Local conservation law and dark radiation in cosmological braneworld

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In the context of the Randall-Sundrum (RS) single-brane scenario, we discuss the bulk geometry and dynamics of a cosmological brane in terms of the local energy conservation law which exists for the bulk that allows slicing with a maximally symmetric three-space. This conservation law enables us to define a local mass in the bulk. We show that there is a unique generalization of the dark radiation on the brane, which is given by the local mass. We find there also exists a conserved current associated with the Weyl tensor, and the corresponding local charge, which we call the Weyl charge, is given by the sum of the local mass and a certain linear combination of the components of the bulk energy-momentum tensor. This expression of the Weyl charge relates the local mass to the projected Weyl tensor, $E_{\mu\nu}$, which plays a central role in the geometrical formalism of the RS braneworld. On the brane, in particular, this gives a decomposition of the projected Weyl tensor into the local mass and the bulk energy-momentum tensor. Then, as an application of these results, we consider a null dust model for the bulk energy-momentum tensor and discuss the black hole formation in the bulk. We investigate the causal structure by identifying the locus of the apparent horizon and clarify possible brane trajectories in the bulk. We find that the brane stays always outside the black hole as long as it is expanding. We also find an upper bound on the value of the Hubble parameter in terms of the matter energy density on the brane, irrespective of the energy flux emitted from the brane.

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I. INTRODUCTION

The braneworld scenario has attracted much attention in recent years [1]. In this scenario, our Universe is assumed to be on a (mem)brane embedded in a higher-dimensional spacetime. There are many models of the braneworld scenario and corresponding cosmologies. One of them that has been extensively studied is the braneworld cosmology based on a model proposed by Randall and Sundrum (RS) [2], in which a single positive tension brane exists in a five-dimensional spacetime (called the bulk) with negative cosmological constant, the so-called RS2 model. In this paper, we focus our discussion on this single-brane model.

In many cases, the five-dimensional bulk geometry is assumed to be anti–de Sitter (AdS) or AdS-Schwarzschild [3–5]:

$$ds^2=-\left(K+\frac{r^2}{\ell^2}-\frac{M_0}{r^2}\right)dt^2+\left(K+\frac{r^2}{\ell^2}-\frac{M_0}{r^2}\right)^{-1}dr^2+r^2d\Omega^2_{(K,3)}, \tag{1.1}$$

where $\ell:=\sqrt{-6/\Lambda_5}$ is the AdS curvature radius, M_0 is the black hole mass, and $d\Omega^2_{(K,3)}$ is the maximally symmetric (constant curvature) three-space with $K=-1$, 0, or $+1$. The brane trajectory in the bulk, $(t,r)=(t(\tau),r(\tau))$, is determined by the junction condition [6]. As usual, we impose the reflection symmetry with respect to the brane. Then, we obtain the effective Friedmann equation on the brane as [4,5]

$$\left(\frac{\dot r}{r}\right)^2+\frac{K}{r^2}=\left(\frac{\kappa_5^4}{36}\sigma^2-\frac{1}{l^2}\right)+\frac{\kappa_5^4}{18}(2\sigma\rho+\rho^2)+\frac{M_0}{r^4}, \tag{1.2}$$

where σ and ρ are the brane tension and energy density of the matter on the brane, respectively, and $\dot r=dr/d\tau$ with τ being the proper time on the brane. The final term is proportional to the mass of the bulk black hole and is often called the ''dark radiation'' since it behaves as the ordinary radiation. Geometrically, it comes from the projected Weyl tensor in the bulk, denoted commonly by $E_{\mu\nu}$ [7]. If we apply Eq. (1.2) to the real Universe, the values of σ, ℓ, and M_0 are constrained by observations of the cosmological parameters [8].

When the bulk ceases to be pure AdS-Schwarzschild, or when there exists a dynamical degree of freedom other than the metric, the parameter M_0 is no longer constant in general, but becomes dynamical. For instance, this is the case of the so-called bulk inflaton model [9–13], or when the brane radiates gravitons into the bulk [15]. In particular, in Ref. [10], the dynamics of a bulk scalar field is investigated in the context of the bulk inflaton model under the assumption that the backreaction of the scalar field on the geometry is small, and it is found that there exists an interesting integral expression for the projected Weyl tensor in terms of the energy-momentum tensor of the scalar field. This suggests the existence of a local conservation law in the bulk that directly relates the dark radiation on the brane to the dynamics in the bulk.

In this paper, we investigate the case when there is nontrivial dynamics in the bulk, and clarify the relation between the bulk geometry and the dynamics of the brane. We focus on the case of isotropic and homogeneous branes and hence assume the existence of slicing by the maximally symmetric three-space as in Eq. (1.1). In this case, we can derive a local energy conservation law in the bulk, in analogy with spherical symmetric spacetimes in four dimensions [16]. Then, this

conservation law can be used to relate the brane dynamics to the geometrical properties of the bulk, especially with the projected Weyl tensor in the bulk.

The paper is organized as follows. In Sec. II, we derive the local energy conservation law in the bulk and discuss the general property of the bulk geometry and cosmology on the brane. We show that there exists a unique generalization of the dark radiation that is directly related to the local mass in the bulk. We also find that there exists another conserved current associated with the Weyl tensor, as a nonlinear version of what was found in Ref. [10]. In a vacuum (Ricci flat) spacetime, the local charge for this current is found to be equivalent to the local mass. Let us call this the Weyl charge. The difference between the local mass and Weyl charge is given by the linear combination of certain components of the bulk energy-momentum tensor, and the projected Weyl tensor that appears in the effective Friedmann equation on the brane is indeed given by this Weyl charge. Thus we have a unique decomposition of the projected Weyl tensor term into the part due to the bulk mass that generalizes the dark radiation term and the part due to the bulk energy-momentum tensor. In Sec. III, as an application of the conservation law derived in Sec. II, we consider a simple null dust model and discuss the black hole formation in the bulk. We identify the location of an apparent horizon and analyze possible trajectories of the brane in the bulk. We show that the brane stays always outside of the apparent horizon of the black hole as long as the brane is expanding. In Sec. IV, we summarize our work and mention future issues.

II. LOCAL CONSERVATION LAW IN A SPACETIME WITH MAXIMALLY SYMMETRIC THREE-SPACE

In this section, we discuss the general property of a dynamical bulk spacetime with a maximally symmetric three-space, and consider cosmology on the brane. First, we derive a local conservation law in the bulk, as a generalization of the local energy conservation law in a spherically symmetric spacetime in four dimensions [16]. Namely, we show that a locally conserved energy flux vector exists in spite of the absence of a timelike Killing vector field. This enables us to define a local mass in the bulk spacetime. We also show that there exists a conserved current associated with the Weyl tensor. This gives rise to a locally defined Weyl charge. It is shown that the Weyl charge and the local mass are closely related to each other.

Next, we introduce the brane as a boundary of the dynamical spacetime. The effective Friedmann equation, is determined via the junction condition, and it is shown that the local mass corresponds to the generalized dark radiation. Finally, we show that the projected Weyl tensor on the brane is uniquely related to the local mass.

A. Local conservation law

We assume that the bulk allows slicing by a maximally symmetric three-space. Then, the bulk metric can written in the double-null form

$$ds^2=\frac{4r_{,u}r_{,v}}{\Phi}dudv+r(u,v)^2d\Omega^2_{(K,3)}, \tag{2.1}$$

where we refer to v and u as the advanced and retarded time coordinates, respectively. In Appendix A, the explicit components of the connection and curvature in an $(n+2)$-dimensional spacetime with maximally symmetric n-space are listed.

The five-dimensional Einstein equations are given by

$$G_{ab}+\Lambda_5 g_{ab}=\kappa_5^2 T_{ab}+S_{ab}\delta(y-y_0), \tag{2.2}$$

where the indices $\{a,b\}$ run from 0 to 3, and 5, and Λ_5 and κ_5^2 are the five-dimensional cosmological constant and gravitational constant, respectively. The brane is introduced as a singular hypersurface located at $y=y_0$, where y denotes a Gaussian normal coordinate in the direction of the extra dimension in the vicinity of the brane, and S_{ab} denotes the energy-momentum tensor on the brane. The spacetime is assumed to be reflection symmetric with respect to the brane.

First, we consider the Einstein equations in the bulk. They are given by

$$3\frac{r_{,u}}{r}\left(\log\left|\frac{r_{,v}}{\Phi}\right|\right)_{,u}=\kappa_5^2T_{uu}, \quad 3\frac{r_{,v}}{r}\left(\log\left|\frac{r_{,u}}{\Phi}\right|\right)_{,v}=\kappa_5^2T_{vv},$$

$$6\frac{r_{,u}r_{,v}}{r^2}\left(1-\frac{K}{\Phi}\right)+3\frac{r_{,uv}}{r}=\kappa_5^2T_{uv}-\frac{2r_{,u}r_{,v}}{\Phi}\Lambda_5, \tag{2.3}$$

$$\left\{\frac{r^2\Phi}{2r_{,u}r_{,v}}\left[\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,uv}+4\frac{r_{,uv}}{r}\right]-(K-\Phi)\right\}\gamma_{ij}$$
$$=\kappa_5^2T_{ij}-r^2\gamma_{ij}\Lambda_5,$$

where γ_{ij} is the intrinsic metric of the maximally symmetric three-space.

Now, we derive the local conservation law. We introduce a vector field in five-dimensional spacetime as

$$\xi^a=\frac{1}{2}\Phi\left(-\frac{1}{r_{,v}}\frac{\partial}{\partial v}+\frac{1}{r_{,u}}\frac{\partial}{\partial u}\right)^a. \tag{2.4}$$

From the form of the metric (2.1), we can readily see that ξ^a is conserved:

$$\sqrt{-g}\,\xi^a{}_{;a}=(\sqrt{-g}\,\xi^a)_{,a}=2\sqrt{\gamma}[(r^3r_{,u})_{,v}-(r^3r_{,v})_{,u}]=0, \tag{2.5}$$

where $\gamma=\det\gamma_{ij}$. Note that, for an asymptotically constant curvature spacetime, the vector field ξ^a becomes asymptotically the timelike Killing vector field $-(\partial/\partial t)^a$.

With this vector field ξ^a, we define a new vector field,

$$\tilde{S}^a=\xi^b\tilde{T}_b{}^a, \tag{2.6}$$

where

$$\tilde{T}_{ab}=T_{ab}-\frac{1}{\kappa_5^2}\Lambda_5 g_{ab}\,. \tag{2.7}$$

Using the Einstein equations, the components of the vector field $\tilde{S}^a$ are given by

$$\kappa_5^2\sqrt{-g}\tilde{S}^v=\frac{3}{2}[r^2(K-\Phi)]_{,u}\sqrt{\gamma},$$

$$\kappa_5^2\sqrt{-g}\tilde{S}^u=-\frac{3}{2}[r^2(K-\Phi)]_{,v}\sqrt{\gamma}. \tag{2.8}$$

Then, we have the local conservation law as

$$\tilde{S}^a{}_{;a}=0. \tag{2.9}$$

Since ξ^a is conserved separately, the conservation of $\tilde{S}^a$ implies that we have another conserved current S^a defined by

$$S^a:=\xi^b T_b{}^a \quad \left(=\tilde{S}^a+\frac{1}{\kappa_5^2}\Lambda_5\xi^a\right). \tag{2.10}$$

Thus we have the local conservation law for the energy-momentum tensor in the bulk.

From Eqs. (2.8), we readily see the local mass corresponding to $\tilde{S}^a$ is given by [16]

$$\tilde{M}:=(K-\Phi)r^2, \tag{2.11}$$

where the factor 3/2 in the original expression for $\tilde{S}^a$ is eliminated for later convenience. Alternatively, corresponding to S^a, we have another local mass that excludes the contribution of the bulk cosmological constant,

$$M:=\tilde{M}-\frac{1}{6}\Lambda_5 r^4=(K-\Phi)r^2-\frac{1}{6}\Lambda_5 r^4. \tag{2.12}$$

In what follows, we focus on the matter part M, rather than on the whole mass $\tilde{M}$. It may be noted, however, that this decomposition of $\tilde{M}$ to the cosmological constant part and the matter part is rather arbitrary, as in the case of a bulk scalar field. Here we adopt this decomposition just for convenience. For example, this decomposition is more useful when we consider small perturbations on the static AdS-Schwarzschild bulk. We note that, in the case of a spherically symmetric asymptotic flat spacetime in four dimensions (hence $K=+1$ and with no cosmological constant), this function M agrees with the Arnowitt-Deser-Misner (ADM) energy or the Bondi energy in the appropriate limits.

B. Local mass and Weyl charge

From the five-dimensional Einstein equations (2.3), we can write down the local conservation equation for M in terms of the bulk energy-momentum tensor explicitly as

$$M_{,v}=\frac{2}{3}\kappa_5^2 r^3(T^u{}_v r_{,u}-T^v{}_v r_{,v}),$$

$$M_{,u}=\frac{2}{3}\kappa_5^2 r^3(T^v{}_u r_{,v}-T^v{}_v r_{,u}), \tag{2.13}$$

or in a bit more concise form,

$$dM=\frac{2}{3}\kappa_5^2 r^3(T^u{}_v r_{,u}dv+T^v{}_u r_{,v}du-T^v{}_v dr). \tag{2.14}$$

Using the above, we can immediately write down two integral expressions for M given in terms of flux crossing the $u=$const hypersurfaces from v_1 to v_2, and flux crossing the $v=$const hypersurfaces from u_1 to u_2, respectively, as

$$M(v_2,u)-M(v_1,u)=\frac{2}{3}\kappa_5^2\int_{v_1}^{v_2}dv\,r^3(T^u_v r_{,u}-T^v_v r_{,v})\Big|_{u=\text{const.}},$$

$$M(v,u_2)-M(v,u_1)=\frac{2}{3}\kappa_5^2\int_{u_1}^{u_2}du\,r^3(T^v_u r_{,v}-T^v_v r_{,u})\Big|_{v=\text{const.}}. \tag{2.15}$$

Finally, let us consider the Weyl tensor in the bulk. In the present case of a five-dimensional spacetime with maximally symmetric three-space, there exists only one nontrivial component of the Weyl tensor, say $C_{vu}{}^{vu}$. The explicit expressions for the components of the Weyl tensor are given in Appendix A, Eqs. (A7). Using the Bianchi identities and the Einstein equations, we have [26]

$$C_{abcd}{}^{;d}=J_{abc}\,, \tag{2.16}$$

where

$$J_{abc}=\frac{2(n-1)}{n}\kappa_{n+2}^2\left(T_{c[a;b]}+\frac{1}{(n+1)}g_{c[b}T_{;a]}\right). \tag{2.17}$$

From this, we can show that there exists a conserved current,

$$Q^a=r\ell_b n_c J^{bca},\quad Q^a{}_{;a}=0, \tag{2.18}$$

where ℓ_a and n_a are a set of two hypersurface orthogonal null vectors,

$$\ell_a=\sqrt{\frac{2}{\Phi}}(-r_{,v}dv)_a,\quad \ell^a=-\sqrt{\frac{1}{2}\Phi}\,\frac{1}{r_{,u}}\left(\frac{\partial}{\partial u}\right)^a,$$

$$n_a=\sqrt{\frac{2}{\Phi}}(r_{,u}du)_a,\quad n^a=\sqrt{\frac{1}{2}\Phi}\,\frac{1}{r_{,v}}\left(\frac{\partial}{\partial v}\right)^a. \tag{2.19}$$

The nonzero components are written explicitly as

$$Q^u=-rJ^{vu}{}_v,\quad Q^v=-rJ^{vu}{}_u, \tag{2.20}$$

and we have

$$(r^4 C_{vu}{}^{vu})_{,v} = r^4 J^v{}_{vv},$$
$$(r^4 C_{vu}{}^{vu})_{,u} = r^4 J^u{}_{uu}. \tag{2.21}$$

These are very similar to Eqs. (2.8). It is clear that $r^4 C_{vu}{}^{vu}$ defines a local charge associated with this conserved current, that is, the Weyl charge.

Using the Einstein equations, we then find that the Weyl charge can be expressed in terms of M and the energy-momentum tensor as

$$r^4 C_{vu}{}^{vu} = 3\tilde{M} + \frac{r^4}{6}(6G^v{}_v - G^i{}_i) = 3M + \frac{\kappa_5^2}{6} r^4 (6T^v{}_v - T^i{}_i). \tag{2.22}$$

This is one of the most important results in this paper. As we shall see below, the Weyl component $C_{vu}{}^{vu}$ is directly related to the projected Weyl tensor $E_{\mu\nu}$, and hence this relation gives explicitly how the local mass M and the local value of the energy-momentum tensor affects the brane dynamics.

C. Apparent horizons

As in the conventional four-dimensional gravity, the gravitational dynamics may lead to the formation of a black hole in the bulk. Rigorously speaking, the black hole formation can be discussed only by analyzing the global causal structure of a spacetime. Nevertheless, we discuss the black hole formation by studying the formation of an apparent horizon.

In four dimensions, an apparent horizon is defined as a closed two-sphere on which the expansion of an outgoing (or ingoing) null geodesic congruence vanishes. Here, we extend the definition to our case and define an apparent horizon as a three-surface on which the expansion of a radial null geodesic congruence vanishes. Note that ''radial'' here means simply those congruences that have only the (v,u) components; hence an apparent horizon will not be a closed surface if $K=0$.

The expansions of the congruence of null geodesics forming the $u=$const and $v=$const hypersurfaces, respectively, are given by [16]

$$\rho_u = -\frac{1}{2} u^{;a}{}_{;a} = -\frac{1}{2r}\frac{\Phi}{r_{,u}}, \quad \rho_v = -\frac{1}{2} v^{;a}{}_{;a} = -\frac{1}{2r}\frac{\Phi}{r_{,v}}. \tag{2.23}$$

Naively, if $\Phi=0$, one might think that both ρ_u and ρ_v vanish. However, from the regularity condition of the metric (2.1), we have

$$-4\frac{r_{,u} r_{,v}}{\Phi} > 0. \tag{2.24}$$

Hence, it must be that $r_{,u}=0$ or $r_{,v}=0$ if $\Phi=0$. If $\Phi = r_{,v}=0$, we have $\rho_u=0$ and an apparent horizon for the outgoing null geodesics is formed, whereas if $\Phi = r_{,u}=0$, we have $\rho_v=0$ and an apparent horizon for the ingoing null geodesics is formed.

D. Brane cosmology

We now consider the dynamics of a brane in a dynamical bulk with maximally symmetric three-space [3]. The brane trajectory is parametrized as $(v,u)=(v(\tau),u(\tau))$. Taking τ to be the proper time on the brane, we have

$$4\frac{r_{,u} r_{,v}}{\Phi}\dot{u}\dot{v} = -1 \tag{2.25}$$

on the brane, where $\dot{u} = du/d\tau$ and so on. The unit vector tangent to the brane (i.e., the five-velocity of the brane) is given by

$$v^a = \left(\dot{v}\frac{\partial}{\partial v} + \dot{u}\frac{\partial}{\partial u}\right)^a, \quad v_a = \frac{2r_{,u} r_{,v}}{\Phi}(\dot{u}dv + \dot{v}du)_a, \tag{2.26}$$

and the unit normal to the brane is given by

$$n^a = \left(-\dot{v}\frac{\partial}{\partial v} + \dot{u}\frac{\partial}{\partial u}\right)^a, \quad n_a = \frac{2r_{,u} r_{,v}}{\Phi}(\dot{u}dv - \dot{v}du)_a. \tag{2.27}$$

The components of the induced metric on the brane are calculated as

$$q_{\mu\nu} = \frac{\partial x^a}{\partial y^\mu}\frac{\partial x^b}{\partial y^\nu} g_{ab}, \tag{2.28}$$

where μ, ν run from 0 to 3 and y^μ are the intrinsic coordinates on the brane with $y^0=\tau$ and $y^i = x^i$ $(i=1,2,3)$. Then the induced metric on the brane is given by

$$ds^2_{(4)} = -d\tau^2 + r(\tau)^2 d\Omega^2_{(K,3)}. \tag{2.29}$$

The trajectory of the brane is determined by the junction condition under the Z_2 symmetry with respect to the brane. The extrinsic curvature on the brane is determined as

$$K_{\mu\nu} = -\frac{\kappa_5^2}{2}\left(S_{\mu\nu} - \frac{1}{3} S q_{\mu\nu}\right), \tag{2.30}$$

where $S_{\mu\nu}$ is assumed to take the form

$$S^\mu_\nu = \mathrm{diag}(-\rho, p, p, p) - \sigma\delta^\mu_\nu, \tag{2.31}$$

with σ and ρ being the tension and energy density of the matter on the brane, respectively, as introduced previously, and p being the isotropic pressure of the matter on the brane. Substituting the induced metric (2.29) in Eq. (2.30), we obtain

$$r_{,u}\dot{u} = -\frac{r}{2}\left[\frac{\kappa_5^2}{6}(\rho+\sigma) - H\right], \tag{2.32}$$

$$r_{,v}\dot{v}=\frac{r}{2}\left[\frac{\kappa_5^2}{6}(\rho+\sigma)+H\right], \tag{2.33}$$

where $H=\dot{r}/r$. Multiplying the above two equations and using the normalization condition (2.25), we then obtain the effective Friedmann equation on the brane:

$$H^2+\frac{K}{r^2}=\left(\frac{\kappa_5^4}{36}\sigma^2-\frac{1}{l^2}\right)+\frac{\kappa_5^4}{18}(2\sigma\rho+\rho^2)+\frac{M}{r^4}. \tag{2.34}$$

We see that M is a natural generalization of the dark radiation in the AdS-Schwarzschild case to a dynamical bulk.

For a dynamical bulk, M varies in time. The evolution of M is determined by Eq. (2.14), and on the brane it gives

$$\begin{aligned}\dot{M}&=M_{,v}\dot{v}+M_{,u}\dot{u}\\&=\frac{2}{3}\kappa_5^2 r^4\left[T_{vv}\left(\frac{1}{6}\kappa_5^2(\rho+\sigma)-H\right)\dot{v}^2\right.\\&\quad\left.-T_{uu}\left(\frac{1}{6}\kappa_5^2(\rho+\sigma)+H\right)\dot{u}^2\right]-\frac{2}{3}\kappa_5^2 r^4 H T^v{}_v.\end{aligned} \tag{2.35}$$

This result is consistent with Refs. [12,15]. From the Codacci equation on the brane [7],

$$D_\nu K^\nu{}_\mu-D_\mu K^\nu{}_\nu=\kappa_5^2 T_{ab}n^b q^a{}_\mu, \tag{2.36}$$

where D_μ is the covariant derivative with respect to $q_{\mu\nu}$ and $K_{\mu\nu}$ is the extrinsic curvature of the brane, we obtain the equation for the energy transfer of the matter on the brane to the bulk,

$$\dot{\rho}+3H(\rho+p)=2(-T_{vv}\dot{v}^2+T_{uu}\dot{u}^2). \tag{2.37}$$

Equations (2.34), (2.35), and (2.37) determine the cosmological evolution on the brane, once the bulk geometry is solved. These equations will be applied to a null dust model in the next section. The case of the Einstein-scalar theory in the bulk is briefly discussed in Appendix B.

Now we relate the above result to the geometrical approach developed in Ref. [7], in particular with the $E_{\mu\nu}$ term on the brane. The projected Weyl tensor

$$E_{\mu\nu}=C_{a\mu b\nu}n^a n^b \tag{2.38}$$

has only one nontrivial component as

$$E_{\tau\tau}=C_{abcd}n^a n^c v^b v^d=4C_{uvuv}\dot{u}^2\dot{v}^2=-C_{vu}{}^{vu}. \tag{2.39}$$

Using Eq. (2.22), this can be uniquely decomposed into the part proportional to M and the part due to the projection of the bulk energy-momentum tensor on the brane. We find

$$E_{\tau\tau}=-\frac{3\tilde{M}}{r^4}+\frac{1}{6}(G^i{}_i-6G^v{}_v)=-\frac{3M}{r^4}+\frac{\kappa_5^2}{6}(T^i{}_i-6T^v{}_v). \tag{2.40}$$

If we eliminate the M/r^4 term from Eq. (2.34) by using this equation, we recover the effective Friedmann equation on the brane in the geometrical approach [7],

$$H^2+\frac{K}{r^2}=\left(\frac{\kappa_5^4}{36}\sigma^2-\frac{1}{l^2}\right)+\frac{\kappa_5^4}{18}(2\sigma\rho+\rho^2)+\kappa_5^2 T^{(b)}_{\tau\tau}-\frac{E_{\tau\tau}}{3}, \tag{2.41}$$

where $T^{(b)}_{\tau\tau}$ comes from the projection of the bulk energy-momentum tensor on the brane and is given in the present case by

$$T^{(b)}_{\tau\tau}=\frac{1}{6}T^i{}_i-T^v{}_v. \tag{2.42}$$

Finally, from the brane point of view, it may be worthwhile to give the expressions for the effective total energy density and pressure on the brane. They are given by

$$\rho^{(\mathrm{tot})}=\rho^{(\mathrm{brane})}+\rho^{(\mathrm{bulk})},\quad p^{(\mathrm{tot})}=p^{(\mathrm{brane})}+p^{(\mathrm{bulk})}, \tag{2.43}$$

where

$$\begin{aligned}\kappa_4^2\rho^{(\mathrm{brane})}&=3\left[\frac{1}{6}\kappa_5^2(\rho+\sigma)\right]^2,\\\kappa_4^2 p^{(\mathrm{brane})}&=\frac{1}{12}\kappa_5^4(\rho+\sigma)(\rho-\sigma+2p),\\\kappa_4^2\rho^{(\mathrm{bulk})}&=\frac{3\tilde{M}}{r^4},\\\kappa_4^2 p^{(\mathrm{bulk})}&=\frac{\tilde{M}}{r^4}+\frac{1}{3}\kappa_5^2\left(-\frac{\dot{u}}{\dot{v}}\tilde{T}^v_u-\frac{\dot{v}}{\dot{u}}\tilde{T}^u_v+2\tilde{T}^v_v\right),\end{aligned} \tag{2.44}$$

where $\tilde{M}$ is given by Eq. (2.11) and $\tilde{T}^a{}_b$ is defined by Eq. (2.7), and both contain the contribution from the bulk cosmological constant. It may be noted that, unlike the effective energy density, the effective pressure contains a part coming from the bulk that cannot be described by the local mass alone. The contracted Bianchi identity implies the conservation law for the total effective energy-momentum on the brane:

$$\begin{aligned}&\dot{\rho}^{(\mathrm{bulk})}+3H(\rho^{(\mathrm{bulk})}+p^{(\mathrm{bulk})})\\&\quad=-\dot{\rho}^{(\mathrm{brane})}-3H(\rho^{(\mathrm{brane})}+p^{(\mathrm{brane})}).\end{aligned} \tag{2.45}$$

This is mathematically equivalent to Eq. (2.35). However, these two equations have different interpretations. From the bulk point of view, Eq. (2.35) is more relevant, which describes the energy exchange between the brane and the bulk, whereas a natural interpretation of Eq. (2.45) is that it describes the energy exchange between two different matters on the brane: the intrinsic matter on the brane and the bulk matter induced on the brane. The important point is, as mentioned above, that the pressure of the bulk matter has contri-

butions not only from the local mass but also from a projection of the bulk energy-momentum tensor, which makes the equation of state different from $p^{(\mathrm{bulk})}=\rho^{(\mathrm{bulk})}/3$, i.e., that of a simple dark radiation.

III. APPLICATION TO THE NULL DUST MODEL

In this section, by using the local mass derived in the preceding section, we discuss the bulk geometry and brane cosmology in the context of a null dust model. Especially, we pay attention to the gravitational collapse due to the emission of energy from the brane. Namely, we consider an ingoing null dust fluid emitted from the brane [15,17,18].

A. Setup

The energy-momentum tensor of a null dust fluid takes the form [24],

$$T_{ab}=\mu_1 \ell_a \ell_b+\mu_2 n_a n_b, \tag{3.1}$$

where ℓ_a and n_a are the ingoing and outgoing null vectors, respectively, introduced in Eqs. (2.19). If we require that the energy-momentum conservation law is satisfied for the ingoing and outgoing null dust independently, we have

$$\mu_1=\frac{\Phi}{(r_{,v})^2 r^3}\frac{f(v)}{2}, \quad \mu_2=\frac{\Phi}{(r_{,u})^2 r^3}\frac{g(u)}{2}, \tag{3.2}$$

where $f(v)$ and $g(u)$ are arbitrary functions of v and u, respectively, and have the dimension $(G_5\times \mathrm{mass})^{-1}$. We assume the positive energy density, i.e., $f(v)\geqslant 0$ and $g(u)\geqslant 0$. Thus, the nontrivial components of the energy-momentum tensor are

$$T_{vv}=\frac{f(v)}{r^3}, \quad T_{uu}=\frac{g(u)}{r^3}. \tag{3.3}$$

To satisfy the local conservation law in an infinitesimal interval $(u,u+du)$ and $(v,v+dv)$, we find that the intensity functions $f(v)$ and $g(u)$ have to satisfy the relation

$$f(v)\left(\frac{\Phi}{r_{,v}}\right)_{,u}=g(u)\left(\frac{\Phi}{r_{,u}}\right)_{,v}. \tag{3.4}$$

In general, if both $f(v)$ and $g(u)$ are nonzero, it seems almost impossible to find an analytic solution that satisfies Eq. (3.4). Hence we choose to set either $f(v)=0$ or $g(u)=0$. In the following discussion, we focus on the case that $g(u)=0$, that is, the ingoing null dust.

B. Bulk geometry of the null dust collapse

For $g(u)=0$, Eqs. (2.14) give

$$M_{,v}=\frac{1}{3}\kappa_5^2\frac{\Phi}{r_{,v}}f(v), \quad M_{,u}=0. \tag{3.5}$$

The second equation implies $M=M(v)$. Substituting Eq. (3.3) into the Einstein equations (2.3), we find

$$\frac{\Phi}{r_{,v}}=e^{F(v)}, \tag{3.6}$$

where the function $F(v)$ describes the freedom in the rescaling off the null coordinate v. This equation is consistent with Eq. (3.4). Thus, we obtain the solution as

$$\Phi=r_{,v}e^{F(v)}=K+\frac{r^2}{\ell^2}-\frac{M(v)}{r^2},$$

$$M(v)=\frac{1}{3}\kappa_5^2\int_{v_0}^{v}dv e^{F(v)}f(v)+M_0, \tag{3.7}$$

where we have assumed that $f(v)=0$ for $v<v_0$, that is, v_0 is the epoch at which the ingoing flux is turned on. For definiteness, we assume that the bulk is pure AdS at $v<v_0$ and set $M_0=0$ in what follows.

Transforming the double-null coordinates (v,u) to the half-null coordinates (v,r) as

$$r_{,u}du=dr-r_{,v}dv, \tag{3.8}$$

the solution is expressed as

$$ds^2=-4\Phi(r,v)e^{-2F(v)}dv^2+4e^{-F(v)}dvdr+r^2 d\Omega^2_{(K,3)}, \tag{3.9}$$

where Φ is given by the first of Eqs. (3.7). This is an ingoing Vaidya solution with a negative cosmological constant [15,17]. For an arbitrary intensity function $f(v)$, this is an exact solution for the bulk geometry. Note that if we rescale v as $dv\to d\overline{v}=e^{-F}dv$, $f(v)$ scales as $f(v)\to\overline{f}(v)=e^{-2F}f(v)$, which manifestly shows the invariance of the solution under this rescaling.

An apparent horizon for the outgoing radial null congruence is located on the three-space, satisfying

$$\Phi=r_{,v}=0, \quad \text{while} \quad r_{,u}=\text{finite}. \tag{3.10}$$

This gives

$$r^2=\frac{\ell^2}{2}\left(\sqrt{K^2+4\frac{M(v)}{\ell^2}}-K\right). \tag{3.11}$$

The direction of the trajectory of the apparent horizon is given by

$$\frac{dr}{dv}=\frac{M_{,v}\ell^2 r}{2(r^4+M\ell^2)}=\frac{\kappa_5^2 f(v)e^{F(v)}\ell^2 r}{6(r^4+M\ell^2)}. \tag{3.12}$$

Thus, for $f(v)>0$, dr/dv is positive, which implies that the trajectory of the apparent horizon is spacelike.

For the case of $K=+1$ or $K=0$, the apparent horizon originates from $r=0$, while it originates from $r=\ell$ for $K=-1$. A schematic view of the null dust collapse is shown in Fig. 1. We assume that the brane emits the ingoing flux during a finite interval (bounded by the dashed lines in the figures) and no naked singularity is formed. For all the cases,

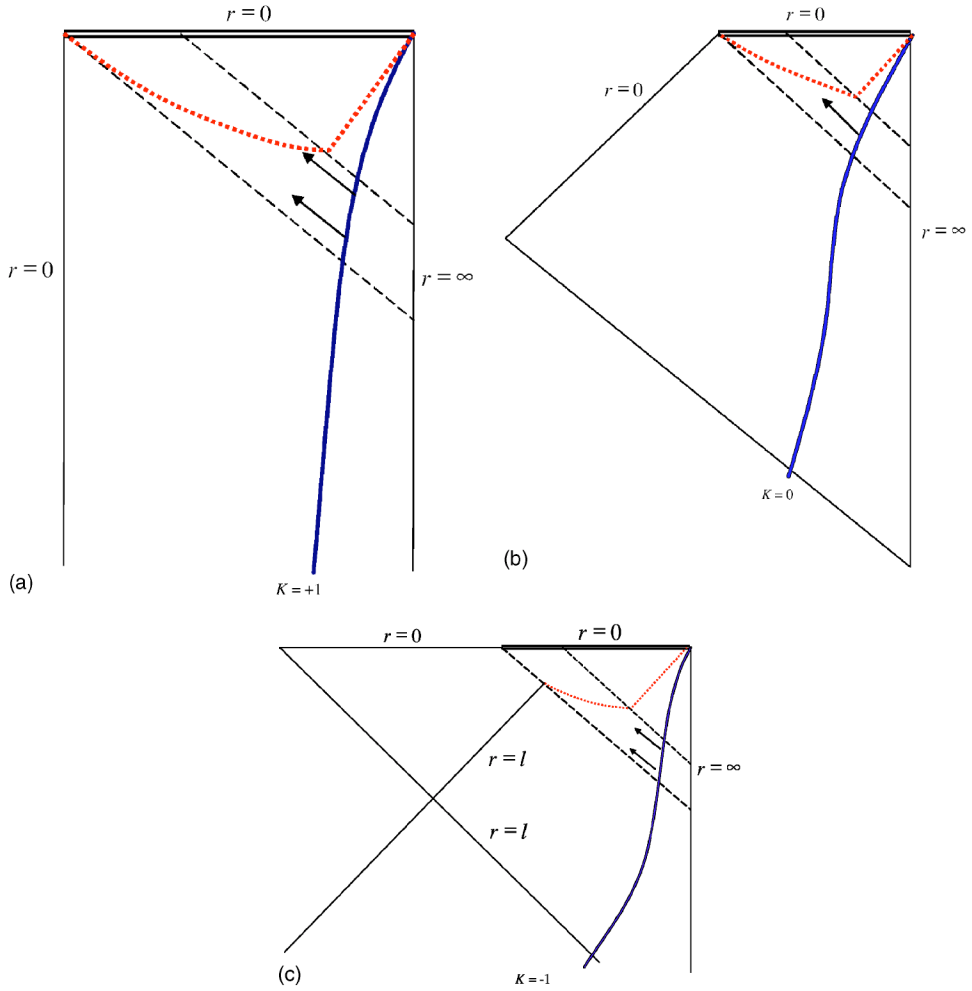

FIG. 1. (Color online) Causal structure of a spacetime with ingoing null dust for the cases of $K=+1$, 0 and -1. In each figure, The (almost vertical) wavy curve represents the brane trajectory and the dotted line is the locus of the apparent horizon. The thick horizontal line at $r=0$ represents the spacelike curvature singularity formed there. The ingoing flux is assumed to be emitted during a finite interval bounded by the dashed lines.

the causal structures after the onset of emission are very similar. The spacelike singularity is formed at $r=0$, but it is hidden inside the apparent horizon.

C. Brane trajectory in the bulk

In the null dust model, using Eq. (2.25), the proper time on the brane is related to the advanced time in the bulk as [18]

$$\dot{v}_{\pm}=e^{F(v)}\frac{\dot{r}\pm\sqrt{\dot{r}^2+\Phi}}{2\Phi}. \tag{3.13}$$

To determine the appropriate sign in the above, we require that the brane trajectory is timelike, hence $\dot{v}>0$, and examine the signs of $\dot{v}_{\pm}$ for all possible cases:

$$(1)\ \ \dot{r}>0,\ \ \Phi>0\rightarrow\dot{v}_{+}>0,\ \ \dot{v}_{-}<0.$$

$$(2)\ \ \dot{r}>0,\ \ \Phi<0\rightarrow\dot{v}_{+}<0,\ \ \dot{v}_{-}<0.$$

$$(3)\ \ \dot{r}<0,\ \ \Phi>0\rightarrow\dot{v}_{+}>0,\ \ \dot{v}_{-}<0.$$

$$(4)\ \ \dot{r}<0,\ \ \Phi<0\rightarrow\dot{v}_{+}>0,\ \ \dot{v}_{-}>0.$$

From these, we can conclude the following. For an expanding brane, $\dot{r}>0$, the brane exists always outside the horizon, $\Phi>0$, and $\dot{v}$ is given by $\dot{v}_{+}$. On the other hand, a contracting brane, $\dot{r}<0$, can exist either outside or inside of the horizon. Thus, if the brane is expanding initially, the trajectory is given by $\dot{v}=\dot{v}_{+}$, and it stays outside the horizon until it starts to recollapse, if ever. If the brane universe starts to recollapse, which is possible only in the case $K=+1$, by continuity, the trajectory is still given by $\dot{v}=\dot{v}_{+}$, and the brane universe is eventually swallowed into the black hole.

From the above result, we find

$$r_{,u}\dot{u}=\dot{r}-r_{,v}\dot{v}=\frac{r-\sqrt{\dot{r}^2+\Phi}}{2}<0. \tag{3.14}$$

Using Eq. (2.32), this gives an upper bound of the Hubble parameter on the brane as

$$H<\frac{1}{6}\kappa_5^2(\rho+\sigma). \tag{3.15}$$

Let us now turn to the effective Friedmann equation on the brane. For simplicity, we tune the brane tension to the Randall-Sundrum value, $\kappa_5^2\sigma=6/\ell$. The effective Friedmann equation on the brane is

$$H^2+\frac{K}{r^2}=\frac{1}{18}\kappa_5^4\rho\sigma+\frac{1}{36}\kappa_5^4\rho^2+\frac{M(\tau)}{r^4}, \tag{3.16}$$

where $M(\tau)=M(v(\tau))$ for notational simplicity. From Eq. (2.37), the energy equation on the brane is given by

$$\dot{\rho}+3\frac{\dot{r}}{r}(\rho+p)=-2\frac{f(\tau)}{r^3}\dot{v}^2, \tag{3.17}$$

where $f(\tau)=f(v(\tau))$. From Eq. (2.35), the time derivative of M is given by

$$\dot{M}=\frac{2}{3}r\kappa_5^2\left[\frac{1}{6}\kappa_5^2(\rho+\sigma)-H\right]f(\tau)\dot{v}^2. \tag{3.18}$$

Thus, from Eq. (3.15), M continues to increase on the brane.

The advanced time in the bulk is related to the proper time on the brane by $\dot{v}_+$ in Eq. (3.13). Specifically, using the equality,

$$\Phi=K+\frac{r^2}{\ell^2}-\frac{M}{r^2}=r^2\left(\frac{\kappa_5^4}{36}(\rho+\sigma)^2-H^2\right) \tag{3.19}$$

on the brane, we have

$$\dot{v}=\frac{e^{F(v)}}{2r}\left(\frac{\kappa_5^2}{6}(\rho+\sigma)-H\right)^{-1}. \tag{3.20}$$

Note that the product $f\dot{v}^2$ is invariant under the rescaling of v. Once $f(\tau)$ is given, we can solve the system of equations (3.16)–(3.18) self-consistently for a given initial condition, and determine the bulk geometry and the brane dynamics at the same time [15]. A quantitative analysis of the brane cosmology is left for future work.

D. Formation of a naked singularity

In the previous subsections, we assumed that there is no naked singularity in the bulk. However, it has been shown that a naked singularity can be formed in the null dust collapse [19–25]. For instance, a naked singularity exists in a Vaidya spacetime when the flux of radiation rises from zero sufficiently slowly. We expect the same is true in the present case.

Without loss of generality, we set $e^{F(v)}=2$. We consider the following situation. For $v<0$, the bulk geometry is purely AdS. The radiative emission from the brane begins at $v=0$. We choose the intensity function as

$$f(v)=\frac{2\lambda}{\kappa_5^2}v, \tag{3.21}$$

where λ is a positive constant. This corresponds to the self-similar Vaidya spacetime if the cosmological constant were absent [19]. The brane ceases to emit radiation at $v=v_0$ and the bulk becomes static AdS-Schwarzschild for $v>v_0$. Thus the local mass is given by

$$M(v)=\begin{cases}0 & (v<0)\\ \frac{2}{3}\lambda v^2 & (0\leqslant v\leqslant v_0)\\ \frac{2}{3}\lambda v_0^2 & (v_0<v).\end{cases} \tag{3.22}$$

The singularity is formed at $(r,v)=(0,0)$, and it is naked if there exists a future-directed radial null geodesic emanating from it. The null geodesics then form a Cauchy horizon. The trajectory of a radial null geodesic is determined by the equation

$$\frac{dr}{dv}=\frac{1}{2}\left(K+\frac{r^2(v)}{\ell^2}-\frac{M(v)}{r^2(v)}\right). \tag{3.23}$$

Let us analyze the above equation in the vicinity of $v=0$. A future-directed radial null geodesic exists if $x:=\lim_{v\to0}dr/dv$ is positive. Using L'Hôpital's theorem, we obtain

$$x=\lim_{v\to0}\frac{r(v)}{v}=\lim_{v\to0}\frac{dr}{dv}=\frac{1}{2}\left(K-\frac{2\lambda}{3x^2}\right). \tag{3.24}$$

It is clear that the above equation has no solution when $K=0$ or $K=-1$. Hence no naked singularity is formed for $K=0$ or $K=-1$. Therefore, we consider the case $K=1$. We introduce a function,

$$Q(x)=3x^3-\frac{3}{2}x^2+\lambda. \tag{3.25}$$

Then, the condition for the naked singularity formation is that $Q(x)=0$ has a solution for a positive x. The function $Q(x)$ has a minimal point at $x=1/3$. Therefore, the singularity is naked if

$$Q(1/3)=-\frac{1}{18}+\lambda\leqslant0, \tag{3.26}$$

that is,

$$0<\lambda\leqslant\frac{1}{18}. \tag{3.27}$$

Thus, the bulk has a naked singularity for small values of λ, i.e., for the flux of radiation which rises slowly enough.

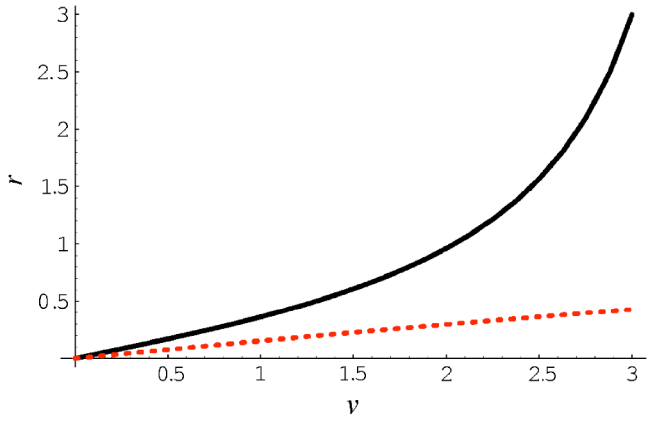

FIG. 2. (Color online) The loci of the null geodesic (the solid curve) and the apparent horizon (the dotted curve) on the (v,r) plane, scaled in units of the AdS radius ℓ, in the critical case $\lambda = 1/18$. Their behaviors are qualitatively the same for all the other values of λ in the range $0<\lambda<1/18$.

Our next interest is whether the naked singularity is local or global. If it is globally naked, it may be visible on the brane. To examine this, we integrate Eq. (3.23). In the vicinity of $v=0$, we find

$$r_{\rm null}(v)=x_0 v\left(1+b\frac{v^2}{\ell^2}+\cdots\right), \tag{3.28}$$

where x_0 is the largest positive root of $Q(x)=0$;

$$x_0=\frac{1}{6}\{1+[1-36\lambda+i6\sqrt{2\lambda(1-18\lambda)}]^{1/3} +[1-36\lambda-i6\sqrt{2\lambda(1-18\lambda)}]^{1/3}\} \tag{3.29}$$

and

$$b=\frac{x_0^2}{2(5x_0-1)}. \tag{3.30}$$

From the form of $Q(x)$, we readily see that x_0 monotonically decreases from 1/2 to 1/3 as λ increases from 0 to 1/18, and hence b is positive definite. We compare this trajectory with the trajectory of the apparent horizon. It is given by Eq. (3.11) with $K=+1$. In the vicinity of $v=0$, it gives

$$r_{\rm app}(v)=\sqrt{\frac{2\lambda}{3}}v\left(1-\frac{\lambda}{8}\frac{v^2}{\ell^2}+\cdots\right). \tag{3.31}$$

Since $x_0>\sqrt{2\lambda/3}$ for all the values of λ in the range $0<\lambda\leqslant 1/18$, and $dr_{\rm app}/dv$ is a decreasing function of v while $dr_{\rm null}/dv$ is an increasing function of v, it follows that the null geodesic lies in the exterior of the apparent horizon and the difference in the radius at the same v increases as v increases, at least when v is small. This suggests that the singularity is globally naked.

In Fig. 2, we plot the loci of the null geodesic and the apparent horizon. The result is clear. The null geodesic always stays outside of the apparent horizon, thus outside of the final event horizon at $v=v_0$. Mathematically, this is due to the cosmological constant term in Eq. (3.23), which strongly drives the null geodesic trajectory to larger values of r. Thus, we conclude that the naked singularity is global and visible on the brane. The causal structure in this case is illustrated in Fig. 3. Investigations on the effect of the visible singularity on the brane are necessary, but they are left for future work.

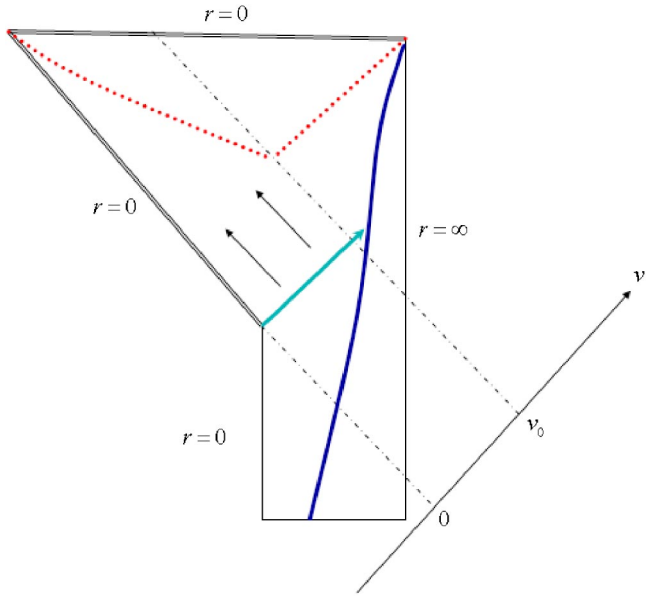

FIG. 3. (Color online) Causal structure of a spacetime with ingoing null dust when a naked singularity is formed. The wavy and almost vertical curve represents the brane trajectory and the dotted line is the locus of the apparent horizon. A naked singularity is formed at $r=0$ along the $v=0$ null line. A radial, future-directed null geodesic originating from the naked singularity (the right-pointed thick line) stays outside of the apparent horizon and reaches the brane.

Finally, let us mention the strength of the naked singularity as we approach it along a radial null geodesic. Let w be an affine parameter of the geodesic, $w=0$ be the singularity, and the tangent vector be denoted by $k^a=dx^a/dw$. We examine $R_{ab}k^ak^b$ and $C_{vu}{}^{vu}$. From Eq. (3.3) and the Einstein equations, we have

$$R_{ab}k^ak^b=\frac{\kappa_5^2 f(v)}{r^3}\left(\frac{dv}{dw}\right)^2=\frac{2\lambda v}{r^3}\left(\frac{dv}{dw}\right)^2 \to_{w\to 0}\frac{2\lambda}{x_0(1-x_0)^2}w^{-2}. \tag{3.32}$$

Also, from Eq. (2.22), we have

$$C_{vu}{}^{vu}=\frac{3M}{r^4}=\frac{2\lambda v^2}{r^4}\to_{w\to 0}\frac{2\lambda}{x_0^4}v^{-2}\propto w^{-2x_0/(1-x_0)}. \tag{3.33}$$

Thus the Ricci tensor and the Weyl tensor diverge as w^{-2} and $w^{-2x_0/(1-x_0)}$, respectively, which is a sign of a strong curvature singularity.

IV. CONCLUSION

In this paper, in the context of the RS2 type braneworld, we discussed the dynamics of the bulk and the effective cosmology on the brane in terms of the local conservation law that exists in the bulk spacetime with a maximally symmetric three-space. First, we formulated the local conservation law in the dynamical bulk. We found that the bulk geometry is completely described by the local mass M and it is directly related to the generalized dark radiation term in the effective Friedmann equation. We also found that there exists a conserved current associated with the Weyl tensor and the projected Weyl tensor that appears in the geometrical approach is just the local charge for this current, and it can be expressed in terms of M and a certain linear combination of the components of the bulk energy-momentum tensor.

Next, as an application of our formalism, we adopted a simple null dust model, in which the energy emitted by the brane is approximated by an ingoing null dust fluid, and investigated the general properties of the bulk geometry and the brane trajectory in the bulk. Usually, the ingoing null dust forms a black hole in the bulk. However, in the case of $K=+1$, a naked singularity can be formed in the bulk when the flux rises from zero slower than a critical rate. We show that the naked singularity is global and thus it can be visible to an observer on the brane. Studies on the implications of a visible naked singularity on the brane is left for future work.

Also, we found that the brane can never enter the black hole horizon as long as it is expanding. In addition, we found an upper bound on the Hubble expansion rate, given by the energy density of the matter on the brane, for arbitrary but non-negative energy flux emitted by the brane. We also presented a set of equations that completely determine the brane dynamics as well as the bulk geometry.

Finally, let us briefly comment on some future issues. In this paper, we only discussed the case of null dust. However, this is too simplified to be realistic. As a realistic situation, it will be interesting to consider a bulk scalar field such as a dilaton or a moduli field. In this case, it will be necessary to solve the bulk and brane dynamics numerically in general. Another interesting issue will be the evaporation of a bulk black hole by the Hawking radiation and its effect on the brane dynamics. We plan to come back to these issues in future publications.

ACKNOWLEDGMENT

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APPENDIX A: GEOMETRICAL QUANTITIES AND LOCAL CONSERVATION LAWS IN $(N+2)$ DIMENSIONS

In this appendix, we give useful formulas in an $(n+2)$-dimensional spacetime with constant curvature n-space, and generalize the expression for the local mass and Weyl charge.

We consider the metric in the double-null form,

$$ds^2=\frac{4r_{,u}r_{,v}}{\Phi}dudv+r(u,v)^2d\Omega^2_{(K,n)}, \tag{A1}$$

where $K=+1$, 0, or -1, corresponding to the sphere, flat space, and hyperboloid, respectively. We denote the metric tensor of the constant curvature space as γ_{ij}. The explicit expressions for the geometrical quantities in this spacetime are as follows.

The Christoffel symbol is

$$\Gamma^u_{uu}=\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,u}, \quad \Gamma^v_{vv}=\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,v},$$

$$\Gamma^u_{ij}=-\frac{r\Phi}{2r_{,u}}\gamma_{ij}, \quad \Gamma^v_{ij}=-\frac{r\Phi}{2r_{,v}}\gamma_{ij}, \tag{A2}$$

$$\Gamma^i_{uj}=\frac{r_{,u}}{r}\delta_{ij}, \quad \Gamma^i_{vj}=\frac{r_{,v}}{r}\delta_{ij}, \quad \Gamma^i_{jk}={}^n\Gamma^i_{jk}.$$

The Riemann tensor is

$$R^u{}_{uuv}=R^v{}_{vvu}=-\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,uv},$$

$$R^u{}_{iuj}=\left[-\frac{1}{2}r\left(\frac{\Phi}{r_{,u}}\right)_{,u}-\frac{r\Phi}{2r_{,u}}\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,u}\right]\gamma_{ij},$$

$$R^v{}_{ivj}=\left[-\frac{1}{2}r\left(\frac{\Phi}{r_{,v}}\right)_{,v}-\frac{r\Phi}{2r_{,v}}\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,v}\right]\gamma_{ij},$$

$$R^i{}_{uju}=\left[-\frac{r_{,uu}}{r}+\frac{r_{,u}}{r}\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,u}\right]\delta^i_j, \tag{A3}$$

$$R^i{}_{vjv}=\left[-\frac{r_{,vv}}{r}+\frac{r_{,v}}{r}\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,v}\right]\delta^i_j,$$

$$R^i{}_{vju}=R^i{}_{ujv}=-\frac{r_{,uv}}{r}\delta^i_j,$$

$$R^i{}_{jkl}=(K-\Phi)(\delta^i_k\gamma_{jl}-\delta^i_l\gamma_{jk}).$$

The Ricci tensor is

$$R_{uu}=n\frac{r_{,u}}{r}\left(\log\left|\frac{r_{,v}}{\Phi}\right|\right)_{,u}, \quad R_{vv}=n\frac{r_{,v}}{r}\left(\log\left|\frac{r_{,u}}{\Phi}\right|\right)_{,v},$$

$$R_{uv}=-\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,uv}-n\frac{r_{,uv}}{r}, \tag{A4}$$

$$R_{ij}=\left[-\frac{rr_{,uv}}{r_{,u}r_{,v}}\Phi+2(n-1)(K-\Phi)\right]\gamma_{ij}.$$

The scalar curvature is

$$R=-\frac{\Phi}{r_{,u}r_{,v}}\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,uv}-2n\frac{\Phi r_{,uv}}{rr_{,u}r_{,v}}+\frac{n(n-1)}{r^2}(K-\Phi). \tag{A5}$$

The Einstein tensor is

$$G_{uu}=R_{uu},G_{vv}=R_{vv},$$

$$G_{uv}=n(n-1)\frac{r_{,u}r_{,v}}{r^2}\left(1-\frac{K}{\Phi}\right)+n\frac{r_{,uv}}{r}, \tag{A6}$$

$$G_{ij}=\left\{\frac{r^2\Phi}{2r_{,u}r_{,v}}\left[\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,uv}+2(n-1)\frac{r_{,uv}}{r}\right]-\frac{(n-2)(n-1)}{2}(K-\Phi)\right\}\gamma_{ij}.$$

The Weyl tensor is

$$C_{uvu}{}^{u}=\frac{n-1}{n+1}\left(\log\left|\frac{r_{,u}r_{,v}}{\Phi}\right|\right)_{,uv}-\frac{r_{,uv}}{r}-\frac{r_{,u}r_{,v}}{r^2\Phi}(K-\Phi),$$

$$C_{iujv}=\frac{1}{n}r^2\gamma_{ij}C_{uvu}{}^{u}, \tag{A7}$$

$$C_{ijkl}=-\frac{2}{n(n-1)}r^4(\gamma_{ik}\gamma_{jl}-\gamma_{il}\gamma_{jk})C_{uv}{}^{vu}.$$

From these formulas, we can show the existence of a conserved current in the same way as given in the text. Namely, with the timelike vector field ξ^a defined by Eq. (2.4), the currents $\tilde{S}^a=\xi^b\tilde{T}_b{}^a$ and $S^a=\xi^bT_b{}^a$ are separately conserved, and the corresponding local masses are given, respectively, by

$$\tilde{M}=r^{n-1}(K-\Phi) \tag{A8}$$

and

$$M=\tilde{M}-\frac{2}{(n-1)(n-2)}\Lambda_{n+2}r^{n-1}. \tag{A9}$$

The v and u derivatives of M are given by the energy-momentum tensor as

$$M_{,v}=\kappa_{n+2}^2\frac{2r^n}{n}(T_v^ur_{,u}-T_v^vr_{,v}),$$

$$M_{,u}=\kappa_{n+2}^2\frac{2r^n}{n}(T_u^vr_{,v}-T_v^vr_{,u}). \tag{A10}$$

Let us now turn to the conserved current associated with the Weyl tensor. We start from the equation that results from the Bianchi identities [26],

$$C_{abcd}{}^{;d}=J_{abc}, \tag{A11}$$

where

$$J_{abc}=\frac{2(n-1)}{n}\kappa_{n+2}^2\left(T_{c[a;b]}+\frac{1}{(n+1)}g_{c[b}T_{;a]}\right). \tag{A12}$$

From this equation, we can show the existence of a locally conserved current Q^a given by

$$Q^a=r\ell_bn_cJ^{bca},\quad Q^a{}_{;a}=0, \tag{A13}$$

where ℓ^a and n^a are the null vectors defined in Eqs. (2.19). The nonzero components are explicitly written as

$$Q^u=-rJ^{vu}{}_{v},\quad Q^v=-rJ^{vu}{}_{u}. \tag{A14}$$

We then find the following relations,

$$(r^{n+1}C_{vu}{}^{vu})_{,v}=r^{n+1}J^v{}_{vv},$$

$$(r^{n+1}C_{vu}{}^{vu})_{,u}=r^{n+1}J^u{}_{uu}. \tag{A15}$$

These relations are generalization of Eqs. (2.21), and imply that the Weyl component $r^{n+1}C_{vu}{}^{vu}$ is the local charge associated with this conserved current.

Using the explicit form of $C_{vu}{}^{vu}$ in Eqs. (A7) and the Einstein equations, we can relate the Weyl charge to the local mass. We find

$$r^{n+1}C_{vu}{}^{vu}=\frac{n(n-1)\tilde{M}}{2}-\frac{n-1}{n(n+1)}r^{n+1}(G^i{}_i-2nG^v{}_v)=\frac{n(n-1)M}{2}-\frac{n-1}{n(n+1)}\kappa_{n+2}^2r^{n+1}(T^i{}_i-2nT^v{}_v). \tag{A16}$$

Finally, we note that this equation implies that the linear combination of the energy-momentum tensor,

$$r^{n+1}(T^i{}_i-2nT^v{}_v), \tag{A17}$$

plays the role of a local charge as well. Therefore, the behavior of this quantity is constrained nonlocally by the integral of the flux given by the corresponding linear combination of the currents S^a and Q^a. Although we do not explore it here, this fact may be useful in an analysis of the behavior of the bulk matter.

APPENDIX B: EINSTEIN-SCALAR THEORY IN THE BULK

In this appendix, we apply the local conservation law to the five-dimensional Einstein-scalar theory. We assume that there is no matter on the brane, but we take account of a coupling of the bulk scalar field to the brane tension. In this

case, the energy exchange between the brane and the bulk, hence the time evolution of M, occurs through the coupling.

We first consider a general bulk scalar field. Then, as a special case, we analyze the local mass on the brane for the exact dilatonic solution discussed by Koyama and Takahashi [13]. Finally, we clarify the relation between the local mass and the term that is identified as the dark radiation term in the effective four-dimensional approach in which the contribution of the scalar field energy-momentum to the brane is required to take the standard four-dimensional form [11].

1. Setup

We consider a theory described by the action

$$S=\int d^5x\sqrt{-g}\left[\frac{1}{2\kappa_5^2}R-\frac{1}{2}\partial_a\phi\partial^a\phi-V(\phi)\right]-\int d^4x\sqrt{-q}\sigma(\phi). \tag{B1}$$

For the bulk with the metric given by Eq. (2.1), the energy-momentum tensor in the bulk is given by

$$T_{vv}=\phi_{,v}^2,\quad T_{uu}=\phi_{,u}^2,$$

$$T_{uv}=-\frac{2r_{,u}r_{,v}}{\Phi}V(\phi),$$

$$T_{ij}=-r^2\left[\frac{\Phi}{2r_{,u}r_{,v}}\phi_{,u}\phi_{,v}+V(\phi)\right]\gamma_{ij}. \tag{B2}$$

On the brane, the first derivatives of the scalar field tangent and normal to the brane are expressed, respectively, as

$$\phi' := \phi_{,a}n^a=-\phi_{,v}\dot{v}+\phi_{,u}\dot{u},$$

$$\dot{\phi} := \phi_{,a}v^a=\phi_{,v}\dot{v}+\phi_{,u}\dot{u}. \tag{B3}$$

The Codacci equation (2.36) gives, via the coupling to the brane tension, the boundary condition at the brane,

$$\phi'=\frac{1}{2}\frac{d}{d\phi}\sigma(\phi). \tag{B4}$$

In the present case, the effective Friedmann equation induced on the brane, Eq. (2.34), becomes

$$3\left[H^2+\frac{K}{r^2}\right]=\frac{1}{12}\kappa_5^2\sigma^2+\frac{3M}{r^4}. \tag{B5}$$

The time evolution of the local mass M on the brane is given by

$$\dot{M}=-\frac{1}{3}\kappa_5^2r^4H\left[\dot{\phi}^2-2V+\frac{1}{4}\left(\frac{d}{d\phi}\sigma\right)^2\right]-\frac{1}{36}\kappa_5^4r^4\dot{\phi}\frac{d}{d\phi}\sigma^2. \tag{B6}$$

From the brane point of view, as given by Eqs. (2.43) in the text, the effective energy density and pressure are composed of the brane tension and the bulk matter induced on the brane as

$$\rho^{(\mathrm{tot})}=\rho^{(\mathrm{T})}+\rho^{(\mathrm{B})},\quad p^{(\mathrm{tot})}=p^{(\mathrm{T})}+p^{(\mathrm{B})}, \tag{B7}$$

where

$$\kappa_4^2\rho^{(\mathrm{T})}=\frac{1}{12}\kappa_5^4\sigma^2,\quad \kappa_4^2p^{(\mathrm{T})}=-\frac{1}{12}\kappa_5^4\sigma^2,$$

$$\kappa_4^2\rho^{(\mathrm{B})}=\frac{3M}{r^4},$$

$$\kappa_4^2p^{(\mathrm{B})}=\frac{M}{r^4}+\frac{1}{3}\kappa_5^2\left[\dot{\phi}^2-2V+\frac{1}{4}\left(\frac{d}{d\phi}\sigma\right)^2\right], \tag{B8}$$

where Eq. (B4) is used. From the Bianchi identity on the brane, the conservation law for the total effective energy-momentum on the brane is obtained as

$$\dot{\rho}^{(\mathrm{B})}+3H(\rho^{(\mathrm{B})}+p^{(\mathrm{B})})=-\dot{\rho}^{(\mathrm{T})}. \tag{B9}$$

The above relation is mathematically equivalent to Eq. (B6).

As discussed after Eq. (2.45) in the text, Eq. (B9) gives the point of view from the brane, and it is naturally interpreted as the equation describing the energy exchange between the brane tension and the bulk matter induced on the brane. On the other hand, the time variation of the local mass along the brane, Eq. (B6), gives the point of view from the bulk, and it contains not only the energy transfer from the brane tension to the bulk (the last term) but also the energy flow of the bulk scalar field at the location of the brane, which is nonvanishing in general even if the scalar field has no coupling to the brane tension.

2. Dilatonic exact solution

In the case $K=0$, and for special forms of $V(\phi)$ and $\sigma(\phi)$, an exact cosmological solution is known, as a realization of the bulk inflaton model [13]. The forms of the potential and brane tension are

$$\kappa_5^2V(\phi)=\left(\frac{\Delta}{8}+\delta\right)\lambda_0^2e^{-2\sqrt{2}b\kappa_5\phi}, \tag{B10}$$

$$\kappa_5^2\sigma(\phi)=\sqrt{2}\lambda_0e^{-\sqrt{2}b\kappa_5\phi}, \tag{B11}$$

where δ, b, and λ_0 are constant and are all assumed to be non-negative, and

$$\Delta=4b^2-\frac{8}{3}. \tag{B12}$$

If $\delta=0$, there exists a static, Minkowski brane solution [14]. In order to avoid the presence of a naked singularity, the dilatonic coupling b^2 is assumed to be smaller than 1/6 [13]. This implies that Δ is negative and is in the range

$$2 \leqslant (-\Delta) \leqslant \frac{8}{3}. \tag{B13}$$

The exact solution takes the form

$$ds^2 = e^{2W(z)}(-d\tau^2 + e^{2\alpha(\tau)}\delta_{ij}dx^i dx^j + e^{2\sqrt{2}b\kappa_5\phi(\tau)}dz^2),$$

$$\phi = \phi(\tau) + \Xi(z), \tag{B14}$$

with the brane located at $z = z_0$ and it is assumed that $\Xi(z_0) = 0$ without loss of generality. The scale factor of the brane and the scalar field on the brane are given by

$$r(\tau) = e^{\alpha(\tau)} = (H_0\tau)^{1/6b^2}, \quad e^{\sqrt{2}b\kappa_5\phi(\tau)} = H_0\tau, \tag{B15}$$

where

$$H_0 = \left(\Delta + \frac{8}{3}\right)\lambda_0\sqrt{\frac{\delta}{-\Delta}} = 4b^2\lambda_0\sqrt{\frac{\delta}{-\Delta}}. \tag{B16}$$

As seen from the first of Eqs. (B15), the power-law inflation is realized on the brane for $b^2 < 1/6$.

Let us consider the time evolution of the energy content in this model. From the brane point of view, the time derivative of the brane tension $\rho^{(\mathrm{T})}$ is always negative:

$$\dot{\rho}^{(\mathrm{T})} = \frac{\Delta}{48b^4\delta\tau^3} < 0. \tag{B17}$$

Thus, from Eq. (B9), for an observer on the brane, there is one-way energy transfer from the brane tension to the bulk matter induced on the brane. From the bulk point of view, however, the situation is slightly more complicated. The time derivative of the local mass (or the generalized dark radiation) on the brane, Eq. (B6), is evaluated as

$$\frac{\dot{M}}{r^4} = \frac{1}{18b^6\delta\tau^3}\left(\frac{1}{3} - b^2\right)\left(\frac{\Delta}{8} + \delta\right). \tag{B18}$$

The sign of $\dot{M}$ is determined by the sign of $\Delta/8 + \delta$. Note that the sign of $\Delta/8 + \delta$ determines the sign of the bulk potential as well, as seen from Eq. (B10). If $\Delta/8 + \delta > 0$, i.e., $\delta > (-\Delta)/8 = (b^2/2) - 1/3$, we have $\dot{M} > 0$. Since M is the total bulk mass integrated up to the location of the brane, the increase in M implies an energy flow from the brane to the bulk. Therefore, in this case, the energy in the brane tension is transferred to the bulk scalar field and it flows out into the bulk. In contrast, if $\delta < (-\Delta)/8$, we have $\dot{M} < 0$. In this case, although there is still energy transfer from the brane tension to the bulk scalar field, the bulk energy flows onto the brane. In other words, there is a localization process of the bulk energy onto the brane that overwhelms the energy released from the brane tension.

3. Local mass and the effective four-dimensional description

In the bulk inflaton model with a quadratic potential [9–12], it has been shown that the bulk scalar field projected on the brane behaves exactly like a four-dimensional field in the low energy limit, $H^2\ell^2 \ll 1$, where H is the Hubble parameter of the brane, and the leading order correction gives the gradual energy loss from the scalar field to the bulk, giving rise to the dark radiation term [9,11]. Here, we discuss the relation between the dark radiation term appearing in this effective four-dimensional description and the generalized dark radiation term given by the local mass in the bulk.

From the geometrical description [7], the induced Einstein equation on the brane is written as

$${}^{(4)}G_{\mu\nu} = -\frac{1}{12}\kappa_5^4\sigma^2 q_{\mu\nu} + \kappa_5^2\tilde{T}^{(b)}_{\mu\nu} - E_{\mu\nu}, \tag{B19}$$

where

$$\tilde{T}^{(b)}_{\mu\nu} = \frac{2}{3}\left[\tilde{T}_{ab}q^a_\mu q^b_\nu - \left(\tilde{T}_{ab}n^a n^b - \frac{1}{4}\tilde{T}_{ab}g^{ab}\right)q_{\mu\nu}\right], \tag{B20}$$

is the projected tensor of the bulk energy-momentum onto the brane that includes the contribution of the cosmological constant; see Eq. (2.7). For a homogeneous and isotropic brane, the nonvanishing components are

$$\tilde{T}^{(b)}_{\tau\tau} = -\tilde{T}^v{}_v + \frac{1}{6}\tilde{T}^i{}_i,$$

$$\tilde{T}^{(b)i}{}_i = \frac{1}{6}\tilde{T}^i{}_i - \frac{\dot{u}}{\dot{v}}\tilde{T}^v{}_u - \frac{\dot{v}}{\dot{u}}\tilde{T}^u{}_v + \tilde{T}^v{}_v. \tag{B21}$$

Let us decompose $E_{\mu\nu}$ as

$$E_{\mu\nu} = E^{(b)}_{\mu\nu} + E^{(d)}_{\mu\nu}, \tag{B22}$$

where $E^{(b)}_{\mu\nu}$ is to be expressed in terms of the bulk energy-momentum tensor in such a way that the effective four-dimensional description is recovered, and $E^{(d)}_{\mu\nu}$ is the part that should be identified as the dark radiation term in this effective four-dimensional approach. To be in accordance with Ref. [11], we choose the components of $E^{(b)}_{\mu\nu}$ as

$$-E^{(b)}_{tt} = -\frac{1}{8}\kappa_5^2\left(\frac{\dot{u}}{\dot{v}}\tilde{T}^v{}_u + \frac{\dot{v}}{\dot{u}}\tilde{T}^u{}_v\right) + \frac{1}{12}\kappa_5^2(\tilde{T}^i{}_i - 3\tilde{T}^v{}_v)$$
$$= -E^{(b)i}{}_i, \tag{B23}$$

and identify the remaining part with the dark radiation term, X,

$$-E^{(d)}_{tt} = X = -E^{(d)i}{}_i. \tag{B24}$$

In the effective four-dimensional description, the Einstein equation on the brane takes the form,

$${}^{(4)}G_{\mu\nu} = \kappa_4^2 T^{(\mathrm{eff})}_{\mu\nu} - E^{(d)}_{\mu\nu}, \tag{B25}$$

where $T^{(\mathrm{eff})}_{\mu\nu}$ is the effective energy-momentum tensor on the brane,

$$\kappa_4^2 T_{\mu\nu}^{(\mathrm{eff})} = -\frac{1}{12}\kappa_5^4\sigma^2 q_{\mu\nu} + \kappa_5^2 \tilde{T}_{\mu\nu}^{(b)} - E_{\mu\nu}^{(b)}, \tag{B26}$$

and κ_4^2 is the four-dimensional gravitational constant that should be appropriately defined to agree with the conventional four-dimensional Einstein gravity in the low energy limit. In the present case of homogeneous and isotropic cosmology, the only nontrivial components are the effective energy density and pressure, which are given explicitly by

$$\begin{aligned}\kappa_4^2\rho^{(\mathrm{eff})} &= -\kappa_4^2 \tilde{T}^{(\mathrm{eff})\tau}{}_{\tau} = \frac{1}{12}\kappa_5^4\sigma^2 - \frac{5}{4}\kappa_5^2\tilde{T}^{v}{}_{v} + \frac{1}{4}\kappa_5^2\tilde{T}^{i}{}_{i} \\ &\quad - \frac{1}{8}\kappa_5^2\left(\frac{\dot{u}}{\dot{v}}\tilde{T}^{v}{}_{u} + \frac{\dot{v}}{\dot{u}}\tilde{T}^{u}{}_{v}\right), \\ \kappa_4^2 p^{(\mathrm{eff})} &= \frac{1}{3}\kappa_4^2 \tilde{T}^{(\mathrm{eff})i}{}_{i} = -\frac{1}{12}\kappa_5^4\sigma^2 + \frac{1}{4}\kappa_5^2\tilde{T}^{v}{}_{v} + \frac{1}{12}\kappa_5^2\tilde{T}^{i}{}_{i} \\ &\quad - \frac{3}{8}\kappa_5^2\left(\frac{\dot{u}}{\dot{v}}\tilde{T}^{v}{}_{u} + \frac{\dot{v}}{\dot{u}}\tilde{T}^{u}{}_{v}\right).\end{aligned} \tag{B27}$$

The effective Friedmann equation on the brane is written as

$$\begin{aligned}3\left[H^2 + \frac{K}{r^2}\right] &= \kappa_4^2\rho^{(\mathrm{eff})} - E_{\tau\tau}^{(d)} \\ &= \frac{1}{12}\kappa_5^4\sigma^2 - \frac{5}{4}\kappa_5^2\tilde{T}^{v}{}_{v} + \frac{1}{4}\kappa_5^2\tilde{T}^{i}{}_{i} \\ &\quad - \frac{1}{8}\kappa_5^2\left(\frac{\dot{u}}{\dot{v}}\tilde{T}^{v}{}_{u} + \frac{\dot{v}}{\dot{u}}\tilde{T}^{u}{}_{v}\right) + X.\end{aligned} \tag{B28}$$

Applying this to a bulk scalar field with the action (B1), we find $\rho^{(\mathrm{eff})}$ and $p^{(\mathrm{eff})}$ are given by those of a four-dimensional scalar field φ with the potential,

$$\kappa_4^2 V^{(\mathrm{eff})}(\varphi) = \kappa_5^2\left(\frac{1}{2}V(\phi) + \frac{\kappa_5^2}{12}\sigma^2(\phi) - \frac{1}{16}\sigma'^{\,2}(\phi)\right), \tag{B29}$$

where $\varphi = \sqrt{\kappa_5^2/\kappa_4^2}\,\phi$. From the contracted Bianchi identity, we obtain

$$D^{\mu}T_{\mu\tau}^{(\mathrm{eff})} = \dot{\rho}^{(\mathrm{eff})} + 3H(\rho^{(\mathrm{eff})} + p^{(\mathrm{eff})}) = -\frac{1}{\kappa_5^2}\frac{1}{r^4}(r^4 X). \tag{B30}$$

Unfortunately, as we can see from Eqs. (B27), there is no simplification in the energy equation in terms of the five-dimensional energy-momentum tensor.

From the effective four-dimensional point of view, what happens is the conversion of the scalar field energy on the brane to the dark radiation via the coupling to the brane tension. From the bulk point of view, a natural interpretation is to regard the local mass M on the brane as the generalized dark radiation. These two different identifications of the dark radiation term on the brane coincide only when the bulk is in vacuum and M is constant. Comparison of the above decomposition of $E_{\tau\tau}$ with Eq. (2.40), we find the difference between the dark radiation in the four-dimensional description and the generalized dark radiation in terms of the local mass M as

$$\frac{3\tilde{M}}{r^4} = \frac{1}{4}\kappa_5^2(\tilde{T}_i^i - 5\tilde{T}_v^v) - \frac{1}{8}\kappa_5^2\left(\frac{\dot{u}}{\dot{v}}\tilde{T}_u^v + \frac{\dot{v}}{\dot{u}}\tilde{T}_v^u\right) + X. \tag{B31}$$

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