Can nonadiabatic perturbations arise after single-field inflation?

Steven Weinberg*

Theory Group, Department of Physics, University of Texas, Austin, Texas 78712, USA
(Received 28 April 2004; published 31 August 2004)

It is shown that nonadiabatic cosmological perturbations cannot appear during the period of reheating following inflation with a single scalar inflaton field.

According to a widely adopted picture [1], the perturbations to the Robertson-Walker cosmology arose from the quantum fluctuations in a slowly rolling scalar "inflaton" field during a period of inflation, then became classical as their wavelength was stretched beyond the horizon, and subsequently were imprinted on the decay products of the inflaton during a period of "reheating." One of the attractions of this picture (and in particular the assumption that there is just one inflaton field) is that it has generally been thought to lead only to adiabatic perturbations, in agreement with current observations [2].

A recent report [3] has raised the question of whether it is possible for nonadiabatic cosmological perturbations to arise during reheating even after inflation with a single inflaton field. This would be very important if true, for then observational limits on nonadiabatic fluctuations in the cosmic microwave background would provide some constraints on the otherwise mysterious era of reheating, and indeed on the whole history of the universe between inflation and the present.

However, there are very general grounds for expecting that single-field inflation can only produce adiabatic fluctuations, whatever happens in reheating or subsequently. It has been shown [4] that, whatever the constituents of the universe, the differential equations for cosmological perturbations in Newtonian gauge *always* have a solution that has certain "adiabatic" properties (spelled out below) for physical wavelengths that are sufficiently far outside the horizon (that is, for physical wave numbers that are much less than the cosmological expansion rate). For this solution, there is a quantity ζ [5], defined by

$$\zeta \equiv -\Psi + \delta\rho/3(\bar{\rho} + \bar{p}), \tag{1}$$

whose rate of change vanishes far outside the horizon. Here $\delta\rho$ is the perturbation to the total energy density in Newtonian gauge; bars denote unperturbed quantities; the perturbed metric is given by

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(1 - 2\Psi)d\mathbf{x}^2; \tag{2}$$

and as usual $H \equiv \dot{a}/a$. (Reference [4] dealt mostly with a quantity [6]

$$\mathcal{R} \equiv -\Psi + H\delta u, \tag{3}$$

where δu is the perturbation to the total velocity potential, but outside the horizon $\mathcal{R}$ and ζ are the same.) Also, for this adiabatic mode, the perturbations to the metric and the total energy density and pressure are given outside the horizon by

$$\Phi = \Psi = \zeta \left[-1 + \frac{H(t)}{a(t)} \int_{t_1}^{t} a(t')dt' \right] \tag{4}$$

$$\delta\rho = -\frac{\zeta \dot{\bar{\rho}}(t)}{a(t)} \int_{t_1}^{t} a(t')dt',$$

$$\delta p = -\frac{\zeta \dot{\bar{p}}(t)}{a(t)} \int_{t_1}^{t} a(t')dt'. \tag{5}$$

This is a solution for any value of t_1, so the difference of adiabatic solutions with different values of t_1 is also a solution, with $\zeta = 0$ and $\Phi = \Psi \propto H(t)/a(t)$, $\delta\rho/\dot{\bar{\rho}} \propto 1/a(t)$, etc. We will adjust t_1 so that the total perturbation takes the form (4), (5), in which case t_1 will be at some early time during inflation to avoid perturbations that are very large at early times. Furthermore, if the total energy-momentum tensor $T^{\mu\nu}$ is given by a sum of tensors $T^{\mu\nu}_\alpha$ for fluids labeled α (not necessarily individually conserved) then Eq. (5) holds for each of the individual perturbations

$$\delta\rho_\alpha = -\frac{\zeta \dot{\bar{\rho}}_\alpha(t)}{a(t)} \int_{t_1}^{t} a(t')dt', \quad \delta p_\alpha = -\frac{\zeta \dot{\bar{p}}_\alpha(t)}{a(t)} \int_{t_1}^{t} a(t')dt', \tag{6}$$

and similarly for any other four-dimensional scalar, such as the inflaton field

$$\delta\varphi = -\frac{\zeta \dot{\bar{\varphi}}(t)}{a(t)} \int_{t_1}^{t} a(t')dt'. \tag{7}$$

In general there may be other solutions, for which ζ may or may not be constant and the perturbations are not given by Eqs. (4)–(7). (Misleadingly, these others are often called isocurvature solutions.) But if only one inflaton field makes a contribution to the energy-momentum tensor during inflation then the perturbations to this field and the metric in Newtonian gauge are governed by a third-order set of differential equations with a single constraint, so only two independent solutions contribute to cosmological perturbations, and since we have found two explicit adiabatic solutions outside the horizon, these are the only solutions for the coupled system

---

*Electronic address: weinberg@physics.utexas.edu

of inflaton and gravitational fields from horizon exit to the end of inflation. During reheating and the subsequent evolution of the universe other fields and fluids become important, but the adiabatic mode is a true solution outside the horizon whatever the contents of the universe, so with the universe in the adiabatic mode at the beginning of reheating it remains in this mode as long as the wavelength remains outside the horizon.

So how did nonadiabatic modes arise in Ref. [3]? The theorem of Ref. [4] assumes that the field equations hold at every moment, which requires that the perturbations are differentiable functions of time. As he suggested might be the case, the reason that Armendáriz-Picón found a nonadiabatic mode during reheating in Ref. [3] was that, although it was assumed that initially during inflation only the inflaton field was present and there was no energy transfer to other fields, he supposed in this reference that the energy transfer rate rose discontinuously to a nonzero value at the beginning of reheating. (The model considered in Ref. [3] actually gave a constant value for ζ, but the perturbations had unequal values of $\delta\rho/\bar{\rho}$ for the inflaton and its decay products, so as recognized by Armendáriz-Picón, this perturbation was not adiabatic.) Of course, a discontinuous change in the energy transfer rate is unphysical.

There is one weak point in the above general argument that nonadiabatic perturbations do not arise during reheating. It is the assumption that there was nothing but the inflaton and gravitation before reheating. Of course, in order for reheating to occur at all, there must be other fields or fluids besides the inflaton, and these do not suddenly come into existence during reheating. The reactions that produce matter during reheating cannot be completely absent beforehand, so the remaining question is whether the transfer of energy from the inflaton to other fields during inflation before reheating excites these other modes.

This question is answered by a further theorem that if the matter energy density during inflation is small, then even if the perturbations to the matter energy density are initially not at all adiabatic, the departures from adiabaticity would decay exponentially fast as soon as the energy transfer rate becomes appreciable.

Here is the proof. The comoving rate per proper volume of energy transfer from the inflaton field φ to "matter" fields (possibly including radiation) is in general some four-scalar function X of all these fields and perhaps their first and higher time derivatives:

$$-u_\mu T_M^{\mu\nu}{}_{;\nu} = X(\varphi, \dots), \tag{8}$$

where u^μ is the four-vector velocity of the total energy-momentum tensor, normalized so that $u^\mu u_\mu = -1$. To zeroth order in all perturbations to the metric and other fields, this reads

$$\dot{\bar{\rho}}_M + 3H(\bar{\rho}_M + \bar{p}_M) = \bar{X}, \tag{9}$$

where bars denote unperturbed values, taken to depend only on time. (We choose signs so that $\bar{u}^0 = +1$.) To first order in perturbations, Eq. (8) gives in Newtonian gauge

$$\delta\dot{\rho}_M + 3H(\delta\rho_M + \delta p_M) - 3(\bar{\rho}_M + \bar{p}_M)\dot{\Psi} = \delta X + \Phi\bar{X}, \tag{10}$$

the last term on the right arising from the perturbation to u_0. We have dropped terms involving spatial gradients, which become negligible outside the horizon.

Now suppose that at some early time during inflation the density and pressure of matter are small. (This is plausible, because the energy density of fermions and gauge fields produced by quantum fluctuations would be quadratic in the fluctuations.) Then the inflaton will provide the chief source of the gravitational field. As mentioned above, under these conditions the perturbations to the inflaton and gravitational potentials will be described by the adiabatic solution

$$\delta\varphi = -\dot{\bar{\varphi}}\mathcal{I}, \quad \Phi = \Psi = -\zeta + H\mathcal{I}, \tag{11}$$

in which for convenience we have introduced the notation

$$\mathcal{I}(t) \equiv \frac{\zeta}{a(t)} \int_{t_1}^{t} a(t')dt'. \tag{12}$$

Also, under these conditions the energy transfer rate will depend only on the inflaton and gravitational fields and their time derivatives, so

$$\delta X = -\dot{\bar{X}}\mathcal{I}. \tag{13}$$

[For instance, if X depends only on φ, then

$$\delta X = \frac{\partial\bar{X}}{\partial\bar{\varphi}}\delta\varphi = -\left(\frac{\partial\bar{X}}{\partial\bar{\varphi}}\dot{\bar{\varphi}}\right)\mathcal{I}.$$

Equation (13) also holds if X depends also on $\dot{\varphi}$, $\ddot{\varphi}$, etc., provided that for each pair of time derivatives there is a factor of g^{00} to keep X a scalar.] Putting Eq. (13) together with Eq. (11), the right-hand side of Eq. (10) is

$$\delta X + \Phi\bar{X} = -\frac{\partial}{\partial t}[\bar{X}\mathcal{I}], \tag{14}$$

and the difference between Eq. (10) and the time-derivative of Eq. (9) gives

$$\frac{\partial}{\partial t}[\delta\rho_M + \dot{\bar{\rho}}_M\mathcal{I}] = -3H(\delta\rho_M + \delta p_M) - 3(\bar{\rho}_M + \bar{p}_M)\dot{\mathcal{I}}$$

$$-3\frac{\partial}{\partial t}[(\bar{\rho}_M + \bar{p}_M)H\mathcal{I}]. \tag{15}$$

This can be simplified by noting that $\mathcal{I}$ satisfies the differential equation

$$\ddot{\mathcal{I}} + \frac{\partial}{\partial t}(H\mathcal{I}) = 0, \tag{16}$$

so

$$\frac{\partial}{\partial t}[\delta\rho_M + \dot{\bar\rho}_M \mathcal{I}] = -3H(\delta\rho_M + \delta p_M + \dot{\bar\rho}_M \mathcal{I} + \dot{\bar p}_M \mathcal{I}). \tag{17}$$

To continue, let us assume that the matter pressure is a function $p_M(\rho_M)$ only of the matter energy density ρ_M, as in the case of pure radiation or pure dust. (We are *not* assuming this for the combined system of matter and inflaton.) This is plausible, because the decay of a single real inflaton field would not generally produce any chemical potentials. Then $\dot{\bar p}_M = p'_M(\bar\rho_M)\dot{\bar\rho}_M$ and $\delta p_M = p'_M(\bar\rho_M)\delta\rho_M$. Equation (17) now reads

$$\frac{\partial}{\partial t}[\delta\rho_M + \dot{\bar\rho}_M \mathcal{I}] = -3H(1 + c_M^2)(\delta\rho_M + \dot{\bar\rho}_M \mathcal{I}), \tag{18}$$

where $c_M^2 \equiv dp_M/d\rho_M$ is the squared sound speed. This shows that $|\delta\rho_M + \dot{\bar\rho}_M \mathcal{I}|$ decreases monotonically and faster than a^{-3}, but that is not good enough, because we have to make sure that this is not just because $|\delta\rho_M|$ and $|\dot{\bar\rho}_M \mathcal{I}|$ are both decreasing. For this purpose, we use Eq. (9) again, and rewrite Eq. (18) as

$$\frac{\partial}{\partial t}\ln N = -\bar{X}\frac{(1 + c_M^2)}{\bar\rho_M + \bar p_M}, \tag{19}$$

where N is a dimensionless measure of the departure of the matter energy density perturbation from its adiabatic value

$$N \equiv \left|\frac{\delta\rho_M + \dot{\bar\rho}_M \mathcal{I}}{\bar\rho_M + \bar p_M}\right|. \tag{20}$$

Since energy is flowing from the inflaton field to matter, X is positive, and Eq. (19) shows that N decreases monotonically. Early in inflation the matter perturbation may be nowhere near adiabatic, with N of the same order of magnitude as the fractional density perturbation. As time passes during inflation, the transfer of energy from the inflaton to matter may make both $\delta\rho_M$ and $\dot{\bar\rho}_M$ large, but the quantity N continues to decrease. Eventually, after the energy transfer rate X becomes large for a sufficiently long time during reheating, the matter density perturbation can no longer be ignored in calculating X and the gravitational field perturbations, but by that time N will have decayed exponentially. With the density

and pressure perturbations of the matter as well as the inflaton satisfying Eq. (6), we will still have $\delta X = -\dot{X}\mathcal{I}$ and $\Phi = \Psi = -\dot{\mathcal{I}}$, and the above analysis will remain valid.

Note that asymptotically, after the transfer of energy from the inflaton to matter ceases, the quantity $-\dot{\bar\rho}_M \mathcal{I}/(\bar\rho_M + \bar p_M)$ approaches $3\zeta/(1 - \dot{H}/H^2)$, so as long as $\bar p_M < \bar\rho_M$, $-\dot{\bar\rho}_M \mathcal{I}/(\bar\rho_M + \bar p_M)$ is bounded below in absolute value by $3|\zeta|/4$. Thus the reason that N approaches zero is *not* that $\delta\rho_M/(\bar\rho_M + \bar p_M)$ and $-\dot{\bar\rho}_M \mathcal{I}/(\bar\rho_M + \bar p_M)$ are both approaching zero. Rather, the ratio of these quantities, $-\delta\rho_M/\dot{\bar\rho}_M \mathcal{I}$, approaches unity.

With the matter pressure a function only of the matter density, the matter pressure perturbation will then also satisfy Eq. (6). Thus the perturbations become adiabatic, in the sense of Eqs. (6) and (7). Also, with the energy density and pressure perturbations of matter as well as of the inflaton all satisfying Eq. (6), the total energy density and pressure perturbations will obviously satisfy Eq. (5). It follows that the perturbations also satisfy the adiabatic condition that ζ is constant, because in general

$$\dot\zeta = \frac{\dot{\bar\rho}\,\delta p - \dot{\bar p}\,\delta\rho}{3(\bar\rho + \bar p)^2}. \tag{21}$$

The same analysis also obviously applies if the inflaton energy goes into several species of matter, with a pressure for each species given by a function of the energy density in that species.

In conclusion, even if the decay of the inflaton during inflation produces a small matter density whose perturbations are not at all adiabatic, the departure from adiabaticity will decay rather than grow as inflation proceeds, and the departures of the perturbations from their adiabatic values will become exponentially small when the matter density becomes large during reheating.

[1] A. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); S. Hawking, Phys. Lett. **115B**, 295 (1982); A. Starobinsky, *ibid.* **117B**, 175 (1982); F. Bardeen, P. Steinhardt, and M. Turner, Phys. Rev. D **28**, 679 (1983). A model in which cosmological perturbations arise during inflation from quantum fluctuations in a scalar field derived from the spacetime curvature had been considered earlier by Y. Mukhanov and G. Chibisov, JETP Lett. **33**, 532 (1981).

[2] H.V. Peiris *et al.*, Astrophys. J., Suppl. Ser. **148**, 213 (2003); P. Crotty, J. Garcia-Bellido, J. Lesgourgues, and A. Riazuelo, Phys. Rev. Lett. **91**, 171301 (2003).

[3] C. Armendáriz-Picón, astro-ph/0312389.

[4] S. Weinberg, Phys. Rev. D **67**, 123504 (2003). A somewhat improved discussion is given in the Appendix to S. Weinberg, *ibid.* **69**, 023503 (2004). Also see S. Bashinsky and U. Seljak, *ibid.* **69**, 083002 (2004), especially Appendix B.

[5] J.M. Bardeen, P.J. Steinhardt, and M.S. Turner, Phys. Rev. D **28**, 679 (1983). This quantity was reintroduced by D. Wands, K.A. Malik, D.H. Lyth, and A.R. Liddle, *ibid.* **62**, 043527 (2000). Also see D.H. Lyth, *ibid.* **31**, 1792 (1985); K.A. Malik, D. Wands, and C. Ungarelli, *ibid.* **67**, 063516 (2003).

[6] J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980); D.H. Lyth, *ibid.* **31**, 1792 (1985). For reviews, see J. Bardeen, in *Cosmology and Particle Physics*, edited by Li-zhi Fang and A. Zee (Gordon & Breach, New York, 1988); A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large Scale Structure* (Cambridge University Press, Cambridge, UK, 2000).