Entropy and universality of the Cardy-Verlinde formula in a dark energy universe

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We study the entropy of a FRW universe filled with dark energy (cosmological constant, quintessence, or phantom). For the general or time-dependent equation of state $p = w\rho$ the entropy is expressed in terms of energy, Casimir energy, and w. The corresponding expression is reminiscent of the 2D conformal field theory (CFT) entropy only for conformal matter. At the same time, the cosmological Cardy-Verlinde formula relating three typical FRW universe entropies remains universal for any type of matter. The same conclusions hold in modified gravity, which represents the gravitational alternative for dark energy and which contains terms that increase at low curvature. It is interesting that black holes in modified gravity are more entropic than those in Einstein gravity. Finally, some hydrodynamical examples testing the new shear viscosity bound, which is expected to be the consequence of the holographic entropy bound, are presented for the early Universe in the plasma era and for the Kasner metric. It seems that the Kasner metric provides a counterexample to the new shear viscosity bound.

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I. INTRODUCTION

There is growing evidence from high redshift surveys of supernovae and from Wilkinson Microwave Anisotropy Probe (WMAP) data analysis that the current Universe is experiencing a phase of cosmic speed-up. The accepted explanation for this behavior is the dominance of some dark energy contributing up to 70% of the critical energy density. Nevertheless, it remains unclear what this dark energy is: a cosmological constant, quintessence, a phantom, an effective gravitational contribution, or something else. In the absence of a completely consistent dark energy model, a good strategy would be to explore the general properties of a Friedmann-Robertson-Walker (FRW) universe with dark energy described as matter with a general (negative or time-dependent) equation of state. Surprisingly, quite a lot of information about the present Universe and the future of such a universe may be obtained.

In particular, a number of issues related to entropy and energy of the Universe and their bounds may be understood. For instance, it seems clear that the FRW equations are not as simple as they look, as they may encode some quantum field theory structure via the holographic principle. In a very interesting work [1], a strong relation among the FRW equations, conformal field theory entropy, and holography was established. First, this work proposed a holographical bound on the subextensive entropy associated with Casimir energy. Second, it showed that the FRW universe entropy may be presented as a type of Cardy entropy in conformal field theory [2]. The corresponding expression is called the Cardy-Verlinde (CV) formula. Moreover, one more relation—the universal cosmological CV formula—may be obtained by rewriting the FRW equations in terms of three holographic entropies (or energies). There is currently much activity in the study of various aspects of the CV formula (see Ref. [4] and references therein): its holographic origin, the relation to the brane-world approach, and the description via anti–de Sitter (AdS) duals within the AdS/CFT setup. It is also remarkable that the CV formula can be generalized for the case of a general (constant) equation of state [5], while the cosmological CV formula remains valid.

The purpose of the present work is to discuss the entropy, Cardy-Verlinde-like formulas, and related consequences of the holographic entropy bound for a (mainly) FRW universe filled with dark energy where the effective equation of state is negative or even time-dependent. In a similar fashion, these questions are studied for modified gravity, which represents a gravitational alternative for dark energy. It is expected that a better understanding of this topic may shed some light on questions about the origin of holographic relations in the early Universe as well as in the current accelerating Universe, and on the origin of dark energy itself.

The paper is organized as follows. In the next section we discuss the thermodynamic system that corresponds to the FRW universe with a general equation of state that can be negative (cosmological constant, phantoms, or quintessence) or time-dependent. The explicit expression for the entropy of such a FRW universe is found and is presented as a CV formula (in terms of energy and Casimir energy). It is remarkable that for a general equation of state, such a formula does not have a simple form, reminiscent of 2D CFT entropy. Another form of the (cosmological) CV formula (which is expected to have a holographic origin and which relates three

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different typical entropies of a FRW universe) is found to be universal, like the 2D CFT entropy. The entropy bounds (including the Bekenstein bound) for a dark energy universe and their dependence on critical radii are briefly mentioned.

Section III is devoted to the study of the same questions for modified gravity, which contains terms that increase with the decrease of curvature. Such a theory describes the current accelerating Universe and represents the gravitational alternative for dark energy. It is shown that the cosmological CV formula is universal, since it remains the same in both frames (Jordan or Einstein) used to describe such a gravity. In Sec. IV the black hole thermodynamics for modified gravity is briefly discussed. It is shown that for Schwarzschild–anti–de Sitter (SAdS) black holes the entropy is related to the area, with a numerical coefficient that is different from the Einstein gravity case. The relation of such an entropy to the CV formula is briefly mentioned. Section V is related more to hydrodynamics and the early Universe. Namely, some universal lower bound on the relation between shear viscosity and entropy density was recently suggested. It is expected that such a bound directly follows from the Bekenstein entropy bound. As shear viscosity is typical for an anisotropic universe, we test the bound for hydrodynamics or a Kasner universe. It seems that an anisotropic universe may give some counterexample for the bound. Finally, a summary and an outlook are given in the last section.

II. THERMODYNAMICS OF A DARK ENERGY UNIVERSE: ENERGY AND ENTROPY

Let us start from the simple thermodynamic system with the free energy $F=F(V,T)$, where V is volume of the system and T is temperature. The pressure p, energy density ρ, and entropy S are given by

$$p=-\frac{\partial F}{\partial V}, \quad \rho=\frac{1}{V}\left(F-T\frac{\partial F}{\partial T}\right), \quad S=-\frac{\partial F}{\partial T}. \tag{1}$$

The first law of the thermodynamics holds automatically: $TdS=dE+pdV$. Here the total energy E is given by $E=\rho V$. The Boltzmann constant k_B is chosen to be unity $(k_B=1)$. The free energy may be chosen in the following form:

$$F=-f_0T^{\alpha}V^{\beta}, \tag{2}$$

with some constants f_0, α, and β. As a result

$$p=\beta f_0 T^{\alpha}V^{\beta-1}, \quad \rho=(\alpha-1)f_0T^{\alpha}V^{\beta-1}, \quad S=\alpha f_0 T^{\alpha-1}V^{\beta}. \tag{3}$$

Defining a parameter w by $p=w\rho$ (equation of state), we obtain

$$w=\frac{\beta}{\alpha-1}. \tag{4}$$

The case of interest is the negative equation of state, which is typical for the current, dark energy, Universe. The free energy can be rewritten as

$$F=-f_0T(T^{1/w}V)^{\beta}, \tag{5}$$

which tells that the general free energy of the matter with w has the following form:

$$F_w(T,V)=T\hat{F}(T^{1/w}V). \tag{6}$$

Here $\hat{F}(x)$ is a function depending on the matter.

For $\alpha=4$ and $\beta=1$, the classical radiation in four-dimensional spacetime is restored: $p=f_0T^4$, $\rho=3f_0T^4$. In order to obtain an ideal gas, the free energy should be written $F=f_0(T^{\alpha}V^{\beta}-T)$. It is interesting that the last term does not contribute to p (ρ) but does contribute to the entropy S. In the limit that $\alpha\to 1$ and $\beta\to 0$ with finite $c_1=f_0(\alpha-1)$ and $c_2=\beta f_0$, we obtain $F=T\ln(T^{c_1}V^{c_2})$, $p=c_2TV$, $\rho=c_1TV$. Then c_2 can be identified with the number N of the molecules in the gas $c_2=N$ and $c_1=\frac{3}{2}N$ for the monatomic molecule. One can also obtain dust by choosing $\beta=0$:

$$p=0, \quad \rho=(\alpha-1)f_0T^{\alpha}V^{-1}. \tag{7}$$

We may consider the case for which the entropy is constant, $S=S_0$, which is typical for an adiabatically expanding universe for which the first law of thermodynamics holds. From (3) it follows that

$$T=(\alpha f_0)^{-1/(\alpha-1)}S_0^{1/(\alpha-1)}V^{-w}. \tag{8}$$

Here w is given in (4).

Let us apply the above considerations to the $(n+1)$-dimensional FRW metric of the form

$$ds^2=g_{\mu\nu}dx^{\mu}dx^{\nu}=-d\tau^2+a^2(\tau)\gamma_{ij}dx^idx^j, \tag{9}$$

where the n-dimensional metric γ_{ij} is parametrized by $k=-1,0,1$. In the following, the $k=1$ case is mainly considered. Since $V=a^n\int d^nx\sqrt{\gamma}$, the temperature of the Universe is $T\propto a^{-nw}$. By combining (3) and (8), the total energy $E=\rho V$ is given by

$$E=(\alpha-1)\alpha^{-\alpha/(\alpha-1)}f_0^{-\alpha/(\alpha-1)}S_0^{\alpha/(\alpha-1)}V^{-w}\propto a^{-nw}. \tag{10}$$

The a dependence in T and (10) reproduces the corresponding results in Ref. [5].

Rescaling the entropy and the volume as $S_0\to\lambda S_0$ and $V\to\lambda V$, from the expression (10), we obtain $E\to\lambda^{1/(\alpha-1)+1-w}E$. If the energy is extensive, $E\to\lambda E$. For the extensive part of the energy it follows that $\alpha=1+1/w$, $\beta=1$. In order to obtain the expression of this β, Eq. (4) should be used.

The following free energy for general equation of state may be considered:

$$F=-f_0T^{1+1/w}V(1+f_1T^{-2/nw}V^{-2/n}). \tag{11}$$

If there is no the second term, the first term gives the extensive energy. Note that $p=w\rho$ even if the second term is included. As a result, the energy and entropy of the thermal universe follow:

$$E=\frac{f_0}{w}T^{1+1/w}V\left[1+\left(1-\frac{2}{n}\right)f_1T^{-2/nw}V^{-2/n}\right],$$

$$S=f_0T^{1/w}V\left[\left(1+\frac{1}{w}\right)+\left(1+\frac{1}{w}-\frac{2}{nw}\right)f_1T^{-2/nw}V^{-2/n}\right]. \qquad (12)$$

As clear from (1) and (2), the entropy becomes negative (unphysical case) if f_0 or α is chosen to be negative. If the terms containing f_1 can be neglected, as is clear from Eqs. (12), the entropy S becomes negative if

(1) $f_0<0$ and $w<-1$: in this case, the energy E is positive.

(2) $f_0>0$ and $0>w>-1$: in this case, E also becomes negative.

We should also note that the energy (if we neglect the terms containing f_1) is positive (negative) if f_0 is positive (negative). The case for negative entropy would be unphysical and should be excluded. Then the case for $w<-1$ and positive energy E, and the case for $0>w>-1$ and negative energy E, should be excluded.

The subextensive part of the energy E_C, which is called the Casimir energy, is given by

$$E_C=n(E+pV-TS)=-nV^2\frac{\partial}{\partial V}\left(\frac{F}{V}\right)=-2f_0f_1T^{1+1/w-2/nw}V^{1-2/n}. \qquad (13)$$

The extensive part of the energy E_E has the following form:

$$E_E=E-\frac{1}{2}E_C=\frac{f_0}{w}T^{1+1/w}V\left[1+\left(1-\frac{2}{n}+w\right)f_1T^{-2/nw}V^{-2/n}\right]. \qquad (14)$$

From the last expression in (12), we obtain

$$T\sim S^w\left[1+\left(1+\frac{1}{w}\right)^{-1}\left(1-\frac{2}{nw}+\frac{1}{w}\right)f_1T^{-2/nw}V^{-2/n}\right] \qquad (15)$$

and

$$E_E\sim S^{w+1}V^{-w}+\mathcal{O}(f_1^2), \quad E_C\sim S^{w+1-2/n}V^{-w}+\mathcal{O}(f_1^2), \qquad (16)$$

which reproduce the behaviors in Ref. [5]. When the size of the universe is large, the second terms in S (12) and in E_E (14) are subdominant and we obtain

$$S\sim f_0T^{1/w}V\left(1+\frac{1}{w}\right), \quad E_E\sim\frac{f_0}{w}T^{1+1/w}V. \qquad (17)$$

Then combining (13) and (17), for the FRW metric (9) with $k=1$, one gets

$$S\sim f_0\left(1+\frac{1}{w}\right)\left(-\frac{2f_0^2f_1}{n}\right)^{-n/2[(w+1)n-1]}\times V_0^{wn/[(w+1)n-1]}[a^{nw}\sqrt{E_EE_C}]^{n/[(w+1)n-1]}=A[a^{nw}\sqrt{(2E-E_C)E_C}]^{n/[(w+1)n-1]}. \qquad (18)$$

Here

$$A\equiv f_0(1+1/w)(-4f_0^2f_1/n)^{-n/2[(w+1)n-1]}V_0^{wn/[(w+1)n-1]}$$

and

$$V_0=\int d^nx\sqrt{\gamma}.$$

Equation (18) reproduces Eq. (20) in Ref. [5] if we identify $A=(2\pi/\sqrt{\alpha\beta})^{n/[(w+1)n-1]}$. This expression represents one of the forms of the Cardy-Verlinde formula [1] for the general equation of state.

As there are astrophysical indications that dark energy currently dominates the thermal universe, our main interest relates to the case where w can be negative. One usually denotes the matter as quintessence if $-\frac{1}{3}>w>-1$ and as phantom [6] if $w<-1$. When $w=-1$, the situation corresponds to the cosmological constant. First we should note that entropy S (18) becomes singular at $w=-1+1/n$, which occurs since the product E_CE_E becomes independent of the temperature. If the entropy S is conserved, Eq. (18) indicates that the product E_CE_E increases if the size of the universe a increases when w is negative. The entropy may be conserved but we may consider the variation of the entropy as a change of the initial condition.

When $0>w>-1+1/n$, if we keep E_CE_E as constant, Eq. (18) shows that S decreases if a increases. When $w<-1+1/n$, S increases if a increases but S decreases if E_CE_E increases. As is seen from (12), the specific heat dE/dT with fixed volume (V is a constant) becomes negative, when $0>w>-1$. For the phantom matter ($w<-1$), the specific heat is positive and for the cosmological constant, the specific heat vanishes.

For the current realistic Universe the case for many kinds of matter (with dark energy dominance) is typical. In such a case the free energy may be written as sum over various contributions

$$F=-\sum_i f_{i0}T^{1+1/w_i}V(1+f_{i1}T^{-2/nw_i}V^{-2/n}). \qquad (19)$$

Then one gets

$$E=\sum_i\frac{f_{i0}}{w_i}T^{1+1/w_i}V\left[1+\left(1-\frac{2}{n}\right)f_{i1}T^{-2/nw_i}V^{-2/n}\right],$$

$$S=\sum_i f_{i0}T^{1/w_i}V\left[\left(1+\frac{1}{w_i}\right)+\left(1+\frac{1}{w_i}-\frac{2}{nw_i}\right)f_{i1}T^{-2/nw_i}V^{-2/n}\right],$$

$$E_C=-2\sum_i f_{i0}f_{i1}T^{1+1/w_i-2/nw_i}V^{1-2/n},$$

$$E_E = \sum_i \frac{f_{i0}}{w_i} T^{1+1/w_i} V \left[1 + \left(1 - \frac{2}{n} + w_i \right) \times f_{i1} T^{-2/nw_i} V^{-2/n} \right]. \tag{20}$$

Thus, in case for several types of matter, we cannot obtain a simple relation (18). Nevertheless, an inequality follows:

$$S \geqslant S_i \sim A_i [a^{nw_i} \sqrt{(2E_i - E_{Ci})E_{Ci}}]^{n/[(w_i+1)n-1]}. \tag{21}$$

Here

$$A_i \equiv f_{0i} \left(1 + \frac{1}{w_i} \right) \left(-\frac{4 f_{0i}^2 f_{1i}}{n} \right)^{-n/2[(w_i+1)n-1]} \times V_0^{w_i n/[(w_i+1)n-1]}. \tag{22}$$

As $S = \Sigma_i S_i$ and $S_i \geqslant 0$, the inequality (21) holds for arbitrary i. With the entropy S (20), at high temperature the matter with small and positive w_i dominates. We now denote the quantities related with the matter with smallest but positive w_i by the index "min." On the other hand, at low temperature as in current Universe, if all the w_i's are positive, the matter with large w_i dominates. We now denote the quantities related with the matter for largest w_i by the index "max." Then we obtain $S \sim A_i [a^{nw_i} \sqrt{(2E_i - E_{Ci})E_{Ci}}]^{n/[(w_i+1)n-1]}$. Here at high temperature, we have $i = \min$ and at low temperature $i = \max$. If there is a dark energy (say, phantom) with negative w, such a matter dominates at low temperature:

$$S \sim A_p [a^{nw_p} \sqrt{(2E_p - E_{pC})E_{pC}}]^{n/[(w_p+1)n-1]}. \tag{23}$$

Here we have denoted the quantities related with the phantom matter by the index p. Note that for a negative equation of state the above universe entropy formula is dissimilar to the well-known Cardy formula in CFT. Since the entropy is given by

$$S \sim f_{p0} T^{1/w_p} V \left[\left(1 + \frac{1}{w_p} \right) + \left(1 + \frac{1}{w_p} - \frac{2}{nw_p} \right) f_{p1} (T^{1/w_p} V)^{-2/n} \right] \tag{24}$$

for conserved entropy, $T^{1/w_p} V$ is a constant: $T^{1/w_p} V = C$. Then the energy E can be rewritten as

$$\begin{aligned} E &\sim \frac{f_{p0}}{w_p} CT \left[1 + \left(1 - \frac{2}{n} \right) f_{p1} C^{-2/n} \right] \\ &= \frac{f_{p0}}{w_p} C^{w_p+1} V_0^{-w_p} a^{-nw_p} \\ &\quad \times \left[1 + \left(1 - \frac{2}{n} \right) f_{p1} C^{-2/n} \right]. \end{aligned} \tag{25}$$

Thus, the energy is linear with the temperature. In the last line, we have considered the FRW metric (9). Generally in the FRW metric, if we have the relation $p = w\rho$, we find $\rho \propto a^{-n(1+w)}$ (energy conservation) and $E = \rho V \propto a^{-nw}$, which is consistent with (25). If there is only dark matter with $w < 0$ in the Universe, we have $T^{1/w_p} V = C$ even at high temperature. When the Universe expands and the radius grows, the temperature increases too and also the energy E and energy density ρ behave as $E \sim a^{-nw_p}$ and $\rho \sim a^{-n(w_p+1)}$, respectively. As a result the density becomes large and might generate some future singularities (like Big Rip, where the scale factor of the universe diverges in a finite time).

As an example the system with dust and quintessence or phantom matter, where w is negative, may be considered. If we assume that there is no internal structure in the dust, the energy of the dust does not depend on the temperature and the free energy, corresponding to (7), becomes a constant: $F = E_{D0}$. Then the total free energy can be assumed to be given by

$$F = E_{D0} - f_{p0} T^{1+1/w_p} V (1 + f_{p1} T^{-2/nw_p} V^{-2/n}). \tag{26}$$

Thus, one obtains

$$\begin{aligned} E &= E_{D0} + \frac{f_{p0}}{w_p} T^{1+1/w_p} V \left[1 + \left(1 - \frac{2}{n} \right) f_{p1} (T^{-1/w_p} V)^{-2/n} \right], \\ S &= f_{p0} T^{1/w_p} V \left[\left(1 + \frac{1}{w_p} \right) + \left(1 + \frac{1}{w_p} - \frac{2}{nw_p} \right) f_{p1} (T^{1/w_p} V)^{-2/n} \right]. \end{aligned} \tag{27}$$

Note that dust does not contribute to the entropy. The energy of the dust is not extensive or subextensive. The extensive and subextensive (Casimir) parts of the energy of the phantom or quintessence matter are given by

$$\begin{aligned} E_{pC} &= -2 f_{p0} f_{p1} T^{1+1/w_p - 2/nw_p} V^{1-2/n}, \\ E_{pE} &= \frac{f_{p0}}{w_p} T^{1+1/w_p} V \left[1 + \left(1 - \frac{2}{n} + w_p \right) \times f_{p1} T^{-2/nw_p} V^{-2/n} \right]. \end{aligned} \tag{28}$$

If we assume the entropy S is conserved, from the expression of S (27), we find $T^{1/w_p} V$ is a constant:

$$T^{1/w_p} V = C. \tag{29}$$

Then the energy E (27) can be rewritten as

$$\begin{aligned} E &= E_{D0} + \frac{f_{p0}}{w_p} CT \left[1 + \left(1 - \frac{2}{n} \right) f_{p1} C^{-2/n} \right] \\ &= E_{D0} + \frac{f_{p0}}{w_p} C^{w_p+1} V_0^{-w_p} a^{-nw_p} \\ &\quad \times \left[1 + \left(1 - \frac{2}{n} \right) f_{p1} C^{-2/n} \right]. \end{aligned} \tag{30}$$

Then energy is again linear in the temperature. In the last line, we have considered the FRW metric (9). Generally in the FRW metric, if we have the relation $p=w\rho$, we find $\rho \propto a^{-n(1+w)}$ and $E=\rho V \propto a^{-nw}$, which is consistent with the last expression for the phantom or quintessence matter in (30).

Taking into account the recent cosmological considerations of variations of fundamental constants, one may start from the case in which w_p depends on time t. Of course, this may be a negative (or sign-changing) function. The energy conservation condition reads $0=\dot{\rho}_p+n(\dot{a}/a)(\rho_p+p_p)$ by assuming $\rho_p=w_p(t)p_p$. Then we obtain the expression

$$\rho_p=a^{-n[1+w_p(t)]}\exp\left(n\int^t \dot{w}_p(t')\ln a(t')dt'\right).$$

The energy in such a universe is

$$E_p=\rho_p V=a^{-nw_p(t)}\exp\left(n\int^t \dot{w}_p(t')\ln a(t')dt'\right)V_0. \tag{31}$$

If the spacetime expansion is adiabatic and thermodynamical quantities can be defined, Eqs. (1) are valid. Thus, if we define a free energy as in the phantom part of Eq. (26), we can obtain the entropy and energy as in (27) and the extensive and subextensive parts of the energy as in (28). Then if we define a variable ξ by $T=V^{-w_p(t)}\xi$, extracting the phantom part E_p from the expression of E in (27), we obtain

$$E_p=\frac{f_{p0}}{w_p(t)}a^{-nw_p(t)}V_0^{-w_p(t)}\xi^{1+1/w_p(t)} \times\left[1+\left(1-\frac{2}{n}\right)f_1\xi^{-2/nw_p(t)}\right]. \tag{32}$$

By comparing (31) with (32), one finds

$$\begin{aligned}\xi=&\left(\frac{w_p(t)}{f_{p0}}\right)^{w_p(t)/[w_p(t)+1]}V_0^{w_p(t)}\exp\left(\frac{nw_p(t)}{w_p(t)+1}\right.\\ &\times\left.\int^t \dot{w}_p(t')\ln a(t')dt'\right)\left\{1-f_1\frac{1-2/n}{1+1/w_p(t)}\right.\\ &\times\left(\frac{w_p(t)}{f_{p0}}\right)^{-2/n[w_p(t)+1]}V_0^{-2/n}\exp\left(-\frac{2}{w_p(t)+1}\right.\\ &\times\left.\left.\int^t \dot{w}_p(t')\ln a(t')dt'\right)\right\}+\mathcal{O}(f_1^2).\end{aligned} \tag{33}$$

From Eqs. (27) and (28), the expressions of the entropy S_p, the extensive part of the energy E_{pE}, and the Casimir energy E_{pC} may be evaluated:

$$\begin{aligned}S_p=&f_{p0}\xi^{1/w_p(t)}\left[\left(1+\frac{1}{w_p(t)}\right)\right.\\ &+\left.\left(1+\frac{1}{w_p(t)}-\frac{2}{nw_p(t)}\right)f_{p1}\xi^{-2/nw_p(t)}\right],\\ E_{pE}=&\frac{f_{p0}}{w_p(t)}a^{-nw_p(t)}V_0^{-w_p(t)}\xi^{1+1/w_p(t)}\\ &\times\left[1+\left(1-\frac{2}{n}+w_p(t)\right)f_{p1}\xi^{-2/nw_p(t)}\right],\\ E_{pC}=&-2f_{p0}f_{p1}a^{-nw_p(t)}V_0^{-w_p(t)}\xi^{1+1/w_p(t)-2/nw(t)}.\end{aligned} \tag{34}$$

As $w_p(t)$ and ξ are time-dependent, the entropy is not constant and not conserved. Nevertheless, from (34) the Cardy-Verlinde [1]–like formula (in the form of Youm [5]) (23) is still valid:

$$S_p\sim A_p[a^{nw_p}\sqrt{(2E_p-E_{pC})E_{pC}}]^{n/[(w_p+1)n-1]}. \tag{35}$$

We should note, however, since

$$A_p=f_{p0}\left(1+\frac{1}{w_p(t)}\right)\left(-\frac{4f_{p0}^2f_{p1}}{n}\right)^{-n/2[(w_p(t)+1)n-1]} \times V_0^{wn/[w_p(t)+1]n-1}$$

and $w_p(t)$ depend on time, A_p is not a constant but a function of the time t. Thus, the entropy of the expanding universe with a (negative) time-dependent equation of state is found.

Now, the FRW equations for a universe filled with matter with pressure p and energy density ρ are given by

$$H^2=\frac{16\pi G}{n(n-1)}\rho-\frac{k}{a^2},\quad \dot{H}=-\frac{8\pi G}{n-1}(\rho+p)+\frac{k}{a^2}. \tag{36}$$

As in Ref. [1], if we define the Hubble entropy S_H, the Bekenstein-Hawking energy E_{BH}, and the Hawking temperature T_H by

$$S_H\equiv\frac{(n-1)HV}{4G},\quad E_{BH}\equiv\frac{n(n-1)V}{8\pi Ga^2},\quad T_H\equiv-\frac{\dot{H}}{2\pi H}, \tag{37}$$

the FRW equations can be rewritten in universal form as

$$S_H=\frac{2\pi a}{n}\sqrt{E_{BH}(2E-kE_{BH})},\quad kE_{BH}=n(E+pV-T_HS_H), \tag{38}$$

Furthermore with the Bekenstein entropy S_B and the Bekenstein-Hawking entropy S_{BH} as

$$S_B\equiv\frac{2\pi a}{n}E,\quad S_{BH}\equiv\frac{(n-1)V}{4Ga}, \tag{39}$$

we obtain well-known relation between entropies:

$$S_H^2=2S_BS_{BH}-kS_{BH}^2. \tag{40}$$

In the case for $k=1$, Eq. (40) can be rewritten as $S_H^2+(S_B-S_{BH})^2=S_B^2$. Then we find $S_H\leqslant S_B$. For the system with limited self-gravity, the Bekenstein bound [3] occurs:

$S \leq S_B$. This bound is useful for the case in which the system has relatively low energy or small volume. Then Bekenstein entropy S_B scales as $S_B \to \lambda^{1+1/n} S_B$ under the scale transformation $V \to \lambda V$ and $E \to \lambda E$ [1].

Equation (38) has a form similar to the second equation in (18) with $w = 1/n$; this equation is called the cosmological Cardy-Verlinde formula. The second equation in (37) has a form similar to (13) and E_{BH} may correspond to the Casimir energy E_C. In Ref. [1], the following cosmological bound has been proposed:

$$E_C \leq E_{BH}. \tag{41}$$

As seen from the definition of E_{BH} in (37), we find $E_{BH} \sim a^{n-2}$. If we consider phantom or quintessence as the matter field, as seen from the last expression in (30), the behavior of the Casimir energy is given by $E_C \sim a^{-nw}$. Then if $w < -1 + 2/n$ and E_C is positive, there is a critical radius a_c where $E_C = E_{BH}$, and if the radius, a of the universe is larger than the critical radius, $a > a_c$, the bound in (41) is violated. Formally a_c is given by

$$a_c = \left[-\frac{16\pi G f_{p0} f_{p1} V_0^{-w_p-1} C^{1-2/n}}{n(n-1)} \right]^{1/[nw_p+n-2]}$$

with the parameters f_{p0}, f_{p1}, and C, which may be determined by some initial conditions. If we consider four-dimensional spacetime $(n=3)$, because $w < -1 + 2/n$, we obtain $w < -\frac{1}{3}$, then for the quintessence $(-1 < w < -\frac{1}{3})$, the cosmological constant $(w = -1)$, and the phantom $(w < -1)$, there is always a critical radius a_c and the bound (41) is violated if $a > a_c$.

Similarly, one can discuss the entropy bounds for the dark energy universe as in Ref. [5] even if w_p depends on time. Although the entropy is not conserved, the expression of the entropy S_p (35) still holds. The quantity $(2E_p - E_p)E_{pC}$ inside the square root of (35) has a maximum E_p^2 when $E_{pC} = E_p$. Then

$$S \leq A_p [a^{nw_p} E_p]^{n/[(w_p+1)n-1]} \quad \text{for} \quad w_p > -1 + \frac{1}{n},$$

$$S \geq A_p [a^{nw_p} E_p]^{n/[(w_p+1)n-1]} \quad \text{for} \quad w_p < -1 + \frac{1}{n}. \tag{42}$$

As w_p depends on time, at some time, we may have $w_p > -1 + 1/n$ and at another time, $w_p < -1 + 1/n$. If we define the Bekenstein entropy S_{pB} for the dark energy as in (39), $S_{pB} \equiv (2\pi a/n) E_p$, we find, even if w_p depends on time, the relation as in Ref. [5]:

$$S \leq S_0 [a^{nw_p - 1} S_B]^{n/[(w_p+1)n-1]} \quad \text{for} \quad w_p > -1 + \frac{1}{n},$$

$$S \geq S_0 [a^{nw_p - 1} S_B]^{n/[(w_p+1)n-1]} \quad \text{for} \quad w_p < -1 + \frac{1}{n}. \tag{43}$$

Here S_0 is given by $S_0 = A_p (n/2\pi)^{n/[(w_p+1)n-1]}$. However, as w_p and A_p depend on time, S_0 also depends on time. If $w < -1 < -1 + 1/n$, the entropy can be negative (unphysical case) even if the energy is positive. If $-1 < w < 1/n$, the entropy becomes negative only when the energy is negative.

III. ENTROPY AND ENERGY IN MODIFIED GRAVITY

In Refs. [7,8], a gravitational alternative was suggested for the dark energy modifying the standard Einstein action at low curvature by the $1/R$ term. Such modified gravity may produce the current cosmic speed-up [7] and may be naturally generated by string/M theory [9]. It represents some kind of higher derivative and nonlocal gravity, and as such it may contain some instabilities [11]. Nevertheless, with some mild modifications at high curvature regions, the theory is shown to be stable [12], which is also supported by quantum field theory [12]. Modified gravity was studied in Palatini form [10], and it seems that it may be viable also in such a version. Classically, its action may be mapped to an equivalent scalar-tensor theory. We discuss below the entropy, the energy, and CV formula for an accelerated universe in modified gravity, which provides the gravitational dark energy.

Let us start from the rather general four-dimensional action:

$$\hat{S} = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} f(R), \tag{44}$$

where $\kappa^2 = 16\pi G$, R is the scalar curvature, and $f(R)$ is some arbitrary function. By using the conformal transformation $g_{\mu\nu} \to e^{\sigma} g_{\mu\nu}$ with $\sigma = -\ln f'(R)$, etc., the action (44) is rewritten as

$$\hat{S}_E = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left(R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) \right),$$

$$V(\sigma) \equiv e^{\sigma} g(e^{-\sigma}) - e^{2\sigma} f(g(e^{-\sigma})) = \frac{A}{f'(A)} - \frac{f(A)}{f'(A)^2}. \tag{45}$$

Here $g(B)$ is given by solving the equation $B = f'(A)$ with respect to A, $A = g(B)$, and A in (45) is given by $A \equiv -e^{2\sigma}$. This is the standard form of the scalar-tensor theories where the scalar field is fictitious [12].

We now consider the FRW cosmology in modified gravity. The FRW metric in the physical (Jordan) frame is given by

$$ds^2 = -dt^2 + \hat{a}(t)^2 \sum_{i,j=1}^{3} \gamma_{ij} dx^i dx^j. \tag{46}$$

The FRW equation in the Einstein frame has the following form:

$$3H_E^2 + \frac{3k}{2\widehat{a_E}^2} = \frac{\kappa^2}{2} (\rho_{(\sigma E)} + \rho_{(m)}). \tag{47}$$

Here $\rho_{(m)}$ is the energy density of the matter but for simplicity, we neglect the matter. We also concentrate on the $k=0$ case but the obtained results are correct even for $k\neq 0$ case if the radius of the universe is large enough. The Hubble constant H_E in the Einstein frame is defined by $H_E \equiv \dot{\hat{a}}_E/\hat{a}_E$ with the scale factor $\hat{a}_E$ in the Einstein frame: $\hat{a}_E = e^{-\sigma/2}\hat{a}$. The contribution from the σ field to the energy-momentum tensor $\rho_{(\sigma E)}$ is given by

$$\rho_{(\sigma E)} \equiv \frac{1}{\kappa^2}\left(\frac{3}{2}\dot{\sigma}^2 + V(\sigma)\right). \tag{48}$$

In the Einstein frame, the equation of motion for σ has the following form:

$$0 = 3(\ddot{\sigma} + 3H_E\dot{\sigma}) + V'(\sigma). \tag{49}$$

Assuming that when the curvature is small the action is given by

$$\hat{S} = \frac{1}{\kappa^2}\int d^4x\sqrt{-g}\left(R - \frac{\tilde{a}}{R}\right), \tag{50}$$

the potential is given by $V(\sigma) \sim (2/\sqrt{\tilde{a}})e^{(3/2)\sigma}$. Since $\sigma = -\ln f'(R) \sim -\ln(\tilde{a}/R^2)$, σ is negative and large. Then the solution of Eqs. (47) and (49) is given by

$$\hat{a}_E = \hat{a}_{E0}\left(\frac{t_E}{t_0}\right)^{4/3}, \quad \sigma = -\frac{4}{3}\ln\frac{t_E}{t_0}, \quad \frac{t_0^2}{\sqrt{\tilde{a}}} = 4. \tag{51}$$

Here t_E is the time coordinate in the Euclidean frame, which is related to the time coordinate t in the (physical) Jordan frame by $e^{\sigma/2}dt_E = dt$. As a result $3t_E^{1/3} = t$ and in the physical (Jordan) frame the power law inflation occurs,

$$\hat{a} = e^{\sigma/2}\hat{a}_E \propto t_E^{2/3} \propto t^2. \tag{52}$$

In general, if $p = w\rho$, the scale factor a behaves as $\hat{a} \sim t^{2/3(w+1)}$. Then as we can see from (52), in the Jordan frame we find $w = -\frac{2}{3}$ and from (52), in the Einstein frame, $w = -2$. In fact, in the Einstein frame one has

$$\rho_{(\sigma E)} \sim \frac{32}{3\kappa^2 t_E^2}, \quad p_{(\sigma E)} \equiv \frac{1}{\kappa^2}\left(\frac{3}{2}\dot{\sigma}^2 - V(\sigma)\right) \sim -\frac{16}{3\kappa^2 t_E^2}. \tag{53}$$

Although the Jordan frame is physical, as the separation of the gravity and the matter is easier in the Einstein frame, we work in the Einstein frame for a while. The FRW equation (47) can be rewritten in the form of the cosmological CV formula with $n=3$ as

$$S_H^E = \frac{2\pi a}{3}\sqrt{E_{BH}^E(2E^E - kE_{BH}^E)} \tag{54}$$

by defining

$$S_H^E \equiv \frac{H_E V_E}{2G}, \quad E^E \equiv \rho_{(\sigma E)}V_E, \quad E_{BH}^E \equiv \frac{3V_E}{4\pi G\hat{a}_E^2},$$

$$V_E \equiv \hat{a}_E^3\int d^3x\sqrt{-\gamma}, \tag{55}$$

and $\kappa^2 = 16\pi G$. The second FRW equation can be given by considering the derivative of the (first) FRW equation (47) with respective Einstein time t_E and can be rewritten as

$$kE_{BH}^E = 3(E^E + p_{(\sigma E)}V_E - T_H^E S_H^E). \tag{56}$$

Here

$$T_H^E \equiv -\frac{1}{2\pi H_E}\frac{dH_E}{dt_E}$$

and we find

$$p_{(\sigma E)} = -\frac{1}{3H_E}\frac{d\rho_{(\sigma E)}}{dt_E} - \rho_{(\sigma E)}. \tag{57}$$

In the physical Jordan frame, since $\hat{a} = e^{\sigma/2}\hat{a}_E$ and $e^{\sigma/2}dt_E = dt$, the Hubble parameter is

$$H \equiv \frac{1}{\hat{a}}\frac{d\hat{a}}{dt} = \frac{1}{\hat{a}_E}\frac{d\hat{a}_E}{dt_E}\frac{dt_E}{dt} + \frac{1}{2}\frac{d\sigma}{dt} = H_E e^{-\sigma/2} + \frac{\dot{\sigma}}{2}. \tag{58}$$

Then in the Jordan frame, the FRW equation can be rewritten as

$$3H^2 + \frac{3k}{\hat{a}^2} = \frac{\kappa^2}{2}\rho_{(\sigma)}, \quad \rho_{(\sigma)} \equiv \rho_{(\sigma E)}\mathrm{e}^{-\sigma} + H\dot{\sigma} - \frac{\dot{\sigma}^2}{4}. \tag{59}$$

Defining $S_H \equiv HV/2G$, $E \equiv \rho_{(\sigma)}V$, $E_{BH} \equiv 3V/4\pi G\hat{a}^2$, $V \equiv \hat{a}^3\int d^3\sqrt{-\gamma}$, we obtain the cosmological Cardy-Verlinde formula:

$$S_H = \frac{2\pi a}{3}\sqrt{E_{BH}(2E - kE_{BH})}. \tag{60}$$

By differentiating the FRW equation (59) with respect to t, one gets the second FRW equation:

$$\frac{dH}{dt} - \frac{k}{\hat{a}^2} = \frac{\kappa^2}{2}(\rho_{(\sigma)} + p_{(\sigma)}), \quad p_{(\sigma)} \equiv -\frac{1}{3H}\frac{d\rho_{(\sigma)}}{dt} - \rho_{(\sigma)}. \tag{61}$$

With the definition of the temperature T_H as

$$T_H \equiv -\frac{1}{2\pi H}\frac{dH}{dt},$$

it follows that

$$kE_{BH} = 3(E + p_{(\sigma)}V - T_H S_H). \tag{62}$$

For the case of $k=0$, by substituting (51) and (52) into the expressions of $\rho_{(\sigma)}$ in (59) and $p_{(\sigma)}$ in (61), we find

$$\rho_{(\sigma)}=\frac{\rho_{(\sigma)0}}{t^2}, \quad \rho_{(\sigma)0}\equiv\frac{22(27)^{2/3}}{3\kappa^2 t_0^{4/3}}-12, \quad p_{(\sigma)}=-\frac{2}{3}\frac{\rho_{(\sigma)0}}{t^2}. \tag{63}$$

Eventually, it follows $w=-\frac{2}{3}$ in the Jordan frame.

At low temperature, as the field with lowest (negative) w dominates, we may have a equation similar to (23) with $n=3$:

$$S\sim A_\sigma[a^{3w_\sigma}\sqrt{(2E_\sigma-E_{\sigma C})E_{\sigma C}}]^{3/(3w_\sigma+2)},$$
$$A_\sigma\equiv f_{\sigma 0}\left(1+\frac{1}{w_\sigma}\right)\left(-\frac{4f_{\sigma 0}^2 f_{\sigma 1}}{3}\right)^{-3/2(3w_\sigma+2)} \times V_0^{3w_\sigma/(3w_\sigma+2)}. \tag{64}$$

Since $w_\sigma=-\frac{2}{3}$, the exponents in (64) diverges. Then when the entropy is finite, the following condition appears:

$$\left(-\frac{4f_{\sigma 0}^2 f_{\sigma 1}}{3}\right)^{-3/2} V_0^{3w_\sigma}[a^{3w_\sigma}\sqrt{(2E_\sigma-E_{\sigma C})E_{\sigma C}}]^3=1. \tag{65}$$

Notice that the solution (51) or (52) is for $k=0$ case. Then the Casimir force should vanish. In order to find the Casimir force, we need to consider the $k\neq 0$ case. As the expansion over k corresponds to the expansion with respect to the inverse of the radius of the universe, we may consider the perturbation with respect to k in order to obtain the Casimir energy. We should also note that, as discussed after (12), since w is now greater than -1 but negative, the entropy S could be negative only if the energy is negative.

IV. BLACK HOLE THERMODYNAMICS

We now consider the black hole solution in the modified gravity, whose action is given by (50). As it will be shown, its thermodynamical properties are also related to the CV formula. If we assume $R_{\mu\nu}\propto g_{\mu\nu}$, the equation of motion is given by

$$0=\left(1+\frac{\tilde{a}}{R^2}\right)R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\left(R-\frac{\tilde{a}}{R}\right). \tag{66}$$

Then $R=\pm\sqrt{3\tilde{a}}$, $R_{\mu\nu}=\pm(\sqrt{3\tilde{a}}/4)g_{\mu\nu}$. A large class of solutions is given by the family of metrics

$$ds^2=-e^{2\rho}dt^2+e^{-2\rho}dr^2+\sum_{i,j=1,2} g_{ij}^{(2)}dx^i dx^j,$$
$$e^{2\rho}=\frac{1}{r}\left(-\mu+k^{(2)}r-\frac{\Lambda r^3}{3}\right), \quad \Lambda=\mp\frac{\sqrt{3\tilde{a}}}{4}, \tag{67}$$

embracing de Sitter (dS) and anti–de Sitter (AdS) black holes with any horizon topology. Here $k^{(2)}$ is the Ricci curvature of the transverse manifold, as given by the Ricci tensor $R_{ij}^{(2)}$ of the metric $g_{ij}^{(2)}$, i.e., $R_{ij}^{(2)}=k^{(2)}g_{ij}^{(2)}$. If $\Lambda<0$ $(\Lambda>0)$, the spacetime is asymptotically anti–de Sitter (de Sitter). In both cases the curvature radius will be defined by $L^2=3/|\Lambda|=12/\sqrt{3\tilde{a}}$.

We shall mainly study the SAdS metric, although our results apply equally well to any horizon topology. The thermodynamical free energy can be obtained according to a quantum gravity tree-level formula involving the Euclidean action I_E: $F(\beta)=\beta^{-1}I_E=(\kappa/2\pi)I_E$, where κ is the surface gravity of the black hole. To pursue this program one has to regularize the volume divergences. In anti–de Sitter gravity one can achieve this, essentially, by two well known methods. One is the counterterm method inspired by the Maldacena duality with conformal field theories, the other a background subtraction chosen to correspond to the vacuum of the CFT. This uniquely identifies it as anti–de Sitter space itself, with no matter inside. The unregularized Euclidean action will be

$$I_E=-\frac{1}{16\pi G}\int d^4x\left(R-\frac{\tilde{a}}{R}\right)|g|^{1/2}-\frac{1}{8\pi G}\oint K|h|^{1/2}d^3x. \tag{68}$$

The Euclidean SAdS solution is given by (67) taking $k^{(2)}=1$ and the metric $g_{ij}^{(2)}$ to be that of a round two-sphere,

$$ds^2=\left(1-\frac{\mu}{r}+\frac{r^2}{L^2}\right)d\tau^2+\left(1-\frac{\mu}{r}+\frac{r^2}{L^2}\right)^{-1}dr^2+r^2d\omega_2^2, \tag{69}$$

where $d\omega_2^2$ is the line element of a two-sphere with unit radius and volume $\omega_2=4\pi$. Moreover, $\tau\simeq\tau+\beta$ is periodically identified up to β and the curvature radius is $L^2=12/\sqrt{3\tilde{a}}$. This is a solution of (68) with $R=-\sqrt{3\tilde{a}}$. Therefore it represents a spherically symmetric black hole immersed in anti–de Sitter space.

The background metric will be (69) with $\mu=0$, i.e., anti–de Sitter space at finite temperature $T=\kappa/2\pi$. This has zero gravitational entropy, since there is no horizon. The action (68) for the metric (69) is easily seen to be

$$I_E=\frac{\sqrt{\tilde{a}}\beta}{3\sqrt{12}G}(R_m^3-r_+^3)+\text{boundary terms}, \tag{70}$$

where R_m is an upper bound for the radial integration and r_+ is the radius of the horizon. The action of the background is

$$I_{EB}=\frac{\sqrt{\tilde{a}}\beta}{3\sqrt{12}G}R_0^3+\text{background boundary terms}, \tag{71}$$

where again R_0 is a radial cutoff. Now a meaningful comparison of the black hole free energy with the vacuum free energy (empty AdS space) requires that the vacuum metric on the surface $r=R_0$ be asymptotically coincident with the actual metric on the surface $r=R_m$. This matching condition ensures that the boundary temperatures in the black hole and the background be equal. A simple check gives the matching condition that, asymptotically, $R_0=R_m-\mu L^2/6R_m^2$. Using this in (71) and subtracting the result from (70) give the regularized action[1]

$$\Delta I_E=\frac{\sqrt{\tilde{a}}\beta}{6\sqrt{12G}}(\mu L^2-2r_+^3). \tag{72}$$

We note that the mass parameter μ and r_+ are functions of β through the defining relations $\mu=r_++r_+^3/L^2$ and $\beta=4\pi L^2 r_+/(\mu L^2+2r_+^3)$. Hence the entropy could be computed by the familiar thermodynamical relation $S=\beta\partial_\beta\Delta I_E-\Delta I_E$. Instead we may use an easier way. We note that both $R-\tilde{a}/R$ and $R-2\Lambda$ are proportional to $\sqrt{\tilde{a}}$, so I_E must be proportional to the action as computed in Einstein gravity. Denoting this as I_{AdS}, a simple computation gives $I_E=\frac{4}{3}I_{AdS}$. We know that the entropy in Einstein gravity is $A/4G$, so we immediately conclude that in $1/R$ gravity the entropy must be

$$S=\frac{4}{3}\frac{A}{4G}=\frac{A}{3G}. \tag{73}$$

So black holes in modified gravity are a little bit more entropic than expected. We may confirm this result by using the Noether charge method. In this case the formula is [13]

$$S=4\pi\int_{S^2}\frac{\partial\mathcal{L}}{\partial R}d^2x,$$

where $\mathcal{L}=\mathcal{L}(R)$ is the Lagrangian density and the integral is over the horizon at $r=r_+$. In our case $\mathcal{L}=\sqrt{g}(R-\tilde{a}/R)/16\pi G$, so

$$S=\frac{A}{4G}\left(1+\frac{\tilde{a}}{R^2}\right)=\frac{4}{3}\frac{A}{4G},$$

as a simple computation will confirm using $R^2=3\tilde{a}$. These calculations can be done in any spacetime dimensions, say d. Then (73) generalizes to

$$S=\frac{2d}{d+2}\frac{A}{4G}. \tag{74}$$

Note that for the black hole with the size of a FRW universe, the entropy is defined by the Bekenstein-Hawking entropy S_{BH} (39). Then the above result indicates that S_{BH} should also be modified by the factor $2d/(d+2)$ if compared with the FRW universe in Einstein gravity.

The higher entropy of black holes in $1/R$ gravity means that they are more massive than those in Einstein theory, since by the first law $dM=TdS$. The precise prediction should just be that M is larger by the factor $z=2d/(d+2)$.

[1]One finds that the boundary terms do not contribute to the final result.

An asymptotic SAdS$_d$ black hole in general relativity has an excitation energy over the AdS vacuum that can be computed by canonical methods through the formula

$$M=-\frac{1}{8\pi G}\oint N(\Theta-\Theta_0)\sqrt{\sigma}d^{d-2}x. \tag{75}$$

Here we integrate, over a $(d-2)$-dimensional sphere at infinity contained in a Cauchy surface of equal time, the lapse function $N=\sqrt{-g_{tt}}$ times the trace of the second fundamental form of the sphere as embedded in the Cauchy surface, after a regularizing subtraction from empty AdS space. For the metric (69) one finds

$$M=\frac{(d-2)\omega_{d-2}}{16\pi G}\mu. \tag{76}$$

This can be expressed as a function of the black hole radius by using the condition $N(r_+)=0$, which is $\mu=r_+^{d-3}+r_+^{d-1}/L^2$. In theories with an AdS dual, this relation can be interpreted as the energy of a CFT ‘‘living’’ on the boundary of AdS spacetime, and leads to a CV formula for AdS black holes. In higher-derivative gravity, and this is just our case, things may be not so straightforward. For a theory whose Lagrangian $L=L(R)$ is a function of the scalar curvature, the above mass can be related to a Noether charge [14] that is proportional to $\partial L/\partial R$, as in the entropy derivation given above. Moreover, it is this Noether charge that enters the formulation of the first law for stationary black holes in diffeomorphism covariant theories of gravity [13,14]. The result is the mass formula (75), except that the integrand gets multiplied with $16\pi G\partial L/\partial R$ evaluated on the background solution, where $L=(R-\tilde{a}/R)/16\pi G$ is the actual Lagrangian. This gives all masses an extra coefficient $1+\tilde{a}/R^2=\frac{4}{3}$. It is therefore clear that the Cardy-Verlinde formula for AdS black holes [15,16], being the square root of a quadratic function of all the relevant energies, will give the entropy the 4/3 coefficient too, in accord with our calculations.

V. HYDRODYNAMICAL EXAMPLES TESTING THE HOLOGRAPHIC ENTROPY BOUND

The suggestion of Kovtun *et al.* [17] that there may exist in cosmology a universal lower bound on η/s—η being the shear viscosity and s the entropy content per unit volume—is interesting, since it may be of fundamental importance. These authors are concerned with the infrared properties of theories whose gravity duals contain a black brane with a nonvanishing Hawking temperature, the point being that the infrared behavior is governed by hydrodynamical laws. If we for definiteness consider a stack of N nonextremal D3 branes in type IIB supergravity, the metric near the horizon is given by

$$ds^2=\frac{r^2}{R^2}[-f(r)dt^2+dx^2+dy^2+dz^2]+\frac{R^2}{r^2f(r)}dr^2+R^2d\Omega_5^2, \tag{77}$$

where $R\propto N^{1/4}$ is a constant and $f(r)=1-r_0^4/r^4$ with r_0 being the horizon. The Hawking temperature of this metric is $T=r_0/\pi R^2$, and η and s are given by $\eta=\frac{1}{8}\pi N^2T^3$, $s=\frac{1}{2}\pi^2N^2T^3$. Thus, in dimensional notation

$$\frac{\eta}{s}=\frac{\hbar}{4\pi k_B}=6.08\times 10^{-13}\text{ K s}. \tag{78}$$

The conjecture of Kovtun *et al.* (see also Ref. [18]) is that the value in Eq. (78) is a *lower bound* for η/s. Since this bound does not involve the speed of light, the authors even conjecture that this bound exists for all systems, including nonrelativistic ones.

The idea has recently been further elaborated in Ref. [19], arguing that the bound follows from the generalized covariant entropy bound. From Eq. (40), there is the Bekenstein (and also the holographic) entropy bound, which is used to prove the new bound to shear viscosity.

The purpose of this section is to elucidate this holographic idea by considering some examples explicitly. We will choose examples from general physics. Our scope is thus wider than in the previous sections; our aim is to investigate the generality of the entropy bound. We will consider three examples, the first taken from ordinary hydrodynamics, the second from the theory of the Universe in the beginning of its plasma era, and finally the third taken from the very early Universe under conditions corresponding to the Kasner metric. The third example is presumably the one of main interest; the shear viscosity concept is after all a concept that relates to an *anisotropic* physical situation. Moreover, we will discuss the validity of the Cardy-Verlinde entropy formula in the case of viscous cosmology, thus elaborating on the previous treatment on this topic in Ref. [20]. The central inequality that we intend to analyze is thus

$$\frac{\eta/s}{\hbar/4\pi k_B}>1. \tag{79}$$

Example 1. Hydrodynamics: Small-Reynolds-number flow. The following setup taken from ordinary hydrodynamics involves both the shear viscosity η and the entropy density s: Assume that a solid sphere with radius R and with high thermal conductivity λ is immersed in a uniform flow passing it at small Reynolds numbers. We take the origin in the center of the sphere, and use spherical coordinates with the polar axis in the direction of the undisturbed velocity $\mathbf{u}$ of the stream. The equation of thermal conduction is $\nabla^2T=-(\eta/2\lambda)(v_{i,k}+v_{k,i})^2$, where $\mathbf{v}$ is the fluid velocity for $r\geqslant R$. Inserting Stokes's formula (applicable at low Reynolds numbers) for $\mathbf{v}$, the solution for the temperature distribution $T(r)$ can be written as [21]

$$T(r)-T_0=\frac{9u^2\eta}{4\lambda}\left\{\left(\frac{3}{4}\frac{R^2}{r^2}-\frac{5}{3}\frac{R^3}{r^3}+\frac{R^4}{r^4}-\frac{1}{12}\frac{R^6}{r^6}\right)\cos^2\theta+\frac{2}{3}\frac{R}{r}-\frac{3}{4}\frac{R^2}{r^2}+\frac{5}{9}\frac{R^3}{r^3}-\frac{1}{6}\frac{R^4}{r^4}-\frac{1}{36}\frac{R^6}{r^6}\right\}, \tag{80}$$

where T_0 is the constant reference temperature at infinity. The boundary conditions are $T=T_1=\text{const}$ and $\int(\partial T/\partial r)r^2\sin\theta d\theta=0$ for $r=R$. From Eq. (80) it is seen that $\Delta T\equiv T_1-T_0=5u^2\eta/8\lambda$.

One may ask: What is the appropriate value to be inserted for the entropy density s? Taking water as an example, one might use the handbook value for s, resulting in $\eta/s=2.3\times 10^{-10}$ K s, as in Ref. [19]. However, in our opinion the physically most natural value to use for s in the present example is the one associated with the temperature difference ΔT. This amounts to setting $s=\rho c_p\int_{T_0}^{T_1}(dT/T)\simeq\rho c_p\Delta T/T_0$, c_p being the specific heat capacity at constant pressure. We then get

$$\frac{\eta}{s}=\frac{8\nu T_0}{5u^2}\frac{1}{\text{Pr}},$$

where $\nu=\eta/\rho$ is the kinematic viscosity and $\text{Pr}=\nu\rho c_p/\lambda$ the Prandtl number. We choose the moderate velocity $u=1$ mm/s to keep the Reynolds number small, and take $T=300$ K. Then, with $\nu=0.010\text{ cm}^2/\text{s}$, Pr=6.75 [21] we get $\eta/s=71$ K s as a typical value. The inequality (79) is obviously satisfied.

Example 2. Plasma era in the early Universe. As the next step we consider the initial stage of the plasma era in the early Universe. This can be taken to occur at about $t=1000$ s after the big bang, when the Universe was characterized by ionized H and He in approximate equilibrium with radiation (cf. Refs. [22–25]). The number densities of electrons and photons are equal, $n\simeq 10^{19}\text{ cm}^{-3}$, the temperature is $T\simeq 4\times 10^8$ K, and the energy density is $\rho c^2=a_rT^4$, where $a_r=\pi^2k_B^4/(15\hbar^3c^3)=7.56\times 10^{-15}\text{ erg cm}^{-3}K^{-4}$ is the radiation constant. The pressure is $p=\rho c^2/3$. The presence of energy dissipation and viscosity coefficients in the cosmic fluid is due to the fact that the thermal equilibrium is not quite perfect. From relativistic kinetic theory one can calculate the viscosity coefficients. Let $x=m_ec^2/k_BT$ be the ratio between electron rest mass and thermal energy; when $x\gg 1$ it is convenient to use the polynomial approximations [26] (cf. also [24]) for the evaluation of the shear viscosity η and the bulk viscosity ζ:

$$\eta=\frac{5m_e^6c^8\zeta(3)}{9\pi^3\hbar^3e^4n}x^{-4},\quad \zeta=\frac{\pi c^2\hbar^3n}{256e^4\zeta(3)}x^3, \tag{81}$$

$\zeta(3)=1.202$ being the Riemann zeta function. At $T=4\times 10^8$ K one has $x=14.8$, leading to

$$\eta=2.8\times 10^{14}\text{ g cm}^{-1}\text{ s}^{-1},\quad \zeta=7.0\times 10^{-3}\text{ g cm}^{-1}\text{ s}^{-1}. \tag{82}$$

We note that both η and ζ now contain $\hbar$, and also that η is enormously larger than ζ.

The entropy density, in view of the radiation dominance, is given by

$$s=\frac{4}{3}a_rT^3=6.45\times 10^{11}\ \text{erg cm}^{-3}\text{K}^{-1}, \tag{83}$$

and so $\eta/s=435$ K s. This value is surprisingly large enough to satisfy the bound in Eq. (79) as the value in *Example 1*. There seems to be no simple reason why this should be so; the physical conditions in the two cases are widely different.

So far, we assumed a radiation-dominated FRW universe. What happens if the universe is instead filled with matter obeying the relation $p=w\rho c^2$, with w constant and negative? To investigate this point let us go back to Eq. (17), in which the subextensive parts are neglected. For the ratio $s/\rho c^2$, where $s=S/V$ and $\rho c^2=E/V$, we obtain

$$\frac{s}{\rho c^2}=\frac{1+w}{T}. \tag{84}$$

This expression is seen to be independent of the prefactor f_0. Let us assume that the energy density at $T=4\times 10^8$ K is the same as before, i.e., $\rho c^2=a_rT^4=1.94\times 10^{20}$ erg cm^{-3}. Then s is found from (84), and taking the shear viscosity to be given by (81) as before, we obtain the following simple equation:

$$\frac{\eta}{s}=\frac{578}{1+w}. \tag{85}$$

We see that except in the case where w is close to -1, the order of magnitude of η/s is roughly the same as above. It is moreover evident that the expression (85) is physically meaningful only when $w>-1$ (the viscosity η has always to be positive, for general thermodynamical reasons). We thus see that the inclusion of shear viscosity implies that it is only the case of quintessence that is of physical interest. The case of phantoms, $w<-1$, leads to negative entropies and is in the present context excluded.

Example 3. The Kasner universe. Our third example is taken from the theory of the very early Universe. From ordinary hydrodynamics we know that the shear viscosity comes into play whenever there are fluid sheets sliding with respect to each other. Correspondingly, in a relativistic formulation, the most natural circumstances under which η is expected to be of significance are when anisotropy is brought into consideration. It becomes natural to focus attention on the anisotropic Kasner metric

$$ds^2=-dt^2+t^{2p_1}dx^2+t^{2p_2}dy^2+t^{2p_3}dz^2, \tag{86}$$

where the numbers p_1,p_2,p_3 are constants. The two numbers P and Q are defined by $P=\Sigma_{i=1}^3 p_i$, $Q=\Sigma_{i=1}^3 p_i^2$. In a vacuum Kasner space, $P=Q=1$. Here, we assume that there is an isotropic fluid with energy density ρ and pressure p immersed in this space. Both ρ and p, as well as the viscosity coefficients η and ζ, are assumed to be dependent on time but independent of position. If $U^\mu=(U^0,U^i)$ is the fluid's four-velocity, the energy-momentum tensor is $T_{\mu\nu}=\rho U_\mu U_\nu+(p-\zeta\theta)h_{\mu\nu}-2\eta\sigma_{\mu\nu}$, where $h_{\mu\nu}=g_{\mu\nu}+U_\mu U_\nu$ is the projection tensor, $\theta=U^\mu{}_{;\mu}$ is the scalar expansion, $\theta_{\mu\nu}=\frac{1}{2}(U_{\mu;\alpha}h^\alpha_\nu+U_{\nu;\alpha}h^\alpha_\mu)$ is the expansion tensor, and $\sigma_{\mu\nu}=\theta_{\mu\nu}-\frac{1}{3}h_{\mu\nu}\theta$ is the shear tensor.

Consider now the Einstein equations, taking the cosmological constant Λ to be zero. With $\kappa^2=16\pi G$ we obtain from $R_{\mu\nu}=\frac{1}{2}\kappa^2(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T^\alpha_\alpha)$ the two equations

$$P-Q+\frac{3}{4}\kappa^2t\zeta P=\frac{1}{4}\kappa^2t^2(\rho+3p), \tag{87}$$

$$p_i(1-P-\kappa^2t\eta)+\frac{1}{4}\kappa^2t\left(\zeta+\frac{4}{3}\eta\right)P=-\frac{1}{4}\kappa^2t^2(\rho-p). \tag{88}$$

The structure of the Einstein equations leads to the time relationships,

$$\rho(t)=\rho_0(t_0/t)^2,\quad p(t)=p_0(t_0/t)^2,$$
$$\zeta(t)=\zeta_0t_0/t,\quad \eta(t)=\eta_0t_0/t, \tag{89}$$

where $\{\rho_0,p_0,\zeta_0,\eta_0\}$ refer to the chosen initial instant $t=t_0$. We can then write the equations such that they contain time-independent quantities only:

$$P-Q+\frac{3}{4}\kappa^2\zeta_0t_0P=\frac{1}{4}\kappa^2t_0^2(\rho_0+3p_0), \tag{90}$$

$$p_i(1-P-\kappa^2\eta_0t_0)+\frac{1}{4}\kappa^2t_0\left(\zeta_0+\frac{4}{3}\eta_0\right)P$$
$$=-\frac{1}{4}\kappa^2t_0^2(\rho_0-p_0). \tag{91}$$

Let us consider the production of entropy. First, for the Bianchi type-I spaces the average expansion anisotropy parameter A is defined as [27]

$$A=\frac{1}{3}\sum_{i=1}^3\left(1-\frac{H_i}{H}\right)^2, \tag{92}$$

where $H_i=\dot{a}_i/a_i$ with $a_i=t^{p_i}$ are the directional Hubble factors and $H=\frac{1}{3}\Sigma_1^3H_i$ is the average Hubble factor. Accordingly, in our case $A=3Q/P^2-1$. Next, the entropy current four-vector is $S^\mu=n\sigma U^\mu$, where n is the baryon number density and $\sigma=s/n$ the nondimensional entropy per baryon. In general,

$$S^\mu{}_{;\mu}=\frac{2\eta}{T}\sigma_{\mu\nu}\sigma^{\mu\nu}+\frac{\zeta}{T}\theta^2, \tag{93}$$

meaning in the comoving frame of reference ($\dot{\sigma}=d\sigma/dt$)

$$\dot{\sigma}=\frac{3P^2}{nTt^2}\left(\zeta+\frac{2}{3}\eta A\right). \tag{94}$$

As we would expect, the anisotropy in general provides a significant contribution to the growth of entropy, in view of the large magnitude of η. However, let us go back to Eq. (91): this equation tells us that all the p_i have to be *equal* in the present case. With $p_1=p_2=p_3\equiv b$ we get for the *isotropic* Kasner space

$$b=\frac{1}{6}\left[1+\frac{3}{4}\kappa^2 t_0\zeta_0 + \sqrt{\left(1+\frac{3}{4}\kappa^2 t_0\zeta_0\right)^2+3\kappa^2 t_0^2(\rho_0-p_0)}\right]. \tag{95}$$

It is seen that the shear viscosity is absent. Equation (94) reduces to

$$\dot{\sigma}=\frac{3P^2}{nk_BTt^2}\zeta, \tag{96}$$

when written in dimensional form.

Let us evaluate the expression (96). Due to the proportionality to the small bulk viscosity we can insert for n and T as if the cosmic fluid were ideal. Thus from conservation of particle number, $n\propto a^{-3}$, and from conservation of entropy, $a\propto T^{-1}$. Moreover, as $t\propto T^{-2}$, we can write Eq. (96) as

$$\dot{\sigma}=\frac{3P^2\zeta_0}{n_0k_BT_0t_0}\,\frac{1}{t}. \tag{97}$$

Thus $\sigma-\sigma_0\propto\ln(t/t_0)$ is the increase in specific entropy when t increases from t_0 to t. Multiplying with the particle density n, we obtain an expression for the corresponding increase $s-s_0$ in entropy density. Recalling the expression in Eq. (89) for η, we then derive as our main result the following expression for the sought ratio:

$$\frac{\eta}{s}=\frac{\eta_0t_0}{s_0t}\left[1+\frac{3P^2\zeta_0}{s_0T_0t_0}\left(\frac{t_0}{t}\right)^{3/2}\ln\frac{t}{t_0}\right]^{-1}. \tag{98}$$

It is of interest to evaluate this expression at $t=t_0$. Let us identify t_0 with the instant at which $T=10^{12}$ K, i.e., at $t=2\times10^{-4}$ s. Then $n_0=6\times10^{29}\ \mathrm{cm}^{-3}$, $\rho_0=4.5\times10^{34}\ \mathrm{erg\ cm}^{-3}$. This temperature is a kind of limit for standard cosmological theory. If $T>10^{12}$ K the universe consists of many kinds of particles and antiparticles, but when T has fallen below this value a large number of hadrons has disappeared, and the universe consists of leptons, antileptons, photons, and nucleons. We then have [26]

$$\eta=\frac{3\pi c\hbar^4}{608m_eG_F^2}x,\qquad \zeta=\frac{\pi c\hbar^4}{7776m_eG_F^2}x^5, \tag{99}$$

which is valid when $x=m_ec^2/k_BT$ is small. Here, the weak coupling constant is given by $G_Fc/\hbar^3=10^{-5}m_p^{-2}$.

With $T_0=10^{12}$ K we get $x=5.94\times10^{-3}$, and so we have at this instant

$$\eta_0=1.8\times10^{23}\ \mathrm{g\ cm^{-1}\ s^{-1}},\quad \zeta_0=6.0\times10^{12}\ \mathrm{g\ cm^{-1}\ s^{-1}}. \tag{100}$$

The entropy density is calculated approximatively by assuming radiation dominance, such as before. Then, from $s=\frac{4}{3}a_rT^3$ we get $s_0=1.0\times10^{22}\ \mathrm{erg\ cm^{-3}\ K^{-1}}$. Thus, $\eta_0/s_0=18$ K s. Once again, we end up with the same order-of-magnitude result for the ratio η/s as before, when we choose to work at the instant $t=t_0$. However, Eq. (98) tells us that η/s *diminishes* with increasing t, and approaches zero when $t\to\infty$. It means that η/s cannot in this case be subject to a lower bound. The Kasner case thus provides a counterexample to the suggestion in Eq. (79). Of course, this can be considered as rather academical as current Universe is not an anisotropic one.

Actually, it follows already from the thermodynamical formalism that the lower bound in Eq. (79) cannot be universal. At least this is so in a phenomenological theory, in which η and ζ are arbitrary input parameters. Namely, from Eq. (93) or (94) it is seen that the specific entropy rate of change involves both η and ζ. Let us imagine that η is kept constant while ζ is changing. Therefore $\dot{\sigma}$ and accordingly σ itself, as well as the ratio η/s, change. If this ratio were subject to a lower bound, this would correspond to the existence of a maximum value of s. However, we may make σ and s as large as we wish, by inserting a sufficiently large value of ζ in Eq. (94). Recall in this context the way in which viscosity coefficients are introduced in fluid mechanics: they are based on the assumption that *first order* velocity gradients are sufficient to construct the contribution to the stress tensor due to deviations from thermal equilibrium. The theory is thus approximate already from the outset.

The discussion of Verlinde [1] about the holographic bound on the subextensive entropy associated with the Casimir energy assumed a radiation-dominated FRW universe. As shown in Ref. [20], the same entropy formula holds if the fluid possesses a constant, though small, bulk viscosity. Similarly, the generalized entropy formula [5] for the case in which the state equation is $p=w\rho$ with w a constant (still assuming a FRW metric) was also found to hold in the presence of the same kind of viscosity [28].

One may ask: How does the entropy formula look if the cosmic fluid possesses both a shear viscosity and a bulk viscosity? The answer is immediate, if the anisotropy is originally introduced via the Kasner metric. As shown above the Einstein equations eliminate the anisotropies, and we are left with an isotropic Kasner metric whose scale factor is t^b, where b is given by Eq. (95). The anisotropy factor A vanishes, and the production of entropy is governed by the bulk viscosity ζ; cf. Eq. (94).

Let us assume that ζ is constant and small, so that we can adopt the same expression for $a(t)$ as in the case of a nonviscous fluid. The argument can be given similarly to that given in Ref. [28]: Taking $n=3$ we see that the quantity $\rho a^{3(w+1)}$ can be considered as a function of $n\sigma$. Since $E\sim\rho a^3$ and $S\sim n\sigma a^3$, it follows that Ea^{3w} is independent of V and is a function of S only. The conventional decomposition of the total energy $E(S,V)$ into an extensive part and a subextensive part, $E(S,V)=E_E(S,V)+\frac{1}{2}E_C(S,V)$, together with the scale transformations $E_E(\lambda S,\lambda V)=\lambda E_E(S,V)$, $E_C(\lambda S,\lambda V)=\lambda^{1/3}E_C(S,V)$, leads to

$$E_E=\frac{\alpha}{4\pi a^{3w}}S^{w+1}, \quad E_C=\frac{\beta}{2\pi a^{3w}}S^{w+1/3}, \tag{101}$$

where α and β are constants. We thus get, when we reinstate $a=t^b$,

$$S=\left[\frac{2\pi t^{3bw}}{\sqrt{\alpha\beta}}\sqrt{(2E-E_C)E_C}\right]^{3/(3w+2)}. \tag{102}$$

This is the Kasner-metric-induced form of our previous expression (18), when $n=3$. It will be of interest to understand better the connection between the CV formula and shear viscosity bound. However, this requires the nontrivial generalization of the CV formula for an anisotropic universe with shear viscosity.

VI. DISCUSSION

In summary, we studied the entropy of a FRW universe filled with dark energy and its representation in the form of the holographic CV formula. This investigation shows that the expression of the entropy in terms of energy and Casimir energy depends on the equation of state in a quite complicated form. Only for a radiation-dominated FRW universe does the corresponding CV formula acquire the form typical for 2D CFT entropy. At the same time, for a negative or time-dependent equation of state, such a formula seems to have nothing in common with 2D CFT, still being related to holography. Nevertheless, there exists another cosmological CV formula that is very useful for deriving the entropy bounds and that is the same for any type of matter under consideration. It is remarkable that the universality of the cosmological CV formula together with the fact that it is predicted by the form of FRW equations proves its holographical origin. Of course, the actual reasons for such a manifestation of the holographic principle in the modern Universe remain obscure. (Some hints may be drawn from the brane-world approach.) Furthermore, all conclusions above remain true for the modified gravity, which is considered as a gravitational alternative for dark energy. This should not seem strange after all, as modified gravity may be rewritten in the classically equivalent form as a type of scalar-tensor gravity with matter described by a scalar field.

The black hole thermodynamics in modified gravity is also considered. The black hole entropy law is slightly different (by numerical factor) from the standard case of the Einstein gravity. In the last section we analyzed the recently proposed bound for the ratio of shear viscosity to entropy density. This bound seems to follow from the Bekenstein entropy bound. As shear viscosity is absent in the current isotropic Universe, we concentrate on the early Universe in the plasma era or the anisotropic Kasner universe, where the newly proposed bound seems to be violated.

The important lesson drawn from this and other studies of the entropy of the FRW universe is that the holographic principle does not distinguish whether dark energy is present or not. For instance, the cosmological CV formula is the same whatever the equation of state. This indicates that the origin of dark energy should be sought within fundamental theory, perhaps within string/M theory.

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