Finite volume effects for non-Gaussian multifield inflationary models

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Models of multifield inflation exhibiting primordial non-Gaussianity have recently been introduced. This is the case, in particular, if the fluctuations of a light field scalar field, transverse to the inflaton direction, with quartic coupling can be transferred to the metric fluctuations. So far in those calculations only the ensemble statistical properties have been considered. We explore here how finite volume effects could affect those properties. We show that the expected non-Gaussian properties survive at a similar level when the finite volume effects are taken into account and also find that they can skew the metric distribution even though the ensemble distribution is symmetric.

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I. INTRODUCTION

The observations of cosmic microwave background (CMB) anisotropies and of the large-scale structure of the universe offer a window into the physics of the inflaton. For instance, a detailed measurement of the shape of the power spectrum can give constraints on the shape of the inflaton potential. On the other hand, the detection of non-Gaussian metric fluctuations could signal the existence of effective couplings between the inflaton and other fields. There have been a series of observational advances toward constraining the deviation from Gaussianity using CMB data on large scales ($\theta \sim 10°$) [1], intermediate ($\theta \sim 1°$) [2], and small scales ($\theta \sim 10'$) [3], as well as large-scale structures [4]. The recent CMB data from the Wilkinson Microwave Anisotropy Probe (WMAP) were also used by Komatsu *et al.* [5], who concluded that the CMB spectrum was compatible with Gaussianity on the basis of an analysis of the bispectrum. Similar analyses using Minkowski functionals [6] and the three-point correlation function [7] reached the same conclusions. More recently, an analysis of the WMAP data using measurements of the genus and its statistics [8] concluded that the Gaussianity of the CMB field was ruled out at a 99% level. As stressed later, non-Gaussianity of primordial origin may still have escaped detection and further investigations are needed.

From a theoretical point of view, Gaussianity is a generic prediction of slow-roll single field inflation [9,10]. A series of models including features in the inflationary potential [11], the existence of seeds such as χ^2 [12], axion [13] or topological defects [14], the curvaton scenario [15], and a varying inflaton decay rate [16] have been shown to be able to generate some primordial non-Gaussianity.

In Ref. [10], we proposed a general mechanism to produce non-Gaussianity in the adiabatic mode, and explicit realizations of such models were presented in Ref. [17]. The mechanism is based on the generation of non-Gaussian isocurvature fluctuations which are then transferred to the adiabatic modes through a bend in the classical inflaton trajectory. Natural realizations were shown to involve quartic self-interaction terms. The statistical properties of the resulting metric fluctuations were then shown to be a superposition of a Gaussian and a non-Gaussian contribution of the same variance, the relative weight of the two contributions being related to the total bending of the trajectory in field space. The non-Gaussian probability distribution function (PDF) was also computed and shown to be described by a single new parameter so that generically only two new parameters suffice in describing this class of models.

In order to infer constraints from, e.g., CMB data on these kind of models predicting non-Gaussianity one needs to go through at least two steps.

(1) One first needs a precise prediction of the statistical properties of the curvature in order to construct an estimator adapted to the detection of this kind of non-Gaussianity. For instance, the analysis of Komatsu *et al.* [5] uses the bispectrum and Minkowski functional and constrains a χ^2 deviation from Gaussianity. The gravitational potential was parametrized as $\Phi = \Phi_L + f_{NL}(\Phi_L^2 - \langle \Phi_L^2 \rangle)$ where Φ_L is the Gaussian linear perturbation of zero mean and it was concluded that $-58 < f_{NL} < 134$ at 95% C.L. This constrains only a very peculiar type of non-Gaussianity and does not apply, e.g., to the models of Refs. [10,17].

(2) Even if the form of the PDF is predicted theoretically, one needs to investigate what is measured and how the measurements are related to the theoretical predictions.

The goal of this article is to investigate the observational implications of the theoretical predictions of Ref. [10]. It requires further investigations, in particular, on the effect of the finite volumes of the survey, while a detailed comparison of our model to the existing data set, beyond the scope of this work, is still left for further study. In particular, we want to show that these finite volume effects do not suppress the predicted non-Gaussianity.

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Section II formulates the problem, focusing on the quantities that can actually be observed. In Sec. III, we recall basics of the models we consider in this paper and give simple consequences. We then develop in Sec. IV some arguments to understand the effect of the mean value of the non-Gaussian component on the size of the survey. Section V is devoted to the computation of the lowest order measured cumulants. In particular, we will show that a nonvanishing skewness appears that is directly induced by the finite volume effects as stressed in our concluding remarks, Sec. VI.

II. AN OVERVIEW OF FINITE VOLUME EFFECTS

Basically, what we want to measure are the statistical properties of a scalar field χ of zero mean

$$\langle\chi\rangle=0,$$

where and from now on angular brackets $\langle\ \rangle$ refer specifically to ensemble averages. Such a scalar field will be identified in the following with the gravitational potential, but the actual observations may be complex linear transforms of that field, such as temperature anisotropies or polarization of the CMB or large-scale cosmic density fluctuations. The general questions raised by precision measurements of statistical properties of cosmic fields can be very intricate. We refer, for instance, to [18, Sec. VI] for extensive developments regarding such issues. We will restrict our discussion to their overall aspects.

In practice, while performing such measurements, two scales enter the problem, the scale $R_{\rm S}$ at which χ is measured (e.g., somehow smoothed) and the size $R_{\rm H}$ of the survey. The smoothed field $\chi_{\rm S}$ can be measured at different locations in the survey. The idea is that its spatial fluctuations should provide us with hints of the actual statistical properties of the field. Ensemble averages, however, are inaccessible as such. The values of $\chi_{\rm S}$ we have access to are in finite number and are all inherited from a single stochastic process, that of our universe. One can, however, get insights into the stochastic properties of χ from geometrical averages. In the following, we denote such a geometrical average $\overline{A}$ and its connected part $\overline{A}^c$. For instance, one can have access to quantities such as

$$X_n\equiv\overline{(\chi_{\rm S}-\overline{\chi})^n}^{\,c}=\overline{\delta\chi_{\rm S}^{\,n}}^{\,c}.$$

The X_n are themselves stochastic quantities that depend on the stochastic field χ. To be more specific, we have

$$\overline{\chi}=\int\chi(\mathbf{y},t)\mathcal{W}_{R_{\rm H}}(\mathbf{y})d^3\mathbf{y} \tag{1}$$

and

$$\chi_{\rm S}(\mathbf{x})=\int\chi(\mathbf{y},t)\mathcal{W}_{R_{\rm S}}(\mathbf{x}-\mathbf{y})d^3\mathbf{y}, \tag{2}$$

where $\mathcal{W}_R$ is a window function of volume unity and for simplicity assumed to be spherically symmetric, e.g., for a top hat filtering we have $\mathcal{W}_R=\Theta(|\mathbf{x}-\mathbf{y}|-R)/V$ where Θ is the Heaviside function and V the volume of the ball, or $\mathcal{W}_R=\exp(-|\mathbf{x}-\mathbf{y}|^2/(2R)^2)/(\sqrt{2\pi}R)^3$ for a Gaussian window function.

Clearly, quantities such as $\langle\chi^2\rangle$ cannot be estimated because they contain contributions from modes of wavelengths larger than the size of the survey and that cannot be observed. Only modes with wavelengths in a given range are measurable. This is the case for a Gaussian field in particular. In this case one can show that, if the two scales $R_{\rm S}$ and $R_{\rm H}$ are in large enough ratio, X_n can be good estimates of $\langle(\chi_{\rm S}-\overline{\chi})^n\rangle$, e.g., the expected dispersion of the measured values X_n decreases with increasing $R_{\rm H}/R_{\rm S}$ to a power that depends on the power spectrum shape. In our case, however, not only can the super-Hubble modes not be measured but the modes that can be observed are correlated and also correlated to the super-Hubble modes.

In the following, we will explore the consequences of both the facts that only modes in specific wavelengths are measurable and that they are correlated. Ideally, cosmological models should be able to predict the PDFs of the measured cumulants,

$$\mathcal{P}(X_n),$$

but such predictions are obviously difficult to do in general. We will see in the following how we can estimate these distributions for some families of multifield inflationary models.

III. MODELS

A. Self-interacting scalar fields in de Sitter space

Before we start to investigate the statistics of the observed quantities let us recall the basics of the models we have in mind. In the models [10,17], the non-Gaussianities are first generated by an auxiliary field χ self-interacting in a potential, typically quartic,

$$V(\chi)=\frac{\lambda}{4!}\chi^4. \tag{3}$$

This field is assumed to be a test field so that it does not affect the dynamics of inflation driven by another scalar field ϕ. Assuming an almost de Sitter inflation and neglecting the gravitational back reaction on the evolution of the universe, its evolution will be dictated by

$$\ddot{\chi}+3H\dot{\chi}-\frac{1}{a^2}\Delta\chi=-\frac{\lambda}{3!}\chi^3 \tag{4}$$

where $H=\dot{a}/a$ is the Hubble constant and Δ the comoving Laplacian. The evolution of the scale factor is given by

$$a(t)=a_0e^{Ht},\quad \eta=-\frac{e^{-Ht}}{H},\quad a(\eta)=-\frac{1}{H\eta}, \tag{5}$$

where t is the cosmic time and η, which is negative, the conformal time.

For a free massless scalar field ($\lambda=0$), the Klein-Gordon equation (4) in the de Sitter background takes the form

$$v''+\left(k^2-\frac{2}{\eta^2}\right)v=0, \tag{6}$$

where we have introduced $v\equiv\chi/a$. The general solution of this equation is

$$\chi_k(t)=-\frac{H\eta}{\sqrt{2k}}\left(1+\frac{1}{ik\eta}\right)e^{-ik\eta}, \tag{7}$$

once the quantum field χ is decomposed in Fourier modes as[1]

$$\chi(\mathbf{x},t)=\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}[\chi_k(t)e^{i\mathbf{k}\cdot\mathbf{x}}\hat{b}_{\mathbf{k}}+\mathrm{H.c.}], \tag{8}$$

where $[\hat{b}_{\mathbf{k}},\hat{b}^{\dagger}_{\mathbf{k}'}]=(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')$.

This implies that the field χ has a correlator given by

$$\langle\chi(\mathbf{k})\chi^*(\mathbf{k}')\rangle=(2\pi)^3P(k)\delta(\mathbf{k}-\mathbf{k}'), \tag{9}$$

where $P(k)$ is the power spectrum, which is equal to $P(k)=H^2/2k^3$ on super-Hubble scales for a free scalar field living in de Sitter space. This is the so-called Harrison-Zel'dovich spectrum.

Moreover, the self-interaction term of χ induces nonzero high order correlation functions. For a quartic potential the odd order correlation functions vanish; the even order ones can be computed from a perturbation theory approach at the tree order. For instance, the four-point function in the super-Hubble limit reads

$$\langle\chi_{\mathbf{k}_1}\ldots\chi_{\mathbf{k}_4}\rangle_c=-\frac{\lambda}{3}(2\pi)^6\frac{\log\left(\eta\sum_i k_i\right)}{H^2}\delta\left(\sum_i \mathbf{k}_i\right)\times[P(k_1)P(k_2)P(k_3)+\text{permutations}], \tag{10}$$

as explicitly shown in Ref. [26].

B. Second and fourth moments

When one wants to relate the expectation values of observable quantities such as the X_n to the statistical properties of the field, filtering effects should properly be taken into account. The observable quantities $\delta\chi_{\mathrm{S}}$ can be written in terms of the Fourier modes $\chi_{\mathbf{k}}$ as

$$\delta\chi_{\mathrm{S}}=\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}e^{i\mathbf{k}\cdot\mathbf{x}}\chi(\mathbf{k})\tilde{W}(k) \tag{11}$$

with

$$\tilde{W}(k)\equiv\hat{W}(kR_{\mathrm{S}})-\hat{W}(kR_{\mathrm{H}}). \tag{12}$$

It follows immediately that

$$\langle\delta\chi_{\mathrm{S}}^2\rangle=4\pi\int k^2P(k)\tilde{W}^2(k)\mathrm{d}k. \tag{13}$$

Such an expression can be easily computed in the case of a Harrison-Zel'dovich spectrum and a k-space top-hat window function. Defining k_{S} and k_{H}, respectively, as $1/R_{\mathrm{S}}$ and $1/R_{\mathrm{H}}$, the second moment of $\delta\chi_{\mathrm{S}}$ reduces to

$$\langle\delta\chi_{\mathrm{S}}^2\rangle=2\pi H^2\ln\left(\frac{k_{\mathrm{S}}}{k_{\mathrm{H}}}\right)\equiv\sigma_\delta^2. \tag{14}$$

It follows from this calculation that the expectation value of the observable quantity X_2 is given by

$$\langle X_2\rangle=\langle\delta\chi_{\mathrm{S}}^2\rangle=\sigma_\delta^2. \tag{15}$$

Its (cosmic) variance $\langle X_2^2\rangle_c^{1/2}$ can similarly be computed and it scales like $R_{\mathrm{S}}/R_{\mathrm{H}}$ in case of a Gaussian field.[2]

Furthermore, the expectation value of X_4 is given by

$$\langle X_4\rangle=\langle\delta\chi_{\mathrm{S}}^4\rangle_c=\int\frac{\mathrm{d}^3\mathbf{k}_1\cdots\mathrm{d}^3\mathbf{k}_4}{(2\pi)^6}\tilde{W}(k_1)\cdots\tilde{W}(k_4)\langle\chi_{\mathbf{k}_1}\cdots\chi_{\mathbf{k}_4}\rangle_c. \tag{16}$$

Expressing the four-point correlator by mean of Eq. (10) and assuming a Harrison-Zel'dovich spectrum and a top-hat window function in k space, this expression finally reads

$$\langle\delta\chi_{\mathrm{S}}^4\rangle_c=-\frac{4\lambda}{3H^2}\left(\frac{H^2}{2}\right)^3\int_{k_{\mathrm{H}}}^{k_{\mathrm{S}}}\frac{\mathrm{d}^3\mathbf{k}_1}{k_1^3}\int_{k_{\mathrm{H}}}^{k_{\mathrm{S}}}\frac{\mathrm{d}^3\mathbf{k}_2}{k_2^3}\int_{k_{\mathrm{H}}}^{k_{\mathrm{S}}}\frac{\mathrm{d}^3\mathbf{k}_3}{k_3^3}\times\tilde{W}(|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|)\times\log[(k_1+k_2+k_3+|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|)\eta], \tag{17}$$

where $\tilde{W}(|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|)$, defined in Eq. (12), simply expresses the condition $k_{\mathrm{H}}<|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|<k_{\mathrm{S}}$ and is inherited for the term $\tilde{W}(k_4)$. When the ratio $k_{\mathrm{S}}/k_{\mathrm{H}}$ is large this expression can be easily computed after noting that, in this case, one of the k_i generically dominates the others (the contributions to the integral being roughly uniform over the integration domain when the wave-vector norms are logarithmically spaced). In this limit it is therefore possible to approximate $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3$ by, say, $\mathbf{k}_1$ and $k_1+k_2+k_3+|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|$ by $2k_1$. Finally, the integral reads

[1]Note that the convention of Eq. (8) differs by a factor of $(2\pi)^{3/2}$ compared to the one of Ref. [26]. It implies that Eq. (10) differs by a factor of $(2\pi)^6$ compared to its original form.

[2]The cosmic variance for the measured X_2 is expected to be strongly affected by the existence of a nonvanishing large-scale four-point function for χ. But it is beyond the scope of this paper to investigate such effects.

$$\langle \delta\chi_{\rm S}^4\rangle_c \simeq -\frac{4}{3}\lambda\,\log(2\,\eta k_{\rm S}^{3/4}k_{\rm H}^{1/4})\,\frac{\sigma_\delta^6}{H^2}, \tag{18}$$

which corresponds to what was found in our previous work provided the number of e-folds is identified with $\log(2\eta k_{\rm S}^{3/4}k_{\rm H}^{1/4})$. This means that the kurtosis of the $\delta\chi_{\rm S}$ field is significant if $\lambda\,\log(2\eta k_{\rm S}^{3/4}k_{\rm H}^{1/4})$ approaches unity. This result demonstrates that the mechanism described in our previous study survives finite volume effects. We can also note that observable quantities are insensitive to the behavior of $P(k)$ in the small-k limit.

Finite volume effects have, however, other consequences due to the fact that the observable modes are correlated. The physical reason for these correlations is that they share the same history, e.g., they have been produced in the same stochastic process. We found that a random walk approach for the evolution of the local χ field values gives precious insights into those more subtle finite volume effects.

IV. LESSONS FROM A RANDOM WALK APPROACH

From the previous definitions, it is clear that $\bar{\chi}$ and the different values of $\chi_{\rm S}$ share the contributions of super-Hubble modes. Those cannot be observed but they shape the values of $\bar{\chi}$ and $\chi_{\rm S}$. Actually, the value of $\chi_{\rm R}$ at a given scale, whether it is $R_{\rm H}$ or $R_{\rm S}$, is dynamically built from the stacking of modes that successively leave the horizon.

A random walk approach can then be used to describe the stochastic growth of $\chi_{\rm R}$. It will allow two things, to get insights into the excursion values of $\bar{\chi}$ and to see how $\chi_{\rm S}$ values are correlated through their common history. In this approach the field value evolution is described in term of a Langevin equation during inflation.

Before we go to this equation, let us sort out how the different scales and the evolution equation are related together.

A. Scales intervening in the problem

Different scales will have to be distinguished in our study.

(1) During the inflationary stage, the super-Hubble modes of the scalar field can be treated as classical. In a de Sitter spacetime, the physical Hubble radius is constant, $R^{(\rm phys)}=H^{-1}$, so that the comoving smoothing scale is time dependent, $R=(aH)^{-1}$. The evolution equation of the classical part will thus contain a stochastic force simulating the effect of the quantum noise due to the modes that are crossing the horizon at each time step to become classical. We thus define a stochastic field $\chi_{\rm H}(t)$, for which the filtering scale is dependent on and always equal to the horizon size, $R_{\rm H}^{\rm phys}=a(t)R_{\rm H}$ with $R_{\rm H}^{\rm phys}\sim H^{-1}$ so that

$$\chi_{\rm H}=\chi_{R=1/aH}. \tag{19}$$

The dynamics of $\chi_{\rm H}$ will follow a Langevin equation. We investigate the dynamics and the statistical properties of $\chi_{\rm H}$ in Sec. IV B.

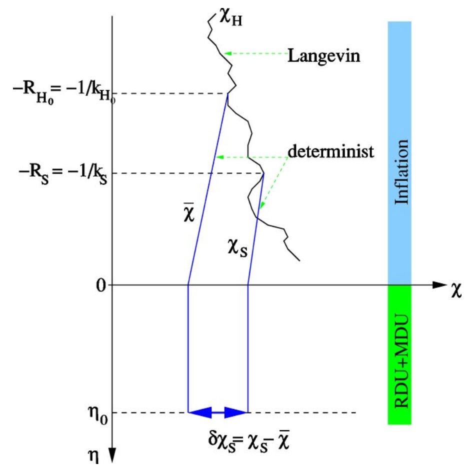

FIG. 1. The different filtered quantities and scales entering our problem. $\chi_{\rm H}$ follows a stochastic dynamics that can be described by a Langevin equation; it has a time-dependent smoothing scale so that more and more modes contribute to the filtered field. The field values $\bar{\chi}$ and $\chi_{\rm S}$ evolve according to a classical Klein-Gordon equation; their smoothing scale is time independent; they coincide with $\chi_{\rm H}$ at horizon crossing times.

(2) On the other hand, the field values $\bar{\chi}$ and $\chi_{\rm S}$ can be seen as time-dependent quantities but they correspond to the filtering of χ at fixed physical scales. They identify, though, with $\chi_{\rm H}$ at precisely the time η at which the scales $R_{\rm H}$ and $R_{\rm S}$, respectively, cross the horizon, e.g., $\bar{\chi}(\eta=-1/R_{\rm H_0})=\chi_{\rm H}(\eta=-1/R_{\rm H_0})$ and $\chi_{\rm S}(\eta=-1/R_{\rm S})=\chi_{\rm H}(\eta=-1/R_{\rm S})$. After these coincidental times the two fields $\bar{\chi}$ and $\chi_{\rm S}$ behave classically, i.e., they follow an inflationary classical Klein-Gordon equation without stochastic source terms.

To summarize, two of the comoving scales $R_{\rm S}$ and $R_{\rm H_0}$ are fixed while one $R_{\rm H}$ is a time-dependent quantity. A sketch of the different sequences that the field dynamics follow is shown on Fig. 1.

B. The late time PDF of $\chi_{\rm H}(\eta)$

We follow a formalism first developed to deal with self-interacting fields in a de Sitter background [19], based on the idea that the infrared part of the scalar field may be treated as a classical spacetime-dependent stochastic field satisfying a Langevin equation [20,21] (see Ref. [22] for the case of a massless free field).

Assuming that χ is slow rolling, the dynamics of $\chi_{\rm H}$ can be obtained by averaging Eq. (4), in which both the second time derivative and the Laplacian term can be neglected, since $\chi_{\rm H}$ contains only long wavelength modes. Using the identity

$$\dot{\chi}_{{\rm H}\,\mathbf{k}}=(\chi_{\mathbf{k}})_{R_{\rm H}}\hat{W}(kR_{\rm H})-kR_{\rm H}\chi_{\mathbf{k}}\hat{W}'(kR_{\rm H}) \tag{20}$$

where $\hat{W}'(u)=d\hat{W}(u)/du$ and where we have used Eq. (5) to express the time derivative of $R_{\rm H}$. It follows that

$$\dot{\chi}_{\rm H}=-\frac{1}{3H}\overline{V'}(\chi_{\rm H}+\delta\chi)+\xi_{\rm Q}(\mathbf{x},t) \tag{21}$$

which reduces ($\delta\chi \ll \chi_{\rm H}$) to

$$\dot{\chi}_{\rm H} = -\frac{1}{3H}\frac{{\rm d}V(\chi_{\rm H})}{{\rm d}\chi_{\rm H}} + \xi_{\rm Q}(\mathbf{x},t). \tag{22}$$

The term $\xi_{\rm Q}(\mathbf{x},t)$ appears from the commutation between $(\dot{\chi})_{R_{\rm H}}$ and $\dot{\chi}_{\rm H}$ using Eq. (20). It is a stochastic noise describing the effects of the small-wavelength (quantum) part exiting the horizon on the classical stochastic part. Using Eq. (8), one obtains its expression as

$$\xi_{\rm Q}(\mathbf{x},t) = -\int \frac{{\rm d}^3\mathbf{k}}{(2\pi)^{3/2}}(kHR_{\rm H})[\chi_k(t)\hat{W}'(kR_{\rm H})e^{i\mathbf{k}\cdot\mathbf{x}}\hat{b}_{\mathbf{k}} + {\rm H.c.}]. \tag{23}$$

Indeed, $\xi_{\rm Q}$ is quantum noise so that we replace it heuristically by a Gaussian stochastic noise ξ with a correlator that matches the quantum expectation value in the standard Bunch-Davies vacuum, that is,

$$\langle \xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle = \langle 0|\xi_{\rm Q}(\mathbf{x},t)\xi_{\rm Q}(\mathbf{x}',t')|0\rangle. \tag{24}$$

It follows that

$$\langle \xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle = \int \frac{k^3{\rm d}k}{4\pi^2}\frac{\sin kr}{r}\frac{\chi_k(t)\chi_k^*(t')}{a(t)a(t')}\hat{W}'\left(\frac{k}{a(t)H}\right) \times \hat{W}'\left(\frac{k}{a(t')H}\right), \tag{25}$$

with $r \equiv |\mathbf{x}-\mathbf{x}'|$, which reduces, using the expression (7) on small scales, to [23]

$$\langle \xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle = \frac{H^4\eta\eta'}{4\pi^2 r}\int {\rm d}k \sin kr(1+ik\eta) \times(1-ik\eta')e^{ik(\eta'-\eta)}\hat{W}'(-k\eta)\hat{W}' \times(-k\eta'). \tag{26}$$

Even though $\chi_{\rm H}$ remains a quantum operator, it was replaced by a stochastic field in such a way that, for all observables, the expectation values of the two fields are in excellent agreement.

In the following and for the sake of simplicity, we consider a window function reducing to a top hat in Fourier space so that $\hat{W}'$ reduces to a Dirac distribution. In that case, using the solution (7) one obtains

$$\langle \xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle = \frac{H^3}{4\pi^2}\frac{\sin a(t)Hr}{a(t)Hr}\delta(t-t'). \tag{27}$$

The case of more realistic window functions (such as a Gaussian) was discussed in Ref. [23].

From a Langevin equation of the form

$$\dot{\chi}_{\rm H} = -\beta V' + \alpha\xi, \quad \langle \xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle = \delta(t-t'), \tag{28}$$

where we assume the two coefficients β and α to be independent of $\chi_{\rm H}$, one can deduce an equation [24], the Fokker-Planck equation, for the PDF of $\chi_{\rm H}$ of the form

$$\partial_t\mathcal{P} = \beta\partial^2_{\chi_{\rm H}}\mathcal{P} + \frac{\alpha}{2}\partial_{\chi_{\rm H}}[V'\mathcal{P}]. \tag{29}$$

It follows from the Langevin equation (22) with a top-hat window function for which the noise is given by Eq. (27) with $\beta = -1/3H$ and $\alpha = H^{3/2}/2\pi$ that the one-point PDF $\mathcal{P}(\chi_{\rm H},t)$ is a solution of

$$\partial_t\mathcal{P} = \frac{H^3}{8\pi^2}\partial^2_{\chi_{\rm H}}\mathcal{P} + \frac{1}{3H}\partial_{\chi_{\rm H}}(V'\mathcal{P}). \tag{30}$$

In a cosmological context, this equation was first derived in Ref. [19] in the case $H = $const, which we are interested in, and then in Ref. [22] in the case $V' = 0$. It was generalized to more involved situations in Ref. [25].

The solution of Eq. (30) was studied in Ref. [21] in which it is shown that $\mathcal{P}$ approaches the static equilibrium solution

$$\mathcal{P}_{\rm eq} = \mathcal{N}^{-1}\exp\left(-\frac{8\pi^2}{3H^4}V(\chi_{\rm H})\right),$$

$$\mathcal{N} \equiv \int_{-\infty}^{\infty}\exp\left(-\frac{8\pi^2}{3H^4}V(\chi_{\rm H})\right){\rm d}\chi_{\rm H}, \tag{31}$$

irrespective of the initial conditions.

For the quartic potential Eq. (3) we find

$$\mathcal{P}_{\rm eq}(\chi_{\rm H}) = \frac{1}{2\Gamma(5/4)H}\left(\frac{\pi^2\lambda}{9}\right)^{1/4}\exp\left[-\frac{\pi^2\lambda}{9H^4}\chi_{\rm H}^4\right]. \tag{32}$$

There are few lessons to learn from this result. The excursion values of $\chi_{\rm H}$ are bounded. Their distribution does not depend on the remote past history of χ and, importantly for the following, the typical value one can expect for $\chi_{\rm H}$, and therefore $\overline{\chi}$, is $H/\lambda^{1/4}$.

C. Consequences for the shape of the PDF of $\delta\chi_{\rm S}$

We can then gain insights into the shape of the probability distribution function of $\delta\chi_{\rm S}$ as a function of $\overline{\chi}$.

Quantitative results can be drawn from the perturbation approach we initially developed in our previous work [10]. We expand the filtered field in terms of the coupling constant as

$$\chi_{\rm S}(\eta) = \chi_{\rm S}^{(0)}(\eta) + \chi_{\rm S}^{(1)}(\eta) + \cdots. \tag{33}$$

$\chi_{\rm S}^{(0)}$ represents the value of the filtered field when the interaction term is switched off. In the slow-roll regime the field $\chi_{\rm S}$ follows the Klein-Gordon equation (22). Contrary to the previously studied case of $\chi_{\rm H}$, the evolution equation does not contain any noise term because the smoothing scale R is now fixed and there are no new modes entering $\chi_{\rm S}$. The free filtered field $\chi_{\rm S}^{(0)}$ is therefore constant and, from the discussion of the previous section, it cannot be assumed to be Gaussian distributed with a zero mean: its expectation value $\overline{\chi_S^{(0)}}$ is actually the value of $\overline{\chi}$ at time $\eta = 1/R_H$. Since this field value is going to be only weakly affected by its subsequent evolution, in the following we will identify $\overline{\chi_S^{(0)}}$ and $\overline{\chi}$. The difference $\delta\chi_S^{(0)} \equiv \chi_S^{(0)} - \overline{\chi_S^{(0)}}$ has been built up from the modes that have left the horizon between $-1/k_H$ and $-1/k_S$. It can be assumed to be Gaussian distributed with a width precisely given by σ_δ.

The first order term in λ, $\chi_S{}^{(1)}$, evolves according to

$$3H\dot{\chi}_S{}^{(1)} = -\frac{\lambda}{3!}[\chi_S^{(0)}]^3. \tag{34}$$

In this approach the treatment of the filtering of the right-hand side of this equation is very crude. The results we are going to find can in any case be checked against more rigorous calculations based on the computed shape of the trispectrum. Our goal now is to capture the essential effects of a nonvanishing $\overline{\chi}$ on the statistical properties of $\delta\chi_S$.

The equation of evolution (34) can be solved to get

$$\chi_S^{(1)}(t) = -\lambda(t - t_H)\frac{(\chi_S^{(0)})^3}{18H}, \tag{35}$$

which also reads

$$\chi_S^{(1)}(t) = -\frac{\lambda N_e}{18H^2}(\chi_S^{(0)})^3, \tag{36}$$

N_e being the number of e-folds between t_H and the end of inflation.

These results imply that

$$\overline{\chi} \approx \overline{\chi_S^{(0)}} - \frac{\lambda N_e}{18H^2}[(\overline{\chi_S^{(0)}})^3 + 3\overline{\chi_S^{(0)}}\sigma_\delta^2], \tag{37}$$

which explicitly shows that $\overline{\chi}$ and $\overline{\chi_S^{(0)}}$ are equal at leading order in λ. It also gives

$$\delta\chi_S = \delta\chi_S^{(0)} - \frac{\lambda N_e}{18H^2}\{(\delta\chi_S^{(0)})^3 + 3([\delta\chi_S^{(0)}]^2 - \sigma_\delta^2)\overline{\chi} + 3\,\delta\chi_S^{(0)}\overline{\chi}^2\}. \tag{38}$$

It is straightforward to see that $\delta\chi_S$ has acquired a nonzero third order moment

$$\overline{\delta\chi_S^3} = -\frac{\lambda N_e}{H^2}\overline{\chi}\sigma_\delta^4 \tag{39}$$

at leading order in λ. This is a finite volume effect in the sense that it exists for a fixed (not ensemble averaged) value of $\overline{\chi}$. This effect cannot be neglected *a priori*. From the study of the previous paragraph, we know that $\overline{\chi}$ should be of the order of $H/\lambda^{1/4}$, which implies that the reduced skewness of $\delta\chi_S$, $\overline{\delta\chi_S}^3/(\overline{\delta\chi_S}^2)^{3/2} \sim \lambda^{3/4} N_e \sigma_\delta / H$, is significant as soon as $\lambda^{3/4} N_e$ approaches unity, a condition similar to that encountered in Sec. III B.

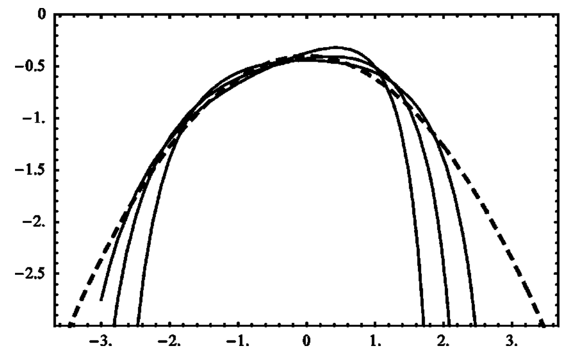

FIG. 2. PDF of $\delta\chi_S = \delta\chi_S - \overline{\chi}$ for different values of $\overline{\chi}$. The dashed line corresponds to a Gaussian distribution; the dot-dashed line to the deformed distribution of $\delta\chi_S$ when $\lambda N_e / H^2 = 1$ and $\overline{\chi} = 0$, and the solid lines to the deformed distribution when $\overline{\chi}$ equals 0.5 and 1.

Actually the evolution equation for χ_S can be solved by

$$\chi_S = \frac{\chi_S^{(0)}}{\sqrt{1 - (\lambda N_e / 9H^2)(\chi_S^{(0)})^2}} \tag{40}$$

and the distribution of χ_S can then be inferred from that of $\chi_S^{(0)}$, assuming that the latter is Gaussian distributed with a nonzero mean value.[3]

In Fig. 2, we present the deformation of the PDF of $\delta\chi_S$ while $\overline{\chi}$ is varied. As expected, it shows that when $\overline{\chi}$ is not zero, the PDF gets skewed in a way that can be easily understood: when $\overline{\chi}$ is positive it gets more difficult to have excursions toward larger values of χ_S, but easier to roll down to smaller values. It is as if the field χ was actually evolving in the potential $\lambda(\chi + \overline{\chi})/4!$. As a result one naturally expects the field χ to have a nonvanishing three-point function. As mentioned before, these calculations treat the smoothing in a rather simplified way but we think that for illustrative purposes it encapsulates the main effects that we want to describe.

It nonetheless shows the way for the computation of the finite volume effects on the expected stochastic properties of the field.

V. FINITE VOLUME EFFECTS ON HIGH ORDER CUMULANTS

What the Langevin picture suggests is that finite volume effects on the observed quantities are not due to the whole

[3] As noted in Ref. [10], such a simple variable change implicitly incorporates ''loop order'' effects that, because of sub-Hubble physics, are not necessarily correctly estimated. In that paper we developed a more elaborate method which allows the reconstruction of the PDF from only the tree order contributions of each cumulant. As the two approaches eventually give the same qualitative results, we here restrict our analysis to the simplest method.

stochastic process that created the observed field but mainly to the value of $\overline{\chi}$ alone. In other words, we expect to have

$$\mathcal{P}(X_n)=\int \mathcal{P}(\overline{\chi})\,\delta(X_n-\langle(\chi_{\rm S}-\overline{\chi})^n\rangle_{\overline{\chi}})\mathrm{d}\overline{\chi}. \tag{41}$$

This form (41) implies in particular that the PDF of X_n, $\mathcal{P}(X_n)$, is expected to be peaked around the expectation value of $\delta\chi_{\rm S}^n$ at $\overline{\chi}$ fixed. In particular, it implies that

$$\langle X_n^p\rangle \simeq \langle(\langle\delta\chi_{\rm S}^n\rangle_{\overline{\chi}})^p\rangle. \tag{42}$$

We will explicitly check this property for the lower order cumulants.

In general, for small enough values of $\overline{\chi}$, the constrained ensemble averages of the form $\langle A(\chi)\rangle_{\overline{\chi}}$ should be given by

$$\langle A(\chi)\rangle_{\overline{\chi}} \simeq \langle A(\chi)\overline{\chi}\rangle\frac{\overline{\chi}}{\sigma^2_{\overline{\chi}}}, \tag{43}$$

where $\sigma^2_{\overline{\chi}}$ is the variance square of the $\overline{\chi}$ fluctuations. This relation, exact for Gaussian fields, is only approximate in general. It can be derived for stochastic variables following a quasi-Gaussian distribution. Here, it will be valid only if the excursion values of $\overline{\chi}$ are modest compared to the fluctuations of $\delta\chi_{\rm S}$.

Not surprisingly, it implies that the even order cumulants are left unchanged.

A. The bispectrum

Nontrivial finite volume effects are then going to appear at the level of the third order cumulants or correlation functions. In particular, it induces a nonvanishing bispectrum, a three-point correlation function of the wave vectors, which is going to be given by

$$\langle\chi(\mathbf{k}_1)\chi(\mathbf{k}_2)\chi(\mathbf{k}_3)\rangle_{\overline{\chi}} \simeq \langle\chi(\mathbf{k}_1)\chi(\mathbf{k}_2)\chi(\mathbf{k}_3)\overline{\chi}\rangle_c\frac{\overline{\chi}}{\sigma^2_{\overline{\chi}}} \tag{44}$$

when $\overline{\chi}$ is small enough. Using Eq. (10) to express $\langle\chi_{\mathbf{k}_1}\cdots\chi_{\mathbf{k}_4}\rangle_c$ and using that Eq. (2) implies that $\overline{\chi_{\mathbf{k}}}=\chi_{\mathbf{k}}\hat{W}(kR_{\rm H})$, the previous expression reduces to

$$\begin{aligned}\langle\chi(\mathbf{k}_1)\chi(\mathbf{k}_2)\chi(\mathbf{k}_3)\rangle_{\overline{\chi}} &= -\frac{\lambda}{3H^2}(2\pi)^{9/2}\frac{\overline{\chi}}{\sigma^2_{\overline{\chi}}}\log\left[\left(\sum_i k_i+\left|\sum_i \mathbf{k}_i\right|\right)\eta\right]\\ &\times\Bigg\{P(k_1)P(k_2)P(k_3)+P\left(\left|\sum_i \mathbf{k}_i\right|\right)\\ &\times[P(k_1)P(k_2)+\text{permutations}]\Bigg\}\hat{W}\left(\left|\sum_i \mathbf{k}_i\right|R_{\rm H}\right).\end{aligned} \tag{45}$$

The factor $\hat{W}(|\mathbf{k}_1+\cdots+\mathbf{k}_3|R_{\rm H})$, which arises from the contribution of modes with $k<1/R_{\rm H}$ to $\overline{\chi}$, ensures that $|\mathbf{k}_1+\cdots+\mathbf{k}_3|$ is small compared to each of the k_i, and thus can be neglected in the log term. It implies that for a Harrison-Zel'dovich type spectrum $P(|\mathbf{k}_1+\cdots+\mathbf{k}_3|)\gg P(k_i)$ so that the first term of Eq. (45) is negligible. As a result we deduce that

$$\begin{aligned}\langle\chi(\mathbf{k}_1)\chi(\mathbf{k}_2)\chi(\mathbf{k}_3)\rangle_{\overline{\chi}} &\simeq -\frac{\lambda\overline{\chi}}{3H^2}(2\pi)^{9/2}\log\left[\left(\sum_i k_i\right)\eta\right][P(k_1)P(k_2)\\ &+\text{permutations}]\frac{P\left(\left|\sum_i \mathbf{k}_i\right|\right)\hat{W}\left(\left|\sum_i \mathbf{k}_i\right|R_{\rm H}\right)}{\sigma^2_{\overline{\chi}}}.\end{aligned} \tag{46}$$

Now, noting that, by definition, $P(k)\hat{W}(kR_{\rm H})\mathrm{d}^3\mathbf{k}/\sigma^2_{\overline{\chi}}$ integrates to unity and that the function $P(|\Sigma_i\mathbf{k}_i|)\hat{W}(|\Sigma_i\mathbf{k}_i|R_{\rm H})$ is, for the modes we are interested in, peaked near the origin, we obtain that this factor is essentially equal to $\delta(\Sigma_i\mathbf{k}_i)$. It therefore implies that

$$\begin{aligned}\langle\chi(\mathbf{k}_1)\chi(\mathbf{k}_2)\chi(\mathbf{k}_3)\rangle_{\overline{\chi}} &= -\frac{\lambda\overline{\chi}}{3H^2}(2\pi)^{9/2}\log\left[\left(\sum_i k_i\right)\eta\right][P(k_1)P(k_2)\\ &+\text{permuations}]\,\delta\left(\sum_i \mathbf{k}_i\right).\end{aligned} \tag{47}$$

Here is one of the main points of this paper: Finite volume effects induce a nonvanishing three-point function although the potential in which χ evolves is symmetric.

From this bispectrum it is possible to compute the third order moment of $\delta\chi_{\rm S}$. Its amplitude will be in agreement with what was obtained from the Langevin equation. It is also the three-point function one expects for a field evolving in the potential $\lambda\overline{\chi}\chi/3!$ (see Ref. [26] for details).

B. The three-point cumulant

We now turn to the lowest order cumulant exhibiting a nontrivial result due to a nonvanishing $\overline{\chi}$, that is, X_3, which reads

$$\begin{aligned}X_3=\int\frac{\mathrm{d}^3\mathbf{k}_1\cdots\mathrm{d}^3\mathbf{k}_3}{(2\pi)^{9/2}}&\hat{W}(|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|R_{\rm H})\\ &\times\tilde{W}(k_1)\tilde{W}(k_2)\tilde{W}(k_3)\chi_{\mathbf{k}_1}\chi_{\mathbf{k}_2}\chi_{\mathbf{k}_3}.\end{aligned} \tag{48}$$

Obviously its ensemble average vanishes,

$$\langle X_3\rangle=0,$$

but not $\langle X_3^2\rangle_c$. Let us check, as expected from our analysis [see Eqs. (41), (42)], that it is well approximated by $\langle(\langle X_3\rangle_{\overline{\chi}})^2\rangle$.

To evaluate the latter expression, we start from the expression of $\langle X_3\rangle_{\bar\chi}$ which is defined by

$$\langle X_3\rangle_{\bar\chi}\simeq\frac{\bar\chi}{\sigma_{\bar\chi}^2}\int\frac{\mathrm{d}^3\mathbf{k}_1\cdots\mathrm{d}^3\mathbf{k}_3}{(2\pi)^{9/2}}\hat W(|\mathbf{k}_1+\cdots+\mathbf{k}_3|R_{\rm H})\times\widetilde W(k_1)\cdots\widetilde W(k_3)\langle\chi_{\mathbf{k}_1}\cdots\chi_{\mathbf{k}_3}\rangle_{\bar\chi}. \quad (49)$$

Using Eq. (47), it reduces after integration over $\mathbf{k}_3$ to

$$\langle X_3\rangle_{\bar\chi}\simeq-\frac{\lambda\bar\chi}{3H^2}\int \mathrm{d}^3\mathbf{k}_1\mathrm{d}^3\mathbf{k}_2\widetilde W(k_1)\widetilde W(k_2)\widetilde W(|\mathbf{k}_1+\mathbf{k}_2|)\times\{P(k_1)P(k_2)+P(|\mathbf{k}_1+\mathbf{k}_2|)[P(k_1)+P(k_2)]\}, \quad (50)$$

where the window function $\hat W(|\mathbf{k}_1+\cdots+\mathbf{k}_3|R_{\rm H})$ was aborbed during the integration over $\mathbf{k}_3$ due to the Dirac distribution. This expression can be computed following the same lines as for the computation of the expectation value of the fourth order cumulant (17),

$$\langle X_3\rangle_{\bar\chi}\simeq-\frac{\lambda\bar\chi}{3H^2}\left(\frac{H^2}{2}\right)^2\int_{k_{\rm H}}^{k_{\rm S}}\frac{\mathrm{d}^3\mathbf{k}_1}{k_1^3}\int_{k_{\rm H}}^{k_{\rm S}}\frac{\mathrm{d}^3\mathbf{k}_2}{k_2^3}3\widetilde W(|\mathbf{k}_1+\mathbf{k}_2|)\times\log[(k_1+k_2+|\mathbf{k}_1+\mathbf{k}_2|)\eta], \quad (51)$$

where, again, $\widetilde W(|\mathbf{k}_1+\mathbf{k}_2|)$ simply expresses the conditions $k_{\rm H}<|\mathbf{k}_1+\mathbf{k}_2|<k_{\rm S}$. A simple expression for this integral can be obtained when $k_{\rm S}$ is much larger than $k_{\rm H}$, where it is possible to replace $\mathbf{k}_1+\mathbf{k}_2$ and $k_1+k_2+|\mathbf{k}_1+\mathbf{k}_2|$ by respectively either $\mathbf{k}_1$ and $2k_1$ or $\mathbf{k}_2$ and $2k_2$. Finally, the integral reads

$$\langle X_3\rangle_{\bar\chi}=-\lambda\bar\chi(4\pi)^2\left(\frac{H^2}{2}\right)^2\log^2\left(\frac{k_{\rm S}}{k_{\rm H}}\right)\log(2\eta k_{\rm S}^{2/3}k_{\rm H}^{1/3}), \quad (52)$$

e.g.,

$$\langle X_3\rangle_{\bar\chi}=-\lambda\bar\chi\log(2\eta k_{\rm S}^{2/3}k_{\rm H}^{1/3})\frac{\sigma_\delta^4}{H^2}, \quad (53)$$

which reproduces the result (39) if N_e is identified with $\log(2\eta k_{\rm S}^{2/3}k_{\rm H}^{1/3})$. In conclusion, we end up with

$$\langle(\langle X_3\rangle_{\bar\chi})^2\rangle^{1/2}\simeq\lambda\log(2\eta k_{\rm S}^{2/3}k_{\rm H}^{1/3})\frac{\sigma_\delta^4\sigma_{\bar\chi}}{H^2}. \quad (54)$$

In this case, it is actually possible to compute $\langle X_3^2\rangle$ from a perturbation theory approach in order to check that its dominant contribution is indeed $\langle(\langle X_3\rangle_{\bar\chi})^2\rangle$. $\langle X_3^2\rangle$ is given in general by

$$\langle X_3^2\rangle=\int\frac{\mathrm{d}^3\mathbf{k}_1\cdots\mathrm{d}^3\mathbf{k}_3}{(2\pi)^{9/2}}\int\frac{\mathrm{d}^3\mathbf{k}_1'\cdots\mathrm{d}^3\mathbf{k}_3'}{(2\pi)^{9/2}}\times\widetilde W(k_1)\cdots\widetilde W(k_3)\widetilde W(k_1')\cdots\widetilde W(k_3')\times\hat W(|\mathbf{k}_1+\cdots+\mathbf{k}_3|R_{\rm H})\hat W(|\mathbf{k}_1'+\cdots+\mathbf{k}_3'|R_{\rm H})\times\langle\chi_{\mathbf{k}_1}\cdots\chi_{\mathbf{k}_3}\chi_{\mathbf{k}_1'}\cdots\chi_{\mathbf{k}_3'}\rangle_c. \quad (55)$$

It involves the expression of the six-point correlation function $\langle\chi_{\mathbf{k}_1}\cdots\chi_{\mathbf{k}_3}\chi_{\mathbf{k}_1'}\cdots\chi_{\mathbf{k}_3'}\rangle_c$ which, in a perturbation theory approach, can be split into two contributions

$$\mathrm{I}\equiv\langle\chi_{\mathbf{k}_1}^{(0)}\chi_{\mathbf{k}_2}^{(0)}\chi_{\mathbf{k}_3}^{(1)}\chi_{\mathbf{k}_1'}^{(0)}\chi_{\mathbf{k}_2'}^{(0)}\chi_{\mathbf{k}_3'}^{(1)}\rangle,$$

$$\mathrm{II}\equiv\langle\chi_{\mathbf{k}_1}^{(0)}\chi_{\mathbf{k}_2}^{(0)}\chi_{\mathbf{k}_3}^{(2)}\chi_{\mathbf{k}_1'}^{(0)}\chi_{\mathbf{k}_2'}^{(0)}\chi_{\mathbf{k}_3'}^{(0)}\rangle,$$

where

$$\chi^{(1)}(\mathbf{x})\sim-\frac{\lambda}{18}\frac{N_e}{H^2}[\chi^{(0)}(\mathbf{x})]^3,\quad \chi^{(2)}(\mathbf{x})\sim\frac{\lambda^2}{8}\frac{N_e^2}{H^4}[\chi^{(0)}(\mathbf{x})]^5.$$

This implies that

$$I=(2\pi)^9\frac{\lambda^2}{18^2}18^2\frac{N_e^2}{H^4}P(k_1)P(k_2)P(\mathbf{k}_3+\mathbf{k}_1+\mathbf{k}_2|)P(k_2')P(k_3')\delta\left(\sum\mathbf{k}_i+\sum\mathbf{k}_i'\right) \quad (56)$$

and

$$II=(2\pi)^9\frac{\lambda^2}{8}6!\frac{N_e^2}{H^4}P(k_1)P(k_2)P(k_1')\times P(k_2')P(k_3')\delta\left(\sum\mathbf{k}_i+\sum\mathbf{k}_i'\right) \quad (57)$$

where 18^2 and 6! are symmetry factors.

Let us evaluate the first contribution:

$$\langle X_3^2\rangle_c^{(I)}=\lambda^2\frac{N_e^2}{H^4}\int\mathrm{d}^3\mathbf{k}_1\mathrm{d}^3\mathbf{k}_2P(k_1)P(k_2)\widetilde W(k_1)\widetilde W(k_2)\times\int\mathrm{d}^3\mathbf{k}_3P(|\mathbf{k}_3+\mathbf{k}_1+\mathbf{k}_2|)\widetilde W(k_3)\times\hat W^2(|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|R_{\rm H})\int\mathrm{d}^3\mathbf{k}_1'\mathrm{d}^3\mathbf{k}_2'P(k_1')P(k_2')\times\widetilde W(k_1')\widetilde W(k_2')\times\widetilde W(|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_1'+\mathbf{k}_2'|). \quad (58)$$

The term $\hat W^2(|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|R_{\rm H})$ implies that $|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|\ll R_{\rm H}$ so that $\widetilde W(k_3)\sim\widetilde W(|\mathbf{k}_1+\mathbf{k}_2|)$ and $\widetilde W(|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_1'+\mathbf{k}_2'|)\sim\widetilde W(|\mathbf{k}_1'+\mathbf{k}_2'|)$. Setting $\mathbf{e}=\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3$, we conclude that

$$\langle X_3^2\rangle_c^{(I)} \sim \lambda^2 \frac{N_e^2}{H^4}\left[\int d^3\mathbf{k}_1 d^3\mathbf{k}_2 P(k_1)P(k_2)\tilde{W}(k_1) \times \tilde{W}(k_2)\tilde{W}(|\mathbf{k}_1+\mathbf{k}_2|)\right]^2 \int d^3\mathbf{e} P(e)\hat{W}^2(eR_{\rm H}). \tag{59}$$

The integral over $\mathbf{e}$ reduces to $\sigma_{\overline{\chi}}^2$ so that

$$\langle X_3^2\rangle_c^{(I)} \sim \lambda^2 \frac{N_e^2}{H^4}\sigma_{\overline{\chi}}^2\left[\int d^3\mathbf{k}_1 d^3\mathbf{k}_2 P(k_1)P(k_2)\tilde{W}(k_1) \times \tilde{W}(k_2)\tilde{W}(|\mathbf{k}_1+\mathbf{k}_2|)\right]^2. \tag{60}$$

The second contribution reduces to

$$\langle X_3^2\rangle_c^{(II)} \sim 90\lambda^2 \frac{N_e^2}{H^4}\int d^3\mathbf{k}_1 d^3\mathbf{k}_2 P(k_1)P(k_2)\tilde{W}(k_1)\tilde{W}(k_2) \times \tilde{W}(|\mathbf{k}_1+\mathbf{k}_2|)\int d^3\mathbf{k}_1' d^3\mathbf{k}_2' d^3\mathbf{k}_3' P(k_1')P(k_2')P(k_3') \times \tilde{W}(k_1')\tilde{W}(k_2')\tilde{W}(k_3')\hat{W}^2(|\mathbf{k}_1'+\mathbf{k}_2'+\mathbf{k}_3'|R_{\rm H}). \tag{61}$$

The second integral reduces to

$$\int d^3\mathbf{k}_1' d^3\mathbf{k}_2' d^3 P(k_1')P(k_2')P(|\mathbf{k}_1'+\mathbf{k}_2'|)\tilde{W}(k_1') \times \tilde{W}(k_2')\tilde{W}(|\mathbf{k}_1'+\mathbf{k}_2'|)\int d^3\mathbf{e}\hat{W}^2(eR_{\rm H}).$$

The integral over $\mathbf{e}$ gives $\sim R_{\rm H}^{-3}$ so that

$$\langle X_3^2\rangle_c^{(II)} \sim 90\lambda^2 \frac{N_e^2}{H^4}R_{\rm H}^{-3}\int d^3\mathbf{k}_1 d^3\mathbf{k}_2 P(k_1)P(k_2)\tilde{W}(k_1) \times \tilde{W}(k_2)\tilde{W}(|\mathbf{k}_1+\mathbf{k}_2|)\int d^3\mathbf{k}_1' d^3\mathbf{k}_2' P(k_1')P(k_2') \times P(|\mathbf{k}_1'+\mathbf{k}_2'|)\tilde{W}(k_1')\tilde{W}(k_2')\tilde{W}(|\mathbf{k}_1'+\mathbf{k}_2'|). \tag{62}$$

Due to the term $\tilde{W}(|\mathbf{k}_1'+\mathbf{k}_2'|)$, we deduce that, in the case of a Harrison-Zel'dovich spectrum, $(R_{\rm S}/R_{\rm H})^3 < |\mathbf{k}_1' + \mathbf{k}_2'|^{-3}R_{\rm H}^{-3} < 1$ so that this contribution is at most equal to that of Eq. (60). As a result we have

$$\langle X_3^2\rangle \simeq \langle X_3^2\rangle_c^{(I)}. \tag{63}$$

From Eq. (60), this reduces to

$$\langle X_3^2\rangle^{1/2} \simeq \lambda N_e (2\pi)^3 \frac{\sigma_\delta^4 \sigma_{\overline{\chi}}}{H^2}, \tag{64}$$

which can be identified with the expectation value of $\langle \delta\chi_{\rm S}^3\rangle_{\overline{\chi}}^2$ over the distribution of $\overline{\chi}$, as obtained in Eq. (54).

This explicit computation shows that, as expected, the fluctuations of the measured values of X_3 are mainly due to the fluctuations of $\overline{\chi}$. It justifies, for instance, that one should expect to see a bispectrum of the form (47) for such inflationary models.

VI. CONCLUSIONS

In this article we have focused on the phenomenology of the non-Gaussianity generated in models developed in Refs. [10,17]. Interestingly, whereas the metric perturbation statistics involve only two microscopic parameters related, respectively, to the weight of the non-Gaussian component and to its PDF, the finite volume effects imply that the statistical properties of any observational quantity will involve a third parameter. This new parameter arises from the fact that the mean value $\overline{\chi}$ of the field over the size of the observable universe does not vanish *a priori*. Obviously $\overline{\chi}$ cannot be determined on the basis of any observation. As described in Sec. IV, this implies that the originally symmetric PDF can be skewed, and that this skewness is directly proportional to $\overline{\chi}$, which needs to be considered as a new parameter of the PDF while dealing with observations.

These results open the way for more detailed phenomenological studies. To see how those properties translate to the temperature and local density fields is not easy. Such an investigation will probably require numerical tools such as those developed in Ref. [27].

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