Late-time cosmology in a (phantom) scalar-tensor theory: Dark energy and the cosmic speed-up

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We consider late-time cosmology in a (phantom) scalar-tensor theory with an exponential potential, as a dark-energy model with equation of state parameter close to -1 (a bit above or below this value). Scalar (and also other kinds of) matter can be easily taken into account. An exact spatially flat FRW cosmology is constructed for such theory, which admits (eternal or transient) acceleration phases for the current universe, in correspondence with observational results. Some remarks on the possible origin of the phantom, starting from a more fundamental theory, are also made. It is shown that quantum gravity effects may prevent (or, at least, delay or soften) the cosmic doomsday catastrophe associated with the phantom, i.e., the otherwise unavoidable finite-time future singularity (Big Rip). A dark-energy model (higher-derivative scalar-tensor theory) is introduced, and it is shown to admit an effective phantom and/or quintessence description with a transient acceleration phase. In this case, gravity favors that an initially insignificant portion of dark energy becomes dominant over the standard matter and radiation components in the evolution process.

I. INTRODUCTION

Recent astrophysical data, ranging from high redshift surveys of supernovae to WMAP observations, indicate that about 70 % of the total energy of our universe is to be attributed to a weird cosmic fluid with large and negative pressure, the dark energy (see [1,2] for a recent review), and that the universe is currently in an accelerating phase. It also turns out that the dark-energy equation of state parameter w is close to -1. So far, the simplest possibility proposed for this kind of dark energy is the use of a scalar field (or a scalar-tensor theory). However, scalar-tensor theories are not free from problems, especially when they are considered directly as dark-energy candidates.

Much attention has been drawn by scalar fields in studies of the early time universe. A variety of scalar potentials has been considered and a number of accelerating (inflationary) cosmologies have been advocated. For instance, the interesting quintessence model [3] with w slightly bigger than -1 is quite popular for the explanation of early- (and late-) time acceleration, especially in the case of exponential potentials [4]. Moreover, exponential scalar potentials often appear naturally after compactification in string and/or M-theory. Needless to say, such a description is model-dependent and is still quite far from the final goal: the formulation of a plausible and consistent dark-energy theory.

Another line of research is related with the case where the dark-energy equation of state parameter is less than -1, since this possibility is not excluded by astrophysical data. Moreover, the recent Supernova data [5] favor the cosmological models with such w. The typical example of a dark energy of this kind is provided by a scalar field with negative kinetic energy, dubbed phantom (see [6,7] and references therein). At first sight, such models may look rather strange and they lead to a number of unpleasant consequences, as a finite-time future singularity (the Big Rip) [8–10]. Nevertheless, the possibility of negative energies seems to be acceptable in classical scalar-tensor theories. Actually, many of them do contain phantoms, as the ones coming from string and/or M-theory compactification, or higher-derivative supergravities, or modifications of Einstein gravity itself. In fact, the issue is somehow delicate, since what looks like a phantom in one reference frame may radically change its nature in another frame (e.g., after a conformal transformation). In this sense, even in the absence of fundamental, physical meaning the phantom can be still useful as a convenient mathematical tool in order to study cosmological models in standard scalar-tensor theories because a phantom-related frame may lead to a simpler formulation of the problem. Finally, there are examples where an effective phantom and/or quintessence description of the late-time universe naturally appears, even if the starting theory does not explicitly exhibit the phantom and/or quintessence structure.

In the present work we study different cases of a late-time spatially flat FRW cosmology in the (phantom) scalar-tensor theory, mainly with exponential potentials. Such scalar is

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considered as a dark energy, and the possibility of deriving the current speed-up is shown also in the presence of matter. Exact FRW cosmologies are constructed for the (phantom) scalar-tensor theory with an exponential potential, a model that can be important for understanding attractors and the stability properties. The possibility of avoiding the unwanted Big Rip by simply taking into account quantum gravity effects, which may become dominant near future singularity, is demonstrated. Finally, the present dark-energy dominance and acceleration, within the effective phantom and/or quintessence description, is discussed in the model where a form of higher-derivative, gravity-matter coupling is introduced.

The organization of the paper is the following: In Sec. II spatially flat FRW cosmological solutions are discussed for scalar-tensor gravity with scalar matter. Explicit examples of accelerating (and decelerating) scale factors are presented for exponential potentials when the theory contains one or two scalar fields. In Sec. III the general solution for a spatially flat FRW cosmology, which includes eternal or transient acceleration, is found in the (phantom) scalar-tensor theory with exponential potential. This is based on the use and extension of a method recently developed by Russo [18]. The comparison with particular solutions of Sec. II is done. Section IV is devoted to the study of the influence of quantum gravity effects on the Big Rip singularity in phantom cosmology. It is shown that taking them into account properly may change the future of the universe, from that with a finite-time singularity to an ordinary de Sitter space. In Sec. V a higher-derivative matter-gravity coupling is suggested as a sort of dark-energy model. It admits an effectively phantom and/or quintessence description and does explain the current dark-energy dominance over standard matter by gravity assistance. Stability analysis of the model demonstrates that the acceleration phase is actually transient. A summary and outlook are given in the discussion. In the Appendix we outline how the (phantom) scalar-tensor theory may originate, via compactification, from a higher-dimensional (super) gravity.

II. EXAMPLES OF THE ACCELERATING UNIVERSE IN (PHANTOM) SCALAR-TENSOR THEORIES

We start from the action of multiscalar-tensor theory. Several illustrative examples of the (accelerating) FRW cosmology will be presented here as simple dark-energy models (see [1,2] for a recent review). A scalar field, ϕ, which may be later regarded as a phantom, couples with gravity. As is typical in these models, a second scalar field, χ, is considered. The string-inspired Lagrangian in the d-dimensional spacetime is

$$S = \frac{1}{\kappa^2} \int d^d x \sqrt{-g}\, e^{\alpha\phi}\left(R - \frac{\gamma}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right)$$

$$+ \int d^d x \sqrt{-g}\left(-\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - U(\chi)\right). \quad (1)$$

Here α and γ are constant parameters and $V(\phi)$ ($U(\chi)$) is the potential for ϕ (χ). If the constant parameter γ is negative, ϕ has a negative kinetic energy and can be regarded as

a phantom [6,7]. We should note, however, that γ need not be negative in order to obtain the accelerated universe as we will later see. As the matter scalar χ does not couple with ϕ directly, the equivalence principle is not violated, although the effective gravitational coupling depends on ϕ as $\kappa e^{-(\alpha\phi/2)}$. One may go to the Einstein frame by the scale transformation

$$g_{\mu\nu} = e^{-[2\alpha/(d-2)]\phi} g_{E\mu\nu}. \quad (2)$$

In the following, the quantities in the Einstein frame are denoted by the index E. After the transformation (2), the action (1) has the following form:

$$S = \frac{1}{\kappa^2} \int d^d x \sqrt{-g_E}\left[R_E - \left(\frac{(d-1)\alpha^2}{d-2} + \frac{\gamma}{2}\right)g_E^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right.$$

$$\left. - e^{-[2\alpha/(d-2)]\phi}V(\phi)\right] + \int d^d x \sqrt{-g_E}$$

$$\times \left(-\frac{e^{-\alpha\phi}}{2}g_E^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - e^{-[d\alpha/(d-2)]\phi}U(\chi)\right). \quad (3)$$

In the Einstein frame, there appears a term coupling the matter χ with ϕ. Even if γ is negative, when

$$\frac{(d-1)\alpha^2}{d-2} + \frac{\gamma}{2} > 0, \quad (4)$$

the kinetic energy of ϕ becomes positive, as for a usual scalar field. Hence, the remarkable observation follows that, what is a phantom in one frame may not be a phantom in a different frame, especially if the coupling is taken into account.

As a first step, one considers the $d=4$ case and assumes $\chi=0$. We now define φ and $\tilde{V}(\varphi)$ as

$$\varphi = \phi\sqrt{\alpha^2 + \frac{\gamma}{3}}, \quad \tilde{V}(\varphi) = e^{-\alpha\phi}V(\phi), \quad (5)$$

and assume that the metric has the FRW form in flat space

$$ds_E^2 = -dt_E^2 + a_E(t_E)^2 \sum_{i=1,2,3}(dx^i)^2. \quad (6)$$

Here t_E is the time coordinate in the Einstein frame. When φ only depends on the time coordinate, the FRW equation and φ equation follow:

$$3H_E^2 = \frac{3}{4}\left(\frac{d\varphi}{dt_E}\right)^2 + \frac{1}{2}\tilde{V}(\varphi), \quad (7)$$

$$0 = 3\left(\frac{d^2\varphi}{dt_E^2} + 3H_E\frac{d\varphi}{dt_E}\right) + \tilde{V}'(\varphi). \quad (8)$$

Here the Hubble parameter (in the Einstein frame) is defined by $H_E \equiv (1/a_E)(da_E/dt_E)$. Note that models of this type may have a double interpretation: as multiscalar-tensor theo-

ries or as matter-scalar-tensor theories. In other words, some scalars may be considered as matter or as part of a gravitational theory. It has been suggested that such models may describe the inflationary early universe as quintessence [3].

Special attention in cosmology has been paid to exponential potentials [4], which often follow from string and/or M-theory compactification. If $\tilde{V}(\varphi)$ behaves as an exponential function of φ

$$\tilde{V}(\varphi) \sim V_0 e^{-2(\varphi/\varphi_0)} \tag{9}$$

(V_0 and φ_0 are constants) during some period, as in the present universe, the solution of (7) and (8) exists[1]

$$a_E = a_{E0}\left(\frac{t_E}{t_{E0}}\right)^{(3/4)\varphi_0^2}, \quad \varphi = \varphi_0 \ln\frac{t_E}{t_{E0}}. \tag{10}$$

Here

$$t_{E0} \equiv \varphi_0 \sqrt{\frac{1}{V_0}\left(\frac{27}{8}\varphi_0^2 - \frac{3}{2}\right)}. \tag{11}$$

In the original (Jordan) frame (1), the time coordinate t and scale factor $a(t)$ in the FRW form of the metric

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3}(dx^i)^2, \tag{12}$$

are related with the corresponding quantities in the Einstein frame (6) by

$$dt = e^{-(\alpha/2)\phi}dt_E = \frac{t_{E0}^{\beta\varphi_0/2}}{1-(\beta\varphi_0/2)}d(t_E^{1-(\beta\varphi_0/2)}), \tag{13}$$

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[1]The action (1) with the potential (9) belongs to the class discussed in [11]. In fact if we redefine the field ϕ by

$$\Phi \equiv \frac{2\sqrt{|\gamma|}}{\alpha}e^{(\alpha/2)\phi},$$

the action (1) can be rewritten as

$$S = \frac{1}{\kappa^2}\int d^dx\sqrt{-g}\left(F(\Phi)R \mp \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - U(\Phi)\right)$$

$$+ \int d^dx\sqrt{-g}\left(-\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - U(\chi)\right).$$

Here

$$F(\Phi) = \frac{\alpha^2}{4|\gamma|}\Phi^2, \quad U(\Phi) = V_0\left(\frac{\alpha\Phi}{2\sqrt{|\gamma|}}\right)^{4[1-\sqrt{1+(\gamma/3\alpha^2)}]}.$$

Then, one can use the same arguments as in [11] in order to fit the parameters so that they satisfy the present cosmological data. The nonminimal scalar-gravitational coupling term, which is required by renormalizability of the quantum-field theory in curved spacetime [12,13], may have very interesting effects on the phantom cosmology (see [14] for a recent discussion).

$$a = e^{-(\alpha\phi/2)}a_E = a_{E0}\left(\frac{t_E}{t_{E0}}\right)^{[(3/4)\varphi_0^2 - \beta\varphi_0]}$$

$$= a_{E0}\left(\frac{t}{t_0}\right)^{\{(3/4)\varphi_0^2 - \beta\varphi_0/[1-(\beta\varphi_0/2)]\}}, \tag{14}$$

$$\phi = \frac{\beta\varphi_0}{\alpha(1-(\beta\varphi_0/2))}\ln\frac{t}{t_0}. \tag{15}$$

Here

$$\beta \equiv \frac{\alpha}{\sqrt{\alpha^2 + (\gamma/3)}}, \quad t_0 \equiv \frac{t_{E0}}{1-(\beta\varphi_0/2)}. \tag{16}$$

Then, cosmic acceleration ($\dot{a}>0$ and $\ddot{a}>0$) occurs when

$$\frac{\frac{3}{4}\varphi_0^2 - \frac{\varphi_0\alpha}{2\sqrt{\alpha^2+(\gamma/3)}}}{1-\left[\frac{\varphi_0\alpha}{2\sqrt{\alpha^2+(\gamma/3)}}\right]} > 1. \tag{17}$$

By properly choosing the parameters α, φ_0, and γ, the present cosmic acceleration can be realized. For the matter with $p=w\rho$, where p is the pressure and ρ is the energy density,

$$a \propto t^{[2/3(w+1)]}. \tag{18}$$

Then, effectively

$$w = -1 + \frac{\frac{2}{3}\left(1 - \frac{\varphi_0\alpha}{2\sqrt{\alpha^2+(\gamma/3)}}\right)}{\frac{3}{4}\varphi_0^2 - \frac{\varphi_0\alpha}{2\sqrt{\alpha^2+(\gamma/3)}}} = -\frac{\varphi_0(2\beta-9\varphi_0)+8}{3(2\beta-3\varphi_0)\varphi_0}. \tag{19}$$

As w diverges when the denominator in the second term vanishes, w can take any value by properly choosing α, φ_0, and γ. For example, if

$$\frac{\varphi_0\alpha}{\sqrt{\alpha^2+(\gamma/3)}} = 2, \tag{20}$$

then $w=-1$. Several interesting cases deserve attention when $\varphi_0^2 > \frac{4}{3}$

$$\begin{cases} w < -1 & \text{if } \frac{3}{2}\varphi_0^2 > \frac{\varphi_0\alpha}{\sqrt{\alpha^2+(\gamma/3)}} > 2 \\ w > -1 & \text{if } \frac{\varphi_0\alpha}{\sqrt{\alpha^2+(\gamma/3)}} > \frac{3}{2}\varphi_0^2 \text{ or } \frac{\varphi_0\alpha}{\sqrt{\alpha^2+(\gamma/3)}} < 2, \end{cases} \tag{21}$$

when $\varphi_0^2 < \dfrac{4}{3}$

$$\begin{cases} w < -1 \quad \text{if} \quad 2 > \dfrac{\varphi_0 \alpha}{\sqrt{\alpha^2 + (\gamma/3)}} > \dfrac{3}{2} \varphi_0^2 \\[4mm] w > -1 \quad \text{if} \quad \dfrac{\varphi_0 \alpha}{\sqrt{\alpha^2 + (\gamma/3)}} < \dfrac{3}{2} \varphi_0^2 \quad \text{or} \quad \dfrac{\varphi_0 \alpha}{\sqrt{\alpha^2 + (\gamma/3)}} > 2. \end{cases} \tag{22}$$

As is clear from (18), cosmic acceleration $(\ddot{a} > 0)$ occurs when

$$w < -\tfrac{1}{3}. \tag{23}$$

When $w < -1$, the universe is not expanding but shrinking, although the universe is still accelerating. In the case $w < -1$, if we change the direction of time as $t \to t_s - t$, the universe is both accelerating and expanding.

In the special case when $w = -\tfrac{1}{3}$, we find

$$\varphi_0^2 = \tfrac{4}{3}. \tag{24}$$

Note that even when γ is positive, e.g., the kinetic energy of ϕ is positive as for usual matter, the effective w can still be less than -1. For example, with the choice

$$\varphi_0 = 4, \quad \frac{\gamma}{\alpha^2} = 3, \tag{25}$$

it follows that $\beta = \frac{1}{\sqrt{2}}$, and from (19),

$$w = -\frac{11\sqrt{2} + 203}{213} = -1.025\ldots < -1. \tag{26}$$

Note that the above considerations are applicable both for the early- as well as for the late-time universe. Indeed, the simple solution under discussion (especially, the no-phantom case) has been considered in different situations. Later on, the general solution for the scalar-tensor theory with an exponential potential will be discussed. With a known general solution, it is much easier to understand the type of situation that appears, be it is a transient (or eternal) acceleration or an attractor or something else.

In the previous example the matter field χ is zero ($\chi = 0$). We now consider the case of $\chi \neq 0$, which is closely related with the so-called double-quintessence model [15]. In the Einstein frame (3), the FRW equation, φ equation, and χ equation have the following form:

$$3H_E^2 = \frac{3}{4}\left(\frac{d\varphi}{dt_E}\right)^2 + \frac{1}{2}\tilde{V}(\varphi) + \frac{\kappa^2}{4}e^{-\beta\varphi}\left(\frac{d\chi}{dt_E}\right)^2$$
$$+ \frac{\kappa^2}{2}e^{-2\beta\varphi}U(\chi), \tag{27}$$

$$0 = 3\left(\frac{d^2\varphi}{dt_E^2} + 3H_E\frac{d\varphi}{dt_E}\right) + \tilde{V}'(\varphi) + \frac{\beta\kappa^2}{2}e^{-\beta\varphi}\left(\frac{d\chi}{dt_E}\right)^2$$
$$- 2\beta\kappa^2 e^{-2\beta\varphi}U(\chi), \tag{28}$$

$$0 = \frac{d}{dt_E}\left(e^{-\beta\varphi}\frac{d\chi}{dt_E}\right) + 3H_E e^{-\beta\varphi}\frac{d\chi}{dt_E} + e^{-2\beta\varphi}U'(\chi). \tag{29}$$

We again assume the exponential potentials

$$\tilde{V}(\varphi) = V_0 e^{-(2/\varphi_0)\varphi}, \quad U(\chi) = U_0 \chi^{4-(4/\beta\varphi_0)}. \tag{30}$$

The following ansatz exists

$$H_E = \frac{h_0}{t_E}, \quad \varphi = \varphi_0 \ln\frac{t_E}{t_{E0}}, \quad \chi = \chi_0\left(\frac{t_E}{t_{E0}}\right)^{\beta\varphi/2}. \tag{31}$$

The parameters h_0, t_{E0}, and χ_0 should be a solution of the following algebraic equations, which can be obtained from (27)–(29):

$$0 = -3h_0^2 + \frac{3}{4}\varphi_0^2 + \frac{1}{2}V_0 t_{E0}^2 + \frac{\kappa^2\beta^2\varphi_0^2\chi_0^2}{16}$$
$$+ \frac{\kappa^2}{2}U_0 t_{E0}^2 \chi_0^{4-(4/\beta\varphi_0)},$$

$$0 = 3(-\varphi_0 + 3h_0\varphi_0) - \frac{2V_0 t_{E0}^2}{\varphi_0} + \frac{\kappa^2\beta^3\varphi_0^2\chi_0^2}{8}$$
$$- 2\kappa^2\beta U_0 t_{E0}^2 \chi_0^{4-(4/\beta\varphi_0)},$$

$$0 = -\frac{\beta\varphi_0}{2}\left(\frac{\beta\varphi_0}{2} + 1\right) + \frac{3h_0\beta\varphi_0}{2}$$
$$+ 4U_0 t_{E0}^2\left(1 - \frac{1}{\beta\varphi_0}\right)\chi_0^{2-(4/\beta\varphi_0)}. \tag{32}$$

For instance, for the special example

$$\varphi_0 = \sqrt{\tfrac{10}{27}}, \quad \beta = 2\sqrt{\tfrac{27}{10}}, \quad V_0 = 0, \tag{33}$$

the explicit solution follows:

$$h_0 = \frac{5}{12}, \quad \chi_0 = \frac{\sqrt{5}}{3\kappa}, \quad t_{E0} = \sqrt{\frac{3}{8U_0}}. \tag{34}$$

Using (16) and (33), one arrives at

$$\gamma = -\tfrac{49}{18}\alpha^2. \tag{35}$$

Then ϕ is surely a phantom with negative kinetic energy in the physical Jordan frame (1). Since $U(\chi) = U_0\chi^2$, $U(\chi)$ corresponds to a mass term and the mass m_χ is given by

$$m_\chi^2 = 2U_0. \tag{36}$$

Also

$$t_{E0} = \frac{\sqrt{3}}{2m_\chi}. \tag{37}$$

Since $dt = \pm e^{-(\alpha\phi/2)}dt_E = \pm e^{-(\beta\varphi/2)}dt_E = \pm (t_{E0}/t_E)dt_E$, we find

$$\frac{t}{t_{E0}} = \pm \ln \frac{t_E}{t_{E0}}. \tag{38}$$

Then, in the physical Jordan frame one gets

$$a = e^{-(\alpha\phi/2)}a_E = a_{E0}\left(\frac{t_E}{t_{E0}}\right)^{-(7/12)} = a_{E0}e^{\mp(7t/12t_{E0})}, \tag{39}$$

$$\phi = \frac{\beta}{\alpha}\varphi = \frac{2}{\alpha}\ln \frac{t_E}{t_{E0}} = \pm \frac{2}{\alpha}\frac{t}{t_{E0}}, \tag{40}$$

$$\chi = \chi_0 \frac{t_E}{t_{E0}} = \chi_0 e^{\pm(t/t_{E0})}. \tag{41}$$

Since the scale factor a behaves as an exponential function of t, the Hubble parameter H is a constant

$$H = \mp \frac{7}{12t_{E0}} = \mp \frac{7m_\chi}{6\sqrt{3}}, \tag{42}$$

which is linear in the mass m_χ of χ. Equation (42) also tells that the effective w is -1 (similar to the cosmological constant).

As a different, more complicated example one can consider the case when $\alpha = 0$, and therefore $\beta = 0$, but the potential depends on both ϕ and χ,

$$S = \int d^4x \sqrt{-g}\left\{\frac{1}{\kappa^2}\left(R - \frac{3}{2}\partial_\mu\varphi\partial^\mu\varphi\right)\right.$$
$$\left. - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - W(\varphi,\chi)\right\}. \tag{43}$$

As $\alpha = 0$, the Einstein frame can be regarded as a truly physical one. Then the equations corresponding to (27)–(29) are

$$3H^2 = \frac{3}{4}\left(\frac{d\varphi}{dt}\right)^2 + \frac{\kappa^2}{4}\left(\frac{d\chi}{dt}\right)^2 + \frac{\kappa^2}{2}W(\varphi,\chi), \tag{44}$$

$$0 = 3\left(\frac{d^2\varphi}{dt^2} + 3H\frac{d\varphi}{dt}\right) + \kappa^2 W_{,\varphi}(\varphi,\chi), \tag{45}$$

$$0 = \frac{d^2\chi}{dt^2} + 3H\frac{d\chi}{dt} + W_{,\chi}(\varphi,\chi). \tag{46}$$

Here the derivative of $W(\varphi,\chi)$ with respect to φ (χ) is expressed as $W_{,\varphi}(\varphi,\chi)$ ($W_{,\chi}(\varphi,\chi)$). The case of the exponential potential $W(\varphi,\chi)$ may be of interest

$$W(\varphi,\chi) = W_0 e^{(\eta/\chi_0)\chi - [(2+\eta)/\varphi_0]\varphi}, \tag{47}$$

where W_0, η, χ_0, and φ_0 are constant parameters. Assuming

$$H = \frac{h_0}{t}, \quad \varphi = \varphi_0 \ln \frac{t}{t_0}, \quad \chi = \chi_0 \ln \frac{t}{t_0}, \tag{48}$$

with constants h_0 and t_0, Eqs. (44)–(46) reduce to the following algebraic equations:

$$0 = -3h_0^2 + \frac{3\varphi_0^2}{4} + \frac{\kappa^2\chi_0^2}{4} + \frac{\kappa^2 W_0 t_0^2}{2},$$

$$0 = 3\varphi_0^2(-1 + 3h_0) - (2+\eta)\kappa^2 W_0 t_0^2,$$

$$0 = 3\chi_0^2(-1 + 3h_0) + \eta\kappa^2 W_0 t_0^2. \tag{49}$$

Equation (49) give

$$\kappa^2\chi_0^2 = -\frac{3\eta}{2+\eta}\varphi_0^2, \tag{50}$$

$$h_0 = \frac{3\varphi_0^2}{2(2+\eta)}, \tag{51}$$

$$\kappa^2 W_0 t_0^2 = \frac{3\varphi_0^2(3h_0 - 1)}{2+\eta}. \tag{52}$$

Equation (50) shows that

$$-2 \leq \eta \leq 0. \tag{53}$$

The effective w is given by

$$w = -1 + \frac{2}{3h_0} = -1 + \frac{4(2+\eta)}{9\varphi_0^2}. \tag{54}$$

As discussed in (53), as $w > -1$ we do not have a phantom. If the second term in (54) is small, one may obtain quintessence.

Other types of matter can be easily considered too. For instance, matter may be dust. The energy density ρ_{dust} in the Jordan frame (1) behaves as $\rho_{\text{dust}} \propto a^{-3}$. Then

$$\rho_{\text{dust}} = \rho_0 a^{-3} = \rho_0 e^{(3\alpha/2)\phi}a_E^{-3} = \rho_0 e^{(3\beta/2)\varphi}a_E^{-3}, \quad p_{\text{dust}} = 0. \tag{55}$$

It is well known that dust has no pressure. It could correspond to the baryon and/or cold dark-matter components. Instead of (27) and (28), we obtain the following equations of motion in the Einstein frame:

$$3H_E^2 = \frac{3}{4}\left(\frac{d\varphi}{dt_E}\right)^2 + \frac{1}{2}\tilde{V}(\varphi) + \frac{\kappa^2}{2}e^{-(\beta/2)\varphi}\rho_0 a_E^{-3}, \tag{56}$$

$$0 = 3\left(\frac{d^2\varphi}{dt_E^2} + 3H_E\frac{d\varphi}{dt_E}\right) + \tilde{V}'(\varphi) - \frac{\beta\kappa^2}{2}e^{-(\beta/2)\varphi}\rho_0 a_E^{-3}. \tag{57}$$

With the form of $\tilde{V}(\varphi)$ as in (30), the solution occurs

$$\varphi = \varphi_0 \ln \frac{t_E}{t_{E0}}, \quad a_E = a_0 \left(\frac{t_E}{t_{E0}}\right)^{(2/3)-(\beta\varphi_0/6)}. \tag{58}$$

Here

$$t_0^2 = \frac{\beta^2 \varphi_0^2 + 9\varphi_0^2 - 9\beta\varphi_0}{6V_0},$$

$$a_0^3 = \frac{\rho_0 \kappa^2 e^{-(\beta\varphi_0/2)}(\beta^2\varphi_0^2 + 9\varphi_0^2 - 9\beta\varphi_0)}{2V_0(8 - 2\beta\varphi_0 - 9\varphi_0^2)}. \tag{59}$$

The time-coordinate t in the Jordan frame is given by

$$\frac{t}{t_0} = \left(\frac{t_E}{t_{E0}}\right)^{1-(\beta\varphi_0/2)}, \quad t_0 \equiv \left|\frac{2t_{E0}}{\beta\varphi_0}\right|, \tag{60}$$

and the scale factor can be obtained

$$a = a_0 \left(\frac{t}{t_0}\right)^{(2/3)[2(1-\beta\varphi_0)/(2-\beta\varphi_0)]}. \tag{61}$$

The effective w is found to be

$$w = -1 + \frac{2-\beta\varphi_0}{2(1-\beta\varphi_0)}. \tag{62}$$

Hence, if

$$1 < \beta\varphi_0 < 2, \tag{63}$$

$w < -1$. If we assume $V_0 > 0$ (and $t_0^2 > 0$ and $\rho_0 a_0^{-3} > 0$), Eq. (59) requires

$$\beta^2\varphi_0^2 + 9\varphi_0^2 - 9\beta\varphi_0 > 0, \quad 8 - 2\beta\varphi_0 - 9\varphi_0^2 > 0, \tag{64}$$

which give $\beta^2\varphi_0^2 - 11\beta\varphi_0 + 8 >$, that is,

$$\beta\varphi_0 < \frac{11-\sqrt{89}}{2} \quad \text{or} \quad \beta\varphi_0 > \frac{11+\sqrt{89}}{2}. \tag{65}$$

Numerically, $\beta\varphi_0 < 0.7830\ldots$ or $\beta\varphi_0 > 14.9339\ldots$, which contradict the results in (63), and there is no accelerating universe. With the assumption $V_0 < 0$, the accelerating universe takes over. For instance, with the choice $\beta\varphi_0 = \frac{3}{2}$ and $\varphi_0^2 = \frac{1}{9}$ ($\beta = \pm\frac{9}{2}$ and $\varphi_0 = \pm\frac{1}{3}$), we obtain $t_0^2 = -(41/24V_0)$ and $a_0^3 = -[\rho_0\kappa^2 e^{-(\beta\varphi_0/2)}/2V_0]$.

Going back to the case of two scalars, by variation over $g_{\mu\nu}$ the Einstein equation follows:

$$\frac{1}{\kappa^2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = \frac{1}{2}(T_{\mu\nu}^\phi + T_{\mu\nu}^\chi),$$

$$T_{\mu\nu}^\phi \equiv \frac{1}{\kappa^2}\left[-\frac{\gamma}{2}\partial_\rho\phi\partial^\rho\phi g_{\mu\nu} + \gamma\partial_\mu\phi\partial_\nu\phi - V(\phi)g_{\mu\nu}\right.$$
$$\left. + 2e^{-\alpha\phi}\nabla_\mu\nabla_\nu(e^{\alpha\phi}) - 2g_{\mu\nu}e^{-\alpha\phi}\nabla^2(e^{\alpha\phi})\right],$$

$$T_{\mu\nu}^\chi = e^{-\alpha\phi}\left(-\frac{1}{2}\partial_\rho\chi\partial^\rho\chi g_{\mu\nu} + \partial_\mu\chi\partial_\nu\chi - U(\chi)g_{\mu\nu}\right). \tag{66}$$

$T_{\mu\nu}^\phi$ and $T_{\mu\nu}^\chi$ may be regarded as the effective energy-momentum tensor of ϕ and χ, respectively.[2] In particular, in the FRW metric (12), we find

$$T_{tt}^\phi = \rho^\phi = \frac{1}{\kappa^2}\left\{\frac{\gamma}{2}\dot{\phi}^2 + V(\phi) - 6\alpha H\dot{\phi}\right\},$$

$$T_{tt}^\chi = \rho^\chi = e^{-\alpha\phi}\left\{\frac{1}{2}\dot{\chi}^2 + U(\chi)\right\},$$

$$T_{ij}^\phi = p^\phi a^2\delta_{ij} = \frac{1}{\kappa^2}\left\{\frac{\gamma}{2}\dot{\phi}^2 - V(\phi) + 2\alpha\ddot{\phi}\right.$$
$$\left. + 2\alpha^2\dot{\phi}^2 + 4\alpha H\dot{\phi}\right\}a^2\delta_{ij},$$

$$T_{ij}^\chi = p^\chi a^2\delta_{ij} = e^{-\alpha\phi}\left\{\frac{1}{2}\dot{\chi}^2 - U(\chi)\right\}a^2\delta_{ij}. \tag{67}$$

The effective w^ϕ and w^χ can be defined as follows:

$$w^\phi = \frac{p^\phi}{\rho^\phi}, \quad w^\chi = \frac{p^\chi}{\rho^\chi}. \tag{68}$$

For the first example in (13)–(15), it follows that

$$\rho^\phi = \frac{3(2\beta - 3\varphi_0)^2\varphi_0^2}{8\kappa^2 t^2\left(1 - \dfrac{\beta\varphi_0}{2}\right)^2},$$

$$p^\phi = \frac{(2\beta - 3\varphi_0)\{\varphi_0^2(2\beta - 9\varphi_0) + 8\varphi_0\}}{8\kappa^2 t^2\left(1 - \dfrac{\beta\varphi_0}{2}\right)^2}, \tag{69}$$

which agrees with $w = p/\rho = p_\phi/\rho_\phi$ in (19). Note that the consideration of various entropies for dark-energy models

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[2]The usual energy-momentum tensors $\mathcal{T}_{\mu\nu}^{\phi,\chi}$ given by the variation over $g_{\mu\nu}$ are related with $T_{\mu\nu}^{\phi,\chi}$ by $\mathcal{T}_{\mu\nu}^{\phi,\chi} = e^{\alpha\phi}T_{\mu\nu}^{\phi,\chi}$.

can be done and then interesting holographic relations among them occur (see [16] for a recent discussion).

For the second example in (39), one gets

$$\rho^\phi = \frac{14}{9\kappa^2 t_{E0}^2}, \quad p^\phi = -\frac{19}{9\kappa^2 t_{E0}^2},$$

$$\rho^\chi = \frac{35}{72\kappa^2 t_{E0}^2}, \quad p^\phi = \frac{5}{72\kappa^2 t_{E0}^2}. \quad (70)$$

Hence,

$$w^\phi = -\tfrac{19}{14}, \quad w^\chi = \tfrac{1}{7}. \quad (71)$$

However, since

$$\rho^\phi + \rho^\chi = -(p^\phi + p^\chi) = \frac{147}{72\kappa^2 t_{E0}^2}, \quad (72)$$

it turns out that $w = -1$, which is consistent with (42). We should also note that, in the Einstein frame (3), ϕ has a positive kinetic energy.

For the case of (48) with (50)–(52) we find

$$\rho = \frac{3}{2\kappa^2}\dot\varphi^2 + \frac{1}{2}\dot\chi^2 + W(\phi,\chi) = \frac{18\varphi_0^2 h_0}{2t^2\kappa^2(2+\eta)},$$

$$p = \frac{3}{2\kappa^2}\dot\varphi^2 + \frac{1}{2}\dot\chi^2 - W(\phi,\chi) = \frac{6\varphi_0^2(2-3h_0)}{2t^2\kappa^2(2+\eta)}, \quad (73)$$

which reproduces (54). Having these various examples of the (accelerated) evolution of the current universe, one can compare it with recent astrophysical data in the way discussed recently in [2,17]. Of course, above illustrative examples of current speed-up may correspond to transient acceleration. In other words, stability of the solutions pretending to be realistic ones should be investigated in detail.

III. EXACT FRW COSMOLOGY FOR THE (PHANTOM) SCALAR-TENSOR THEORY WITH AN EXPONENTIAL POTENTIAL

In this section the exact FRW cosmology in the (phantom) scalar-tensor theory with an exponential potential will be discussed. First, the method that is appropriate to obtain the exact FRW solutions will be reviewed [18] (for introduction of similar variables in quantum cosmology, see [19]). Subsequently, the formulation is extended to the phantom case with a negative kinetic term.

The action of a scalar field ϕ coupled with gravity is

$$S = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\left(R - \frac{\gamma}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right). \quad (74)$$

This action can be regarded as the action with $\alpha = 0$ in (1) or that obtained by replacing $[(d-1)\alpha^2/d-2] + \gamma/2$ and $e^{-(2\alpha/d-2)\phi}V(\phi)$ in (3) with $\gamma/2$ and $V(\phi)$, respectively.

For the standard scalar $\gamma > 0$ and one can normalize ϕ to be $\gamma = 1$, but for the phantom field with negative kinetic term, we have $\gamma < 0$.

For the FRW metric

$$ds^2 = -dt^2 + a(t)^2\sum_{i=1,2,3}(dx^i)^2, \quad (75)$$

the action (74) can be rewritten as

$$S = \frac{1}{\kappa^2}\int d^3x dt\left\{-6a\dot a^2 + a^3\left(\frac{\gamma}{2}\dot\phi^2 - V(\phi)\right)\right\}. \quad (76)$$

The potential $V(\phi)$ is chosen to be

$$V(\phi) = V_0 e^{-(2\phi/\phi_0)}, \quad (77)$$

with constants V_0 and ϕ_0.

First we review the standard case with $\gamma > 0$ following to [18]. The field variables a and ϕ are written in terms of new fields v and u as

$$a = e^{(v+u)/3}, \quad \phi = \frac{2(v-u)}{\sqrt{3\gamma}}, \quad (78)$$

and a new time variable τ is defined by

$$d\tau = dt\sqrt{\frac{3V_0}{8}}e^{-[2(v-u)/\phi_0\sqrt{3\gamma}]}. \quad (79)$$

Then, the action (76) acquires the following form:

$$S = -\frac{1}{\kappa^2}\sqrt{\frac{8V_0}{3}}\int d^3x d\tau\left[\frac{dv}{d\tau}\frac{du}{d\tau} + 1\right]e^{v+u-\bar\alpha(v-u)},$$

$$\bar\alpha \equiv \frac{2}{\phi_0\sqrt{3\gamma}}. \quad (80)$$

Varying over v and u, the equations of motion follow:

$$0 = \frac{d^2u}{d\tau^2} + (1+\bar\alpha)\left(\frac{du}{d\tau}\right)^2 - (1-\bar\alpha), \quad (81)$$

$$0 = \frac{d^2v}{d\tau^2} + (1-\bar\alpha)\left(\frac{du}{d\tau}\right)^2 - (1+\bar\alpha). \quad (82)$$

Since the Hamiltonian H conjugate to τ is given by

$$H = -\frac{1}{\kappa^2}\sqrt{\frac{8V_0}{3}}\int d^3x\left[\frac{dv}{d\tau}\frac{du}{d\tau} - 1\right]e^{v+u-\bar\alpha(v-u)}, \quad (83)$$

the Hamiltonian constraint $H = 0$ yields

$$\frac{dv}{d\tau}\frac{du}{d\tau} = 1. \quad (84)$$

In terms of the new variables $\mathcal{V}$ and $\mathcal{U}$, which are given by

$$\mathcal{V} \equiv e^{(1-\bar{\alpha})v}, \quad \mathcal{U} \equiv e^{(1+\bar{\alpha})u}, \tag{85}$$

the equations of motion are

$$\frac{d^2\mathcal{U}}{d\tau^2} = (1-\bar{\alpha}^2)\mathcal{U}, \quad \frac{d^2\mathcal{V}}{d\tau^2} = (1-\bar{\alpha}^2)\mathcal{V}. \tag{86}$$

When $|\bar{\alpha}| < 1$, the solution of (86) is given by

$$\mathcal{U} = u_+ e^{\tau\sqrt{1-\bar{\alpha}^2}} + u_- e^{-\tau\sqrt{1-\bar{\alpha}^2}},$$

$$\mathcal{V} = v_+ e^{\tau\sqrt{1-\bar{\alpha}^2}} + v_- e^{-\tau\sqrt{1-\bar{\alpha}^2}}, \tag{87}$$

with constants of the integration $u_\pm$ and $v_\pm$. The Hamiltonian constraint (84) restricts the constants as

$$u_+ v_- = -u_- v_+. \tag{88}$$

Then the spacetime metric has the following form [18]:

$$ds^2 = -\frac{8}{3V_0}(v_+ e^{\tau\sqrt{1-\bar{\alpha}^2}} + v_- e^{-\tau\sqrt{1-\bar{\alpha}^2}})^{\bar{\alpha}/(1-\bar{\alpha})}$$

$$\times (u_+ e^{\tau\sqrt{1-\bar{\alpha}^2}} + u_- e^{-\tau\sqrt{1-\bar{\alpha}^2}})^{-[(\bar{\alpha}/(1+\bar{\alpha})]} d\tau^2$$

$$+ (v_+ e^{\tau\sqrt{1-\bar{\alpha}^2}} + v_- e^{-\tau\sqrt{1-\bar{\alpha}^2}})^{2/[3(1-\bar{\alpha})]}$$

$$\times (u_+ e^{\tau\sqrt{1-\bar{\alpha}^2}} + u_- e^{-\tau\sqrt{1-\bar{\alpha}^2}})^{2/[3(1+\bar{\alpha})]} \sum_{i=1,2,3} (dx^i)^2. \tag{89}$$

When $\tau \to +\infty$, Eqs. (78) and (79) give

$$a \to v_+^{1/3(1-\bar{\alpha})} u_+^{1/3(1+\bar{\alpha})} e^{2\tau/(3\sqrt{1-\bar{\alpha}^2})},$$

$$t \to t_{0+} + \frac{\sqrt{1-\bar{\alpha}^2}}{\bar{\alpha}^2}\sqrt{\frac{8}{3V_0}}$$

$$\times v_+^{\bar{\alpha}/[2(1-\bar{\alpha})]} u_+^{-\bar{\alpha}/[2(1+\bar{\alpha})]} e^{\bar{\alpha}^2\tau/\sqrt{1-\bar{\alpha}^2}}. \tag{90}$$

Here t_{0+} is a constant of integration. Hence, $a \propto t^{2/3\bar{\alpha}^2}$ and the universe is accelerating ($d^2a/dt^2 > 0$) if

$$\bar{\alpha}^2 < \tfrac{2}{3}. \tag{91}$$

Note that if $v_- = u_- = 0$, the behavior in (90) is exact even if τ is not large. The case with $v_- = u_- = 0$ corresponds to the solution in Sec. II. On the other hand, when $\tau \to -\infty$

$$a \to v_-^{1/3(1-\bar{\alpha})} u_-^{1/3(1+\bar{\alpha})} e^{-[(2\tau/(3\sqrt{1-\bar{\alpha}^2})]},$$

$$t \to t_{0-} - \frac{\sqrt{1-\bar{\alpha}^2}}{\bar{\alpha}^2}\sqrt{\frac{8}{3V_0}}$$

$$\times v_-^{\bar{\alpha}/[2(1-\bar{\alpha})]} u_-^{-\bar{\alpha}/[2(1+\bar{\alpha})]} e^{-(\bar{\alpha}^2\tau/\sqrt{1-\bar{\alpha}^2})}. \tag{92}$$

Here t_{0-} is again a constant of integration. Thus, we find $a \propto (-t)^{2/3\bar{\alpha}^2}$, then the universe is shrinking but accelerating ($d^2a/dt^2 > 0$) if (91) is satisfied. To summarize the $|\bar{\alpha}| < 1$ case, if $v_- = u_- = 0$ we find an eternal expanding solution as in Sec. II. In the general case, the solution is a bouncing universe, where first the universe shrinks and then it expands.

When $|\bar{\alpha}| > 1$, the solution of (86) can be written as

$$\mathcal{U} = u_c \cos(\tau\sqrt{\bar{\alpha}^2 - 1}) + u_s \sin(\tau\sqrt{\bar{\alpha}^2 - 1}),$$

$$\mathcal{V} = v_c \cos(\tau\sqrt{\bar{\alpha}^2 - 1}) + v_s \sin(\tau\sqrt{\bar{\alpha}^2 - 1}), \tag{93}$$

with constants u_c, u_s, v_c, and v_s, which satisfy

$$v_c u_c + v_s c_s = 0. \tag{94}$$

The solution of (94) can be given by means of three independent parameters v_0, u_0, θ_0 as

$$v_s = v_0 \sin\theta_0, \quad v_c = v_0 \cos\theta_0, \quad u_s = u_0 \cos\theta_0,$$

$$u_c = -u_0 \sin\theta_0. \tag{95}$$

As a result, Eq. (93) simplifies

$$\mathcal{V} = v_0 \cos(\tau\sqrt{\bar{\alpha}^2 - 1} - \theta_0), \quad \mathcal{U} = u_0 \sin(\tau\sqrt{\bar{\alpha}^2 - 1} - \theta_0). \tag{96}$$

As θ_0 can be absorbed into the constant shift of τ, in the following we choose $\theta_0 = 0$. The space-time metric looks as follows:

$$ds^2 = -\frac{8}{3V_0} v_0^{-[2\bar{\alpha}/(1-\bar{\alpha})]} u_0^{2\bar{\alpha}/(1+\bar{\alpha})}$$

$$\times \cos^{-[2\bar{\alpha}/(1-\bar{\alpha})]}(\tau\sqrt{\bar{\alpha}^2 - 1})$$

$$\times \sin^{2\bar{\alpha}/(1+\bar{\alpha})}(\tau\sqrt{\bar{\alpha}^2 - 1}) d\tau^2 + v_0^{2\bar{\alpha}/[3(1-\bar{\alpha})]} u_0^{2\bar{\alpha}/1+\bar{\alpha}}$$

$$\times \cos^{2/[3(1-\bar{\alpha})]}(\tau\sqrt{\bar{\alpha}^2 - 1}) \sin^{2/[3(1+\bar{\alpha})]}(\tau\sqrt{\bar{\alpha}^2 - 1}). \tag{97}$$

There are singularities when

$$\tau\sqrt{\bar{\alpha}^2 - 1} = n\pi, \quad \text{or} \quad (n + \tfrac{1}{2})\pi. \tag{98}$$

Here n is an integer. If we write τ as $\tau\sqrt{\bar{\alpha}^2 - 1} = n\pi + \delta\tau$ and assume $\delta\tau$ is small, we find, by neglecting numerical factors,

$$t \sim (\delta\tau)^{(2\bar{\alpha}+1)/(1+\bar{\alpha})}, \quad a \sim (\delta\tau)^{1/[3(1+\bar{\alpha})]} \sim t^{1/[3(2\bar{\alpha}+1)]}. \tag{99}$$

Note that $(2\bar{\alpha}+1)/(1+\bar{\alpha}) > 0$ as $|\bar{\alpha}| > 1$. Then $\tau = 0$ corresponds to $t = 0$. At $t = 0$, the size of the universe diverges (vanishes) when $2\bar{\alpha} + 1 < 0$ ($2\bar{\alpha} + 1 > 0$). On the other hand, if we write τ as $\tau\sqrt{\bar{\alpha}^2 - 1} = (n + \pi/2)\pi + \delta\tau$ and assume $\delta\tau$ is small, we find, again neglecting numerical factors,

$$t \sim (\delta\tau)^{(1-2\bar{\alpha})/(1-\bar{\alpha})}, \quad a \sim (\delta\tau)^{1/[3(1-\bar{\alpha})]} \sim t^{1/[3(1-2\bar{\alpha})]}. \tag{100}$$

Note that $(1-2\bar{\alpha}+1)/(1-\bar{\alpha})>0$ once more, and $\tau=0$ corresponds to $t=0$. Then, at $t=0$ the size of the universe diverges (vanishes) when $1-2\bar{\alpha}<0$ $(1-2\bar{\alpha}>0)$. These results for the standard scalar with $\gamma>0$ are given in [18].

We now extend the above formulation to the case of a phantom with $\gamma<0$. For this situation, we define a complex field z and its complex conjugate z^* by

$$a=e^{(z+z^*)/3}, \quad \phi=-\frac{2i(z-z^*)}{\sqrt{-3\gamma}}, \tag{101}$$

and define a new time variable τ as in (79) by

$$d\tau=\pm dt\sqrt{\frac{3V_0}{8}}e^{-[2i(z-z^*)/\phi_0\sqrt{-3\gamma}]}. \tag{102}$$

The action (76) becomes

$$S=\mp\frac{1}{\kappa^2}\sqrt{\frac{8V_0}{3}}\int d^3xd\tau\left[\frac{dz}{d\tau}\frac{dz^*}{d\tau}+1\right]e^{z+z^*-i\tilde{\tilde{\alpha}}(z-z^*)}. \tag{103}$$

Here

$$\tilde{\tilde{\alpha}}\equiv\frac{2}{\phi_0\sqrt{-3\gamma}}. \tag{104}$$

The sign $\mp$ in (103) corresponds to the sign in (102). Varying over z^*, one obtains the following equation:

$$0=\frac{d^2z}{d\tau^2}+(1-i\tilde{\tilde{\alpha}})\left(\frac{dz}{d\tau}\right)^2-(1+i\tilde{\tilde{\alpha}}). \tag{105}$$

Now, the Hamiltonian H conjugate to τ is given by

$$H=\mp\frac{1}{\kappa^2}\sqrt{\frac{8V_0}{3}}\int d^3x\left[\frac{dz}{d\tau}\frac{dz^*}{d\tau}-1\right]e^{v+u-\bar{\alpha}(v-u)}, \tag{106}$$

and the Hamiltonian constraint has the following form:

$$\frac{dz}{d\tau}\frac{dz^*}{d\tau}=1. \tag{107}$$

By defining a new variable $\mathcal{Z}$ as

$$\mathcal{Z}\equiv e^{(1-i\tilde{\tilde{\alpha}})z}, \tag{108}$$

Eq. (105) can be rewritten as

$$\frac{d^2\mathcal{Z}}{d\tau^2}=(1+\tilde{\alpha}^2)\mathcal{Z}. \tag{109}$$

The solution of (109) is given by

$$\mathcal{Z}=z_+e^{\tau\sqrt{1+\tilde{\alpha}^2}}+z_-e^{-\tau\sqrt{1+\tilde{\tilde{\alpha}}^2}}, \tag{110}$$

with complex constants of integration $z_\pm$. The Hamiltonian constraint (107) restricts the constants to satisfy

$$z_+z_-^*=-z_-z_+^*. \tag{111}$$

By using three real independent parameters $b_\pm$ and θ_0, the solution of (111) is given by

$$z_+=b_+e^{i\theta_0} \quad z_-=ib_-e^{-i\theta_0}. \tag{112}$$

By using $z_\pm$ in (112), the metric of the spacetime has the following form:

$$ds^2=-\frac{8}{3V_0}|\{z_+e^{\tau\sqrt{1+\tilde{\alpha}^2}}+z_-e^{-\tau\sqrt{1+\tilde{\tilde{\alpha}}^2}}\}^{\alpha/(1-i\alpha)}|^2d\tau^2$$

$$+|\{z_+e^{\tau\sqrt{1+\tilde{\alpha}^2}}+z_-e^{-\tau\sqrt{1+\tilde{\tilde{\alpha}}^2}}\}^{2/[3(1-i\alpha)]}|^2$$

$$\times\sum_{i=1,2,3}(dx^i)^2. \tag{113}$$

When $\tau\rightarrow\infty$, from (101) and (102), it follows that

$$a\sim e^{\{2\tau/(3\sqrt{1+\tilde{\tilde{\alpha}}^2})+2/[3(1+\tilde{\tilde{\alpha}}^2)]\}\Re\{(1+i\tilde{\tilde{\alpha}})\ln z_+\}},$$

$$t\sim t_+\mp\frac{\sqrt{1+\tilde{\tilde{\alpha}}^2}}{\tilde{\tilde{\alpha}}^2}\sqrt{\frac{8}{3V_0}}e^{\{-[\tilde{\alpha}^2/(1+\tilde{\tilde{\alpha}}^2)]\}\Im\{(1+i\tilde{\alpha})\ln z_+\}}$$

$$\times e^{-(\tilde{\tilde{\alpha}}^2\tau/\sqrt{1+\tilde{\tilde{\alpha}}^2})}. \tag{114}$$

Here t_+ is a constant of the integration. Hence, $a\propto[\mp(t-t_+)]^{-(2/3\tilde{\tilde{\alpha}}^2)}$, which tells that the universe is accelerating since $\ddot{a}\propto\frac{2}{3}\tilde{\tilde{\alpha}}^2(\frac{2}{3}\tilde{\tilde{\alpha}}^2+1)[\pm(t-t_+)]^{-(2/3\tilde{\tilde{\alpha}}^2)-2}>0$. The effective w is given by

$$w=-1-\tilde{\tilde{\alpha}}^2<-1. \tag{115}$$

The case corresponds to a phantom. In general,

$$\ddot{a}=\frac{3V_0}{8}\mathcal{Z}^{\{1/[3(1-i\tilde{\tilde{\alpha}})]\}-i\tilde{\tilde{\alpha}}/(1-i\tilde{\tilde{\alpha}})-2}\mathcal{Z}^{*1/[3(1+i\tilde{\tilde{\alpha}})]+i\tilde{\tilde{\alpha}}/(1+i\tilde{\alpha})-2}$$

$$\times\left[\frac{4(1+\tilde{\tilde{\alpha}}^2)}{9}\mathcal{Z}^2\mathcal{Z}^{*2}-\frac{2(1+i\tilde{\tilde{\alpha}})(2-3i\tilde{\tilde{\alpha}})}{9(1-i\tilde{\tilde{\alpha}})}z_+z_-\mathcal{Z}^{*2}\right.$$

$$\left.-\frac{2(1-i\tilde{\tilde{\alpha}})(2+3i\tilde{\tilde{\alpha}})}{9(1+i\tilde{\tilde{\alpha}})}z_+^*z_-^*\mathcal{Z}^2\right]. \tag{116}$$

If $|\mathcal{Z}|$ is large, then $\ddot{a}>0$. We should note that when $z_-=0$, Eq. (114) gives an exact solution corresponding to those in Sec. II. In this case, Eq. (114) is valid even if $|t|$ is not small. The solution can be regarded as an attractor. In fact, when $t\sim t_+$, all the solutions behave as this one.

On the other hand, when $\tau\rightarrow-\infty$ one gets

$$a \sim e^{-[2\tau/(3\sqrt{1+\tilde{\tilde{\alpha}}^2})]+\{2/[3(1+\tilde{\tilde{\alpha}}^2)]\}\Re\{(1-i\tilde{\tilde{\alpha}})\ln z_-\}},$$

$$t \sim t_- \pm \frac{\sqrt{1+\tilde{\tilde{\alpha}}^2}}{\tilde{\tilde{\alpha}}^2}\sqrt{\frac{8}{3V_0}}$$

$$\times e^{-[\tilde{\tilde{\alpha}}^2/(1+\tilde{\tilde{\alpha}}^2)]\Im\{(1-i\tilde{\tilde{\alpha}})\ln z_-\}}e^{\tilde{\tilde{\alpha}}^2\tau/\sqrt{1+\tilde{\tilde{\alpha}}^2}}. \qquad (117)$$

Here t_- is a constant of integration again. Thus, $a \propto (\pm(t-t_-))^{-(2/3\tilde{\tilde{\alpha}}^2)}$ and, once more, the universe is accelerating.

The solution (113) is almost given by the analytic continuation $\bar{\alpha} \to i\tilde{\tilde{\alpha}}$ of that (89), which corresponds to the standard (nonphantom) scalar. The behavior obtained for the solution (113) is, however, rather different from the nonphantom case. In the case of a nonphantom with $\gamma > 0$ in (89), which has been investigated in [18], when $|\bar{\alpha}| < 0$, the behavior in (90) or (92) shows that there is a singularity only in the infinite future or past, since $\tau \to \pm\infty$ corresponds to $t \to \pm\infty$. On the other hand, when $|\bar{\alpha}| > 0$, the behavior in (99) or (100) indicates that there might be Big Rip [8–10] or Big Crunch singularity in the finite future. Even for non-phantom matter, when the strong energy condition is applied, a finite-time future singularity may occur [20]. In the phantom case in Eq. (114) or (117), $\tau \to \pm\infty$ corresponds to $t \to 0$. The singularity occurs in the *finite* future or past. Thus, if the universe is expanding, there should be a finite-time future singularity. This singularity is nothing but the Big Rip [8–10]. If there is a singularity in the past, the universe is not expanding but shrinking. In this sense, the solution with a past singularity is related to that with a future singularity, by reversing the direction of time.

It is possible to relate the action (74) to the action (3) in the Einstein frame by identifying $g_{\mu\nu}$, γ, and $V(\phi)$ with $g_{E\mu\nu}$, $[(d-1)\alpha^2]/(d-2) + \gamma/2$ (with $d=4$), and $e^{-[2\alpha/(d-2)\phi]}V(\phi)$ (with $d=4$ again), respectively. The physical metric is obtained by rescaling the metric as the reverse of (2). Instead of (75), we now assume that the physical metric is given by

$$ds^2 = e^{\bar{\beta}\phi}\left(-dt^2 + a(t)^2\sum_{i=1,2,3}(dx^i)^2\right). \qquad (118)$$

For $\tau \to \pm\infty$, ϕ behaves as

$$\phi \sim \mp\frac{4}{\tilde{\tilde{\alpha}}\sqrt{-3\gamma}}\ln|t-t_\pm|. \qquad (119)$$

Since a behaves as $a \sim |t-t_\pm|^{-(2/3\tilde{\tilde{\alpha}}^2)}$, if

$$\bar{\beta} = -\frac{1}{\tilde{\tilde{\alpha}}}\sqrt{-\frac{\gamma}{3}}, \qquad (120)$$

the singularity corresponding to $\tau \to +\infty$ is cancelled, but there remains a singularity corresponding to $\tau \to -\infty$. On the other hand, if

$$\bar{\beta} = \frac{1}{\tilde{\tilde{\alpha}}}\sqrt{-\frac{\gamma}{3}}, \qquad (121)$$

the singularity corresponding to $\tau \to -\infty$ is cancelled, but there remains a singularity corresponding to $\tau \to +\infty$. In the metric (118), the cosmological time $\tilde{t}$ is defined by $d\tilde{t} = \pm e^{(\beta/2)\phi}dt$, then in case of (120), we find $\tilde{t} \propto \tilde{t}_0 + |t-t_\pm|^{(2/3\tilde{\tilde{\alpha}}^2)+1}$. Here $\tilde{t}_0$ is a constant of integration. The limit $t \to t_\pm$ ($\tau \to \pm\infty$) corresponds to $\tilde{t} \to \tilde{t}_0$. On the other hand, in the case of (121), $\tilde{t} \propto \tilde{t}_0' + |t-t_\pm|^{-(2/3\tilde{\tilde{\alpha}}^2)+1}$. Here $\tilde{t}_0'$ is a constant of integration. If $\tilde{\tilde{\alpha}}^2 > \frac{3}{2}$, $t \to t_\pm$ ($\tau \to \pm\infty$) corresponds to $\tilde{t} \to \tilde{t}_0$, again. If $\tilde{\tilde{\alpha}}^2 > \frac{3}{2}$, however, $t \to t_\pm$ ($\tau \to \pm\infty$) corresponds to $\tilde{t} \to \pm\infty$. Hence, the singularity does not occur within finite time. This example shows that the type (or even the presence itself) of the singularity is also related with the choice of physical metric (frame).

IV. QUANTUM EFFECTS MAY CHANGE THE FINITE TIME FUTURE SINGULARITY

Let us again start from the scalar-tensor theory with a single scalar that can be an effective phantom,

$$L = \frac{1}{\kappa^2}\left(R + \frac{\tilde{\gamma}}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right), \qquad (122)$$

where $\tilde{\gamma} = \pm 1$. It would be interesting to investigate the quantum properties of such scalar-tensor gravity. Indeed, it is known that the phantom theory develops a catastrophic instability at the quantum level. Hence the point is that taking into account quantum gravity effects (or, simply quantum effects) could improve the situation.

The calculation of the one-loop effective action in the former, non-renormalizable theory may indeed be performed (using the above parametrization and some choice for the gauge condition). The result is

$$W_{\text{1-loop}} = -\frac{1}{2}\ln\frac{L^2}{\mu^2}\int d^4x\sqrt{-g}\left\{\frac{5}{2}V^2 - \tilde{\gamma}(V')^2 + \frac{1}{2}(V'')^2\right.$$

$$+\left[\frac{\tilde{\gamma}}{2}V - 2V''\right]\phi_{,\mu}\phi^{,\mu} - \left[\frac{13}{3}V + \frac{\tilde{\gamma}}{12}V''\right]R + \frac{43}{60}R_{\alpha\beta}^2$$

$$\left.+\frac{1}{40}R^2 - \frac{\tilde{\gamma}}{6}R\phi_{,\mu}\phi^{,\mu} + \frac{5}{4}(\phi_{,\mu}\phi^{,\mu})^2\right\}. \qquad (123)$$

The above one-loop action is found in Ref. [21]. In order to consider this effective action as a finite quantum correction to the classical one, the cut-off L should be identified with the corresponding physical quantity. For instance, when the universe is in the (almost) de Sitter phase, the natural choice is $L^2 = |R|$, as the curvature is strong enough and constant [12,13]. At the same time, in the region where $|V| \gg |R|$, L^2 should be identified with $|V|$.

Hence, even in the situation with $V=0$, starting from the action (122) with the usual scalar, the phantom terms may be

induced. This happens if the universe goes through a region with negative curvature. With account to the potential, for some fine-tuning of V one can again arrive to the QG-induced phantom theory, which subsequently can change the universe evolution.

Here, we consider the action where L^2 is replaced with $|R|$ as a simple example

$$W_{1\text{-loop}} = -\frac{1}{2}\int d^4x\sqrt{-g}\ln\frac{|R|}{\mu^2}\left\{\frac{5}{2}V^2 - \tilde{\gamma}(V')^2 + \frac{1}{2}(V'')^2\right.$$

$$+\left[\frac{\tilde{\gamma}}{2}V - 2V''\right]\phi_{,\mu}\phi^{,\mu} - \left[\frac{13}{3}V + \frac{\tilde{\gamma}}{12}V''\right]R + \frac{43}{60}R_{\alpha\beta}^2$$

$$\left.+\frac{1}{40}R^2 - \frac{\tilde{\gamma}}{6}R\phi_{,\mu}\phi^{,\mu} + \frac{5}{4}(\phi_{,\mu}\phi^{,\mu})^2\right\}. \tag{124}$$

The variations of this action are given by

$$\frac{1}{\sqrt{-g}}\frac{\delta W_{1\text{-loop}}}{\delta\phi} = -\frac{1}{2}\ln\frac{|R|}{\mu^2}\left\{\left[\frac{5}{2}V^2 - \tilde{\gamma}(V')^2 + \frac{1}{2}(V'')^2\right]'\right.$$

$$+\left[\frac{\tilde{\gamma}}{2}V - 2V''\right]'\phi_{,\mu}\phi^{,\mu} - 2\nabla_\mu$$

$$\times\left[\left(\frac{\tilde{\gamma}}{2}V - 2V''\right)\phi^{,\mu}\right] - \left[\frac{13}{3}V + \frac{\tilde{\gamma}}{12}V''\right]'R$$

$$\left.+\frac{\tilde{\gamma}}{3}\nabla_\mu(R\phi^{,\mu}) - 5\nabla_\mu((\phi_{,\rho}\phi^{,\rho})\phi^{,\mu})\right\}, \tag{125}$$

$$\frac{1}{\sqrt{-g}}\frac{\delta W_{1\text{-loop}}}{\delta g_{\mu\nu}} = -\frac{1}{2}\ln\frac{|R|}{\mu^2}\left\{\frac{1}{2}g^{\mu\nu}\left(\frac{5}{2}V^2 - \tilde{\gamma}(V')^2 + \frac{1}{2}(V'')^2 + \left(\frac{\tilde{\gamma}}{2}V - 2V''\right)\phi_{,\mu}\phi^{,\mu} - \left(\frac{13}{3}V + \frac{\tilde{\gamma}}{12}V''\right)R + \frac{43}{60}R_{\alpha\beta}^2 + \frac{1}{40}R^2\right.\right.$$

$$-\frac{\tilde{\gamma}}{6}R\phi_{,\mu}\phi^{,\mu} + \frac{5}{4}(\phi_{,\mu}\phi^{,\mu})^2\right) - \left(\frac{\tilde{\gamma}}{2}V - 2V''\right)\phi^{,\mu}\phi^{,\nu} + \left(\frac{13}{3}V + \frac{\tilde{\gamma}}{12}V''\right)R^{\mu\nu} - (\nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^2)$$

$$\times\left(\frac{13}{3}V + \frac{\tilde{\gamma}}{12}V''\right) - \frac{43}{30}R^\mu_{\ \rho}R^{\nu\rho} + \frac{43}{60}\{(\nabla_\alpha\nabla^\nu R^{\alpha\mu} + \nabla_\alpha\nabla^\mu R^{\alpha\nu}) - \nabla^2 R^{\mu\nu} - g^{\mu\nu}\nabla_\rho\nabla_\sigma R^{\rho\sigma}\} + \frac{1}{20}RR^{\mu\nu}$$

$$+\frac{1}{20}(\nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^2)R + \frac{\tilde{\gamma}}{6}R^{\mu\nu}\phi_{,\rho}\phi^{,\rho} - \frac{\tilde{\gamma}}{6}(\nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^2)(\phi_{,\rho}\phi^{,\rho}) + \frac{\tilde{\gamma}}{6}R\partial^\mu\phi\partial^\nu\phi - \frac{5}{2}\phi_{,\rho}\phi^{,\rho}\phi^{,\mu}\phi^{,\nu}$$

$$+(-R^{\mu\nu} + \nabla^\mu\nabla^\nu - g^{\mu\nu}\nabla^2)\left[-\frac{1}{2R}\left(\frac{5}{2}V^2 - \tilde{\gamma}(V')^2 + \frac{1}{2}(V'')^2 + \left[\frac{\tilde{\gamma}}{2}V - 2V''\right]\phi_{,\rho}\phi^{,\rho} - \left[\frac{13}{3}V + \frac{\tilde{\gamma}}{12}V''\right]R\right.\right.$$

$$\left.\left.\left.+\frac{43}{60}R_{\alpha\beta}^2 + \frac{1}{40}R^2 - \frac{\tilde{\gamma}}{6}R\phi_{,\rho}\phi^{,\rho} + \frac{5}{4}(\phi_{,\rho}\phi^{,\rho})^2\right)\right]\right\}. \tag{126}$$

In the case of occurrence of the Big Rip singularity, the curvature quickly grows. However, this means that quantum effects (e.g., quantum gravity effects) become important not only for the early universe, but also for the future universe. These quantum effects may even become dominant when the universe approaches the Big Rip. Suppose that the quantum correction becomes dominant owing to the fact that $W_{1\text{-loop}}$ contains higher-derivative terms. In this case one can neglect the classical terms. To simplify the situation even more, we assume that the curvature and the scalar field ϕ are constant

$$R_{\mu\nu} = \frac{3}{l^2}g_{\mu\nu}, \quad R = \frac{12}{l^2}, \quad \phi = c. \tag{127}$$

The potential $V(\phi)$ is chosen as the exponential function of ϕ,

$$V(\phi) = V_0 e^{-2(\phi/\phi_0)}. \tag{128}$$

Then from (125) and (126), we obtain

$$0 = \frac{1}{\sqrt{-g}}\frac{\delta W_{1\text{-loop}}}{\delta\phi}$$

$$= -\frac{1}{2}\ln\frac{|R|}{\mu^2}\left[-\frac{4}{\phi_0}\left(\frac{5}{2} - \frac{4\tilde{\gamma}}{\phi_0^2} + \frac{8}{\phi_0^4}\right)V_0^2 e^{-(4c/\phi_0)}\right.$$

$$\left.+\frac{2}{\phi_0}\left(\frac{13}{3} + \frac{\tilde{\gamma}}{3\phi_0^2}\right)V_0 e^{-(2c/\phi_0)}\frac{12}{l^2}\right], \tag{129}$$

$$0 = \frac{1}{\sqrt{-g}} \frac{\delta W_{\text{1-loop}}}{\delta g_{\mu\nu}}$$

$$= g^{\mu\nu} \left[-\frac{1}{4} \ln\left(\frac{12}{l^2 \mu^2}\right) \left\{ \left(\frac{5}{2} - \frac{4\tilde{\gamma}}{\phi_0^2} + \frac{8}{\phi_0^4}\right) V_0^2 e^{-(4c/\phi_0)} \right. \right.$$

$$\left. - \left(\frac{13}{3} + \frac{\tilde{\gamma}}{3\phi_0^2}\right) V_0 e^{-(2c/\phi_0)} \frac{12}{l^2} + \frac{147}{5l^4} \right\}$$

$$+ \frac{1}{8} \left(\frac{5}{2} - \frac{4\tilde{\gamma}}{\phi_0^2} + \frac{8}{\phi_0^4}\right) V_0^2 e^{-(4c/\phi_0)}$$

$$+ \frac{3}{2l^2} \left(\frac{13}{3} + \frac{\tilde{\gamma}}{3\phi_0^2}\right) V_0 e^{-(2c/\phi_0)} - \frac{441}{40l^4} \right]. \tag{130}$$

Equation (129) can be solved with respect to l^2:

$$R = \frac{12}{l^2} = 2 \left(\frac{5}{2} - \frac{4\tilde{\gamma}}{\phi_0^2} + \frac{8}{\phi_0^4}\right) \left(\frac{13}{3} + \frac{\tilde{\gamma}}{3\phi_0^2}\right)^{-1} V_0 e^{-(2c/\phi_0)}. \tag{131}$$

We should note, however, that Eq. (130) is not consistent with the expression in (131) in general. Then, Eq. (130) might be regarded as an equation determining μ. We should also note that the right-hand side (rhs) in (131) is not always positive. In the case $\tilde{\gamma} > 0$, when $\tilde{\gamma}^2 < 5$, the rhs in (131) is positive, but when $\tilde{\gamma}^2 > 5$, it is positive if $\phi_0^2 > \frac{4}{5}(\tilde{\gamma} + \sqrt{\tilde{\gamma}^2 - 5})$ or $\phi_0^2 < \frac{4}{5}(\tilde{\gamma} - \sqrt{\tilde{\gamma}^2 - 5})$. On the other hand, in a phantom case $\tilde{\gamma} < 0$, the rhs in (131) is positive if $\phi_0^2 > -(\tilde{\gamma}/13)$. Anyway, there may occur a (asymptotically) de Sitter solution. Thus, before entering the Big Rip singularity, the universe becomes a quantum de Sitter space. This qualitative discussion indicates that the finite-time future singularity may never occur (or, at least may become milder) under the conjecture that quantum effects become dominant just before the Big Rip. Due to the sharp increase of the curvature invariants near the Big Rip, such a conjecture looks quite natural.

A similar phenomenon occurs even without quantum gravity. Indeed, let us consider again the phantom theory of Sec. III with the same potential. For the FRW background, if we assume that ϕ only depends on time, the equation of motion for ϕ is given by

$$0 = -\gamma \left(\frac{d^2\phi}{dt^2} + 3H \frac{d\phi}{dt}\right) - V'(\phi). \tag{132}$$

The energy density ρ_ϕ is

$$\rho_\phi = \frac{\gamma}{2} \left(\frac{d\phi}{dt}\right)^2 + V(\phi), \tag{133}$$

and the FRW equation has the following form:

$$\frac{6}{\kappa^2} H^2 = \rho_\phi. \tag{134}$$

Here $H = \dot{a}/a$. Then a solution of (132) and (134) is

$$\phi = \phi_0 \ln \left|\frac{t_s - t}{t_1}\right|, \quad H = -\frac{\gamma \kappa^2}{4(t_s - t)}. \tag{135}$$

Here t_s is a constant of integration and t_1 is given by

$$t_1^2 = -\frac{\gamma \phi_0^2 \left(1 - \frac{3\gamma\kappa^2}{4}\right)}{2V_0}. \tag{136}$$

Equation (135) shows

$$a = a_0 \left|\frac{t_s - t}{t_1}\right|^{-(\gamma\kappa^2/4)}. \tag{137}$$

For a phantom with $\gamma < 0$, a grows up to infinity at $t = t_s$, which is the Big Rip singularity [8–10].

In general, Eq. (132) shows that

$$\frac{d\rho_\phi}{dt} = -3\gamma H \left(\frac{d\phi}{dt}\right)^2, \tag{138}$$

which is positive if $\gamma < 0$, $H > 0$, and $\dot{\phi} \neq 0$. Hence, for the phantom with negative γ, the energy density increases in general. We should also note that since the contribution to ρ_ϕ from the kinetic term is negative if $\gamma < 0$, we find

$$\rho_\phi \leq V(\phi). \tag{139}$$

For the case $V(\phi) = 0$, the energy density ρ_ϕ is not positive. In general, the Big Rip singularity occurs due to the rapid increase of the energy density of the phantom scalar. When $V(\phi) = 0$, the singularity does not occur. To be concrete, we also consider here matter to be dust, whose energy density is given by

$$\rho_d = \frac{\rho_0}{a^3}, \tag{140}$$

with a constant ρ_0, which we assume to be positive. When $V(\phi) = 0$, the solution of (132) is given by

$$\frac{d\phi}{dt} = \frac{c}{a^3}. \tag{141}$$

Here c is a constant. Then the FRW equation has the form:

$$\frac{6}{\kappa^2} H^2 = \frac{\gamma c^2}{2a^6} + \frac{\rho_0}{a^3}. \tag{142}$$

Equation (142) can be solved easily as

$$a^3 = -\frac{\gamma c^2}{2\rho_0} + \frac{9\kappa^2}{4\rho_0^2}(t - t_s). \tag{143}$$

Here t_s is a constant of the integration. Then, there is no singularity and no acceleration, either.

The Big Rip singularity in (137) occurs because the potential is unbounded and goes to positive infinity when $\phi \to -\infty$. Equation (139) tells that if $V(\phi)$ is bounded from above and has a maximum V_m as for $V(\phi) = 0$ case, the

energy density does not grow up infinitely and the Big Rip singularity does not occur. We now assume for the large negative ϕ, the potential approaches a constant,

$$V(\phi) \to V_m \text{ (constant)} \quad \text{when} \quad \phi \to -\infty. \quad (144)$$

In the region, Eq. (132) reduces to

$$0 = -\gamma \left(\frac{d^2\phi}{dt^2} + 3H \frac{d\phi}{dt} \right), \quad (145)$$

which can be solved as in (141). Then the energy density ρ_ϕ (133) has the following form:

$$\rho_\phi = \frac{\gamma c^2}{2a^6} + V_m. \quad (146)$$

The FRW equation becomes

$$\frac{6}{\kappa^2} H^2 = \frac{\gamma c^2}{2a^6} + V_m. \quad (147)$$

The first term in the rhs could be neglected for a large universe. Then, for large a, one gets the de Sitter space as a solution

$$H^2 \to \frac{V_m}{6\kappa^2}. \quad (148)$$

Thus, one way to avoid the singularity might be that, in the present universe for large negative ϕ there is an upper bound in the potential. In this respect, it is also interesting to note that there are phantomlike models, such as generalized Chaplygin gas, where finite-time future singularity does not occur [22].

Another way to argue this is to take into account quantum effects, say, for conformally invariant matter. Then the contributions coming from the conformal anomaly to the energy density ρ_A and pressure p_A are (see [23])

$$\rho_A = -6b' H^4 - \left(\frac{2}{3} b + b'' \right) \left\{ -6H \frac{d^2 H}{dt^2} - 18H^2 \frac{dH}{dt} \right.$$

$$\left. + 3 \left(\frac{dH}{dt} \right)^2 \right\}, \quad (149)$$

$$p_A = b' \left\{ 6H^4 + 8H^2 \frac{dH}{dt} \right\} + \left(\frac{2}{3} b + b'' \right) \left\{ -2 \frac{d^3 H}{dt^3} \right.$$

$$\left. -12H \frac{d^2 H}{dt^2} - 18H^2 \frac{dH}{dt} - 9 \left(\frac{dH}{dt} \right)^2 \right\}. \quad (150)$$

In general, with N scalar, $N_{1/2}$ spinor, N_1 vector fields, N_2 ($=0$ or 1) gravitons, and N_{HD} higher-derivative conformal scalars, b, b', and b'' are given by

$$b = \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{HD}}{120(4\pi)^2},$$

$$b' = -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{HD}}{360(4\pi)^2}, \quad b'' = 0. \quad (151)$$

Near the Big Rip singularity, the scale factor a blows up, as in (137), at $t = t_s$. Then the curvatures behave as $R \propto |t - t_s|^{-2}$, and they become large. Since the quantum correction includes the square of the curvatures, the correction becomes large and important near the Big Rip singularity. Now the FRW equation has the following form:

$$\frac{6}{\kappa^2} H^2 = \frac{\gamma}{2} \left(\frac{d\phi}{dt} \right)^2 + V(\phi) + \rho_A. \quad (152)$$

We now write H and ϕ as

$$H = h_0 + \delta h, \quad \phi = \phi_0 \ln \left| \frac{t_s - t}{t_1} \right| + \delta\phi. \quad (153)$$

Here h_0, t_s, and t_1 are constants. We assume that when $t \to t_s$, δh and $\delta\phi$ become very small compared with the first terms, respectively. In H, however, as the first term is a constant, only the second term contributes to dH/dt and $d^2 H/dt^2$. From (132) with the same exponential potential, one obtains

$$0 = -\gamma \left(-\frac{\phi_0}{(t_s - t)^2} - \frac{3h_0}{t_s - t} \right) + \frac{2V_0 t_1^2}{\phi_0(t_s - t)^2} \left(1 - \frac{2}{\phi_0} \delta\phi \right)$$

$$+ o[(t_s - t)^{-1}], \quad (154)$$

which gives

$$V_0 t_1^2 = -\frac{\gamma \phi_0^2}{2}, \quad \delta\phi = -\frac{3}{2}(t_s - t). \quad (155)$$

Then from (152), we have

$$0 = \frac{3\gamma h_0 \phi_0}{t_s - t} + 6h_0 \left(\frac{2}{3} b + b'' \right) \frac{d^2 \delta h}{dt^2} + o[(t_s - t)^{-1}], \quad (156)$$

and

$$\delta h = \frac{\gamma \phi_0}{2 \left(\frac{2}{3} b + b'' \right)} (t_s - t) \ln \left| \frac{t_s - t}{t_2} \right|. \quad (157)$$

Here t_2 is a constant of integration. Since $H = \dot{a}/a$, the scale factor is

$$a = a_0 \left| \frac{t_s - t}{t_2} \right|^{\{\gamma\phi_0/4[(2/3)b + b'']\}(t_s - t)^2}$$

$$\times e^{-h_0(t_s - t) - \{\gamma\phi_0/8([(2/3)b + b'']\}(t_s - t)^2 + o[(t_s - t)^2]}. \quad (158)$$

In d^2a/dt^2 or dH/dt, there appears a logarithmic singularity. The behavior of the singularity, however, becomes rather milder than the case without quantum correction, where due to the singularity at $t=t_s$, the universe cannot develop beyond the singularity. Since the singularity becomes mild due to the quantum correction and a and H are finite at $t=t_s$, the universe can develop to the region $t>t_s$. Then, essentially the Big Rip singularity is removed due to the quantum correction.

V. GRAVITY-ASSISTED DARK-ENERGY DOMINANCE AND EFFECTIVE PHANTOM AND/OR QUINTESSENCE COSMOLOGY

Recently, an effective phantom and/or quintessence description of the late time universe was obtained via the introduction of a new higher-derivative coupling between matter and gravity [24]. It was shown that such a model may explain the gravity-assisted dark-energy dominance. In this section we will consider a simple (scalar) example of this kind of model when standard matter is also included.

The starting action is

$$S=\int d^4x\sqrt{-g}\left\{\frac{1}{\kappa^2}R+R^\alpha L_d+L_m\right\}. \quad (159)$$

Here L_d is the matterlike Lagrangian density (dark energy) and L_m the Lagrangian density of the (standard) matter. By variation over $g_{\mu\nu}$, the equation of motion follows:

$$0=\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}=\frac{1}{\kappa^2}\left\{\frac{1}{2}g^{\mu\nu}R-R^{\mu\nu}\right\}+\tilde{T}^{\mu\nu}+T_m^{\mu\nu}. \quad (160)$$

Here the effective energy momentum tensor $\tilde{T}_{\mu\nu}$ is defined by

$$\tilde{T}^{\mu\nu}\equiv-\alpha R^{\alpha-1}R^{\mu\nu}L_d+\alpha(\nabla^\mu\nabla^\nu-g^{\mu\nu}\nabla^2)(R^{\alpha-1}L_d)$$
$$+R^\alpha T^{\mu\nu}, \quad (161)$$

and $T^{\mu\nu}$ is given by

$$T^{\mu\nu}\equiv\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\left(\int d^4x\sqrt{-g}L_d\right). \quad (162)$$

The standard matter part of the energy momentum tensor $T_m^{\mu\nu}$ is also defined as

$$T_m^{\mu\nu}\equiv\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\left(\int d^4x\sqrt{-g}L_m\right). \quad (163)$$

For simplicity, the Lagrangian density of a free massless scalar is considered as L_d,

$$L_d=-\frac{1}{2}\partial_\mu\phi\partial_\nu\phi. \quad (164)$$

Note that for the above L_d choice and with a higher-derivative scalar curvature term in the gravitational sector

(also without standard matter), model of this kind was discussed, with different purposes, in Ref. [25].

The equation of motion has the following form:

$$0=\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\phi}=\frac{1}{\sqrt{-g}}\partial_\mu(R^\alpha\sqrt{-g}g^{\mu\nu}\partial_\nu\phi). \quad (165)$$

The metric is again chosen to describe a FRW universe with flat three-space,

$$ds^2=-dt^2+a(t)^2\sum_{i=1,2,3}(dx^i)^2. \quad (166)$$

If one assumes that ϕ depends only on t [$\phi=\phi(t)$], the solution of the scalar-field equation (165) is given by

$$\dot{\phi}=qa^{-3}R^{-\alpha}. \quad (167)$$

Here q is a constant of integration. Hence

$$R^\alpha L_d=\frac{q^2}{2a^6R^\alpha}, \quad (168)$$

which becomes dominant when R is small (large) compared with the Einstein term $(1/\kappa^2)R$ if $\alpha>-1$ ($\alpha<-1$). Thus, one arrives at the remarkable possibility that dark energy grows to asymptotic dominance over the usual matter with decrease of the curvature.

Combining (159) and (160), one gets $S\sim\int d^4x\sqrt{-g}\{(1/\kappa^2)R+[q^2/(2a^6R^\alpha)]\}$, which may indicate $R\sim a^{-[6/(\alpha+1)]}$. Then the curvature R might be stabilized to have a nontrivial minimum due to the second term in (159).

Substituting (167) into (160), the $(\mu,\nu)=(t,t)$ component of the equation of motion has the following form:

$$0=-\frac{3}{\kappa^2}H^2+\rho_d+\rho_m,$$

$$\rho_d\equiv\frac{36q^2}{a^6}(6\dot{H}+12H^2)^{-\alpha-2}\left\{\frac{\alpha(\alpha+1)}{4}\ddot{H}H+\frac{\alpha+1}{4}\dot{H}^2\right.$$
$$\left.+\left(1+\frac{13}{4}\alpha+\alpha^2\right)\dot{H}H^2+\left(1+\frac{7}{2}\alpha\right)H^4\right\}. \quad (169)$$

Here ρ_m is the energy density of the standard matter. Specifically, when $\alpha=-1$, Eq. (169) looks like

$$0=-\left(\frac{3}{\kappa^2}+\frac{15q^2}{2a^6}\right)H^2+\rho_m. \quad (170)$$

If $\rho_m=0$, this equation has only the trivial solution $H=0$ (a is a constant).

When $\rho_m=0$, we can easily find the accelerating solution of (169) [24],

$$a = a_0 t^{(\alpha+1)/3} \quad \left(H = \frac{\alpha+1}{3t} \right),$$

$$a_0^6 \equiv \frac{\kappa^2 q^2 (2\alpha-1)(\alpha-1)}{3(\alpha+1)^{\alpha+1} \left(\frac{2}{3}(2\alpha-1) \right)^{\alpha+2}}. \tag{171}$$

Equation (171) tells that the universe accelerates, that is, $\ddot{a} > 0$ if $\alpha > 2$. Even for $\alpha < -1$, by changing the time variable by $t \to t_0 - t$ (t_0 is a constant), the universe is expanding and accelerating. In this case, however, there is a Big Rip singularity at $t = t_0$.

For the matter satisfying the relation $p = w\rho$, where p is the pressure and ρ the energy density, from the usual FRW equation, one has

$$a \propto t^{2/[3(w+1)]}. \tag{172}$$

For $a \propto t^{h_0}$ it follows that

$$w = -1 + \frac{2}{3h_0}, \tag{173}$$

and an accelerating expansion ($h_0 > 1$) of the universe occurs if

$$-1 < w < -\tfrac{1}{3}. \tag{174}$$

For the case of (171), one finds

$$w = \frac{1-\alpha}{1+\alpha}. \tag{175}$$

Then if $\alpha < -1$, $w < -1$, what corresponds to an effective phantom. In this case, changing t as $t_0 - t$ in (171), there appears a Big Rip singularity at $t = t_0$. In [26], however, it has been shown that the phantom energy with $w < -1$ makes the radius of the wormhole spacetime (when it does occur) to increase in time and thus before the Big Rip the radius becomes infinite and, as a result, the Big Rip singularity may be avoided.

It is interesting to investigate the stability of the solution in (171). For this purpose, we write the scale factor a as

$$a = a_0 t^{(\alpha+1)/3}(1+\delta) \quad (|\delta| \ll 1). \tag{176}$$

Here a_0 is given in (171). From (169), it follows that

$$0 = -\frac{2(2\alpha-1)(\alpha+1)(\alpha-1)}{t^4}\delta$$

$$-\frac{2(2\alpha^3 - 18\alpha^2 - 33\alpha - 7)}{t^3}\frac{d\delta}{dt}$$

$$+\frac{9\alpha(\alpha+4)}{t^2}\frac{d^2\delta}{dt^2} + \frac{9\alpha}{t}\frac{d^3\delta}{dt^3}. \tag{177}$$

The solutions of (177) are given as

$$\delta \propto t^{x_0}. \tag{178}$$

Here x_0 is a constant. Then Eq. (177) gives

$$0 = 9\alpha x_0^3 + 9\alpha(\alpha+1)x_0^2 + (-4\alpha^3 + 27\alpha^2 + 48\alpha + 14)x_0$$

$$-2(\alpha+1)(2\alpha-1)(\alpha-1). \tag{179}$$

As clear from (175), when $\alpha \to +\infty$, $w \gtrsim -1$, which corresponds to quintessence and, when $\alpha \to -\infty$, we find $w \lesssim -1$, which corresponds to a phantom. Imagine that parameter $\alpha \to \pm\infty$. If we assume $x_0 = \mathcal{O}(1)$, Eq. (179) reduces to

$$0 = -4\alpha^3(1+x_0) + \mathcal{O}(\alpha^2). \tag{180}$$

Hence

$$x_0 \sim -1. \tag{181}$$

With $x_0 = \mathcal{O}(\alpha)$, one obtains

$$0 = -4\alpha^3 x_0 + 9\alpha^2 x_0^2 + 9\alpha x_0^3 + \mathcal{O}(\alpha^3). \tag{182}$$

As a result

$$x_0 \sim 0, \frac{\alpha}{3}, -\frac{4\alpha}{3}. \tag{183}$$

Note that the first solution $x_0 \sim 0$ corresponds to the solution in (181) because we have assumed $x_0 = \mathcal{O}(\alpha)$. The perturbation looks as

$$\delta \sim \delta_1 t^{-1} + \delta_2 t^{\alpha/3} + \delta_3 t^{-(4\alpha/3)}. \tag{184}$$

Here $\delta_{1,2,3}$ are constants. The second term in (184) may indicate the instability of the solution (171). When $\alpha \to +\infty$, which corresponds to quintessence, the second term may become dominant when $t \to \infty$. On the other hand, since the case $\alpha \to -\infty$ corresponds to the phantom, we replace $t \to t_0 - t$. The second term may become dominant near the Big Rip $t \to t_0$. This might, however, indicate that the Big Rip never occurs, since the Big Rip solution is unstable. In other words, even if the present universe equation-of-state parameter looks as $w \lesssim -1$, the universe might transit to another solution corresponding to the second term in (184). However, it is difficult to find the nonperturbative behavior of the solution corresponding to the second term in (184).

The $\alpha \to -1$ case, which corresponds to $w \to -\infty$, can be also considered. Let us write

$$\alpha = -(1+\epsilon) \quad (\epsilon \ll 1). \tag{185}$$

It is natural to assume that ϵ is positive. Equation (179) gives

$$0 = -9(1+\epsilon)x_0^3 + 9\epsilon x_0^2 + (-3+18\epsilon)x_0 + 12\epsilon + \mathcal{O}(\epsilon^2). \tag{186}$$

Its approximate solution is

$$x_0 = 4\epsilon, \quad -\frac{3}{2}\epsilon \pm \frac{i}{\sqrt{3}}\left(1 - \frac{7}{2}\epsilon\right). \tag{187}$$

The perturbation δ is found to be given by

$$\delta = \delta_0(t_0-t)^{4\epsilon} + (t_0-t)^{-3/2\,\epsilon}\{\delta_c\cos[\tfrac{1}{\sqrt{3}}(1-\tfrac{7}{2}\epsilon)\ln(t_0-t)]$$

$$+ \delta_s\sin[\tfrac{1}{\sqrt{3}}(1-\tfrac{7}{2}\epsilon)\ln(t_0-t)]\}. \tag{188}$$

Here δ_0, δ_c, and δ_s are constants. The first term decreases near the Big Rip at $t=t_0$, but the other terms oscillate rapidly near the Big Rip and the amplitude becomes large, which also shows the instability of the (transient acceleration) solution (171). This may be quite an acceptable result, on the condition that it lasts sufficiently long enough to comply with observational data. The way to avoid a finite-time future singularity due to the instability of the accelerating cosmology may deserve some attention.

The case $\alpha \gtrsim 1$ corresponds to $w \lesssim 0$. With

$$\alpha = 1 + \epsilon \quad (0 < \epsilon \ll 1), \tag{189}$$

Eq. (179) gives

$$0 = 9(1+\epsilon)x_0^3 + (18+27\epsilon)x_0^2 + (85+90\epsilon)x - 4\epsilon + \mathcal{O}(\epsilon^2). \tag{190}$$

Its solution is

$$x_0 = \frac{4}{85} + \mathcal{O}(\epsilon^2), \quad -1 \pm \frac{2i}{9} + \mathcal{O}(\epsilon). \tag{191}$$

Therefore δ is given by

$$\delta \sim \hat\delta_0 t^{4\epsilon/85} + \frac{1}{t}\left[\hat\delta_c\cos\left(\frac{2}{9}\ln t\right) + \hat\delta_s\sin\left(\frac{2}{9}\ln t\right)\right]. \tag{192}$$

Here $\hat\delta_0$, $\hat\delta_c$, and $\hat\delta_s$ are constants, too. The first term increases with t, and thus the solution (171) may be unstable again.

Even with numerical calculation for general α, there seems to be instabilities in the solution (171) if $w<0$ although the solution could be stable if $w>0$. This may suggest again that the acceleration of such a dark-energy universe might be transient.

So far the standard matter contribution has been neglected. When $\rho_m \neq 0$, the simple assumption is that ρ_m behaves as

$$\rho_m = \rho_0 a^{-\beta}. \tag{193}$$

Here ρ_0 and β are constant. From Eq. (170) it follows

$$t = \sqrt{\frac{3}{\rho_0\kappa^2}}\int da\, a^{(\beta/2)-4}\sqrt{a^6 + \frac{5q^2\kappa^2}{2}}. \tag{194}$$

For the case of a general α, if the dark-energy density is dominant: $\rho_d \gg \rho_m$, the solution should behave as (171). Hence, the energy density ρ_m of the standard matter evolves as

$$\rho_m = \rho_0 a_0^{-\beta} t^{-[(\alpha+1)\beta/3]}. \tag{195}$$

On the other hand, ρ_d behaves as

$$\rho_d \propto t^{-2}. \tag{196}$$

Since $\beta=4$ for the radiation and $\beta=3$ for the dust and the acceleration of the universe occurs when $\alpha>2$, we may assume $\beta>2$ and $\alpha>2$. As a result,

$$\frac{(\alpha+1)\beta}{3} > 2. \tag{197}$$

When time t grows, ρ_m decreases further as compared with ρ_d.

The essence of a gravity-assisted dark-energy dominance is clearly seen in the example below. Let the standard matter dominate, as compared with the dark energy $\rho_m \gg \rho_d$. Then the scale factor is given by that of the standard FRW equation

$$a = \left(\frac{\beta^2\rho_0\kappa^2}{12}\right)^{1/\beta} t^{2/\beta}. \tag{198}$$

The energy density of the standard matter ρ_m behaves as $\rho_m \sim t^{-2}$. On the other hand, the dark energy behaves as

$$\rho_d \sim t^{2\alpha - (12/\beta)}. \tag{199}$$

Hence if

$$2\alpha - \frac{12}{\beta} > -2, \tag{200}$$

the dark energy ρ_d becomes larger as time passes. Equation (200) can be satisfied if $\alpha, \beta > 2$ as in the dark-matter dominant case.

Let us assume that in the early universe, the standard matter and/or radiation is dominant. The universe evolves according to (198). From (200), dark energy increases with time growth. When $\rho_d \gtrsim \rho_m$, as in the present universe, the acceleration of the universe begins. Thus, in the future the accelerating universe evolves with the scale factor (171). However, that is most probably a transient acceleration. This is not strange owing to the fact that the above effective phantom and/or quintessence description is achieved by means of a higher-derivative coupling between dark matter and gravity. It is known that higher-derivative gravities (see [13] for a review) may have problems with unitarity (stability) when they are considered as fundamental theories. Hence, the model under discussion should be treated as a kind of effective matter-gravity theory. It would be interesting to analyze some astrophysical predictions of our model (say, rotation curves of galaxies) in the way it was recently discussed, e.g., in [27].

VI. DISCUSSION

In summary, (phantom) scalar-tensor cosmology with an exponential scalar potential suggests the comely possibility of a dark-energy universe with an equation-of-state parameter w, which is negative and very close to -1. Convenient

choices of the theoretical parameters may shift the explicit value of w from above to below -1. Moreover, such a universe naturally admits a (transient or eternal) acceleration phase. A very nice property of this theory is that what appears as a phantom in one frame may appear as a standard scalar in another frame. It is demonstrated that in the situation when a finite-time future singularity is predicted by the growing phantom-energy density, the consideration of quantum gravity effects might drastically change the future of our universe, removing the singularity in a quite natural way. From another side, it is also shown that a higher-derivative gravity-matter coupling term being not the phantom may in fact provide an effective phantom and/or quintessence description of the late-time universe, suggesting the possibility of a dark-energy model of a brand new type. In this case, gravity makes dark energy become the (evolving) main contribution to the total energy density—as compared with the standard matter and/or radiation, which was initially dominant—which leads to the appearance of the phase of transient acceleration.

Current attention to phantom models as dark-energy candidates is not driven by the internal consistency and/or beauty of this theory, which still contains a number of partially resolved problems, as we have already mentioned. Rather, it is the lack of a good theoretical understanding of the present universe coming from more usual theories that calls for alternative explanations to be considered, on one hand. On the other hand, one sees also that the cosmic-zoo structure, which emerges from these alternatives, is so rich and suitable at times, that some concepts, which so far seemed to be strange (like the idea of negative energy itself), deserve to be investigated with care and to the end. In this respect, even the mild indications that have been reported of a possible phantom origin coming from string and/or M-theory or on the chance (described above) to avoid the Big Rip catastrophe by taking properly into account the quantum effects seem indeed very promising.

APPENDIX

In this Appendix several remarks are made about the possible origin of the phantom-related models coming from the higher-dimensional theories considered in the paper. This topic was widely investigated in the Kaluza-Klein context (in relation with the string and/or M-theory), and for a recent discussion the reader is addressed to [28]. We start from a $4+n$-dimensional spacetime, whose metric is given by

$$ds^2 = \sum_{\mu,\nu=0,1,2,3} g_{\mu\nu}dx^\mu dx^\nu + e^{2\phi(x^\mu)}\sum_{i,j=1}^n \tilde{g}_{ij}d\xi^i d\xi^j.$$

$$(A1)$$

For simplicity, one may assume that the metric $\tilde{g}_{ij}$ corresponds to an Einstein manifold, where the Ricci tensor $\tilde{R}_{ij}$ constructed from $\tilde{g}_{ij}$ is proportional to $\tilde{g}_{ij}$: $\tilde{R}_{ij}=k\tilde{g}_{ij}$. Here k is a constant. When $n=1$, k always vanishes $(k=0)$. When $n \geq 3$, the above metric is given as the solution of the n-dimensional Euclidean Einstein equation. When $n=2$, since the two-dimensional Einstein equation is trivial, in the conformal gauge the above condition for the Ricci tensor is the Liouville equation. Under the above assumptions, the $4+n$ dimensional Einstein action with matter field χ (bosonic sector of some higher-dimensional supergravity) can be written as

$$S_{4+n} = \frac{1}{\kappa^2}\int d^{4+n}x\sqrt{-g^{(4+n)}}\left(R^{(4+n)} - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - U(\chi)\right)$$

$$= \frac{V_n}{\kappa^2}\int d^4x\sqrt{-g}e^{n\phi}\left(R + n(n-1)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right.$$

$$\left. + nke^{-2\phi} - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - U(\chi)\right).$$

$$(A2)$$

Here V_n is the volume of the n-dimensional manifold whose metric tensor is given by $\tilde{g}_{ij}$. We should note that the kinetic energy of ϕ becomes negative, as for the phantom. Rescaling the four-dimensional metric $g_{\mu\nu}$ by $g_{\mu\nu}\rightarrow e^{-n\phi}g_{\mu\nu}$, the action (A2) can be rewritten as

$$S_{4+n} = \frac{V_n}{\kappa^2}\int d^4x\sqrt{-g}\left(R - \frac{n(n+2)}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right.$$

$$\left. + nke^{-(n+2)\phi} - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - e^{-n\phi}U(\chi)\right).$$

$$(A3)$$

Now the kinetic energy of ϕ is positive. If we further rescale ϕ by

$$\phi = \varphi\sqrt{\frac{3}{n(n+2)}},$$

$$(A4)$$

the four-dimensional action looks like

$$S_{4+n} = \frac{V_n}{\kappa^2} \int d^4x \sqrt{-g} \left(R - \frac{3}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + nke^{-\varphi\sqrt{[3(n+2)]/n}} \right.$$

$$\left. - \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - e^{-\varphi\sqrt{3n/(n+2)}} U(\chi) \right). \tag{A5}$$

The above action belongs to the same class as (43) by identifying

$$W(\varphi,\chi) = e^{-\varphi\sqrt{3n/(n+2)}} U(\chi). \tag{A6}$$

Comparing the above expression with (47), one obtains

$$\sqrt{\frac{3n}{n+2}} = \frac{2+\eta}{\varphi_0}. \tag{A7}$$

Then, if

$$0 < \varphi_0 < 2\sqrt{\frac{n+2}{3n}}, \tag{A8}$$

Eq. (53) follows. If

$$U(\chi) = W_0 e^{(\eta/\chi_0)\chi}, \tag{A9}$$

we can obtain for w a value less than -1, and correspondigly the universe expands with acceleration, by a proper choice of the parameters.

One may consider the product compactification to be more general than (A1),

$$ds^2 = \sum_{\mu,\nu=0}^{d-1} g_{\mu\nu} dx^\mu dx^\nu + e^{2\phi^{(1)}(x^\mu)} \sum_{i,j=1}^{n} g_{ij}^{(1)} d\xi^i d\xi^j$$

$$+ e^{\phi^{(2)}(x^\mu)} \sum_{I,J=1}^{N} g_{IJ}^{(2)} d\xi^i d\xi^j. \tag{A10}$$

Since the scalar curvature $R^{(d+n+N)}$ in $(d+n+N)$-dimensional spacetime is given by

$$R^{(d+n+N)} = R^{(d)} + k^{(1)} e^{-2\phi^{(1)}} - n(n+1) \partial_\mu \phi^{(1)} \partial^\mu \phi^{(1)}$$

$$- 2n\nabla^2 \phi^{(1)} + k^{(2)} e^{-2\phi^{(2)}} - N(N+1) \partial_\mu \phi^{(2)}$$

$$\times \partial^\mu \phi^{(2)} - 2N\nabla^2 \phi^{(2)} - 2nN \partial_\mu \phi^{(1)} \partial^\mu \phi^{(2)}, \tag{A11}$$

if we start from the $(d+n+N)$-dimensional action coupled with the scalar field χ with potential $U(\chi)$, we obtain

$$S^{(d+n+N)} = \frac{1}{\kappa^2} \int d^{d+n+N} x \sqrt{-g^{(d+n+N)}}$$

$$\times \left[R^{(d+n+M)} - \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - U(\chi) \right]$$

$$= \frac{V_n V_N}{\kappa^2} \int d^d x \sqrt{-g^{(d)}} e^{n\phi^{(1)} + N\phi^{(2)}}$$

$$\times \left[R^{(d)} - n \partial_\mu \phi^{(1)} \partial^\mu \phi^{(1)} - N \partial_\mu \phi^{(2)} \partial^\mu \phi^{(2)} \right.$$

$$+ \partial_\mu (n\phi^{(1)} + N\phi^{(2)}) \partial^\mu (n\phi^{(1)} + N\phi^{(2)})$$

$$\left. + k^{(1)} e^{-2\phi^{(1)}} + k^{(2)} e^{-2\phi^{(2)}} - \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - U(\chi) \right]. \tag{A12}$$

If $U(\chi)$ is a constant, one may regard it as the cosmological constant. Further rescaling the metric tensor by

$$g_{\mu\nu} \to e^{-[2/(d-2)](n\phi^{(1)} + N\phi^{(2)})} g_{\mu\nu}, \tag{A13}$$

we get

$$S^{(d+n+N)} = \frac{V_n V_N}{\kappa^2} \int d^d x \sqrt{-g^{(d)}} \left[R^{(d)} - n \partial_\mu \phi^{(1)} \partial^\mu \phi^{(1)} \right.$$

$$- N \partial_\mu \phi^{(2)} \partial^\mu \phi^{(2)} - \frac{1}{d-2} \partial_\mu$$

$$\times (n\phi^{(1)} + N\phi^{(2)}) \partial^\mu (n\phi^{(1)} + N\phi^{(2)})$$

$$+ k^{(1)} e^{-[2/(d-2)](n\phi^{(1)} + N\phi^{(2)}) - 2\phi^{(1)}}$$

$$+ k^{(2)} e^{-[2/(d-2)](n\phi^{(1)} + N\phi^{(2)}) - 2\phi^{(2)}} - \frac{1}{2} \partial_\mu \chi \partial^\mu \chi$$

$$\left. - e^{-[2/(d-2)](n\phi^{(1)} + N\phi^{(2)})} U(\chi) \right]. \tag{A14}$$

In this frame, the kinetic term of the matter field χ does not directly couple to the scalar fields $\phi^{(1)}$ and $\phi^{(2)}$. Then the Newton law is not violated in the leading order of perturbation. The kinetic terms of the fields $\phi^{(1)}$ and $\phi^{(2)}$ can be diagonalized by

$$\phi^{(1)} = y^- \phi^+ + y^+ \phi^-, \quad \phi^{(2)} = y^+ \phi^+ - y^- \phi^-. \tag{A15}$$

Here

$$y^\pm \equiv \sqrt{\frac{1}{2} \left(1 \pm \frac{(d-2)^2(n-N)^2 \left(1 + \frac{n+N}{d-2}\right)^2}{4n^2 N^2} \right)}. \tag{A16}$$

Hence

$$S^{(d+n+N)} = \frac{V_n V_N}{\kappa^2} \int d^d x \sqrt{-g^{(d)}} \left[R^{(d)} - x^+ \partial_\mu \phi^+ \partial^\mu \phi^+ - x^- \partial_\mu \phi^- \partial^\mu \phi^- \right.$$

$$+ k^{(1)} e^{-\{[2n/(d-2)+2]y^- + 2N/(d-2)y^+\}\phi^+ - \{[2n/(d-2)+2]y^+ - [2N/(d-2)]y^-\}\phi^-}$$

$$+ k^{(2)} e^{-\{[2N/(d-2)+2]y^+ + [2n/(d-2)]y^-\}\phi^+ - \{-[2N/(d-2)+2]y^- - [2n/(d-2)]y^+\}\phi^-}$$

$$\left. - \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - e^{-[2/(d-2)]\{(ny^- + Ny^+)\phi^+ + (ny^+ - Ny^-)\phi^-\}} U(\chi) \right]. \tag{A17}$$

Here,

$$x^\pm \equiv \frac{1}{2}\left(n + N + \frac{n^2 + N^2}{d-2} \pm \sqrt{D} \right) > 0,$$

$$D \equiv \left(n + N + \frac{n^2 + N^2}{d-2} \right)^2 - 4nN\left(1 + \frac{n+N}{d-2} \right)$$

$$= (n-N)^2 \left(1 + \frac{n+N}{d-2} \right)^2 + \frac{4n^2 N^2}{(d-2)^2} > 0. \tag{A18}$$

For simplicity, the case $k^{(2)} = U(\chi) = \chi = 0$ and $d = 4$ is considered. Comparing the action (A17) with (43), we may identify

$$\varphi \leftrightarrow \sqrt{\frac{2x^+}{3}} \phi^+, \quad \kappa\chi \leftrightarrow \sqrt{2x^-}\, \phi^-. \tag{A19}$$

From (47), one sees that

$$W_0 = -k^{(1)},$$

$$\frac{2+\eta}{\varphi_0} = \{(n+2)y^- + Ny^+\} \sqrt{\frac{3}{2x^+}},$$

$$\frac{\eta}{\chi_0} = -\frac{(n+2)y^+ - Ny^-}{\sqrt{2x^-}}. \tag{A20}$$

Using (50), an expression for η can be found

$$\eta = -\frac{2}{1+\zeta}, \quad \zeta \equiv \frac{x^- \{(n+2)y^- + Ny^+\}^2}{x^+ \{(n+2)y^+ - Ny^-\}^2}. \tag{A21}$$

Since $x^\pm > 0$ (A18), $\zeta > 0$ and therefore η satisfies (53). Then $w > -1$ and there is no phantom. More explicitly w is given by

$$w = -1 + \frac{2\{(n+2)y^- + Ny^+\}}{3x^+(2+\eta)}. \tag{A22}$$

Numerically, with $n=2$ and $N=5$, $w = 0.002\,695\,54$; with $n=3$ and $N=4$, $w = -0.058\,669\,1$. Although not realistic, if we choose $n=6$ and $N=31$, it follows that $w = -0.353\,435$.

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