alpha-2g_{\nu\beta}R_{\mu\alpha}\Box+\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta$$
$$+3g_{\nu\beta}R_{\mu\tau\rho\lambda}R_\alpha{}^{\tau\rho\lambda}+\delta_{\mu\nu,\alpha\beta}R_{\rho\lambda}\nabla^\rho\nabla^\lambda-g_{\mu\nu}(2R_{\rho\alpha\lambda\beta}\nabla^\lambda\nabla^\rho+R_{\rho\lambda\tau\alpha}R^{\rho\lambda\tau}{}_\beta)+\frac{1}{8}R^2_{\rho\sigma\lambda\tau}(g_{\mu\nu}g_{\alpha\beta}-2\delta_{\mu\nu,\alpha\beta})$$
$$+2R_{\mu\nu}(\nabla_\nu\nabla_\beta+\nabla_\beta\nabla_\nu)\Big\}h^{\alpha\beta}, \tag{A4}$$

$$(\sqrt{-g}R^2_{\mu\nu})^{(2)}=\sqrt{-g}h^{\mu\nu}\Big\{\frac{1}{2}\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta-\frac{1}{2}g_{\mu\nu}\nabla_\alpha\Box\nabla_\beta-\frac{1}{2}g_{\nu\beta}\nabla_\mu\Box\nabla_\alpha+\frac{1}{4}(\delta_{\mu\nu,\alpha\beta}+g_{\mu\nu}g_{\alpha\beta})\Box^2+\frac{1}{2}R_{\nu\beta}\nabla_\alpha\nabla_\mu$$
$$-\frac{1}{2}g_{\nu\beta}R_{\rho\mu}(\nabla^\rho\nabla_\alpha+3\nabla_\alpha\nabla^\rho)+\frac{3}{2}g_{\alpha\beta}R_{\rho\mu}\nabla^\rho\nabla_\nu-R_{\mu\alpha\beta\nu}\Box+\frac{1}{4}(2\delta_{\mu\nu,\alpha\beta}-g_{\mu\nu}g_{\alpha\beta})R^{\rho\lambda}\nabla_\rho\nabla_\lambda+R_{\lambda\mu\nu\alpha}R_\beta{}^\lambda$$
$$+R_{\mu\rho\lambda\nu}R_\alpha{}^{\rho\lambda}{}_\beta-\frac{1}{8}(2\delta_{\mu\nu,\alpha\beta}-g_{\mu\nu}g_{\alpha\beta})R^2_{\rho\lambda}+R_{\mu\alpha}R_{\nu\beta}+2g_{\nu\beta}R_{\mu\rho}R^\rho{}_\alpha-g_{\alpha\beta}R_{\mu\rho}R^\rho{}_\nu\Big\}h^{\alpha\beta}, \tag{A5}$$

$$(\sqrt{-g}R^2)^{(2)}=\sqrt{-g}h^{\mu\nu}\Big\{\nabla_\mu\nabla_\nu\nabla_\alpha\nabla_\beta-2g_{\alpha\beta}\nabla_\mu\Box\nabla_\nu+g_{\alpha\beta}R_{\lambda\nu}\nabla_\mu\nabla^\lambda+g_{\mu\nu}R_{\lambda\beta}\nabla^\lambda\nabla_\alpha+g_{\mu\nu}g_{\alpha\beta}\Box^2-g_{\nu\beta}R\nabla_\alpha\nabla_\mu-2R_{\mu\nu}\nabla_\alpha\nabla_\beta$$
$$+2g_{\alpha\beta}R_{\mu\nu}\Box+g_{\mu\nu}R\nabla_\alpha\nabla_\beta+\frac{1}{2}(\delta_{\mu\nu,\alpha\beta}-g_{\mu\nu}g_{\alpha\beta})R\Box-g_{\mu\nu}RR_{\alpha\beta}+\frac{1}{8}(g_{\mu\nu}g_{\alpha\beta}-2\delta_{\mu\nu,\alpha\beta})R^2$$
$$+2Rg_{\nu\beta}R_{\mu\alpha}+R_{\mu\nu}R_{\alpha\beta}\Big\}h^{\alpha\beta}. \tag{A6}$$

APPENDIX B: PARTICULAR RESULTS OF THE CALCULATIONS IN THE BACKGROUND FIELD METHOD

In this appendix one can find the results for the particular elements of the expression (37). One can easily find the contribution of the commutator (38)

$$\frac{1}{6}\,\mathrm{tr}\,\hat{\mathcal{R}}_{\mu\nu}\hat{\mathcal{R}}^{\mu\nu}=-\frac{n+2}{6}R^2_{\mu\nu\alpha\beta}. \tag{B1}$$

After tedious algebra, we arrive at the following result:

$$\mathrm{tr}\,\hat{U}=aR^2_{\mu\nu\alpha\beta}+bR^2_{\mu\nu}+cR^2, \tag{B2}$$

where

$$\begin{pmatrix} a\\ b\\ c\end{pmatrix}=\frac{1}{2n(y+4x)}\begin{pmatrix} 5xn^2+26xn-24x-xn^3+6yn\\ 5yn^2+10yn-24y-yn^3+24xn+8zn\\ 2yn+4xn+6zn-24z+5zn^2-zn^3\end{pmatrix}. \tag{B3}$$

Other relevant traces are the following:

$$\mathrm{tr}(R\hat{V}^\rho{}_\rho)=(n+2)[(n-1)(a_1+na_2)-a_3]R^2, \tag{B4}$$

where

$$\begin{pmatrix} a_1\\ a_2\\ a_3\end{pmatrix}=\frac{2}{n(y+4x)}\begin{pmatrix} nb_1+b_4+b_8-b_9\\ nb_2+b_5+b_6\\ nb_3-b_7\end{pmatrix};$$

also

$$\mathrm{tr}(R_{\rho\sigma}\hat{V}^{\rho\sigma})=a_4R^2_{\mu\nu}+a_5R^2, \tag{B5}$$

where

$$\begin{bmatrix} a_4\\ a_5\end{bmatrix}=\frac{2}{n(y+4x)}\begin{bmatrix}(n-2)b_4+(n^2+n-2)(nb_6+b_8-b_9)+(n+2)b_7+2nb_{10}\\ (n^2+n-2)(b_1+nb_2+b_5)-(n+2)b_3+nb_4-2b_{10}\end{bmatrix},$$

$$\mathrm{tr}(\hat{V}^\rho{}_\rho)^2=12\frac{(3nx+ny+4x)^2}{(y+4x)^2}R^2_{\mu\nu\alpha\beta}+\frac{16}{(y+4x)^2n}\{n^4x^2+8n^3x^2+6n^2x^2+4n^3xy+6n^2xy+y^2n^2+10y^2n-8y^2$$
$$-16nx^2-32x^2+32yx\}R^2_{\mu\nu}+\frac{2}{(y+4x)^2n^2}\{-24n^3xy-48n^2xy+32nxy-8z^2n^2+12z^2n^3+n^6z^2$$
$$-3n^5z^2+n^4y^2-11n^3y^2-128yx-48n^2x^2+32y^2-4n^4x^2-36n^3x^2-2y^2n^2-16y^2n-14n^4zy$$
$$+128x^2-2z^2n^4-4n^5zx-8n^4zx-32zny+56zn^2y+2n^5zy+64znx+32zn^2x-4n^4xy\}R^2, \tag{B6}$$

and finally

$$\mathrm{tr}(\hat{V}_{\rho\sigma}\hat{V}^{\rho\sigma})=c_1R^2_{\mu\nu\alpha\beta}+c_2R^2_{\mu\nu}+c_3R^2, \tag{B7}$$

where the constants c_1, c_2 and c_3 are given by

$$c_1=\frac{12(24nx^2+8nxy+12n^2x^2-32x^2+6n^2xy+y^2n^2)}{(y+4x)^2n}, \tag{B8}$$

$$c_2=\frac{2}{(y+4x)^2n^2}\{24n^4x^2+96n^3x^2+72n^2x^2-192nx^2+48y^2n-10y^2n^2-128yx-192nxy+40n^2xy-32zn^3x+128znx$$
$$+32zn^2x-64zny+16zn^3y+16zn^2y+52n^3xy+12n^4xy+128x^2+3n^4y^2+n^3y^2+8z^2n^4+8z^2n^3+32y^2\}, \tag{B9}$$

and

$$c_3=\frac{4}{(y+4x)^2n^2}\{-4n^3xy+2n^2xy-4nxy+8zn^3x-zn^3y-4z^2n^2-9z^2n^3+n^5z^2+n^3y^2+32x^2+16yx-16n^2x^2-16y^2$$
$$-8n^3x^2-8y^2n-4nx^2+3n^4zy-4zny-48znx+24zn^2x-24zn^2y+4nz^2+48zy-96zx\}. \tag{B10}$$

---

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