Nonlinear coupled Alfvén and gravitational waves

Andreas Källberg,* Gert Brodin,† and Michael Bradley‡
Department of Physics, Umeå University, SE-901 87 Umeå, Sweden
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In this paper we consider nonlinear interaction between gravitational and electromagnetic waves in a strongly magnetized plasma. More specifically, we investigate the propagation of gravitational waves with the direction of propagation perpendicular to a background magnetic field and the coupling to compressional Alfvén waves. The gravitational waves are considered in the high-frequency limit and the plasma is modeled by a multifluid description. We make a self-consistent, weakly nonlinear analysis of the Einstein-Maxwell system and derive a wave equation for the coupled gravitational and electromagnetic wave modes. A WKB-approximation is then applied and as a result we obtain the nonlinear Schrödinger equation for the slowly varying wave amplitudes. The analysis is extended to 3D wave pulses, and we discuss the applications to radiation generated from pulsar binary mergers. It turns out that the electromagnetic radiation from a binary merger should experience a focusing effect, that in principle could be detected.

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I. INTRODUCTION

Recently much work has been devoted to the study of gravitational waves, largely due to the increased possibility of detection by facilities in operation such as LIGO (Laser Interferometer Gravitational Wave Observatory), or by ambitious detector projects under development such as LISA (Laser Interferometer Space Antenna) [1]. The possibility of interaction between electromagnetic and gravitational fields has also led to alternative proposals for gravitational wave detectors, see e.g., Refs. [2,3] and references therein. Closer to the source, in an astrophysical context, the gravitational waves often propagate in a plasma medium, and the amplitudes can be much larger, which increases the number of possible interaction mechanisms, see e.g. Refs. [4–22]. Linear gravitational wave theory in a magnetized plasma has been studied by for example Refs. [4,5], including the back reaction from the plasma on the gravitational wave. In Refs. [6–9] the authors have studied nonlinear responses to the gravitational wave by the plasma medium, although the back reaction has been neglected. The nonlinear response gives raise to effects, such as parametric instabilities [7,9,11,12], large density fluctuations [8,10], and photon acceleration [8]. The application of gravitational wave processes to astrophysics has been discussed by, for example, Refs. [13–15], and to cosmology by Refs. [16–18]. A number of works studying nonlinear propagation of gravitational waves, including the back reaction from the plasma, have also been written, see e.g., [10,11,19].

In Refs. [20,21] geometrical nonlinearities from the Einstein tensor were considered, and a nonlinear evolution equation was derived. However, it was found that the nonlinear coefficient was proportional to the small difference of the phase velocity and the velocity of light in vacuum. In the present paper we neglect nonlinearities from the Einstein tensor and instead focus on the nonlinear response from matter. In particular we consider coupled gravitational and electromagnetic waves propagating in a magnetized plasma, with the direction of propagation perpendicular to the background magnetic field. For this wave coupling to be efficient, the interaction should be (almost) resonant, i.e., the electromagnetic wave propagation velocity in the plasma should be close to the speed of light. This increases the interaction strength and allows us to neglect the effect of the background curvature, in comparison with the direct interaction with matter.

The plasma is modeled by a multifluid description, and we perform a self-consistent weakly nonlinear analysis of the Einstein-Maxwell system of equations for 1D spatial variations in the high-frequency limit. This system is reduced to a single nonlinear wave equation for the coupled gravitational and electromagnetic waves. We then apply a WKB-approximation to this wave equation, and it turns out that the slowly varying wave amplitude obey the well-known nonlinear Schrödinger (NLS) equation [23]. In Sec. IV B, the analysis is expanded to include a 3D spatial dependence, allowing us to consider diffraction and/or nonlinear self-focusing of the wave. For certain conditions the sign of the nonlinear coefficient is of focusing type, which for sufficient initial amplitudes implies solutions that undergo wave collapse. The conditions for collapse and the possible applications to astrophysics are discussed.

II. OVERVIEW OF THE APPROXIMATION SCHEME

The problem treated in this paper contains a number of small parameters used in the different approximations and expansions made throughout the paper. The aim of this section is to give an overview of these parameters, at what stages of the calculations they are introduced, and to what order the expansions need to be made.

In Sec. III, we introduce the small gravitational wave amplitudes $h_{+,\times}$ and in calculating Maxwell and fluid equations, (27)–(33), we only keep terms to $\mathcal{O}(h_{+,\times})$. However, for the case when the gravitational wave interacts with an

*Email address: andreas.kallberg@physics.umu.se
†Email address: gert.brodin@physics.umu.se
‡Email address: michael.bradley@physics.umu.se

almost resonant Alfvén wave, nonlinearities become important. The reason is that the resonance magnifies $|\delta\mathbf{B}|/|\mathbf{B}_0|$ where $\delta\mathbf{B}$ and $\mathbf{B}_0$ are the wave magnetic field and background magnetic field, respectively. The magnification is of the order $|\delta\mathbf{B}|/|\mathbf{B}_0|\sim h_+\omega/\delta\omega$ where $\delta\omega$ is the frequency mismatch as compared to the Alfvén mode. Therefore, we keep terms up to order $\mathcal{O}(|\delta\mathbf{B}|^3/|\mathbf{B}_0|^3)$, in the expansion of the wave equation (41), in order to obtain the nonlinear amplitude modulation. Other nonlinearities originating from the Einstein tensor and the effective currents in (7)–(12) will not be magnified due to the resonance, and may be neglected in our treatment, see also the discussion following Eq. (20).

Furthermore, in Sec. III we introduce the high-frequency approximation, valid for wave numbers $k\gg 1/r_c$, where r_c is a characteristic radius of curvature of the background. This condition may also be written as the small parameter $\kappa B_0^2/k^2\ll 1$, where $\kappa=8\pi G$ (we use units where $c=\mu_0=\epsilon_0=1$). We have to keep terms to $\mathcal{O}(\kappa B_0^2/k^2)$, in order to include the weak coupling between the gravitational mode and the Alfvén mode.

In Sec. IV we first use the approximations $\partial_t\approx-\partial_z$, except in the linear wave operators, in order to simplify our system of equations. This is justified in the high-frequency approximation. Furthermore, we use $\partial_t\ll\omega_c$, i.e., we assume that the frequency of variations of fields is much smaller than the cyclotron frequency of all particle species in the plasma. In the expansion of Eq. (36), we must include terms to $\mathcal{O}(\partial_t/\omega_c)$. This is because the equations of motion to zeroth order in ∂_t/ω_c gives just the $\mathbf{E}\times\mathbf{B}$ drift of all particle species, and we must include terms of one order higher to obtain a nonzero current in the plasma.

Next, in Sec. IV A, we introduce a weakly modulated wave (WKB-approximation), with the underlying assumption that the characteristic length scale for amplitude modulations obeys $L_\parallel\gg\lambda$. In deriving Eq. (50), we need terms to $\mathcal{O}(\lambda^2/L_\parallel^2)$ in order to include the dispersive term in (50). We also use $1/C_A^2\ll 1$, where C_A is the Alfvén velocity, and only keep terms to lowest nonvanishing order to simplify the equations. Then, in Sec. IV B, we extend the problem to include small variations in the directions perpendicular to the direction of propagation. We apply the condition $L_\perp\gg\lambda$, where $L_\perp$ is the characteristic length for perpendicular variations. In subsequent calculations, we must keep terms to $\mathcal{O}(\lambda^2/L_\perp^2)$ to obtain the diffractive term in Eq. (67), see the discussion at the beginning of Sec. IV B for more details.

The parameters in this section are, in principle, independent of each other with the sole exception that $\delta\omega/\omega\sim 1/C_A^2$, which is our main motivation for considering the regime $C_A^2\gg 1$. In addition to these parameters there are a few other small parameters introduced throughout the paper, but those quantities are related in different ways to the ones presented in this section, and the relations are explained in the text. The only restrictions on the parameters in this section, besides their smallness, is due to physical applicability, i.e., the values of the parameters should be chosen to fit environments close to emitters of reasonably strong gravitational waves. An example of such parameter values is presented in the final section.

III. BASIC EQUATIONS

The gravitational and electromagnetic fields are governed, respectively, by the Einstein field equations

$$G_{ab}=\kappa T_{ab} \tag{1}$$

and the Maxwell field equations

$$\nabla_a F^{ab}=j^b \tag{2}$$

$$\nabla_a F_{bc}+\nabla_b F_{ca}+\nabla_c F_{ab}=0 \tag{3}$$

where G_{ab} is the Einstein tensor, T_{ab} is the energy-momentum tensor, F_{ab} is the electromagnetic field tensor, j^b is the four-current density, and ∇ denotes covariant differentiation. We use the spacelike signature $(-+++)$ for the metric.

For the plasma we use a multifluid description, where the plasma is seen as a number of charged fluids, one for each plasma species, and collisions between particles are neglected. The total energy-momentum tensor is then $T_{ab}=T_{ab}^{(fl)}+T_{ab}^{(em)}$. Here the fluid part is

$$T_{ab}^{(fl)}=\sum_s \{[\mu_{(s)}+p_{(s)}]u_{(s)a}u_{(s)b}+p_{(s)}g_{ab}\} \tag{4}$$

where $\mu_{(s)}, p_{(s)}$, and $u_{(s)a}$ are the internal energy density, pressure, and four-velocity for each plasma species, s, g_{ab} is the metric tensor and the electromagnetic part is

$$T_{ab}^{(em)}=F_a{}^cF_{bc}-\tfrac{1}{4}g_{ab}F^{cd}F_{cd} \tag{5}$$

In the absence of collisions, the evolution equations for each fluid species can be written [24]

$$\nabla_b T_{(s)}^{ab}=F^{ab}j_{b(s)} \tag{6}$$

as is consistent with Eq. (1).

With these preliminaries, we now follow the approach applied in Refs. [8,9,13,19,25,26] and introduce an observer four-velocity V^a, so that the electromagnetic field can be decomposed relative to this into an electric and a magnetic part, $E_a=F_{ab}V^b$ and $B_a=\frac{1}{2}\epsilon_{abc}F^{bc}$, respectively. Here $\epsilon_{abc}=V^d\epsilon_{abcd}$ where ϵ_{abcd} is the four-dimensional volume element with $\epsilon_{0123}=\sqrt{|\det g_{ab}|}$. Next we introduce an orthonormal frame (ONF) with basis $\{\mathbf{e}_a=e_a^{\mu}\partial_{x^\mu}\}$, where $\mathbf{e}_0=\mathbf{V}=V^a\mathbf{e}_a$, (i.e. $V^a=\delta_0^a$), and write the fluid four-velocity as $u^a=(\gamma,\gamma\mathbf{v})$, where $\gamma=(1-v_\alpha v^\alpha)^{-\frac{1}{2}}$, $\alpha=1,2,3$, and $\mathbf{v}$ is the fluid three velocity. Dividing the four-current $j^a=\Sigma_s q_{(s)}n_{(s)}u_{(s)}^a$ in the same manner, the Maxwell equations (2) and (3) and fluid equations (6) can be written [13,19,26]

$$\nabla\cdot\mathbf{E}=\rho+\rho_E \tag{7}$$

$$\nabla\cdot\mathbf{B}=\rho_B \tag{8}$$

$$\mathbf{e}_0\mathbf{E}-\nabla\times\mathbf{B}=-\mathbf{j}-\mathbf{j}_E \tag{9}$$

$$\mathbf{e}_0\mathbf{B}+\nabla\times\mathbf{E}=-\mathbf{j}_B \tag{10}$$

$$\mathbf{e}_0(\gamma n)+\nabla\cdot(\gamma n\mathbf{v})=\Delta n \tag{11}$$

$$(\mu+p)(\mathbf{e}_0+\mathbf{v}\cdot\nabla)\gamma\mathbf{v}=-\gamma^{-1}\nabla p-\gamma\mathbf{v}(\mathbf{e}_0+\mathbf{v}\cdot\nabla)p+qn(\mathbf{E}+\mathbf{v}\times\mathbf{B})+(\mu+p)\mathbf{g} \tag{12}$$

where we have omitted the fluid species index *s*, and it is understood that we have one pair of fluid equations (11) and (12) for each plasma fluid species. We have introduced the three-vector notation $\mathbf{E}\equiv(E^\alpha)=(E^1,E^2,E^3)$ etc., and $\nabla\equiv(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$. The charge density is $\rho=\Sigma_s q_{(s)}\gamma n_{(s)}$ and n is the proper particle number density. The effective charges, currents, and forces originating from the inclusion of the gravitational field are given by

$$\rho_E\equiv-\Gamma^\alpha_{\beta\alpha}E^\beta-\epsilon^{\alpha\beta\gamma}\Gamma^0_{\alpha\beta}B_\gamma \tag{13}$$

$$\rho_B\equiv-\Gamma^\alpha_{\beta\alpha}B^\beta+\epsilon^{\alpha\beta\gamma}\Gamma^0_{\alpha\beta}E_\gamma \tag{14}$$

$$\mathbf{j}_E\equiv[-(\Gamma^\alpha_{0\beta}-\Gamma^\alpha_{\beta0})E^\beta+\Gamma^\beta_{0\beta}E^\alpha-\epsilon^{\alpha\beta\gamma}(\Gamma^0_{\beta0}B_\gamma+\Gamma^\delta_{\beta\gamma}B_\delta)]\mathbf{e}_\alpha \tag{15}$$

$$\mathbf{j}_B\equiv[-(\Gamma^\alpha_{0\beta}-\Gamma^\alpha_{\beta0})B^\beta+\Gamma^\beta_{0\beta}B^\alpha+\epsilon^{\alpha\beta\gamma}(\Gamma^0_{\beta0}E_\gamma+\Gamma^\delta_{\beta\gamma}E_\delta)]\mathbf{e}_\alpha \tag{16}$$

$$\Delta n\equiv-\gamma n(\Gamma^\alpha_{0\alpha}+\Gamma^\alpha_{00}v_\alpha+\Gamma^\alpha_{\beta\alpha}v^\beta) \tag{17}$$

$$\mathbf{g}\equiv-\gamma[\Gamma^\alpha_{00}+(\Gamma^\alpha_{0\beta}+\Gamma^\alpha_{\beta0})v^\beta+\Gamma^\alpha_{\beta\gamma}v^\beta v^\gamma]\mathbf{e}_\alpha \tag{18}$$

where Γ^a_{bc} are the Ricci rotation coefficients associated with the tetrad $\{\mathbf{e}_a\}$.

From now on we will assume that the plasma is cold, so that we can neglect the pressure terms in Eq. (12) and let $\mu_{(s)}=m_{(s)}n_{(s)}$, where $m_{(s)}$ is the mass of each particle species. We will also apply the high-frequency approximation [27] for the gravitational waves. Thus we will assume that the background gravitational field and the unperturbed plasma and electromagnetic fields fulfill Einstein's field equations (1), and we introduce perturbations to the metric, $g_{\mu\nu}=g^{(0)}_{\mu\nu}+h_{\mu\nu}$, (greek indices are coordinate indices), and to the energy-momentum tensor, $T_{ab}=T^{(0)}_{ab}+\delta T_{ab}$, which should then fulfill

$$\delta G_{ab}=\kappa\delta T_{ab} \tag{19}$$

where δG_{ab} is the (linearized) perturbation to the Einstein tensor caused by the metric perturbation.

It was recently shown by Ref. [19] that in the high-frequency approximation the gravitational wave can be taken to be in the transverse and traceless (TT) gauge even in the presence of matter. In this gauge the metric of a linearized gravitational wave propagating in the *z* direction is given by [28]

$$ds^2=-dt^2+(1+h_+)dx^2+(1-h_+)dy^2+2h_\times dxdy+dz^2 \tag{20}$$

where $h_+\equiv h_+(z,t)$ and $h_\times\equiv h_\times(z,t)$ denotes the two polarization modes of weak gravitational waves in the TT gauge, and $|h_+|,|h_\times|\ll1$. The quantities h_+ and $h_\times$ differ from the corresponding quantities in the vacuum case, $h_+\equiv h_+(z-t),h_\times\equiv h_\times(z-t)$, because of the weak interaction with the plasma. However, this difference is very small, which will allow us to use $\partial_t\approx-\partial_z$ in many of the subsequent approximations. We note that components of the metric tensor corresponding to the quadratically nonlinear contribution from the pseudoenergy momentum tensor, as well as higher-order cubic nonlinearities are neglected in (20). The basic motivation for this is that we neglect nonlinearities from the Einstein tensor and focus on the nonlinear response from matter. When determining the regime of validity for this approach, a couple of things should be noted, as follows:

(i) As shown by Ref. [20], for unidirectional propagation of a weakly modulated plane wave in a flat background (in the absence of the gravitational wave), the nonlinearities in the Einstein tensor cancel up to cubic order as far as the evolution of the amplitude function is considered. (Although the background curvature due to the pseudoenergy momentum tensor of the wave is affected, this does not lead to amplitude modulations. We note that this result is consistent with the exact so called PP-wave solutions where the amplitude function is unaffected by the nonlinearities in the Einstein tensor, although the background curvature is modified due to the wave.)

(ii) If the background in the absence of the gravitational wave is weakly curved (with a characteristic radius of curvature r_c fulfilling $kr_c\gg1$, where k is the gravitational wave number), nonlinearities in the Einstein tensor lead to amplitude modulations of the order of $\partial_t h_{+,\times}\sim\omega h^3_{+,\times}/(k^2r_c^2)$, see Ref. [21].

(iii) If there are wave perturbations in matter close to resonance (i.e., assuming a wave mode in matter fulfilling $\omega=k+\delta\omega$, where $\delta\omega\ll\omega$), the matter response due to the gravitational wave is magnified. As will be shown below, for our case the nonlinear contribution, close to resonance, to the amplitude modulation scales as $\partial_t h_{+,\times}\sim\kappa T_0h^3_{+,\times}\omega/\delta\omega^2$, where T_0 is the magnitude of the background energy-momentum tensor. We note that the nonlinear matter response dominates over that associated with the combined effect of the background curvature and the Einstein tensor nonlinearities, provided $k^{-2}r_c^{-2}\ll\kappa T_0/\delta\omega^2$.

(iv) Using the high-frequency approximation ($kr_c\gg1$, see [27,29]) and assuming a weakly nonlinear response from matter, it is always possible to use the TT gauge [19]. As far as the direct interaction with the medium is concerned, i.e., the contribution that is magnified due to the resonance, the background metric can then be considered as flat (i.e., $g^{(0)}_{ab}=\eta_{ab}$), see [29].

Next we choose our tetrad basis for the metric (20) as

$$\mathbf{e}_0=\partial_t,\ \mathbf{e}_1=(1-\tfrac{1}{2}h_+)\partial_x-\tfrac{1}{2}h_\times\partial_y,$$
$$\mathbf{e}_2=(1+\tfrac{1}{2}h_+)\partial_y-\tfrac{1}{2}h_\times\partial_x,\ \mathbf{e}_3=\partial_z. \tag{21}$$

Using (20) and (21) in the linearized Einstein field equations

(19), subtracting the $_{11}$ and $_{22}$, and adding the $_{12}$ and $_{21}$ components of the equations one gets [8,19,29]

$$(\partial_t^2-\partial_z^2)h_+=\kappa(\delta T_{11}-\delta T_{22}) \tag{22}$$

$$(\partial_t^2-\partial_z^2)h_\times=\kappa(\delta T_{12}+\delta T_{21}) \tag{23}$$

In the basis (21) the non-zero terms in (13)–(18) are

$$\mathbf{j}_E=-\tfrac{1}{2}(E_1\partial_t h_+ + B_2\partial_z h_+ + E_2\partial_t h_\times - B_1\partial_z h_\times)\mathbf{e}_1 + \tfrac{1}{2}(E_2\partial_t h_+ - B_1\partial_z h_+ - E_1\partial_t h_\times - B_2\partial_z h_\times)\mathbf{e}_2 \tag{24}$$

$$\mathbf{j}_B=-\tfrac{1}{2}(B_1\partial_t h_+ - E_2\partial_z h_+ + B_2\partial_t h_\times + E_1\partial_z h_\times)\mathbf{e}_1 + \tfrac{1}{2}(B_2\partial_t h_+ + E_1\partial_z h_+ - B_1\partial_t h_\times + E_2\partial_z h_\times)\mathbf{e}_2 \tag{25}$$

$$\mathbf{g}=-\tfrac{1}{2}\gamma(v_1\partial_t h_+ + v_1v_3\partial_z h_+ + v_2\partial_t h_\times + v_2v_3\partial_z h_\times)\mathbf{e}_1 + \tfrac{1}{2}\gamma(v_2\partial_t h_+ + v_2v_3\partial_z h_+ - v_1\partial_t h_\times - v_1v_3\partial_z h_\times)\mathbf{e}_2 + \tfrac{1}{2}\gamma((v_1^2-v_2^2)\partial_z h_+ + 2v_1v_2\partial_z h_\times)\mathbf{e}_3. \tag{26}$$

The generalization to a finite gravitational wave pulse width, i.e., including a weak spatial dependence perpendicular to the direction of propagation, leads to additional terms in (24)–(26) and (22) and (23). An outline of this case is found in Sec. IV B.

IV. COUPLED ALFVÉN AND GRAVITATIONAL WAVES

In Ref. [8] a test fluid approach was taken to show that a gravitational wave can drive the amplitude of an electromagnetic wave to a nonlinear regime, and in Ref. [19] a linear analysis of the interaction between a gravitational wave and the extraordinary electromagnetic wave was made using the equations presented in Sec. III. Here we intend to take a similar approach, but guided by the results in Ref. [8], we will include a nonlinear response of matter and fields, while still assuming the metric perturbation to remain small.

We assume the presence of a background magnetic field, $\mathbf{B}_0=B_0\mathbf{e}_1$, and introduce the perturbations $n=n_0+\delta n$, $\mathbf{B}=(B_0+B_x)\mathbf{e}_1$, $\mathbf{E}=E_y\mathbf{e}_2+E_z\mathbf{e}_3$, and $\mathbf{v}=v_y\mathbf{e}_2+v_z\mathbf{e}_3$ [30]. We also note that in the case of gravitational waves propagating in a magnetized plasma, with the magnetic field perpendicular to the direction of propagation, only the h_+-polarization part of the gravitational wave couples effectively [31] to the electromagnetic wave, see also Ref. [19]. Thus we put $h_\times=0$ in order to simplify the algebra. As in Ref. [8], we assume $v_y\ll 1$, (because $v_y\sim h_+$), and therefore neglect terms of the type v_y^2 and $v_y h_+$, but we allow for $v_z\sim 1$. We will also consider slow variations such that $\partial_t\ll\omega_c\equiv qB_0/m$ for each plasma species. With these restrictions the Maxwell and fluid equations (7)–(12) can be reduced to the following set of equations [32]:

$$\partial_z E_z=\sum_s q\gamma(n_0+\delta n) \tag{27}$$

$$\partial_t E_y-\partial_z B_x=-\sum_s[q\gamma(n_0+\delta n)v_y]-\tfrac{1}{2}E_y\partial_t h_+ + \tfrac{1}{2}(B_0+B_x)\partial_z h_+ \tag{28}$$

$$\partial_t E_z=-\sum_s q\gamma(n_0+\delta n)v_z \tag{29}$$

$$\partial_t B_x-\partial_z E_y=-\tfrac{1}{2}E_y\partial_z h_+ + \tfrac{1}{2}(B_0+B_x)\partial_t h_+ \tag{30}$$

$$\partial_t[\gamma(n_0+\delta n)]=-\partial_z[\gamma(n_0+\delta n)v_z] \tag{31}$$

$$\partial_t(\gamma v_y)+v_z\partial_z(\gamma v_y)=\frac{q}{m}[E_y+v_z(B_0+B_x)] \tag{32}$$

$$\partial_t(\gamma v_z)+v_z\partial_z(\gamma v_z)=\frac{q}{m}[E_z-v_y(B_0+B_x)] \tag{33}$$

and the linearized Einstein equations, (22) and (23), become

$$(\partial_t^2-\partial_z^2)h_+=\kappa(E_y^2-2B_0B_x-B_x^2). \tag{34}$$

For the electromagnetic wave to exchange energy effectively with the gravitational wave, the two waves must be almost resonant, so we will consider all perturbations to be of the form $B_x\approx B_x(z-t)$, which allows us to use $\partial_t\approx-\partial_z$ in simplifying our system of equations. A certain care must be taken here though; this approximation may of course not be used directly in the operator $(\partial_t^2-\partial_z^2)$. Equations (27)–(33) can then be reduced to

$$(\partial_t^2-\partial_z^2)B_x+\sum_s\frac{m^2\omega_p^2}{q^2}\partial_z\left(\frac{\partial_z(\gamma v_z)}{B_0+B_x}\right)=\partial_t[(B_x+E_y)\partial_t h_+]+B_0\partial_t^2 h_+ \tag{35}$$

$$\partial_t(\gamma v_y)+v_z\partial_z(\gamma v_y)=\frac{q}{m}[v_z(B_0+B_x)+E_y] \tag{36}$$

where $\omega_p=\sqrt{n_0q^2/m}$ is the plasma frequency. In order to eliminate E_y we can start with the linearized version of Eq. (30). Using $\partial_t\approx-\partial_z$ and integrating we obtain

$$E_y=-B_x+\tfrac{1}{2}B_0h_+. \tag{37}$$

Reinserting this into (30) we find that the equation is fulfilled to $\mathcal{O}(h_+)$, which means that we can use the expression (37) for E_y even when the electromagnetic amplitude is large, i.e., comparable to B_0.

We now make an expansion of Eq. (36) in the small parameter ∂_t/ω_c and use (37) to obtain v_z. The result is

$$v_z = \frac{B_x - \frac{1}{2}B_0 h_+}{B_0 + B_x} + \mathcal{O}\left(\frac{\partial_t}{\omega_c}\right) \approx \frac{B_x}{B_0 + B_x}. \tag{38}$$

We see that v_z does not depend on the charge of the particles, but is the same for all particle species, which means that the z component of the current is zero, and the current is in the y direction. Inserting v_z, E_y and expanding the wave operator in Eq. (35) as $\partial_t^2 - \partial_z^2 = (\partial_t - \partial_z)(\partial_t + \partial_z) \approx 2(\partial_t + \partial_z)\partial_t$, we are left with the following coupled system of equations:

$$[\partial_t + \mathcal{V}(B_x)\partial_z]B_x = \tfrac{1}{2}B_0 \partial_t h_+ \tag{39}$$

$$(\partial_t^2 - \partial_z^2)h_+ = -2\kappa B_0 B_x \tag{40}$$

where $\mathcal{V}(B_x) = 1 - (1/2C_A^2)[B_0/(B_0 + 2B_x)]^{3/2}$ and we have introduced the Alfvén velocity $C_A \equiv (1/\Sigma_s \omega_p^2/\omega_c^2)^{1/2}$. We note from (39) that our previous assumption $\partial_t \approx -\partial_z$ requires the background parameters to satisfy $C_A^2 \gg 1$. In arriving at Eq. (40) we have also neglected a small term $-\kappa h_+ B_0 B_x$, resulting from the substitution of (37) into (34). The left-hand side of Eq. (39) contains the wave operator for compressional Alfvén waves (also called fast magnetosonic waves), which in the considered regime propagates close to the speed of light. Furthermore, the left-hand side of Eq. (40) describes gravitational waves in vacuum. The right-hand sides of Eqs. (39) and (40) are the mutual interaction terms, which may provide a comparatively effective energy exchange, because the propagation velocities of the wave modes are close to each other. Now combining these two equations and again expanding $\partial_t^2 - \partial_z^2$ we obtain, after one time integration, the following wave equation for the combined electromagnetic and gravitational wave mode:

$$(\partial_t + \partial_z)[\partial_t + \mathcal{V}(B_x)\partial_z]B_x = -\frac{\kappa B_0^2}{2}B_x. \tag{41}$$

A. WKB-approximation for quasimonochromatic waves

Here we intend to show that for quasimonochromatic waves, the wave equation (41) leads to the nonlinear Schrödinger (NLS) equation [23] for the weakly varying amplitude. We begin the analysis of Eq. (41) by first considering the linear case. Thus replacing $\mathcal{V}(B_x)$ with $\mathcal{V}(B_0) = 1 - 1/2C_A^2$ we get

$$(\partial_t + \partial_z)\left[\partial_t + \left(1 - \frac{1}{2C_A^2}\right)\partial_z\right]B_x = -\frac{\kappa B_0^2}{2}B_x. \tag{42}$$

From a plane wave ansatz $B_x = \bar{B}\exp[i(kz - \omega t)]$, we then directly obtain the linear dispersion relation

$$D(\omega,k) = (\omega - k)\left(\omega - k + \frac{k}{2C_A^2}\right) - \frac{\kappa B_0^2}{2} = 0. \tag{43}$$

This is in agreement with the dispersion relation presented by Ref. [8], if we in their result make the appropriate approximations corresponding to $\partial_t \approx -\partial_z$. Solving the dispersion relation we get the following result:

$$\omega = k\left(1 - \frac{1}{4C_A^2}\right) \pm \sqrt{\frac{k^2}{16C_A^4} + \frac{\kappa B_0^2}{2}}. \tag{44}$$

We see that there are two roots of (43) and will hereafter refer to these two modes as the fast mode, for the root with positive sign, and the slow mode, for the root with negative sign. We see that for large k, i.e., $k \gg \sqrt{\kappa B_0^2}\, C_A^2$, we can neglect the small quantity $\kappa B_0^2/2$, and the dispersion relation for the fast mode takes the form $\omega \approx k$, while for the slow mode we get $\omega \approx k(1 - 1/2C_A^2)$. We note that in this regime, the main part of the energy is in the form of gravitational wave energy for the fast mode, whereas most of the energy is electromagnetic for the slow mode. This can be seen from the approximate dispersion relations above, together with the coupled Eqs. (39) and (40). For very long wavelengths, $k \lesssim \sqrt{\kappa B_0^2}\, C_A^2$, we of course still have two roots of the dispersion relation, but in this case the two modes divide their energies roughly equal between the gravitational and electromagnetic form. Therefore, we refer to this regime as that of mixed modes.

The next step is to include higher order terms in the amplitude expansion of $\mathcal{V}(B_x)$ and let the wave amplitude vary weakly in space and time, as compared to ω and k. Keeping terms up to third order in B_x, Eq. (41) now reads

$$(\partial_t + \partial_z)\left[\partial_t + \left(1 - \frac{1}{2C_A^2} + \frac{3}{2C_A^2 B_0}B_x - \frac{15}{4C_A^2 B_0^2}B_x^2\right)\partial_z\right]B_x = -\frac{\kappa B_0^2}{2}B_x. \tag{45}$$

We note that the nonlinear terms induces second-harmonic (SH) and low-frequency (LF) perturbations, and we must therefore modify our ansatz according to

$$B_x = B(z,t)e^{i(kz-\omega t)} + B_{SH}(z,t)e^{2i(kz-\omega t)} + c.c. + B_{LF}(z,t) \tag{46}$$

where c.c. stands for complex conjugate of the preceding terms, and we define ω and k in (46) to fulfill the dispersion relation (43) exactly. Inserting this into (45), we get a long equation involving both SH and LF terms as well as terms of the original frequency ω. To analyze this we use standard techniques for nonlinear wave equations, which we give an outline of here (see e.g., Ref. [23] for details):

(i) First we note that the induced SH and LF terms are one order smaller in an amplitude expansion than terms of the original frequency, which allows us to neglect all terms of higher order in B_{SH} and B_{LF}.

(ii) From linear theory we know that the envelope of the original perturbation $B(z,t)$ travels with the group velocity v_g. Since quadratic perturbations of the original amplitude acts as a driver for both second harmonic and low-frequency perturbations, we observe that B_{SH} as well as B_{LF} will propagate with the group velocity. Thus we may use the approximation $\partial_t \approx -v_g \partial_z$ for derivatives acting on the *amplitudes*. Note also that in general we must use this approximation here instead of $\partial_t \approx -\partial_z$. This is because of the operator $(\partial_t + \partial_z)$, which otherwise will give zero to lowest order when acting on the wave perturbation.

(iii) The different time scales in the equation (second harmonics, low frequency, original frequency) are picked out by multiplying with the appropriate exponential function and then averaging over many wavelengths and period times in order to omit the rapidly oscillating terms in the resultant equation, see [23]. (E.g., to find the equation governing the wave of original frequency, ω, we multiply by $\exp[-i(kz-\omega t)]$ and take an average over several wavelengths and period times.)

To lowest order, the equation for the SH terms becomes

$$D(2\omega,2k)B_{SH}=\frac{3(\omega k-k^2)}{B_0 C_A^2}B^2 \tag{47}$$

where, from (43), we note that $D(2\omega,2k)=3\kappa B_0^2/2$. Similarly, the equation for the low frequency wave is

$$\left[\partial_z^2 - C_A^4\left(4\omega-4k+\frac{k}{C_a^2}\right)^2\right]B_{LF} = \frac{3(4\omega-4k+k/C_A^2)}{2B_0(\omega-k+k/2C_A^2)}\partial_z^2|B|^2 \tag{48}$$

where, in arriving at (48), we have used $\partial_t \approx -v_g\partial_z$ and v_g is given by

$$v_g=\frac{2\omega(1-1/4C_A^2)-2k(1-1/2C_A^2)}{2\omega-2k(1-1/4C_A^2)}. \tag{49}$$

Note that if $C_A^2 \rightarrow \infty$, i.e., the plasma density goes to zero, $v_g \rightarrow 1$, although, as can be seen from (43), $\omega - k \neq 0$.

The back reaction at the original frequency is determined by the equation

$$\left(i(\partial_t+v_g\partial_z)+\frac{v_g'}{2}\partial_z^2\right)B=\frac{F}{B_0}\left(B_{SH}B^*+B_{LF}B+\frac{5}{2B_0}|B|^2B\right) \tag{50}$$

where the group dispersion, v_g', is given by

$$\frac{v_g'}{2}=\frac{\kappa B_0^2}{C_A^4(4\omega-4k+k/C_A^2)^3}, \tag{51}$$

$$F=\frac{3(k^2-\omega k)}{2C_A^2(2k-2\omega-k/2C_A^2)} \tag{52}$$

and the star denotes complex conjugate.

In order to simplify the system of Eqs. (47)–(50), we first consider the LF equation (48). We see that for the different modes we can compare the solution of Eq. (48) with the SH term and get $B_{LF}/B_{SH} \sim \partial_z^2|B|^2/k^2$. Now remembering that the wave amplitudes vary weakly in space, as compared to k, we then conclude that this quantity is very small and we therefore neglect the term involving the low frequency field from now on.

Let us now write Eq. (50) in a more simple form by using (47) to eliminate B_{SH} and making the following coordinate transformations:

$$\xi=\beta(z-v_g t) \tag{53}$$

$$\tau=\frac{|v_g'|\beta^2}{2}t \tag{54}$$

where

$$\beta=C_A^2(4\omega-4k+k/C_A^2) \tag{55}$$

and the absolute value in the transformation (54) has been introduced in order to keep τ positive at all positive times t. Note that the sign of v_g' depends on the roots of the dispersion relation; for the mode with positive sign in (44) v_g' is positive and for the mode with negative sign v_g' is negative. Furthermore we use the linearized equations to relate B to the gravitational perturbation h_+. Keeping only lowest order terms in Eq. (39) we get

$$h_+=\frac{2(\omega-k+k/2C_A^2)}{B_0\omega}B. \tag{56}$$

Transforming the gravitational perturbation amplitude according to

$$\tilde{h}_+=\Psi_1 h_+ \tag{57}$$

where

$$\Psi_1^2=\frac{3\omega^2k(5\omega-5k+9k/2C_A^2)}{16C_A^2(\omega-k+k/2C_A^2)^4} \tag{58}$$

then allows us to rewrite Eq. (50) as a standard NLS equation for the rescaled gravitational perturbation amplitude:

$$(i\partial_\tau \pm \partial_\xi^2)\tilde{h}_+ = \pm|\tilde{h}_+|^2\tilde{h}. \tag{59}$$

Here the plus and minus signs correspond to the fast and the slow mode, respectively. Equation (59) can be solved by the inverse scattering technique, as discussed by for example Ref. [23]. Asymptotically in time, the solution is a train of solitons.

B. Generalization to a 3D spatial dependence

In this section we address the fact that in reality we will not have exact plane wave solutions to the linearized Einstein and Maxwell equations. Rather the involved quantities will also depend on the x and y coordinates if we have wave propagation in the z direction. We will here assume that this deviation from plane waves is small and use a perturbative treatment. The underlying assumption is that ∂_x and ∂_y is of order of $1/L_\perp$, where $1/L_\perp$ is the characteristic width of the pulse fulfilling $\lambda/L_\perp \ll 1$ (λ is the wavelength), while ∂_z is of order $1/\lambda$.

Assuming an x and y dependence in the metric perturbation function, h_+, will induce perturbations in the form of additional components in the Einstein tensor, as seen from the Lorentz gauge condition (60). We thus note that there must be corresponding perturbations in the energy-momentum tensor in order to fulfill Einstein's equations (19). The metric must of course be adjusted correspondingly, and this leads to the addition of components in the metric perturbation, h_{ab}; specifically the following components (of order $\lambda/L_\perp$) must be added: h_{01}, h_{02}, h_{13}, and h_{23}. Inserting this metric perturbation into (19) and using the Lorentz-condition

$$\partial_b h^{ab}=0 \tag{60}$$

we get the following modified version of Eq. (22):

$$(\partial_t^2-\partial_z^2-\nabla_\perp^2)h_+=\kappa(\delta T_{11}-\delta T_{22}) \tag{61}$$

where we again have subtracted the $_{11}$ and $_{22}$ components of Eq. (19), and we have introduced $\nabla_\perp^2\equiv\partial_x^2+\partial_y^2$.

The variation of the perturbed fields in the x and y directions will also induce other components of the electromagnetic and velocity field perturbations, so we introduce additional terms in the perturbed fields according to $\mathbf{E}=E_x\mathbf{e}_1+E_y\mathbf{e}_2+E_z\mathbf{e}_3$ and correspondingly for other fields. These extra perturbations are small, of order $\lambda/L_\perp$. Note that we now have several small quantities here and we neglect terms of the type $\partial_x h_+$, $\partial_x E_x$, $v_x B_y$, etc., i.e., we neglect all terms that include more than one small quantity [with the important exception of the $\nabla_\perp^2$-operator in (61) because this operator compares with the operator $(\partial_t^2-\partial_z^2)$, which is small in itself]. We also make the same approximations made previously.

With these approximations we can calculate the effective charges, currents, and forces (13)–(18) and work out the Maxwell and fluid equations (7)–(12). The result is a large set of equations that can be reduced using $\partial_z\approx-\partial_t$, and the first-order results (37) and (38). We finally arrive at the following equation for the magnetic field perturbation

$$(\partial_t^2-\partial_z^2-\nabla_\perp^2)B_x+\frac{1}{C_A^2}\partial_z\left[\left(\frac{B_0}{B_0+2B_x}\right)^{3/2}\partial_z B_x\right]=B_0\partial_t^2 h_+ \tag{62}$$

and Eq. (61) reduces to

$$(\partial_t^2-\partial_z^2-\nabla_\perp^2)h_+=-2\kappa B_0 B_x. \tag{63}$$

In order to simplify this system further we expand the operator $(\partial_t^2-\partial_z^2-\nabla_\perp^2)\approx 2\partial_t(\partial_t+\partial_z)-\nabla_\perp^2$ and insert this in (62) and (63). Let us also introduce the notation ∂_t^{-1} for the inverse operation of ∂_t. Using (63) we can then combine these two equations into the following modified version of the wave equation (41)

$$(\partial_t+\partial_z-\tfrac{1}{2}\partial_t^{-1}\nabla_\perp^2)(\partial_t+\mathcal{V}(B_x)\partial_z-\tfrac{1}{2}\partial_t^{-1}\nabla_\perp^2)B_x=-\frac{\kappa B_0^2}{2}B_x. \tag{64}$$

Next we note that for perturbations on the form $B_x\sim B(t)\exp i(kz-\omega t)$ we can expand the operator ∂_t as $\partial_t=-i\omega+\tilde{\partial}_t$, where $\tilde{\partial}_t$ acts only on the amplitude. If $B(t)$ varies slowly compared to the exponential part (i.e., $\tilde{\partial}_t/\omega\ll 1$), we can then expand the inverse operator, ∂_t^{-1} as

$$\partial_t^{-1}=\frac{1}{-i\omega+\tilde{\partial}_t}=\frac{i}{\omega}\left(1-i\frac{\tilde{\partial}_t}{\omega}+\cdots\right)\approx\frac{i}{\omega}. \tag{65}$$

Using this result and expanding the velocity function $\mathcal{V}(B_x)$ up to second order in B_x we get the equation

$$(\partial_t+\partial_z)[\partial_t+\mathcal{V}_{exp}(B_x)\partial_z]B_x-\left[\frac{i}{\omega}\left(\partial_t+\partial_z-\frac{1}{4C_A^2}\partial_z\right)\nabla_\perp^2\right]B_x=-\frac{\kappa B_0^2}{2}B_x \tag{66}$$

where

$$\mathcal{V}_{exp}(B_x)=1-(1/2C_A^2)+(3B_x/2C_A^2B_0)-(15B_x^2/4C_A^2B_0^2).$$

We see that this is just Eq. (45) with an extra term on the left-hand side corresponding to the small variation in the x and y directions. In order to analyze this equation, we make a small modification to our ansatz (46) in order to include the weak spatial dependence in the amplitudes and insert this into (66). Working out the algebra, we see that the equations for the SH and LF terms are unaltered to the desired accuracy. This means that the only alteration to our previous results in Sec. IV A will be an extra term in the operator on the left-hand side of Eq. (50), corresponding to the $\nabla_\perp^2$-term in (66). Making the same transformations made in Sec. IV A, the final result for the transformed gravitational perturbation amplitude $\tilde{h}_+$ is

$$(i\partial_\tau\pm\partial_\xi^2+\Upsilon\nabla_\perp^2)\tilde{h}_+=\pm|\tilde{h}_+|^2\tilde{h}_+ \tag{67}$$

where the coefficient Υ is given by

$$\Upsilon=\frac{\left|\left(4\omega-4k+\dfrac{k}{C_A^2}\right)\right|}{4\omega(\omega-k)\left(\omega-k+\dfrac{k}{2C_A^2}\right)} \tag{68}$$

and where the plus and minus signs again refer to the fast and slow mode, respectively.

V. SUMMARY AND DISCUSSION

We have studied the weakly nonlinear propagation of coupled Alfvén and gravitational waves (AGW) propagating perpendicular to an external magnetic field in a plasma with $C_A^2\gg 1$, using the coupled Einstein and Maxwell equations. One of our main results is the general evolution equation (41), which holds for a broadband spectrum. A thorough study of this equation is a project for further study. In order to simplify (41), we have made a WKB-ansatz, and showed that this leads to the well-known NLS equation (59), where the generalization to a 3D spatial dependence has been described in Sec. IV B.

Possible astrophysical sources for large amplitude AGW are binary pulsars, particularly in the later stages of their evolution, i.e., not too far from merging. Clearly such a system produces gravitational waves, and as discussed by, for example, Ref. [13], this leads also to large amplitude electromagnetic perturbations. Through the back reaction in Einstein's equations the electromagnetic perturbation is coupled to the gravitational one, and linearly this is described by the dispersion relation (43). For typical wavelengths and background parameters (i.e., for $k \gg \sqrt{\kappa B_0^2 C_A^2}$), the dispersion relation approximately separates in the fast mode, $\omega \approx k$, and the slow mode, $\omega \approx k(1-1/2C_A^2)$, where the former mainly has gravitational character and the later one electromagnetic character. However, very close to collapsing binaries we have magnetic fields $B_0 \sim 10^8$T, in which case there is no clear distinction in character between the modes. On the other hand, for moderate values of B_0 further from the source, the modes will be clearly separated. However, due to the strong coupling in the near region of pulsars, both the fast and slow mode will be represented at larger distances, although the energy density of the later will be smaller. The amount of energy given to the slow mode can be estimated from the results of Ref. [13] and correspond to an electromagnetic power of the order 10^{25} W close to merging.

In order to discuss the nonlinear evolution, we consider pulses including a perpendicular dependence, which is described by Eq. (67). For simplicity we study long pulses with a shape that depends only on a (normalized) cylindrical radius, $\tilde{r} = \sqrt{(x^2+y^2)/\Upsilon}$, thus neglecting dispersive effects. On reinstating the speed of light c, the condition for this is $16\pi G B_0^2 C_A^2 L_\perp^2 / c^6 \mu_0 \ll k^2 L_\parallel^2$, where $L_\perp$ and $L_\parallel$ are the characteristic length scales for variations in the perpendicular and parallel directions respectively. In this case we get a cylindrically symmetric NLS equation

$$\left[i\partial_\tau + \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \frac{\partial}{\partial \tilde{r}} \right) \right] \tilde{h}_+ = \pm |\tilde{h}_+|^2 \tilde{h}_+ . \tag{69}$$

Contrary to Eq. (59), there are no exact solutions known for the cylindrical version (69), except for some physically uninteresting special cases. Still Eq. (69) has been studied in some detail, both analytically, using approximate variational techniques [33], and numerically [34]. Of most interest to us is the case with the minus sign, corresponding to the slow, electromagnetically dominated mode. The main motivation for considering this mode is that this choice gives a nonlinearity of focusing type. For strong enough nonlinearity the solutions to (69) then show wave collapse, i.e., the pulse focuses indefinitely and the pulse radius $\tilde{r}_p \to 0$ in a finite time. The main characteristics of the collapse can be described within an approximate variational framework. Following Ref. [33] we use a trial function $\tilde{h}_+(\tilde{r},\tau) = A(\tau)\mathrm{sech}[\tilde{r}/a(\tau)]\exp[ib(\tau)\tilde{r}^2]$ together with Rayleigh-Ritz optimization in order to derive a single ordinary differential equation for the pulse width $a(\tau)$. Pulse energy conservation then follows from the optimization scheme according to $a(\tau)A(\tau) = A(0)a(0)$. Furthermore, the evolution of the width scales as $[a(\tau)/a(0)]^2 - 1 \propto [1 - A(0)^2 a(0)^2 / I_c]\tau^2$ where $I_c \approx 1.35$. Thus if $A(0)^2 a(0)^2 > 1.35$, the pulse will collapse to zero width in a finite time, and in the opposite case linear diffraction will dominate to spread out the pulse indefinitely. Of course, for the case of collapse, higher-order nonlinearities neglected in the derivation of (69) will eventually become important, which will change the later stages of the given scenario.

Let us study whether the electromagnetically dominated slow mode, excited by binary pulsars close to collapse may undergo wave collapse. Here a word of caution is at hand. In a real system of this type a number of effects outside the model equation (69) are likely to play important roles for the pulse dynamics. For the sake of simplicity we will here exclude such effects. From the variational approach mentioned above, which agrees with numerical works [34], we find that collapse takes place if $|\tilde{h}_+|^2 > I_c / \tilde{r}_p^2$, which can be written

$$\frac{B^2}{B_0^2} \frac{c^6}{C_A^6} \frac{3\mu_0 k^2 c^4}{8\pi G B_0^2} k^2 r_p^2 > 1.35 \tag{70}$$

where we have reintroduced dimensional quantities. The electromagnetic fields leaving the binary system is excited by gravitational quadrupole radiation, which has a certain directionality, but not a very pronounced one. Thus before significant focusing takes place, the pulse width can be replaced by the radial distance from the source r_{dist} as a rough order of magnitude estimate for the pulse width. Thus we substitute $r_p \to r_{\mathrm{dist}}$ in (70), use $B^2 r_{\mathrm{dist}}^2 = 10^{25} \mu_0 / c 4\pi$ T^{2}m^2 and take $k = 10^{-5}$ m^{-1} corresponding to the parameter values mentioned above, discussed in more detail in Ref. [13]. The only extra parameters we need to specify are B_0 and ρ_0. For a broad range of relevant background values 10^{-20} kg/m$^3 < \rho_0 < 10^{-5}$ kg/m^3 and 10^{-10} T$< B_0 < 0.1$ T, the condition (70) is always fulfilled. The main concern is to find values fulfilling the previous assumption $C_A^2/c^2 = B_0^2/\mu_0 \rho_0 c^2 \gg 1$, which holds for a low-density plasma $\rho_0 \sim 10^{-20}$ kg/m^3, provided $B_0 > 10^{-4}$ T. However, we note that the condition (70) can be misleading, since for the strong powers considered, the level of the second harmonic fields typically obeys $B_{SH} > B_0$ at relevant distances from the source. This level lies outside the regime of validity for weakly nonlinear waves. We note that the left-hand side of (70) is proportional to B_{SH}. Replacing the expression for B_{SH} found from (47) by its limit of validity $B_{SH} \sim B_0$ (relevant for more moderate powers than 10^{25} W), we note that collapse will occur when

$$\frac{c^2 k^2 r_{\mathrm{dist}}^2}{C_A^2} \gtrsim 1 \tag{71}$$

which is easily fulfilled at reasonable distances ($r_{\mathrm{dist}} \gtrsim 10^7$ m) from the source. The characteristic timescale for significant focusing corresponding to the example given above (with $B_{SH} \sim B_0$) is given by $T_{\rm foc} \sim C_A^2/c^2\omega$.

The nonlinearity of the fast (gravitationally dominated) mode is not of focusing type. We note that from Eq. (59), there is still the possibility of other nonlinear effects, such as (dark) soliton formation that might, in principle, be detected. The characteristic time scale for such nonlinear processes is typically very large [35], which can be seen from Eqs. (54)–(59) inserting realistic background parameter values for the relevant quantities. This means that for nonlinear effects to be important for this mode, we must assume extreme parameter values that cannot be justified by astrophysical observations.

The condition (71) does not contain gravitational parameters, which is a consequence of the electromagnetic dominance of the slow mode. However, we note that the considered process is induced by the gravitational-electromagnetic coupling, and thus it still has a gravitational origin. The condition for wave collapse of this mode can be fulfilled for a reasonable range of parameters, which opens up for the possibility of structure formation of the electromagnetic radiation pattern. In the example considered above, the focusing takes place on fractions of a second (unless C_A^2 is extremely large), and thus the nonlinearities may cause noticeable structures in the later stages of binary merging. We wish to point out here, though, that this example is to be seen as a somewhat crude estimation of the focusing effect, and that this example does not express all of the physics involved. In the absolute vicinity of a binary merger, other significant effects are almost certain to appear. A more complete treatment is a project for future research.

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