

Neutrino propagation in a strongly magnetized medium

E. Elizalde*

Institute for Space Studies of Catalonia, IEEC/CSIC, Edifici Nexus, Gran Capità 2-4, 08034 Barcelona, Spain

Efrain J. Ferrer and Vivian de la Incera

Physics Department, State University of New York at Fredonia, Fredonia, New York 14063, USA

(Received 26 May 2004; published 27 August 2004)

We derive general expressions at the one-loop level for the coefficients of the covariant structure of the neutrino self-energy in the presence of a constant magnetic field. The neutrino energy spectrum and index of refraction are obtained for neutral and charged media in the strong-field limit ($M_W \gg \sqrt{B} \gg m_e, T, \mu, |\mathbf{p}|$) using the lowest Landau level approximation. The results found within the lowest Landau level approximation are numerically validated, summing in all Landau levels, for strong $B \gg T^2$ and weakly strong $B \gtrsim T^2$ fields. The neutrino energy in leading order of the Fermi coupling constant is expressed as the sum of three terms: a kinetic-energy term, a term of interaction between the magnetic field and an induced neutrino magnetic moment, and a rest-energy term. The leading radiative correction to the kinetic-energy term depends linearly on the magnetic field strength and is independent of the chemical potential. The other two terms are only present in a charged medium. For strong and weakly strong fields, it is found that the field-dependent correction to the neutrino energy in a neutral medium is much larger than the thermal one. Possible applications to cosmology and astrophysics are considered.

DOI: 10.1103/PhysRevD.70.043012

PACS number(s): 95.30.Cq, 13.15.+g, 14.60.Pq, 98.80.Cq

I. INTRODUCTION

There are many astrophysical systems on which the physics of neutrinos in a magnetic field plays an important role. Let us recall that proto-neutron stars typically possess very strong magnetic fields. Large magnetic fields $B = 10^{12} - 10^{14}$ G have been associated with the surface of supernovas [1] and neutron stars [2], and fields perhaps as large as 10^{16} G with magnetars [3]. Even larger fields could exist in the star's interior. It is presumed from the scalar virial theorem [4] that the interior field in neutron stars could be as high as 10^{18} G. A magnetic field such as this ($\sim 10^{18}$ G) in the interior of a compact star will be larger in two orders than the chemical potential characterizing its quark matter density.

Unveiling the interconnection between the star magnetic field and its particle current flows could shed new light on the question of the star evolution. For example, it is well known that neutrinos drive supernova dynamics from beginning to end. Neutrino emission and interactions play a crucial role in core collapse supernovae [5]. Their eventual emission from the proto-neutron star contains nearly all the energy released in the star explosion. Neutrino luminosity, emissivity and the specific heat of the densest parts of the star are governed by charged and neutral current interactions involving matter at high densities and in the presence of strong magnetic fields. Thus, a total understanding of the star cooling mechanism in a strongly magnetized medium is crucial for astrophysics.

On the other hand, the explanation of large-scale magnetic fields observed in a number of galaxies, and in clusters of

galaxies [6] seems to require the existence of seed fields of primordial origin [7]. According to several mechanisms [8], strong primordial fields $\sim 10^{24}$ G could be generated at the electroweak transition. Even larger fields have been associated with superconducting magnetic strings, which would generate fields $\sim 10^{30}$ G in their vicinity if created after inflation [9].

Were primordial magnetic fields present in the early universe, they would have had nontrivial consequences for particle-physics cosmology. For instance, as it is well known, oscillations between neutrino flavors may change the relative abundance of neutrino species and may thereby affect primordial nucleosynthesis (for a recent review on neutrinos in cosmology see [10]). Therefore, if a strong magnetic field ($M_W \gg \sqrt{B} \gg m_e, T, \mu, |\mathbf{p}|$, with M_W and m_e the W-boson and electron masses respectively) modifies the neutrino energy spectrum of different flavors in different ways, a primordial magnetic field can consequently influence the oscillation process in the primeval plasma [11].

The propagation of neutrinos in magnetized media has been previously investigated by several authors [12–15]. Weak-field calculations were done for magnetized vacuum in Refs. [12,13], and at $T \neq 0$ and $\mu = 0$ (μ is the electric chemical potential) in Ref. [14]. In the $\mu = 0$ case, as long as $B < T^2$, both the field- and temperature-dependent leading contributions to the neutrino energy resulted of $1/M_W^4$ -order [12–14]. In the charged plasma [15], the field-dependent terms were much larger, $\sim 1/M_W^2$, but they vanished in the ($\mu \rightarrow 0$)-limit. The weak-field results of papers [12–15] led to the idea that the magnetic-field effects could be significant in astrophysics, because of the field- and μ -dependent terms of order $1/M_W^2$, but were irrelevant ($\sim 1/M_W^4$ -order) in the early universe due to its charge-symmetric character ($\mu = 0$). Hence, it has been assumed that in cosmology the

*On leave at Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA.

main contribution to neutrino energy was the purely thermal term of order T^4/M_W^4 [16]. However, as we will show below, a strong magnetic field ($M_W \gg \sqrt{B} \gg m_e, T, \mu, |\mathbf{p}|$) gives rise to a new contribution to the neutrino energy that is linear in the field, independent of the chemical potential, and that is of the same order ($1/M_W^2$) as the largest terms found in the weak-field charged-medium case. This new result can turn magnetic-field effects relevant for cosmology.

In recent papers [17,18], we investigated the effects of a strong magnetic field on neutrinos in magnetized vacuum (i.e. with $T=0$ and $\mu=0$). There, to facilitate the calculations in the strong-field limit, we extended the Ritus' Epeigenfunction method of diagonalization of the Green functions of spin-1/2 charged particle in electromagnetic field [19,20], to the case of spin-1 charged particles [17]. This formulation, which is particularly advantageous for strong-field calculations, provides an alternative method to the Schwinger approach to address QFT problems in electromagnetic backgrounds [21]. The use of Ritus' method made it very convenient to study the neutrino-self-energy in magnetized media, since it allowed to diagonalize in momentum space both the electron and the W-boson Green's functions in the presence of a magnetic field. Ritus' formalism has also been recently applied to investigate nonperturbative QFT in electromagnetic backgrounds [22].

From the above considerations it is clear that strong magnetic fields can play a significant role in a variety of astrophysical systems, and possibly also in the early universe. For these applications, the analysis has to be carried out in the presence of a medium. Thus, in the present paper we extend the results obtained in papers [17] and [18] to include finite temperature and density, performing a detailed study of the effects of a strong magnetic field on neutrino propagation in neutral and charged media, and discussing possible applications to astrophysics and cosmology.

We stress that in calculating the neutrino self-energy in a magnetized medium, we should consider, as usual, its vacuum and statistical parts. In this case the vacuum part depends on the magnetic field, and for strong fields it can make important contributions even at high temperatures $T^2 \gg eB$. The reason is that the vacuum and statistical terms have different analytical behaviors, due to the lack of the statistic ultraviolet cutoff in the vacuum part. This fact gives rise to a field-dependent vacuum contribution ($1/M_W^2$ -order) which is larger than the thermal one ($1/M_W^4$ -order), into the self-energy. Therefore, as shown in this paper, a strong magnetic field can become more relevant than temperature for neutrino propagation in neutral media.

The plan of the paper is as follows. In Sec. II we consider the radiative correction to the neutrino dispersion relation in the presence of a constant magnetic field. We introduce the general covariant structure of the neutrino self-energy in the presence of an external field, and find the dispersion relation as a function of the coefficients of each independent term in the covariant structure. The general form of the found dispersion relation goes beyond any given approximation and medium characteristic and serves as a guidance for particular applications. The general expressions for the coefficients of

the self-energy structures in the presence of the magnetic field are found in the one-loop approximation in Sec. III. The leading behavior in the $1/M_W^2$ expansion of those coefficients is then calculated in Sec. IV in the strong-field limit [i.e. in the lowest Landau level (LLL) approximation] for neutral and charged magnetized media. These results are then used in Sec. V to find the corresponding neutrino dispersion relations and the indexes of refraction in neutral and charged strongly magnetized media. In Sec. VI, the LLL-approximation is numerically corroborated by summing in all Landau levels and finding the values of the coefficients for parameter ranges corresponding to strong $B \gg T^2$ and weakly strong $B \gtrsim T^2$ fields. Possible applications to cosmology and astrophysics are discussed in Sec. VII. Finally, in Sec. VIII we summarize the main outcomes of the paper and make some final remarks.

II. NEUTRINO SELF-ENERGY GENERAL STRUCTURE IN A MAGNETIC FIELD

The neutrino field equation of motion in a magnetized medium, including radiative corrections, is

$$\left[\not{p} + \Sigma \right] \Psi_L = 0 \quad (1)$$

where the neutrino self-energy operator $\Sigma(p)$ depends on the parameters characterizing the medium, as for instance, temperature, magnetic field, particle density, etc.

The operator $\Sigma(p)$ is a Lorentz scalar that can be formed in the spinorial space taking the contractions between the characteristic vectors and tensors of the system with all the independent elements of the Dirac ring. Explicit chirality reduces it to

$$\Sigma = R \overline{\Sigma} L, \quad \overline{\Sigma} = V_\mu \gamma^\mu \quad (2)$$

where $L, R = \frac{1}{2}(1 \pm \gamma_5)$ are the chiral projector operators, and V_μ is a Lorentz vector that can be spanned as a superposition of four basic vectors that can be formed from the characteristic tensors of the problem. In a magnetized medium, besides the neutrino four-momentum p_μ , we have to consider the magnetic-field strength tensor $F_{\mu\nu}$, to form the covariant structures

$$\overline{\Sigma}(p, B) = a \not{p}_\parallel + b \not{p}_\perp + c p^\mu \hat{F}_{\mu\nu} \gamma^\nu + i d p^\mu \hat{F}_{\mu\nu} \gamma^\nu. \quad (3)$$

In Eq. (3) we introduced the notations $\hat{F}_{\mu\nu} = (1/|B|) F_{\mu\nu}$, and $\hat{F}_{\mu\nu} = (1/2|B|) \varepsilon_{\mu\nu\rho\lambda} F^{\rho\lambda}$. In the covariant representation the magnetic field can be expressed as $B_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} u^\nu F^{\rho\lambda}$, where u_μ is the vector four-velocity of the center of mass of the magnetized medium. The presence of the magnetic field, and hence of the dimensionless magnetic field tensor $\hat{F}_{\mu\nu}$ and its dual $\hat{F}_{\mu\nu}$, allows the covariant separation in Eq. (3) between longitudinal and transverse momentum terms that appears naturally in magnetic backgrounds

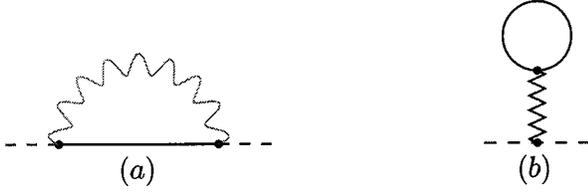


FIG. 1. One-loop contributions to the neutrino self-energy in a magnetized medium. (a) Bubble graph: The dashed line represents the test neutrino, the solid line a charged lepton of the same family of the test neutrino, and the wiggly line the W-boson. (b) Tadpole graph: The dashed line represents the test neutrino, the solid line a charged lepton of any species, and the zigzag line the Z-boson.

$$\not{p}_{\parallel} = p^{\mu} \hat{F}_{\mu}^{\rho} \hat{F}_{\rho\nu} \gamma^{\nu}, \quad \not{p}_{\perp} = p^{\mu} \hat{F}_{\mu}^{\rho} \hat{F}_{\rho\nu} \gamma^{\nu}. \quad (4)$$

Coefficients a , b , c , and d are Lorentz scalars that depend on the parameters of the theory and the approximation used. Notice that finite temperature and/or density would not introduce any new term into Eq. (3). This formulation is indeed the most general one for a magnetized medium. In magnetized space, even if $T = \mu = 0$, the vector u_{μ} is present, since the presence of a constant magnetic field fixes a special Lorentz frame: the rest frame [on which $u_{\mu} = (1, 0, 0, 0)$] where the magnetic field is defined ($u_{\mu} F^{\mu\nu} = 0$). Thus, the inclusion of a medium does not break any additional symmetry. To express $\bar{\Sigma}(p, B)$ in terms of u_{μ} only means, therefore, to make a change in the selected basis vectors.¹

The neutrino energy spectrum in a magnetic background can be found from Eq. (1) as the nontrivial solution of

$$\det \left[\not{p} + \sum (p, B) \right] = 0. \quad (5)$$

Using the covariant structure (3), the dispersion relation (5) takes the form

$$p_0^2 = p_3^2 + \frac{(1+b)^2 - d^2}{(1+a)^2 - c^2} p_{\perp}^2. \quad (6)$$

One of the main goals of this paper is to find the coefficients a , b , c , and d , in the one-loop approximation, and hence, the dispersion relations for different systems in the presence of strong magnetic-field backgrounds.

III. ONE-LOOP NEUTRINO SELF-ENERGY

Let us consider the one-loop corrections to the neutrino self-energy in the presence of a constant magnetic field. To leading order in the Fermi coupling constant the main field-dependent contribution to the self-energy comes from the bubble diagram [Fig. 1(a)] with internal lines of virtual charged leptons and W-bosons, and from the tadpole diagram [Fig. 1(b)] with virtual loop of charged leptons.

In a neutral medium only the bubble diagram contributes,

while in a charged medium both diagrams should be considered. However, only the bubble distinguishes between neutrino flavors, since the flavor of the internal charged lepton is directly associated to the flavor of the propagating neutrino (i.e., the one appearing in the external legs). For the tadpole, because the internal boson is the neutral Z-boson, the charged leptons corresponding to different families are linked to the same neutrino flavor. Therefore, with regard to magnetic field effects on neutrino oscillations, the bubble contribution is the essential one. Thus, because the ultimate goal of our study is to elucidate the possible effect of a strong magnetic field on neutrino flavor oscillations, henceforth we only consider the bubble diagram contribution, represented in Fig. 1(a), into the neutrino self-energy.

The bubble contribution to the one-loop self-energy is

$$\Sigma(x, y) = \frac{ig^2}{2} R \gamma_{\nu} S(x, y) \gamma^{\mu} G_F(x, y)_{\mu}{}^{\nu} L \quad (7)$$

where $S(x, y)$ and $G_F(x, y)_{\mu}{}^{\nu}$ are the Green's functions of the electron and W-boson, in the presence of the magnetic field, respectively. Since both virtual particles are electrically charged, the magnetic field interacts with both of them producing the Landau quantization of the corresponding transverse momenta [17,18].

The electron Green's function, diagonal in momentum space at arbitrary field strength, was obtained in Refs. [19,20] by Ritus, using what has become an alternative method to the Schwinger approach [21] to deal with QFT problems on electromagnetic backgrounds. In this method the electron Green's function in configuration space is given by

$$S(x, y) = \int_i \frac{d^3 \hat{q}}{(2\pi)^4} E_q(x) \frac{1}{\gamma \cdot \hat{q} + m_e} \bar{E}_q(y) \quad (8)$$

where $\hat{q}_{\mu} = (q_0, 0, q_2, q_3)$, $\bar{E}_q = \gamma^0 E_q^{\dagger} \gamma^0$, and the magnetic field has been specialized in the rest frame along the Z -direction (i.e. given in the Landau gauge as $A_{\mu}^{ext} = B x_1 \delta_{\mu 2}$).

The transformation functions $E_q(x)$ in Eq. (8) play the role, in the presence of magnetic fields, of the usual Fourier functions e^{iqx} in the free case and are given by

$$E_q(x) = \sum_{\sigma} E_{q\sigma}(x) \Delta(\sigma), \quad (9)$$

with spin matrix

$$\Delta(\sigma) = \text{diag}(\delta_{\sigma 1}, \delta_{\sigma -1}, \delta_{\sigma 1}, \delta_{\sigma -1}), \quad \sigma = \pm 1, \quad (10)$$

and eigenfunctions

$$E_{q\sigma}(x) = N(\kappa) \exp i(\hat{q} \cdot \hat{x}) D_{\kappa}(\rho) \quad (11)$$

where $N(\kappa) = (4\pi |eB|)^{1/4} / \sqrt{\kappa!}$ is a normalization factor and $D_{\kappa}(\rho)$ denotes the parabolic cylinder functions [23] with argument $\rho = \sqrt{2|eB|} [x_1 - (q_2/eB)]$ and positive integer index

¹This point will turn clear in Sec. IV when expressing $\bar{\Sigma}(p, B)$ in terms of u_{μ} in the charged medium.

$$\kappa = \kappa(l, \sigma) \equiv l + \frac{\sigma}{2} - \frac{1}{2}, \quad \kappa = 0, 1, 2, \dots \quad (12)$$

The integer l in Eq. (12) labels the Landau levels $l = 0, 1, 2, \dots$

The electron momentum eigenvalue in the presence of the magnetic field, \bar{q} , is given by

$$\bar{q}_\mu = (q_0, 0, -\text{sgn}(eB)\sqrt{2|eB|l}, q_3) \quad (13)$$

Ritus' technique for spin-1/2 particles was recently extended to the spin-1 particle case in Refs. [17,18]. In this case the corresponding W-boson Green's function can be diagonalized in momentum space as

$$G_F(k, k')_\mu{}^\nu = (2\pi)^4 \hat{\delta}^{(4)}(k - k') \frac{\delta_\mu{}^\nu}{\bar{k}^2 + M_W^2} \quad (14)$$

where $\hat{\delta}^{(4)}(k - k') = \delta^3(k - k') \delta_{nn'}$, and the W-boson momentum eigenvalue in the presence of the magnetic field, \bar{k} , is given by the relation

$$\bar{k}^2 = -k_0^2 + k_3^2 + 2(n - 1/2)eB, \quad n = 0, 1, 2, \dots \quad (15)$$

with n denoting the W-boson Landau levels. Considering the mass-shell condition $\bar{k}^2 = -M_W^2$ in Eq. (15), we can identify at $eB > M_W^2$ the so called "zero-mode problem" [24,25]. As known, at those values of the magnetic field a vacuum instability appears giving rise to W-condensation [25]. In our calculations we restrict the magnitude of the magnetic field to $eB < M_W^2$, so as to avoid the presence of any tachyonic mode.

The W-boson Green's function in configuration space is then

$$G_F(x, y)_\mu{}^\nu = \sum_n \frac{d^3 \hat{k}}{(2\pi)^4} \Gamma_{k\mu}^\alpha(x) \frac{\delta_\alpha^\beta}{\bar{k}^2 + M_W^2} \Gamma_{k\beta}^\dagger{}^\nu(y) \quad (16)$$

where the transformation functions are

$$(2\pi)^4 \delta^{(4)}(p - p') \Sigma(p, B) = \frac{ig^2}{2} \int d^4x \int d^4y e^{-i(p \cdot x - p' \cdot y)} \left\{ R \left[\gamma_\nu \left(\sum_l \frac{d^3 \hat{q}}{(2\pi)^4} E_q(x) \frac{1}{\gamma \cdot \bar{q} + m_e} \bar{E}_q(y) \right) \right. \right. \\ \left. \left. \times \gamma^\mu \left(\sum_n \frac{d^3 \hat{k}}{(2\pi)^4} \frac{\Gamma_{k\mu}^\alpha(x) \Gamma_{k\alpha}^\dagger{}^\nu(y)}{\bar{k}^2 + M_W^2} \right) \right] L \right\}. \quad (24)$$

The coefficients of the covariant expression for $\Sigma(p, B)$ can be found using the explicit form (24) in the covariant expression

$$(2\pi)^4 \delta^{(4)}(p - p') \Sigma(p, B) = a' \not{p}_\parallel + b' \not{p}_\perp + c' p^\mu \hat{F}_{\mu\nu} \gamma^\nu + id' p^\mu \hat{F}_{\mu\nu} \gamma^\nu. \quad (25)$$

Introducing in Eq. (24) the transformation functions E_q and $\Gamma_{k\mu}^\alpha$, and taking into account the following properties of the spinor matrices:

$$\Gamma_{k\mu}^\alpha(x) = P^\alpha_\gamma [\mathcal{F}_k(x)]^\gamma_\lambda P^{-1\lambda}{}_\mu \quad (17)$$

with

$$P^\alpha_\gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix} \quad (18)$$

and

$$[\mathcal{F}_k(x)]^\gamma_\lambda = \sum_{\eta=0, \pm 1} \mathcal{F}_{k\eta}(x) [\Omega^{(\eta)}]^\gamma_\lambda. \quad (19)$$

The matrices $\Omega^{(\eta)}$ are explicitly given in terms of the W-boson spin projections η by

$$\Omega^{(\eta)} = \text{diag}(\delta_{\eta,0}, \delta_{\eta,1}, \delta_{\eta,-1}, \delta_{\eta,0}), \quad \eta = 0, \pm 1 \quad (20)$$

and the eigenfunctions $\mathcal{F}_{k\eta}$ are

$$\mathcal{F}_{k\eta}(x) = N(\kappa') \exp(i(\hat{k} \cdot \hat{x}) D_{\kappa'}(\xi)) \quad (21)$$

with $N(\kappa')$ a normalization factor similar to that appearing in Eq. (11), and $D_{\kappa'}(\xi)$ denoting the parabolic cylinder functions with argument $\xi = \sqrt{2|eB|(x_1 - k_2/eB)}$ and positive integer index κ' given in terms of the W-boson Landau numbers n and spin projections η as

$$\kappa' = \kappa'(n, \eta) \equiv n - \eta - 1, \quad \kappa' = 0, 1, 2, \dots \quad (22)$$

As the neutrino is an electrically neutral particle, the transformation to momentum space of its self-energy can be carried out by the usual Fourier transform

$$(2\pi)^4 \delta^{(4)}(p - p') \Sigma(p, B) = \int d^4x d^4y e^{-i(p \cdot x - p' \cdot y)} \Sigma(x, y). \quad (23)$$

Substituting with Eqs. (7), (8), and (16) on the right-hand side (RHS) of Eq. (23) we obtain

$$\begin{aligned}
\Delta(\pm)^\dagger &= \Delta(\pm), \quad \Delta(\pm)\Delta(\pm) = \Delta(\pm), \quad \Delta(\pm)\Delta(\mp) = 0 \\
\gamma^\parallel \Delta(\pm) &= \Delta(\pm)\gamma^\parallel, \quad \gamma^\perp \Delta(\pm) = \Delta(\mp)\gamma^\perp, \\
L\Delta(\pm) &= \Delta(\pm)L, \quad R\Delta(\pm) = \Delta(\pm)R
\end{aligned} \tag{26}$$

where the notation $\gamma^\parallel = (\gamma^0, \gamma^3)$ and $\gamma^\perp = (\gamma^1, \gamma^2)$ was used, we find for the coefficients appearing in Eq. (25) the following general expressions in the one-loop approximation [henceforth we consider $\text{sgn}(eB) > 0$, to simplify the notation]:

$$\begin{aligned}
a' &= \frac{ig^2}{2p_\parallel^2} \sum_{l,n} \int d^4x \int d^4y \int \frac{d^3\hat{q}}{(2\pi)^4} \int \frac{d^3\hat{k}}{(2\pi)^4} \frac{e^{-i(p \cdot x - p' \cdot y)}}{(\bar{q}^2 + m_e^2)(\bar{k}^2 + M_W^2)} \{(q_0 + q_3)(p_3 - p_0)I_{n-2,l-1}(x)I_{n-2,l-1}^*(y) \\
&\quad + (q_3 - q_0)(p_3 + p_0)I_{n,l}(x)I_{n,l}^*(y)\}
\end{aligned} \tag{27}$$

$$\begin{aligned}
b' &= \frac{ig^2}{2p_\perp^2} \sum_{l,n} \int d^4x \int d^4y \int \frac{d^3\hat{q}}{(2\pi)^4} \int \frac{d^3\hat{k}}{(2\pi)^4} \frac{e^{-i(p \cdot x - p' \cdot y)}}{(\bar{q}^2 + m_e^2)(\bar{k}^2 + M_W^2)} \{(p_2 + ip_1)\bar{q}_2 I_{n-1,l}(x)I_{n-1,l-1}^*(y) \\
&\quad + (p_2 - ip_1)\bar{q}_2 I_{n-1,l-1}(x)I_{n-1,l}^*(y)\}
\end{aligned} \tag{28}$$

$$\begin{aligned}
c' &= \frac{ig^2}{2p_\parallel^2} \sum_{l,n} \int d^4x \int d^4y \int \frac{d^3\hat{q}}{(2\pi)^4} \int \frac{d^3\hat{k}}{(2\pi)^4} \frac{e^{-i(p \cdot x - p' \cdot y)}}{(\bar{q}^2 + m_e^2)(\bar{k}^2 + M_W^2)} \{(q_0 + q_3)(p_3 - p_0)I_{n-2,l-1}(x)I_{n-2,l-1}^*(y) \\
&\quad - (q_3 - q_0)(p_3 + p_0)I_{n,l}(x)I_{n,l}^*(y)\}
\end{aligned} \tag{29}$$

$$\begin{aligned}
d' &= \frac{-ig^2}{2p_\perp^2} \sum_{l,n} \int d^4x \int d^4y \int \frac{d^3\hat{q}}{(2\pi)^4} \int \frac{d^3\hat{k}}{(2\pi)^4} \frac{e^{-i(p \cdot x - p' \cdot y)}}{(\bar{q}^2 + m_e^2)(\bar{k}^2 + M_W^2)} \{(p_2 + ip_1)\bar{q}_2 I_{n-1,l}(x)I_{n-1,l-1}^*(y) \\
&\quad - (p_2 - ip_1)\bar{q}_2 I_{n-1,l-1}(x)I_{n-1,l}^*(y)\}
\end{aligned} \tag{30}$$

where

$$I_{n,l}(x) = 2\pi\sqrt{eB} \exp[i\hat{x} \cdot (\hat{k} + \hat{q})] \varphi_n(\xi/\sqrt{2}) \varphi_l(\rho/\sqrt{2}) \tag{31}$$

and $\varphi_m(x)$ are the orthonormalized harmonic oscillator wave functions, defined in terms of the Hermite polynomials $H_m(x)$ as

$$\varphi_m(x/\sqrt{2}) = \frac{2^{-m/2}}{[\sqrt{\pi m!}]^{1/2}} H_m(x/\sqrt{2}) \exp-(x^2/4). \tag{32}$$

Expressions (27)–(30), when substituted in Eq. (25), give the general formula for the one-loop neutrino self-energy in a constant magnetic field of arbitrary strength. Note that, in this approach, the W-boson/magnetic-field interaction is kept in the poles of the self-energy operator through the effective momentum \bar{k}^2 , as well as in the harmonic oscillator wave functions $\varphi_n(\xi/\sqrt{2})$.

IV. NEUTRINO SELF-ENERGY IN THE STRONG-FIELD LIMIT

Since in the strong-field limit ($M_W \gg \sqrt{B} \gg m_e, T, \mu, |\mathbf{p}|$) the gap between the electron Landau levels is larger than the rest of the parameters entering in the electron energy, it is

consistent to use the LLL approximation in the electron Green's function, while in the W-boson Green's function, because $M_W \gg \sqrt{B}$, we must sum in all W-boson Landau levels.

The neutrino self-energy in a magnetized vacuum was found within this approximation in Ref. [17]. In what follows, we extend that result to neutral and charged media, introducing finite temperature and density effects. However, since the vacuum contribution is always present in the statistical calculations of the self-energy, we summarize below the results found in Ref. [17].

In the strong field approximation ($l=0$) the covariant structure of the neutrino self-energy reduces to

$$(2\pi)^4 \delta^{(4)}(p-p') \bar{\Sigma}_0(p, B) = a'_0 \not{p}_\parallel + c'_0 p^\mu \hat{F}_{\mu\nu} \gamma^\nu. \tag{33}$$

That is, b' and d' are zero and

$$\begin{aligned}
a'_0 &= \frac{ig^2(p_3 + p_0)}{2p_\parallel^2} \sum_n \int d^4x \int d^4y \int \frac{d^3\hat{q}}{(2\pi)^4} \\
&\quad \times \int \frac{d^3\hat{k}}{(2\pi)^4} \frac{(q_3 - q_0)I_{n,0}(x)I_{n,0}^*(y) e^{-i(p \cdot x - p' \cdot y)}}{(q_\parallel^2 + m_e^2)(\bar{k}^2 + M_W^2)}
\end{aligned} \tag{34}$$

$$c'_0 = -a'_0. \quad (35)$$

After summing in n and integrating in x , y , and \hat{k} we obtain

$$a'_0 = -c'_0 = \frac{ig^2\pi(p_3+p_0)}{2p_{\parallel}^2} e^{-p_{\perp}^2/2eB} \times \int d^2q_{\parallel} \frac{(q_3-q_0)}{[q_{\parallel}^2+m_e][(q_{\parallel}-p_{\parallel})^2 - eB + M_W^2]}. \quad (36)$$

Integrating in q_{\parallel} and considering that $M_W^2 \gg eB, m_e^2, p_{\parallel}^2$, we obtain for the neutrino self-energy in the strongly magnetized vacuum

$$\bar{\Sigma}_0(p, B) = a_0 \not{p}_{\parallel} + c_0 p^{\mu} \hat{F}_{\mu\nu} \gamma^{\nu} \quad (37)$$

with coefficients given by

$$a_0 = -c_0 \simeq \frac{-g^2 eB}{2(4\pi)^2 M_W^2} e^{-p_{\perp}^2/2eB}. \quad (38)$$

Notice the $1/M_W^2$ -order of the leading contribution in the LLL approximation. Using the zero-temperature weak-field results of Ref. [12] we can identify the scalar coefficients of the general structure (3) for that case as

$$\tilde{a}_0 = \tilde{b}_0 \simeq 0, \quad \tilde{c}_0 = -\frac{3}{2} \tilde{d}_0 \simeq \frac{6g^2 eB}{(4\pi)^2 M_W^2}. \quad (39)$$

If we compare the vacuum results at weak-field, Eq. (39), with those at strong field, Eq. (38), we can see that in both cases the nonzero coefficients have field-dependent leading

contributions of $1/M_W^2$ -order. However, as we will show in Sec. V, they will play very different roles in the neutrino energy spectrum, since they are associated to different self-energy structures. While the strong-field results (38) produce a magnetic-field dependence in the neutrino energy spectrum which is linear in the Fermi coupling constant, the weak-field results (39) generate a smaller second-order contribution.

A. Neutral medium in strong magnetic field

In a neutral medium the self-energy operator can be written as

$$\bar{\Sigma}(p, B) = \bar{\Sigma}_0(p, B) + \bar{\Sigma}_T(p, B), \quad (40)$$

where $\bar{\Sigma}_0$ is the vacuum contribution [given in the strong-field approximation by Eqs. (37) and (38)], and $\bar{\Sigma}_T$ is the statistical part which depends on temperature and magnetic field (in a charged medium the statistical part can also depend, of course, on the chemical potential). Our goal now is to find $\bar{\Sigma}_T(p, B)$ in the parameter range $M_W \gg \sqrt{B} \gg m_e, T, |\mathbf{p}|$. Here we can also assume that the electrons are confined to the LLL. Thus, performing the same calculations as in Eqs. (34)–(36) we arrive at a self-energy operator $\bar{\Sigma}(p, B)$ with a structure identical to Eq. (33), and coefficients given by an expression similar to Eq. (36), but with the Matsubara replacement, $\int dq_0 \rightarrow (2\pi i/\beta) \sum_{m=-\infty}^{\infty} [q_4 = (2m+1)\pi/\beta, m=0, \pm 1, \pm 2, \dots]$, corresponding to the discretization of the fourth component q_4 of the fermion momentum, obtained after introducing the Wick rotation to Euclidean space ($q_0 \rightarrow iq_4$), β being the inverse of the temperature $\beta = 1/T$.

After summing in m and extracting the vacuum part (36), we obtain that the coefficients corresponding to the thermal contribution $\bar{\Sigma}_T(p, B)$ are given by

$$a'_0(T) = -c'_0(T) = \frac{-i2\pi^2 g^2 eB(p_3 + ip_4)}{p_{\parallel}^2} e^{-p_{\perp}^2/2eB} \delta^{(4)}(p-p') \int dq_3 \left\{ \frac{q_3(p_4^2 + \varepsilon_2^2 - \varepsilon_1^2) + 2ip_4 \varepsilon_1^2}{\varepsilon_1[(p_4^2 + \varepsilon_2^2 - \varepsilon_1^2)^2 + 4\varepsilon_1^2 p_4^2]} f_F(\varepsilon_1) \right. \\ \left. + \frac{(q_3 - ip_4)(p_4^2 - \varepsilon_2^2 + \varepsilon_1^2) - 2ip_4 \varepsilon_2^2}{\varepsilon_2[(p_4^2 - \varepsilon_2^2 + \varepsilon_1^2)^2 + 4\varepsilon_2^2 p_4^2]} f_B(\varepsilon_2) \right\} \quad (41)$$

where

$$f_F(\varepsilon_1) = \frac{1}{1 + e^{\beta\varepsilon_1}}, \quad f_B(\varepsilon_2) = \frac{1}{1 - e^{\beta\varepsilon_2}} \quad (42)$$

are the fermion and boson distribution functions respectively, with effective energies

$$\varepsilon_1 = \sqrt{q_3^2 + m_e^2},$$

$$\varepsilon_2 = \sqrt{(q_3 - p_3)^2 + M_W^2 - 2eB}. \quad (43)$$

To do the integral in Eq. (41) we take into account the approximation $M_W \gg \sqrt{B} \gg T \gg m_e, |\mathbf{p}|$. Thus, to leading order we can neglect the contribution of the boson distribution,

since it is damped by the exponential factor $e^{-M_W/T}$. Integrating in momentum and expanding in powers of $1/M_W^2$, we obtain, up to leading order,

$$\bar{\Sigma}_T(p, B) = a_0(T) \not{p}_{\parallel} + c_0(T) p^{\mu} \hat{F}_{\mu\nu} \gamma^{\nu} \quad (44)$$

$$a_0(T) = -c_0(T) = \frac{-g^2 e B T^2}{24 M_W^4} e^{-p_{\perp}^2/2eB}. \quad (45)$$

Hence, the thermal contribution is one order smaller in powers of $1/M_W^2$ than the field-dependent vacuum contribution (38). Notice that in the strong-field approximation the thermal contribution (45) has the same $1/M_W^4$ -order as the corresponding thermal contributions at weak [14] and zero [16] field.

We should call attention to the fact that when calculating the thermal contribution to the e-W bubble by using the Schwinger proper-time method [21], some authors [14] have considered the following expansion:

$$G(k, B)_{\mu\nu} \approx \frac{\delta_{\mu\nu}}{M_W^2} + \frac{\delta_{\mu\nu} \Delta^2 - \Delta_{\mu} \Delta_{\nu}}{M_W^4} + \mathcal{O}\left(\frac{\Delta^2 F_{\mu\nu}}{M_W^6}\right) \quad (46)$$

within the W-boson Green function in the presence of a constant magnetic field

$$G(x, y)_{\mu\nu} = \phi(x, y) \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot (x-y)} G(k, B)_{\mu\nu}, \quad (47)$$

where

$$\phi(x, y) = \exp i \frac{e}{2} y_{\mu} F^{\mu\nu} x_{\nu} \quad (48)$$

is the well-known phase factor depending on the applied field [27], and $\Delta(k)$ is the energy-momentum transfer.

Based on this expansion, it has been argued that the nonlocal lower contribution to the bubble diagram should be of order $1/M_W^4$, and that the local ($1/M_W^2$)-order term only contributes at finite density. While this argument is correct when applied to the statistical part of the self-energy operator [it explains the $1/M_W^4$ order appearing in the thermal coefficients $a_0(T)$ and $c_0(T)$ in Eq. (45)], it is not valid for the self-energy vacuum contribution. For the vacuum ($T=0$, $\mu=0$) contribution the situation is different, since we cannot neglect the internal momenta as compared with M_W in the poles of the W-boson Green's function, due to the lack of the ultraviolet cutoff which is present in statistics. Thus, when calculating the vacuum part, the first term in Eq. (46) should be replaced by $\delta_{\mu\nu}/(M_W^2 + k^2)$. Such momentum depen-

dence in the pole of the W-boson Green's function makes a nontrivial contribution of ($1/M_W^2$)-order into the nonlocal part of the bubble diagram. Observe that the W-boson Green's function obtained in our approach, Eq. (14), only has such structure indeed, and that the vacuum results at strong (38) and weak (39) fields give rise to nonlocal ($1/M_W^2$)-order contributions to neutrino self-energy.

B. Charged medium in strong magnetic field

We consider now a medium with a finite density of electric charge. As usual, a finite density is reflected through the shift $q_0 \rightarrow q_0 - \mu$ in the momenta of the particles with non-zero charge associated to μ . The chemical potential, depending on the value of the charge density, plays the role of an external parameter. In stellar medium, for example, the electric charge is fixed by the net charge of the baryons, which, due to their large masses, are usually treated as a classical background. Here the following comment is in order: in electroweak matter at finite density there exists the possibility to induce additional ‘‘chemical potentials’’ [28]. These chemical potentials are nothing but dynamically generated background fields given by the average $\langle W_0^3 \rangle$ and $\langle B_0 \rangle$ of the zero components of the gauge fields. They are known to appear, for instance, in the presence of finite lepton/baryon density [28] (for recent applications related to this effect see Refs. [29]). In our case, a possible consequence of the condensation of such average fields could be the modification of the effective chemical potential of the W-boson, which then might be different from the electron chemical potential. Nevertheless, that modification, even if present in the case under study, will have no relevant consequence, as the W-boson contribution will be always suppressed by the factor $e^{-M_W/T}$.

In the strong-field approximation [i.e. after considering the LLL approximation in the neutrino self-energy (25)–(30)] the nonzero coefficients of the neutrino self-energy in the charged medium are given in Euclidean space by

$$a'_0 = -c'_0 = \frac{ig^2 \pi^2 (p_3 + ip_4)}{\beta p_{\parallel}^2} e^{-p_{\perp}^2/2eB} \times \sum_m \int dq_3 \frac{(q_3 - iq_4^*)}{[(q_4^*)^2 + \varepsilon_1^2][(q_4^* - p_4)^2 + \varepsilon_2^2]}, \quad (49)$$

This expression is similar to Eq. (36) after taking into account the Matsubara replacement $\int dq_0 \rightarrow (i/\beta) \sum_{m=-\infty}^{\infty} [q_4 = (2m+1)\pi/\beta, m=0, \pm 1, \pm 2, \dots]$, with the additional change $q_4 \rightarrow q_4^*$, where $q_4^* = q_4 + i\mu$.

After carrying out the temperature sum in Eq. (49), and subtracting the vacuum part (36), we arrive at

$$a'_0(T, \mu) = -c'_0(T, \mu) = \frac{-i\pi^2 g^2 e B (p_3 + ip_4)}{2p_{\parallel}^2} e^{-p_{\perp}^2/2eB} \delta^{(4)}(p-p') \int dq_3 \left\{ \frac{(q_3 + \varepsilon_1)}{\varepsilon_1 [(i\varepsilon_1 - p_4)^2 + \varepsilon_2^2]} f_F^-(\varepsilon_1) \right. \\ \left. + \frac{(q_3 - \varepsilon_1)}{\varepsilon_1 [(i\varepsilon_1 + p_4)^2 + \varepsilon_2^2]} f_F^+(\varepsilon_1) + \frac{(q_3 + \varepsilon_2 - ip_4)}{\varepsilon_2 [(i\varepsilon_2 + p_4)^2 + \varepsilon_1^2]} f_B^-(\varepsilon_2) + \frac{(q_3 - \varepsilon_2 - ip_4)}{\varepsilon_2 [(i\varepsilon_2 - p_4)^2 + \varepsilon_1^2]} f_B^+(\varepsilon_2) \right\} \quad (50)$$

where

$$f_F^{\mp}(\varepsilon_1) = \frac{1}{1 + e^{\beta(\varepsilon_1 \mp \mu)}}, \quad f_B^{\mp}(\varepsilon_2) = \frac{1}{1 - e^{\beta(\varepsilon_2 \mp \mu)}} \quad (51)$$

are the fermion/antifermion and boson/antiboson distributions respectively. Here, we obtain an equal energy split for both distributions since both charged particles have the same electric charge and thus are characterized by the same chemical potential.

Assuming the approximation $M_W \gg \sqrt{B} \gg \mu, T, m_e, |\mathbf{p}|$, Eq. (50) is reduced in leading order to

$$a'_0(T, \mu) = -c'_0(T, \mu) = \frac{-i\pi^4 g^2 (p_3 + ip_4)}{M_W^2 p_{\parallel}^2} \\ \times [N_0^- - N_0^+] e^{-p_{\perp}^2/2eB} \delta^{(4)}(p-p') \quad (52)$$

where the notation N_0^{\mp} for the electron/positron number densities in the LLL

$$N_0^{\mp} = |eB| \int \frac{dq_3}{2\pi^2} f_F^{\mp}(\varepsilon_1) \quad (53)$$

was introduced.

It is interesting to consider the situation where $\mu \gg T$, which is a natural condition in many astrophysical environments. In this case Eq. (52) becomes

$$a'_0(T, \mu) = -c'_0(T, \mu) \\ = \frac{-i2\pi^2 g^2 e B (p_3 + ip_4) \mu}{M_W^2 p_{\parallel}^2} e^{-p_{\perp}^2/2eB} \delta^{(4)}(p-p'). \quad (54)$$

After analytic continuation to Minkowski space, we find that the leading contribution to the statistical part of the self-energy in a strongly magnetized charged medium is

$$\bar{\Sigma}_{T, \mu}(p, B) = a_0(T, \mu) \not{p}_{\parallel} + c_0(T, \mu) p^{\mu} \hat{F}_{\mu\nu} \gamma^{\nu} \quad (55)$$

$$a_0(T, \mu) = -c_0(T, \mu) = \frac{g^2 e B (p_3 + p_0) \mu}{2(2\pi)^2 M_W^2 p_{\parallel}^2} e^{-p_{\perp}^2/2eB}. \quad (56)$$

The apparent dependence on the momentum components p_0, p_3 of the coefficients $a_0(T, \mu), c_0(T, \mu)$ is deceiving, as we can easily rewrite Eqs. (55),(56) in a more convenient

way using the base formed by the four-velocity u_{μ} and the covariant magnetic field vector $B_{\mu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} u^{\nu} F^{\rho\lambda}$, as

$$\bar{\Sigma}_{T, \mu}(p, B) = \bar{a}_0(T, \mu) \not{u} + \bar{c}_0(T, \mu) \not{B} \quad (57)$$

$$\bar{a}_0(T, \mu) = \frac{-g^2 e B \mu}{2(2\pi)^2 M_W^2} e^{-p_{\perp}^2/2eB},$$

$$\bar{c}_0(T, \mu) = \frac{g^2 e \mu}{2(2\pi)^2 M_W^2} e^{-p_{\perp}^2/2eB}. \quad (58)$$

From Eqs. (57),(58) we see that the leading term in the statistical part of the self-energy in the charged medium in strong magnetic field is independent of the momentum and proportional to $1/M_W^2$, thus of larger order than the statistical contribution of the neutral case [Eq. (45)].

Comparing our result (57),(58) with those found at weak field [15], we see that the coefficient of the structure \not{B} has similar qualitative characteristics. That is, both c coefficients are linear in μ and have the same order, $1/M_W^2$. Nevertheless, the coefficients of the structure \not{u} significantly differ in the strong and weak cases. In the weak-field limit \bar{a} only depends on the density, thus its leading contribution is proportional to μ^3 (this is the characteristic term of the MSW effect [26]). However, for fields larger than μ , the relation $\mu B \gg \mu^3$ holds, and the leading becomes proportional to μB , as in Eq. (58). Thus, in the strong field case it is the field the parameter that drives the main dependence of both structures in $\bar{\Sigma}_{T, \mu}$.

V. NEUTRINO INDEX OF REFRACTION IN STRONG MAGNETIC FIELD

The neutrino index of refraction, defined by

$$n = \frac{|\mathbf{p}|}{E(|\mathbf{p}|)} \quad (59)$$

where \mathbf{p} and E are the neutrino momentum and energy respectively, is a quantity that plays a significant role in neutrino flavor oscillations in a medium [26]. A flavor-dependent index of refraction would enhance the periodical change between different flavors of neutrinos travelling through the medium.

To find the index of refraction we need to know the dispersion relation $E = E(|\mathbf{p}|)$. In the presence of a medium the energy spectra of the massless left-handed neutrinos are modified due to their weak interaction with the particles of the background. In a magnetized medium, even though the neutrinos are electrically neutral, they feel the magnetic field indirectly through their interaction with the charged particles of the medium whose properties are affected by the field.

As discussed in Sec. II, in a magnetized medium the general form of the neutrino dispersion relation is given by Eq. (5), with the self-energy depending on the medium characteristics. As follows, we present the neutrino energy spectrum and index of refraction for magnetized neutral and charged media.

A. Neutrino index of refraction in strongly magnetized neutral medium

In the strongly magnetized neutral medium the dispersion relation is obtained from

$$\det[\not{p} + \bar{\Sigma}_0(p, B) + \bar{\Sigma}_T(p, B)] = 0, \quad (60)$$

with $\bar{\Sigma}_0(p, B)$ and $\bar{\Sigma}_T(p, B)$ explicitly given in Eqs. (37), (38) and (44), (45) respectively. The solution of Eq. (60) in leading order in the Fermi coupling constant is

$$\begin{aligned} p_0 &= \pm |\mathbf{p} + \sqrt{2/a^{(0)}}(\mathbf{p} \times \hat{\mathbf{B}})| \\ &\simeq \pm |\mathbf{p}| [1 - a^{(0)} \sin^2 \alpha]. \end{aligned} \quad (61)$$

In Eq. (61), α is the angle between the direction of the neutrino momentum and that of the applied magnetic field, and the coefficient $a^{(0)} = a_0 + a_0(T)$ is explicitly given by

$$a^{(0)} = \frac{-g^2 e B}{8M_W^2} \left[\frac{1}{(2\pi)^2} + \frac{T^2}{3M_W^2} \right] \exp(-p_\perp^2/eB). \quad (62)$$

From Eq. (62) it is clear that in the strong-field limit the thermal correction $a_0(T)$ to the neutrino energy is much smaller than the vacuum correction a_0 . As we will discuss in Sec. VIII, this result can be significant for cosmological applications in the case that a strong primordial magnetic field could exist in the primeval plasma.

The neutrino (+)/antineutrino (-) energies are given by $E_\pm = \pm p_0$ respectively. Substituting Eq. (61) into Eq. (59) we obtain for the neutrino (antineutrino) index of refraction in a neutral medium

$$n \simeq 1 + a^{(0)} \sin^2 \alpha. \quad (63)$$

Equation (63) implies that the index of refraction depends on the neutrino direction of motion. The order of the anisotropy is $g^2(|eB|/M_W^2)$. Maximum field effects take place for neutrinos propagating perpendicularly to the magnetic field. That is, the maximum departure of the neutrino phase velocity from the light velocity c , occurs at $\alpha = \pi/2$. Notice that in the neutral magnetized medium the magnetic field effect does not differentiate between neutrinos and antineutrinos.

Here the following comment is in order. If we consider the weak-field results (39) obtained in Ref. [12] for the neutrino self-energy, in the dispersion relation (6) we obtain

$$E'_\pm = |\mathbf{p}| \left[1 + \frac{5}{18} \tilde{c}_0^2 \sin^2 \alpha \right]. \quad (64)$$

In Eq. (64), as the energy depends on \tilde{c}_0^2 , we can see that the weak field produces a negligible second order correction in term of the Fermi coupling constant expansion (i.e. a $1/M_W^4$ -order effect). It can be corroborated that the inclusion of temperature in this approximation also produces a second order correction [14].

Thus, we conclude that the strong field produces an effect qualitatively larger, which is even more important than the thermal one. We call the reader's attention to the fact that the field-dependent vacuum contribution to the neutrino energy (61) has the same order in the Fermi coupling constant as the ones found in a charged medium at zero [26] and weak [15] fields.

B. Neutrino index of refraction in a strongly magnetized charged medium

Now we consider the effect of a strong magnetic field in neutrino propagation in a charged medium. In this case the dispersion relation is

$$\det[\not{p} + \bar{\Sigma}_0(p, B) + \bar{\Sigma}_{T,\mu}(p, B)] = 0, \quad (65)$$

with $\bar{\Sigma}_0(p, B)$ and $\bar{\Sigma}_{T,\mu}(p, B)$ given in Eqs. (37), (38) and (57), (58) respectively. The solution of Eq. (65) is

$$E_\pm = \pm \frac{4a_0\mu}{(1-2a_0)} + \sqrt{\frac{|\mathbf{p}|^2 - 2a_0[p_3 - 4\mu]p_3}{1-2a_0}}. \quad (66)$$

In leading order in the Fermi coupling constant, expression (66) is approximated by

$$E_\pm \simeq |\mathbf{p}| [1 - a_0 \sin^2 \alpha] - \mathcal{M} \cdot \mathbf{B} \pm E_0 \quad (67)$$

where

$$\mathcal{M} \equiv \frac{-E_0}{|\mathbf{B}|} \frac{\mathbf{p}}{|\mathbf{p}|}, \quad (68)$$

and

$$E_0 = \sqrt{2} G_F e^{-p_\perp^2/2eB} [N_0^- - N_0^+]. \quad (69)$$

On the RHS of Eq. (67), the first term is the modified neutrino kinetic energy, with the field-dependent radiative correction already found in the neutral case [see Eq. (61)]; the second term can be interpreted as a magnetic-moment/magnetic-field interaction energy, with \mathcal{M} playing the role of

a neutrino effective magnetic moment; and the third term is a rest energy induced by the magnetized charged medium.

We underline that, contrary to the case of the neutrino anomalous magnetic moment found beyond the standard model by including right-chiral neutrinos [30], the induced magnetic moment here obtained does not require a massive neutrino. We remind the reader that we are only considering left-chiral Dirac neutrinos, so the magnetic moment in Eq. (67) cannot be associated with the anomalous magnetic moment structural term $\sigma^{\mu\nu}F_{\mu\nu}$ in the self-energy. This structure is forbidden in the present case by the explicit chirality of the standard model. In the case under study we have that although the neutrino is a neutral massless particle, the charged medium can endow it, through quantum corrections, with a magnetic moment proportional to the induced neutrino rest energy. Thus, we are in the presence of a peculiar magnetic moment created thanks to the particle-antiparticle imbalance of the charged medium. Such a charge asymmetry permits the formation of a net virtual current that, due to the magnetic field, circulates in a plane perpendicular to the neutrino propagation and therefore produces an effective magnetic moment in the direction of the neutrino momentum.

In a charged medium CP -symmetry is violated [31], and a common property of electroweak media with CP violation is a separation between neutrino and antineutrino energies. In the present case this property is manifested in the double sign of E_0 in Eq. (67). Although this difference depends on the magnetic field entering in N_0^\pm , we stress that it depends on the magnitude of the magnetic field but not on its direction. Thus, the energy anisotropy connected to the first two terms of Eq. (67) is the same for neutrinos and antineutrinos.

The neutrino/antineutrino energy in the charged medium in the presence of a weak field has been found [15] to be

$$E''_{\pm} = |\mathbf{p}| - b' \frac{\mathbf{p} \cdot \mathbf{B}}{|\mathbf{p}|} \pm a', \quad (70)$$

with

$$a' = \frac{g^2}{4M_W^2} (N_- - N_+),$$

$$b' = \frac{eg^2}{2M_W^2} \int \frac{d^3p}{(2\pi)^3 2E} \frac{d}{dE} (f_- - f_+), \quad (71)$$

where N_{\pm} are the electron/positron number densities, and f_{\pm} the electron/positron distributions.

Comparing the strong-field (67) and the weak-field (70) dispersion relations, we see that the field-dependent correction to the kinetic energy appearing in the strong-field approximation is absent in the leading order of the weak-field case. For a strong-field, that correction turns out to be important for neutrinos propagating perpendicular to the magnetic field. Another difference between Eqs. (67) and (70), is that in the last dispersion relation the rest energy a' does not depend on the magnetic field. This is the pure medium contribution that gives rise to the well known MSW effect [26]. As discussed above, in the strong-field case, where all pa-

rameters, including μ and $|\mathbf{p}|$ are assumed smaller than \sqrt{eB} , such a term is negligible compared to the contribution E_0 which is proportional to the field. In both approximations there is an additional anisotropy related to the induced magnetic moment term, which is linear in the magnetic field and depends on the imbalance between particles and antiparticles. We should notice that this anisotropic term changes its sign when the neutrino reverses its motion. This property is crucial for a possible explanation of the peculiar high pulsar velocities [32]. Nevertheless, for the anisotropic term that modifies the kinetic energy in the strong-field approximation, the neutrino energy-momentum relation is invariant under the change of α by $-\alpha$.

Substituting Eq. (67) into Eq. (59) we obtain that the neutrino/antineutrino index of refraction in the charged medium is given by

$$n_{\pm} = 1 + a_0 \sin^2 \alpha + \frac{\mathcal{M} \cdot \mathbf{B} \mp E_0}{|\mathbf{p}|}. \quad (72)$$

The result (72) implies that the index of refraction depends on the neutrino direction of motion relative to the magnetic field. Moreover, neutrinos and antineutrinos have different index of refractions and therefore different phase velocities even if they move in the same direction. For neutrinos, if $N_0^- > N_0^+$ (i.e. if there is a larger number of electrons than positrons) the index of refraction for $B \neq 0$ is smaller than one, so their phase velocities will be smaller than light velocity. Thus, in the charged medium with strong magnetic field, neutrinos behave as massive particles. This is in agreement with the behavior they have in a dense medium, even in the absence of magnetic field [26]. On the other hand, for antineutrinos the index of refraction [the one with the positive sign in front of E_0 in Eq. (72)] can be larger than one. For instance, for antineutrinos moving opposite to the field-line directions the index of refraction is

$$n_- = 1 + \frac{2E_0}{|\mathbf{p}|} > 1. \quad (73)$$

Thus, antineutrinos will have phase velocities larger than c and depending on the magnetic field strength and electron density. In this regard we should mention that particles with zero rest mass can even have group velocities that exceed c in anomalously dispersive media [33]. Moreover, in non-trivial backgrounds, particles with superluminal propagations are not intrinsically forbidden in quantum field theories. For instance, superluminal photons appear in curved spaces [34], Casimir vacua [35] and in QED with compactified spatial dimensions [36]. Discussions about the nonviolation of microcausality by the existence of such superluminal velocities, as well as the lack of a contradiction between such a phenomenon with the bases of special theory of relativity, can be found in Refs. [33,34,37]. There, it was noted that the ‘‘front velocity,’’ the one related to the index of refraction in the infinite frequency limit, is the one that cannot exceed c , since it is related to the signal transmission.

VI. STRONG FIELD AND LLL APPROXIMATION: NUMERICAL TEST

In the calculation of the neutrino self-energy in the strong-field limit ($M_W \gg \sqrt{B} \gg m_e, T, \mu, |\mathbf{p}|$) done in Sec. IV, we assumed that the main contribution to the self-energy loop diagram came from electrons in the LLL, since in the strong-field limit the energy gap between electron Landau levels is much larger than the electron average energy in the medium. In this section we will check the validity of these arguments with the help of numerical calculations.

Taking into account that the relevant physical quantity in this study is the neutrino energy, we can concentrate our analysis on the self-energy coefficients that give the largest contribution to the dispersion relation. It is easy to check, from the analytical structure of Eqs. (27)–(30), that the leading term in the expansion of the coefficients in powers of $1/M_W^2$ can be at most of order $1/M_W^2$. Keeping in mind that the only coefficients that could contribute with a $1/M_W^2$ -order term in the dispersion relation (6) are a and b [coefficients c and d appear squared in Eq. (6) and their contribution to the dispersion relation should start at least with terms of order $1/M_W^4$], we can approximate the dispersion relation (6) by

$$p_0^2 \approx |\mathbf{p}|^2 - 2(a-b)p_\perp^2. \quad (74)$$

Then, to validate the LLL approximation we should numerically study the ratio

$$\kappa = \frac{a-b}{a_0-b_0}, \quad (75)$$

where a and b are respectively obtained from Eqs. (27) and (28), summing in all the Landau levels and taking parameter values in the strong-field region $M_W \gg \sqrt{B} \gg m_e, T, |\mathbf{p}|$. The coefficients a_0 and b_0 are the corresponding values at $T=0$ in the LLL. In the denominator of Eq. (74) we are neglecting the thermal contributions $a_0(T)$ and $b_0(T)$ since they would make an insignificant contribution ($1/M_W^2$ -order smaller) as compared with the vacuum ones in the parameter range $T \ll M_W$. In Sec. IV, we showed that $b_0=0$, and that a_0 is given by Eq. (38).

The strong-field conditions can be naturally found in many astrophysical systems like magnetars, neutron stars, etc. In cosmological applications, however, the viable primordial fields can never be much larger than the temperature,² as according to the equipartition principle, the magnetic energy can only be a small fraction of the universe energy density. This argument leads to the relation $eB/T^2 \sim \mathcal{O}(1)$. Clearly, this is not a situation very consistent with a strong-field approximation. However, even under this condition, it is natural to expect that the thermal energy should be

²In cosmology the electric chemical potential is so small that we can always assume it is zero.

barely enough to induce the occupation of just a few of the lower electron Landau levels, and one would not be surprised if the LLL approximation still gives the leading contribution to the dispersion relation.

To investigate Eq. (75) it will be more convenient to independently study the variation range for each coefficient ratio,

$$\begin{aligned} \kappa_a &= \frac{a(T=0) + a(T)}{a_0} - \kappa_a(B=0, T=0), \\ \kappa_b &= \frac{b(T=0) + b(T)}{a_0} - \kappa_b(B=0, T=0), \end{aligned} \quad (76)$$

where $a(T=0)[b(T=0)]$ and $a(T)[b(T)]$ are respectively the vacuum and thermal contributions of each coefficient $a(b)$. Here we subtract the ultraviolet divergent zero-field, zero-temperature parts $\kappa_a(B=0, T=0)$, $\kappa_b(B=0, T=0)$, as they can only contribute, after renormalization, with negligible zero-field terms. The validity of the LLL approximation should imply that $\kappa_a \approx 1$ and $\kappa_b \ll 1$ for the range of parameters considered. In the first part of this section we examine the validity of the LLL in the zero-temperature case obtaining $\kappa_a(T=0) = a(T=0)/a_0 - \kappa_a(B=0, T=0) \approx 1$, $\kappa_b(T=0) = b(T=0)/a_0 - \kappa_b(B=0, T=0) \ll 1$. In the second part, we do a similar analysis, for the thermal part of a , finding that $\kappa_a(T) = a(T)/a_0 \ll 1$, as expected from the theoretical considerations given in Sec. IV [see the discussion below Eq. (48)]. As this last numerical calculation is performed for magnetic fields that can even be a few orders smaller than T^2 , the result $\kappa_a(T) \ll 1$ confirms the appropriateness of the LLL-approximation for a rather wide range of primordial fields. The analysis of the finite temperature part of coefficient b [$\kappa_b(T) = b(T)/a_0$] is not explicitly done in the paper, however, it is not hard to see that it gives rise to a similar result, that is, $\kappa_b(T) \ll 1$.

A. Vacuum contribution

If we introduce in expressions (27) and (28) the integral representations

$$\frac{1}{\bar{q}^2 + m_e^2} = \int_0^\infty d\alpha \exp(-(\bar{q}^2 + m_e^2)\alpha), \quad (77)$$

$$\frac{1}{\bar{k}^2 + M_W^2} = \int_0^\infty d\beta \exp(-(\bar{k}^2 + M_W^2)\beta), \quad (78)$$

and take into account the recurrence relation [39]

$$(2l)^{1/2} \varphi_{l-1}(\xi) = (\partial_\xi + \xi) \varphi_l(\xi), \quad (79)$$

in Eq. (28), it is possible to rewrite coefficients a' and b' as

$$\begin{aligned}
a' &= \frac{-ig^2 eB}{2(2\pi)^6 p_{\parallel}^2} \int d^4x \int d^4y \int d^3\hat{q} \int d^3\hat{k} e^{-i(p_1x_1 - p'_1 \cdot y_1)} e^{i\hat{x} \cdot (\hat{k} + \hat{q} - \hat{p}) + i\hat{y} \cdot (\hat{p}' - \hat{k} - \hat{q})} \\
&\quad \times \int_0^{\infty} d\alpha \int_0^{\infty} d\beta \{ e^{-(q_{\parallel}^2 + m_e^2)\alpha - (k_{\parallel}^2 + M_W^2 - eB)\beta} [(q_0 - q_3)(p_3 + p_0) - (q_0 + q_3)(p_3 - p_0)r^2 t] S_{\alpha} S_{\beta} \} \quad (80)
\end{aligned}$$

$$\begin{aligned}
b' &= \frac{ig^2 (eB)^{3/2}}{(2\pi)^6 p_{\perp}^2} \int d^4x \int d^4y \int d^3\hat{q} \int d^3\hat{k} e^{-i(p_1x_1 - p'_1 \cdot y_1)} e^{i\hat{x} \cdot (\hat{k} + \hat{q} - \hat{p}) + i\hat{y} \cdot (\hat{p}' - \hat{k} - \hat{q})} \\
&\quad \times \int_0^{\infty} d\alpha \int_0^{\infty} d\beta \left\{ e^{-(q_{\parallel}^2 + m_e^2)\alpha - (k_{\parallel}^2 + M_W^2 - eB)\beta} \left[\frac{\rho + \rho'}{1+t} p_2 + i \frac{\rho - \rho'}{1-t} p_1 \right] r t S_{\alpha} S_{\beta} \right\}. \quad (81)
\end{aligned}$$

Here, S_{α} and S_{β} are found doing the sum in Landau levels with the help of Mehler's formula [38]

$$S_{\alpha} = \sum_{l=0}^{\infty} \varphi_l(\rho) \varphi_l(\rho') t^l = [\pi(1-t^2)]^{-1/2} \exp\left(-\frac{\rho^2 - \rho'^2}{2}\right) \exp\left(-\frac{(\rho' - \rho t)^2}{1-t^2}\right) \quad (82)$$

where $t = \exp(-2eB\alpha)$, $\rho = \sqrt{2eB}(x_1 + q_2/eB)$ and $\rho' = \sqrt{2eB}(y_1 + q_2/eB)$. S_{β} is obtained from Eq. (82) by replacing $t \rightarrow r = \exp(-2eB\beta)$, $\rho \rightarrow \xi = \sqrt{2eB}(x_1 - k_2/eB)$ and $\rho' \rightarrow \xi' = \sqrt{2eB}(y_1 - k_2/eB)$.

Integrating in \hat{x} , \hat{y} , \hat{q} and k_2 , and introducing the variable changes $x_1 \rightarrow \xi$ and $y_1 \rightarrow \xi'$, we find

$$\begin{aligned}
a' &= \frac{-ig^2 eB \pi}{p_{\parallel}^2} \delta^4(p - p') \int_{-\infty}^{\infty} d^2k_{\parallel} \int_0^{\infty} d\alpha \int_0^{\infty} d\beta \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\xi' \left\{ e^{-[(q_{\parallel} - p_{\parallel})^2 + m_e^2]\alpha - (k_{\parallel}^2 + M_W^2 - eB)\beta} e^{-i(\xi - \xi')p_1/\sqrt{eB}} [(1+r^2)t] \right. \\
&\quad \left. \times (p_0^2 - p_3^2) - (k_0 - k_3)(p_3 + p_0) + (k_0 + k_3)(p_3 - p_0)r^2 t \right\} S'_{\alpha} S_{\beta} \quad (83)
\end{aligned}$$

$$\begin{aligned}
b' &= \frac{2ig^2 (eB)^{3/2} \pi}{p_{\perp}^2} \delta^4(p - p') \int_{-\infty}^{\infty} d^2k_{\parallel} \int_0^{\infty} d\alpha \int_0^{\infty} d\beta \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\xi' \left\{ e^{-[(q_{\parallel} - p_{\parallel})^2 + m_e^2]\alpha - (k_{\parallel}^2 + M_W^2 - eB)\beta} \left[\frac{\xi + \xi'}{1+t} p_2 + i \frac{\xi - \xi'}{1-t} p_1 \right. \right. \\
&\quad \left. \left. + \frac{2p_2^2}{\sqrt{eB}(1+t)} \right] e^{-i(\xi - \xi')p_1/\sqrt{eB}} r t S'_{\alpha} S_{\beta} \right\} \quad (84)
\end{aligned}$$

where

$$S'_{\alpha} = [\pi(1-t^2)]^{-1/2} \exp\left(-\frac{[\xi + p_2/\sqrt{eB}]^2 - [\xi' + p_2/\sqrt{eB}]^2}{2}\right) \exp\left(-\frac{[(\xi' + p_2/\sqrt{eB}) - (\xi + p_2/\sqrt{eB})t]^2}{1-t^2}\right). \quad (85)$$

After Wick rotating to Euclidean space and doing the Gaussian integrals in k_3 , k_4 , ξ and ξ' , we finally obtain

$$a' = \delta^4(p - p') g^2 \pi^2 \int_0^{\infty} d\alpha \int_0^{\infty} d\beta \frac{\beta}{(\alpha + \beta)^2} \frac{\cosh(\alpha + 2\beta)}{\sinh(\alpha + \beta)} \exp(-\hat{m}_e^2 \alpha - \hat{M}_W^2 \beta) \exp\left[-\frac{\alpha\beta}{\alpha + \beta} \hat{p}_{\parallel}^2 + \frac{1}{\cosh(\alpha) + \cosh(\beta)} \hat{p}_{\perp}^2\right] \quad (86)$$

$$b' = -\delta^4(p - p') g^2 \pi^2 \int_0^{\infty} d\alpha \int_0^{\infty} d\beta \frac{1}{(\alpha + \beta)} \frac{\sinh(\beta)}{\sinh^2(\alpha + \beta)} \exp(-\hat{m}_e^2 \alpha - \hat{M}_W^2 \beta) \exp\left[-\frac{\alpha\beta}{\alpha + \beta} \hat{p}_{\parallel}^2 + \frac{1}{\cosh(\alpha) + \cosh(\beta)} \hat{p}_{\perp}^2\right] \quad (87)$$

where we introduced the normalized parameters $\hat{m}_e = m_e/\sqrt{eB}$, $\hat{p}_{\mu} = p_{\mu}/\sqrt{eB}$, and $\hat{M}_W = M_W/\sqrt{eB}$. Equations (86) and (87) are as far as we can go in the calculation of a' and b' after summing in all Landau levels without using any approximation.

In the zero-field limit the coefficients (86) and (87) reduce to

$$a'_{B=0} = b'_{B=0} = -\delta^4(p - p') g^2 \pi^2 \int_0^{\infty} d\alpha \int_0^{\infty} d\beta \frac{\beta}{(\alpha + \beta)^3} \exp\left(-\alpha + \frac{\alpha\beta}{\alpha + \beta} \hat{p}_{\mu}^2 + \bar{M}_W^2 \beta\right) \quad (88)$$

where $\bar{M}_W = M_W/m_e$, $\bar{p}_\mu = p_\mu/m_e$.

Our goal now is to investigate the validity of the LLL approximation for the zero-temperature contribution of the coefficients in the parameters' range: $M_W \gg \sqrt{B} \gg m_e, |\mathbf{p}|$. With this purpose, in Figs. 2 and 3 we plot,

$$\kappa_a(T=0) = \frac{a - a_{B=0}}{a_0} \quad \text{vs} \quad \log \hat{m}^{-1} \in [-2, 2] \quad (89)$$

and

$$\kappa_b(T=0) = \frac{b - b_{B=0}}{a_0} \quad \text{vs} \quad \log \hat{m}^{-1} \in [-2, 2] \quad (90)$$

respectively. Here a_0 is the LLL coefficient defined in Eq. (38). Notice that primed and unprimed variables are related through $a' = (2\pi)^4 \delta^{(4)}(p - p') a$, $b' = (2\pi)^4 \delta^{(4)}(p - p') b$.

From Fig. 2 it can be seen that $\kappa_a(T=0) \approx 1$ when $\sqrt{eB} \gg m_e, |\mathbf{p}|$, while, from Fig. 3, we see that $\kappa_b(T=0) \ll 1$ in all the range $10^{-2} m_e \leq \sqrt{eB} \leq 10^2 m_e$. This result corroborates the initial assumption that the LLL approximation gives the leading contribution to the vacuum part of the neutrino self-energy at strong fields ($M_W \gg \sqrt{B} \gg m_e, |\mathbf{p}|$).

B. Thermal contribution

Let us consider now the finite temperature contributions. The finite temperature part of the coefficient a' can be found in the following way. We start from Eq. (83), perform the Wick rotation to Euclidean space, replace the integral in the four-momentum by the temperature sum $\int dk_4 \rightarrow (2\pi T) \sum_{n=-\infty}^{\infty}$ (with $k_4 = 2\pi n T$), and integrate in k_3 , to obtain

$$\begin{aligned} a' = & \delta^4(p - p') \frac{g^2 e B \pi \sqrt{\pi}}{p_{\parallel}^2} \int_0^{\infty} d\alpha \int_0^{\infty} d\beta \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\xi' \left(S'_\alpha S_\beta \frac{e^{\alpha^2 p_3^2 / (\alpha + \beta)}}{\sqrt{\alpha + \beta}} e^{-[(p_{\parallel})^2 + m_e^2] \alpha - (M_W^2 - eB) \beta} e^{-i(\xi - \xi') p_1 / \sqrt{eB}} \right. \\ & \times (2\pi T) \sum_{n=-\infty}^{\infty} \left\{ (1 + r^2 t)(p_0^2 - p_3^2) + \frac{\alpha p_3}{\alpha + \beta} [(p_3 + p_0) + (p_3 - p_0) r^2 t] + i k_4 [(p_3 - p_0) r^2 t - (p_3 + p_0)] \right\} \\ & \left. \times e^{-(\alpha + \beta) k_4^2 - 2i\alpha p_0 k_4} \right). \end{aligned}$$

After integrating in ξ and ξ' , and introducing the elliptic theta functions representation [38]

$$\theta_3(u/\tau) = \sum_{n=-\infty}^{\infty} \exp[i\pi(\tau n^2 + 2nu)] \quad (91)$$

we find

$$a' = a^{(1)} + a^{(2)} \quad (92)$$

where

$$\begin{aligned} a^{(1)} = & \delta^4(p - p') \frac{g^2 \pi^2 \sqrt{\pi} \hat{T}}{\hat{p}_{\parallel}^2} \int_0^{\infty} d\alpha \int_0^{\infty} d\beta \sinh^{-1}(\alpha + \beta) \frac{\exp\left(\frac{\alpha^2}{\alpha + \beta}\right) \hat{p}_3^2}{\sqrt{\alpha + \beta}} \exp[-[\hat{p}_{\parallel}^2 + \hat{m}_e^2] \alpha] \exp[-\hat{M}_W^2 \beta] \exp\left[-\frac{\hat{p}_{\perp}^2}{\coth \alpha + \coth \beta}\right] \\ & \times \left[\frac{\alpha \hat{p}_3}{\alpha + \beta} [r^{-1} t^{-1/2} (\hat{p}_0 + \hat{p}_3) + r t^{1/2} (\hat{p}_3 - \hat{p}_0)] + (\hat{p}_0^2 - \hat{p}_3^2) (r^{-1} t^{-1/2} + r t^{1/2}) \right] \theta_3(-2\hat{p}_0 \hat{T} \alpha / i 4 \pi \hat{T}^2 (\alpha + \beta)) \end{aligned} \quad (93)$$

and

$$\begin{aligned}
a^{(2)} = & \delta^4(p-p') \frac{g^2 \pi^2 \sqrt{\pi} \hat{T}}{\hat{p}_\parallel^2} \int_0^\lambda d\gamma \int_0^\infty d\lambda \sinh^{-1}(\lambda) \frac{\exp\left(\frac{\gamma^2}{\lambda}\right) \hat{p}_3^2}{\sqrt{\lambda}} \exp\left[-\hat{p}_\parallel^2 + \hat{m}_e^2\right] \gamma \exp\left[-\hat{M}_W^2(\lambda - \gamma)\right] \\
& \times \exp\left[-\frac{\hat{p}_\perp^2}{\coth \gamma + \coth(\lambda - \gamma)} \left[\bar{r} \bar{t}^{1/2} (\hat{p}_3 - \hat{p}_0) - \bar{r}^{-1} \bar{t}^{-1/2} (\hat{p}_0 + \hat{p}_3)\right] \frac{\partial_\gamma \theta_3(-2\hat{p}_0 \hat{T} \gamma / i4 \pi \hat{T}^2 \lambda)}{-2\hat{p}_0}\right]. \quad (94)
\end{aligned}$$

In Eq. (94) we introduced the notation

$$\bar{r} = \exp(-2(\lambda - \gamma)), \quad \bar{t} = \exp(-2\gamma). \quad (95)$$

Integrating Eq. (94) by parts and adding the result with Eq. (93), we obtain

$$\begin{aligned}
a' = & \delta^4(p-p') \frac{g^2 \pi^2 \sqrt{\pi} \hat{T}}{\hat{p}_0 \hat{p}_\parallel^2} \int_0^\infty d\lambda \frac{\sinh^{-1}(\lambda)}{\lambda} [\hat{p}_3 \sinh(\lambda) + \hat{p}_0 \cosh(\lambda)] (\exp[-\hat{p}_0^2 + \hat{m}_e^2] \lambda) \theta_3(-2\hat{p}_0 \hat{T} \lambda / i4 \pi \hat{T}^2 \lambda) \\
& - \frac{g^2 \pi^2 \sqrt{\pi} \hat{T}}{\hat{p}_0 \hat{p}_\parallel^2} \int_0^\infty d\lambda \frac{\sinh^{-1}(\lambda)}{\lambda} [\hat{p}_3 \sinh(2\lambda) + \hat{p}_0 \cosh(2\lambda)] [\exp - \hat{M}_W^2 \lambda] \theta_3(0 / i4 \pi \hat{T}^2 \lambda) \\
& + \frac{g^2 \pi^2 \sqrt{\pi} \hat{T}}{\hat{p}_0 \hat{p}_\parallel^2} \int_0^\infty d\alpha \int_0^\infty d\beta \frac{\sinh^{-1}(\alpha + \beta)}{\sqrt{\alpha + \beta}} \exp - \hat{M}_W^2 \beta \exp - [\hat{p}_\parallel^2 + \hat{m}_e^2] \alpha \exp\left(\frac{\alpha^2}{\alpha + \beta}\right) \hat{p}_3^2 \\
& \times \exp - \frac{\hat{p}_\perp^2}{\coth \alpha + \coth \beta} \left\{ \frac{2\alpha \hat{p}_3 \hat{p}_0}{\alpha + \beta} [\hat{p}_3 \cosh(\alpha + 2\beta) + \hat{p}_0 \sinh(\alpha + 2\beta)] - \hat{p}_\parallel^2 \cosh(\alpha + 2\beta) + \left[(\hat{p}_\parallel^2 + \hat{m}_e^2 - \hat{M}_W^2) - \frac{2\alpha \hat{p}_3^2}{\alpha + \beta} \right. \right. \\
& \left. \left. + \left(\frac{1}{\sinh^2 \alpha} - \frac{1}{\sinh^2 \beta} \right) \frac{\hat{p}_\perp^2}{[\coth \alpha + \coth \beta]^2} \right] [\hat{p}_3 \sinh(\alpha + 2\beta) + \hat{p}_0 \cosh(\alpha + 2\beta)] + [\hat{p}_3 \cosh(\alpha + 2\beta) + \hat{p}_0 \sinh(\alpha + 2\beta)] \right\} \\
& \times \theta_3(-2\hat{p}_0 \hat{T} \alpha / i4 \pi \hat{T}^2 (\alpha + \beta)). \quad (96)
\end{aligned}$$

To isolate the temperature part of a' , we have to subtract the zero-temperature (vacuum) contribution from Eq. (96). With this aim, we use the Jacobi imaginary transformation [38]

$$\theta_3(u/\tau) = (\sqrt{-i\tau})^{-1} \exp[-i\pi u^2/\tau] \theta_3\left(\frac{u}{\tau} - \frac{1}{\tau}\right), \quad (97)$$

to write

$$\theta_3(-2\hat{p}_0 \hat{T} \alpha / i4 \pi \hat{T}^2 (\alpha + \beta)) = \frac{\exp[-\hat{p}_0^2 \alpha^2 / (\alpha + \beta)]}{2\hat{T} \sqrt{\pi} \sqrt{(\alpha + \beta)}} \left[1 + \sum_{n=-\infty}^{\infty}{}' \exp\left(-\frac{n^2}{4\hat{T}^2 (\alpha + \beta)} + \frac{\hat{p}_0 \alpha n}{\hat{T} (\alpha + \beta)}\right) \right] \quad (98)$$

where the symbol $\sum_{n=-\infty}^{\infty}{}'$ means that the term $n=0$ was taken out.

Then, to subtract the vacuum term from a' is equivalent to make in Eq. (96) the following substitution:

$$\theta_3(-2\hat{p}_0 \hat{T} \alpha / i4 \pi \hat{T}^2 (\alpha + \beta)) \rightarrow \theta_3(-2\hat{p}_0 \hat{T} \alpha / i4 \pi \hat{T}^2 (\alpha + \beta)) - \frac{\exp[-\hat{p}_0^2 \alpha^2 / (\alpha + \beta)]}{2\hat{T} \sqrt{\pi} \sqrt{(\alpha + \beta)}}, \quad (99)$$

$$\theta_3(-2\hat{p}_0 \hat{T} \lambda / i4 \pi \hat{T}^2 \lambda) \rightarrow \theta_3(-2\hat{p}_0 \hat{T} \lambda / i4 \pi \hat{T}^2 \lambda) - \frac{\exp[-\hat{p}_0^2 \lambda]}{2\hat{T} \sqrt{\pi} \sqrt{\lambda}}, \quad (100)$$

$$\theta_3(0 / i4 \pi \hat{T}^2 \lambda) \rightarrow \theta_3(0 / i4 \pi \hat{T}^2 \lambda) - \frac{1}{2\hat{T} \sqrt{\pi} \sqrt{\lambda}}, \quad (101)$$

which leads to

$$\begin{aligned}
 a'(T) = & \delta^4(p-p') \frac{g^2 \pi^2 \sqrt{\pi} \hat{T}}{\hat{p}_0 \hat{p}_\parallel^2} \left(\int_0^\infty d\lambda \frac{\sinh^{-1}(\lambda)}{\lambda} [\hat{p}_3 \sinh(\lambda) + \hat{p}_0 \cosh(\lambda)] [\exp - [-\hat{p}_0^2 + \hat{m}_e^2] \lambda] \left[\theta_3(-2\hat{p}_0 \hat{T} \lambda / i4 \pi \hat{T}^2 \lambda) \right. \right. \\
 & - \frac{\exp - [\hat{p}_0^2 \lambda]}{2\hat{T} \sqrt{\pi} \sqrt{\lambda}} \left. \left. - \int_0^\infty d\lambda \frac{\sinh^{-1}(\lambda)}{\lambda} [\hat{p}_3 \sinh(2\lambda) + \hat{p}_0 \cosh(2\lambda)] [\exp - \hat{M}_W^2 \lambda] \left[\theta_3(0/i4 \pi \hat{T}^2 \lambda) - \frac{1}{2\hat{T} \sqrt{\pi} \lambda} \right] \right. \right. \\
 & + \int_0^\infty d\alpha \int_0^\infty d\beta \frac{\sinh^{-1}(\alpha + \beta)}{\sqrt{\alpha + \beta}} \exp - [\hat{M}_W^2 \beta] \exp - [\hat{p}_\parallel^2 + \hat{m}_e^2] \alpha \exp \left(\frac{\alpha^2}{\alpha + \beta} \right) \hat{p}_3^2 \exp - \frac{\hat{p}_\perp^2}{\coth \alpha + \coth \beta} \left[\frac{2\alpha \hat{p}_3 \hat{p}_0}{\alpha + \beta} + 1 \right] \\
 & \times [\hat{p}_3 \cosh(\alpha + 2\beta) + \hat{p}_0 \sinh(\alpha + 2\beta)] - \hat{p}_\parallel^2 \cosh(\alpha + 2\beta) + \left[(\hat{p}_\parallel^2 + \hat{m}_e^2 - \hat{M}_W^2) - \frac{2\alpha \hat{p}_3^2}{\alpha + \beta} + \left(\frac{1}{\sinh^2 \alpha} - \frac{1}{\sinh^2 \beta} \right) \right. \\
 & \left. \left. \times \frac{\hat{p}_\perp^2}{[\coth \alpha + \coth \beta]^2} \right] [\hat{p}_0 \sinh(\alpha + 2\beta) + \hat{p}_3 \cosh(\alpha + 2\beta)] \left. \right] \left[\theta_3(-2\hat{p}_0 \hat{T} \alpha / i4 \pi \hat{T}^2 (\alpha + \beta)) - \frac{\exp - [\hat{p}_0^2 \alpha^2 / (\alpha + \beta)]}{2\hat{T} \sqrt{\pi} (\alpha + \beta)} \right] \right). \tag{102}
 \end{aligned}$$

Equation (102) gives the thermal part of the a' taking into account all Landau levels. To compare the thermal and the LLL contributions we plot $\log a(T)/a_0$ vs $\log \hat{T}^{-1}$ in Fig. 4.

From Fig. 4 we see that in all the range of temperatures considered, the thermal contribution is consistently smaller in several orders than the LLL contribution. This result, along with those obtained in Figs. 2 and 3, imply that $\kappa \approx 1$, thereby validating the LLL approximation in the strong and weakly strong field cases ($M_W \gg \sqrt{B} \gg m_e, |\mathbf{p}|, eB \gtrsim T^2$). In addition, we would like to underline the consistency of these numerical results with the arguments used in Sec. IV [see discussion after Eq. (45)], in the sense that the finite temperature part should be much smaller than the vacuum contribution.

VII. APPLICATIONS TO COSMOLOGY AND ASTROPHYSICS

Two natural environments where the results of the present paper can be of interest are, on one side, stars possessing large magnetic fields (neutron stars, magnetars, etc.), and, on the other, the early universe, which presumably was perme-

ated by large primordial magnetic fields that eventually became the seeds [8,40] of the galactic fields that are observed today at scales of 100 Kpc [6] and larger.

The effects of primordial magnetic fields on neutrino propagation are of special interest during the neutrino decoupling era, on which a large number of these particles escaped the electroweak plasma with the possibility to develop significant oscillations. The strength of the primordial magnetic field in the neutrino decoupling era can be estimated from the constraints derived from the successful nucleosynthesis prediction of primordial ${}^4\text{He}$ abundance [41], as well as on the neutrino mass and oscillation limits [42]. These constraints, together with the energy equipartition principle, lead to the relations

$$m_e^2 \leq eB \leq m_\mu^2, \quad B/T^2 \sim 2, \tag{103}$$

with m_μ being the muon mass. Then, it is reasonable to assume that between the QCD phase transition epoch and the end of nucleosynthesis a primordial magnetic field in the above range could have been present [17].

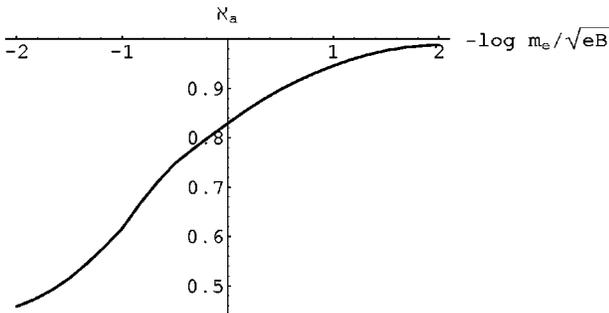


FIG. 2. Plot of $\kappa_a(T=0)$ vs $\log \hat{m}_e^{-1}$ for field range $10^{-2} m_e \leq \sqrt{eB} \leq 10^2 m_e$ and parameter values $|\hat{p}_\parallel| = |\hat{p}_\perp| = 10^{-1} \sqrt{2} \hat{m}_e$, $\hat{M}_W = 10^5 \hat{m}_e$.

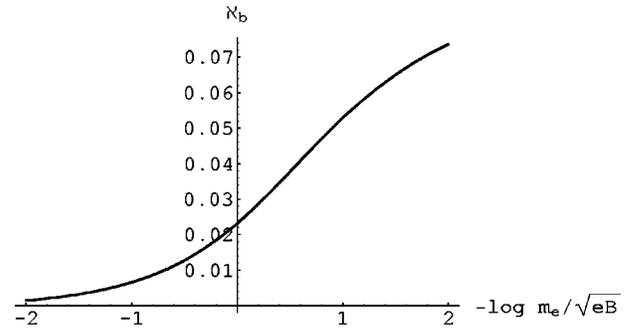


FIG. 3. Plot of $\kappa_b(T=0)$ vs $\log \hat{m}_e^{-1}$ for field range $10^{-2} m_e \leq \sqrt{eB} \leq 10^2 m_e$ and parameter values $|\hat{p}_\parallel| = |\hat{p}_\perp| = 10^{-1} \sqrt{2} \hat{m}_e$, $\hat{M}_W = 10^5 \hat{m}_e$.

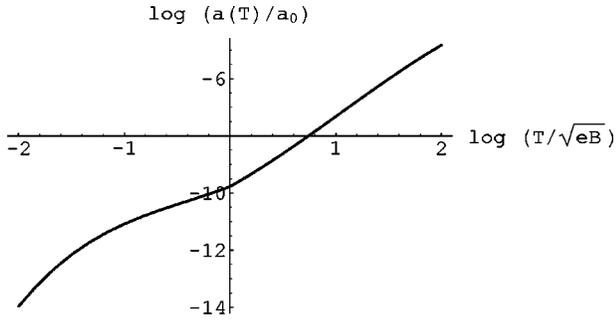


FIG. 4. Plot of $\log(a(T)/a_0)$ vs $\log(\hat{T})$ for field range $10^{-2}T \leq \sqrt{eB} \leq 10^2 T$ and parameter values $|\hat{p}_{\parallel}| = |\hat{p}_{\perp}| = 10^{-1} \sqrt{2} \hat{m}_e$, $\hat{M}_W = 10^5 \hat{m}_e$, $\hat{m}_e = 0.1$.

Unlike the stellar material, whose density can be of order one, we know that the early universe was almost charge symmetric, with a particle-antiparticle asymmetry of only 10^{-9} – 10^{-10} . Therefore, when considering possible consequences of our results for the early universe, we should restrict the discussion to the neutral case ($\mu=0$).

For the neutral case, it is known that at weak field all the corrections in the neutrino energy density, whether they depend on the field and/or temperature, are second order in the Fermi coupling constant [16,12,14], therefore negligible small. Nevertheless, it can be seen from Eqs. (61),(62) that if sufficiently strong magnetic fields were present in the primeval plasma, they would yield corrections to the energy density that are linear in the Fermi coupling constant. Moreover, as shown in Sec. VI, the strong-field approximation that led to these corrections is reliable even for magnetic fields in a more realistic range [i.e. as those satisfying condition (103)].

Notice that a field satisfying Eq. (103) would be effectively strong with respect to the electron-neutrino, but weak for the remaining neutrino flavors. If such a field existed during the decoupling era, it could significantly affect the $\nu_e \leftrightarrow \nu_{\mu}, \nu_{\tau}$ and $\nu_e \leftrightarrow \nu_s$ resonant oscillations [11], as the field would differently modify the energy of ν_e compared to those of ν_{μ}, ν_{τ} and ν_s . The interesting new thing here is that despite the fact that these oscillations would take place in an essentially neutral medium, because of the strong field they will be as significant as those produced by the MSW mechanism [26] in a dense medium. A peculiarity of the strong-field effects on the neutrino energy density is to give rise to anisotropic resonant oscillations. That is, the oscillation probability depends on the direction of the neutrino propagation with respect to the magnetic field.

Another interesting question related to primordial magnetic fields is whether they influenced neutrino propagation prior to the electroweak phase transition, since some of the mechanisms of primordial magnetic field generation allow their existence at very early epochs [7,8]. Before the electroweak transition a primordial magnetic field could only exist in the form of a U(1) hypermagnetic field [45]. Any non-Abelian “magnetic” field would decay at high temperatures because it would acquire a nonperturbative infrared magnetic mass $g^2 T$. The implications of primordial hypermagnetic fields in neutrino propagation before the electroweak phase transition have been studied in Refs. [43,44].

In [43] the strong-field effects led to a large anisotropy in the leptons’ energies. The anisotropy was due to the degeneracy in the energy, which in the leading order does not depend on the transverse momentum component. It is in contrast with the magnetic field case, which takes place after the electroweak phase transition, where, although an anisotropy in the neutrino energy is also present, no degeneracy occurs [see Eqs. (61),(62)]. This different behavior is a direct consequence, on one hand, of the nonzero neutrino hypercharge and hence of the minimal coupling of these particles to the hypermagnetic field; and on the other, of their electrical neutrality, which implies that they can interact with a magnetic field through radiative corrections only.

We underline that the anisotropy in the energy spectrum of neutrinos in background fields could provide an independent way to verify the existence of primordial magnetic fields, since a field-induced anisotropy would be reflected as an imprint in a yet undetected and elusive cosmic background of neutrinos produced during the decoupling era.

For astrophysical applications we should turn our attention to the star interiors characterized by high densities of charged particles under strong magnetic fields. As shown in Sec. V, in the charged medium case the strong-field approximation gives rise to a modification of the kinetic part of the neutrino energy [first term in Eq. (67)], that is not present in leading order at weak field [Eq. (70)] [15]. Due to this new term the largest field-dependent contribution to the neutrino kinetic energy comes from neutrinos propagating perpendicularly to the field.

It is worth noticing that the anisotropic term associated with the effective magnetic moment in the dispersion relations (67) and (70) changes sign when neutrinos reverse their direction of motion. On the other hand, the kinetic energy term in Eq. (67) remains unchanged when the neutrinos move in opposite direction, even though it depends on the direction of propagation of the neutrinos with respect to the magnetic field. As known [32], the anisotropy associated to the magnetic-moment contribution to the neutrino energy can be relevant for a possible explanation of the peculiar high pulsar velocities. It would be interesting to consider the combined effects of the two different anisotropies to understand whether they could affect the dynamics of proto-neutron stars.

Another possible ground of applications of our results is supernova neutrinos. Core collapse supernovae are dominated by neutrinos and their transport properties. In addition, the observation of neutrinos from supernova, which is essential to confirm the basic picture of supernova explosion, can be affected by neutrino oscillations. As we have pointed out in this paper, neutrino transport and oscillations can be both modified by the presence of a strong magnetic field. Magnetic fields as strong as 10^{14} to 10^{16} G could exist in the first seconds of neutrino emission inside the supernova core [46]. Thus, the electron-neutrino energy spectrum found in this paper, within the strong-field limit for the charged medium, should be considered for any study of neutrino oscillations under those conditions.

VIII. CONCLUDING REMARKS

In this paper we carried out a thorough study of the propagation of neutrinos in strongly magnetized neutral and charged media. We started from the most general structure of the neutrino self-energy in a magnetic field, expressing it as the sum of four independent covariant terms with coefficients that are functions of the physical variables of the theory and whose values depend on the approximation considered. General expressions of the four coefficients at one-loop approximation were given in Eqs. (27)–(30).

The coefficients were then calculated in the strong-field limit using the LLL approximation for the electrons. The LLL was assumed to be valid in the parameter range $M_W \gg \sqrt{B} \gg m_e, |\mathbf{p}|, eB \gg T^2$. To justify it one should keep in mind that under these conditions most electrons would not have enough energy to overcome the gap between the Landau levels. Hence, they will be mainly confined to their lower levels and the leading contribution would come from the LLL. This assumption was also corroborated for the above parameters' range by numerical calculations summing in all Landau levels.

The dispersion relation of the neutrinos was written as a function of the four coefficients of the self-energy structures, allowing in this way to straightforwardly obtain the neutrino's energy in the strong-field limit for each physical case.

In concordance with results previously obtained in charged media at weak fields [15,14], in the strong-field case an energy term associated with the interaction between the magnetic field and the effective magnetic moment was also found at leading order in G_F . This interaction energy disappears in the neutral medium, since in a charged-symmetric plasma the contribution to the effective magnetic moment coming from electrons and positrons cancels out.

A main outcome of our investigation was to show that in strongly magnetized systems a term of different nature emerges in both charged and neutral media. The new term, which is linear in the magnetic field and of first order in G_F , enters as a correction to the neutrino kinetic energy in the presence of a strong-magnetic field. This correction is present even in a strongly magnetized vacuum, since it is related to the vacuum part ($T=0, \mu=0$) of the neutrino self-energy at $B \neq 0$.

A characteristic of the field-dependent corrections to the neutrino energy is that they produce an anisotropic index of refraction, since neutrinos moving along different directions have different field-dependent dispersion relations. We should underline that while the magnetic moment interaction term produces a maximum field effect for neutrinos propagating along the field lines, the field correction to the kinetic energy does not contribute to those propagation modes, but on the contrary, the maximum kinetic-energy effect takes place for neutrinos propagating perpendicularly to the field direction. We stress that the anisotropy does not differentiate between neutrinos and antineutrinos.

The charged medium results reported in the current work can be of interest for the astrophysics of neutrinos in stars with large magnetic fields. On the other hand, our finding for the neutral medium can have applications in cosmology, if the existence of high primordial magnetic fields is finally confirmed. Contrary to some authors' belief [14,10] that, regardless of the field intensity, the neutrino dispersion relation in the early universe is well approximated by the dispersion relation in the zero field medium, our results indicate that strong, and even weakly strong, magnetic fields can give rise to a contribution to the neutrino energy that is several orders larger than the pure thermal contribution.

The field-dependent correction to the neutrino energy in a neutral medium with strong magnetic field can have an impact in neutrino flavor-oscillations in the primeval plasma [11] and therefore affect primordial nucleosynthesis. Hence, this new effect could be important to establish possible limits to the strength of the primordial magnetic field.

ACKNOWLEDGMENTS

E.E. is indebted to the members of the Mathematics Dept. of MIT, and especially to Dan Freedman, for the warm hospitality. E.J.F. and V.I. would like to thank the Institute of Space Studies of Catalonia (Spain), where part of this work was accomplished, for the warm hospitality during their visit. The work of E.E. has been supported by DGICYT (Spain), Contract No. PR-2003-0352 and project BFM2003-00620. The work of E.J.F. and V.I. was supported by NSF Grant No. PHY-0070986.

-
- [1] V. Ginzburg, *High Energy Gamma Ray Astrophysics* (North-Holland, Amsterdam, 1991); D. Bhattacharya and G. Srinivasan, in *X-ray Binaries*, edited by W.H.G. Lewin, J. van Paradijs, and E.P.J. van den Heuvel (Cambridge Univ. Press, Cambridge, 1995), p. 495.
- [2] I. Fushiki, E.H. Gudmundsson, and C.J. Pethick, *Astrophys. J.* **342**, 958 (1989); T.A. Mihara *et al.*, *Nature* (London) **346**, 250 (1990); G. Chanugam, *Annu. Rev. Astron. Astrophys.* **30**, 143 (1992).
- [3] R.C. Duncan and C. Thompson, *Astrophys. J. Lett.* **392**, L9 (1992); B. Paczynski, *Acta Astron.* **42**, 145 (1992); V.V. Usov, *Nature* (London) **357**, 472 (1992); C. Thompson and R.C. Duncan, *Astrophys. J.* **473**, 322 (1996); C. Kouveliotou *et al.*, *Nature* (London) **393**, 235 (1998); *Astrophys. J. Lett.* **510**, L115 (1999).
- [4] D. Lai and S.L. Shapiro, *Astrophys. J.* **383**, 745 (1991) and references therein.
- [5] For a recent review on neutrino propagation in dense astrophysical systems, see M. Prakash *et al.*, *Annu. Rev. Nucl. Part. Sci.* **51**, 295 (2001).
- [6] Y. Sofue, M. Fujimoto, and R. Wielebinski, *Annu. Rev. Astron. Astrophys.* **24**, 459 (1986); P.P. Kronberg, *Rep. Prog. Phys.* **57**, 325 (1994); R. Beck *et al.*, *Annu. Rev. Astron. Astrophys.* **34**, 153 (1996).
- [7] T. Vachaspati, *Phys. Lett. B* **265**, 258 (1991); K. Enqvist and P. Olesen, *ibid.* **319**, 178 (1993); B. Cheng and A.V. Olinto, *Phys.*

- Rev. D **50**, 2421 (1994); G. Baym, D. Bodeker, and L. McLerran, *ibid.* **53**, 662 (1996); A. Brandenburg, K. Enqvist, and P. Olesen, *ibid.* **54**, 1291 (1996); Phys. Lett. B **392**, 395 (1997); G. Sigl, A.V. Olinto, and K. Jedamzik, Phys. Rev. D **55**, 4582 (1997); M. Joyce and M.E. Shaposhnikov, Phys. Rev. Lett. **79**, 1193 (1997); K. Enqvist, Int. J. Mod. Phys. D **7**, 331 (1998); M. Giovannini and M.E. Shaposhnikov, Phys. Rev. D **57**, 2186 (1998); O. Törnkvist, *ibid.* **58**, 043501 (1998); M. Hindmarsh and A. Everett, *ibid.* **58**, 103505 (1998); D. Grasso and A. Riotto, Phys. Lett. B **418**, 258 (1998); J. Ahonen and K. Enqvist, Phys. Rev. D **57**, 664 (1998); P. Cea and L. Tedesco, Phys. Lett. B **450**, 61 (1999); Int. J. Mod. Phys. D **12**, 663 (2003); K. Dimopoulos, T. Prokopec, O. Törnkvist, and A.C. Davis, Phys. Rev. D **65**, 063505 (2002); D. Boyanovsky, H.J. de Vega, and M. Simionato, *ibid.* **67**, 123505 (2003); Z. Berezhiani and A.D. Dolgov, Astropart. Phys. **21**, 59 (2004); A.D. Dolgov, astro-ph/0306443; M. Giovannini, Int. J. Mod. Phys. D **13**, 391 (2004).
- [8] D. Grasso and H.R. Rubinstein, Phys. Rep. **348**, 163 (2001).
- [9] J.P. Ostriker, C. Thompson, and E. Witten, Phys. Lett. B **180**, 231 (1986).
- [10] A.D. Dolgov, Phys. Rep. **370**, 333 (2002).
- [11] E.J. Ferrer and V. de la Incera, *Neutrinos under Strong Magnetic Fields, Proceedings of Neutrinos, Flavor Physics and Precision Cosmology, Fourth Tropical Workshop*, edited by J.F. Nieves and R.R. Volkas, AIP Conf. Proc. No. 689 (AIP, Melville, NY, 2003), p. 372.
- [12] G. McKeon, Phys. Rev. D **24**, 2744 (1981).
- [13] A. Erdas and G. Feldman, Nucl. Phys. **B343**, 597 (1990).
- [14] V.B. Semikoz and J.W.F. Valle, Nucl. Phys. **B425**, 651 (1994); P. Elmfors, D. Grasso, and G. Raffelt, *ibid.* **B479**, 3 (1996); A. Erdas, C.W. Kim, and T.H. Lee, Phys. Rev. D **58**, 085016 (1998); D. Grasso, Nucl. Phys. B (Proc. Suppl.) **70**, 267 (1999); A. Erdas and C. Isola, Phys. Lett. B **494**, 262 (2000).
- [15] J.C. D'Olivo, J.F. Nieves, and P.B. Pal, Phys. Rev. D **40**, 3679 (1989).
- [16] N. Notzold and G. Raffelt, Nucl. Phys. **B307**, 924 (1988).
- [17] E. Elizalde, E.J. Ferrer, and V. de la Incera, Ann. Phys. (N.Y.) **295**, 33 (2002).
- [18] E. J. Ferrer, in *Proceedings of the International Conference Quantization, Gauge Theory, and Strings, Moscow*, edited by A. Semikhatov, M. Vasiliev, and V. Zaikin (World Scientific, Singapore, 2001), Vol. II, p. 301; E. Elizalde, E.J. Ferrer, and V. de la Incera, in *Neutrino Dispersion in Intense Magnetic Field, Proceedings of Particle Physics and Cosmology Second Tropical Workshop*, edited by J.F. Nieves, AIP Conf. Proc. No. 540 (AIP, Melville, NY, 2000), p. 372.
- [19] V.I. Ritus, Ann. Phys. (N.Y.) **69**, 555 (1972); Zh. Eksp. Teor. Fiz. **75**, 1560 (1978).
- [20] V. I. Ritus, in *Issues in Intense-Field Quantum Electrodynamics*, edited by V.L. Ginzburg, Proceedings of P.N. Lebedev Physical Institute, Vol. 111, p. 5, Moscow (Nauka, Moscow, 1979), Vol. 111, p. 5. [English translation, Vol. 168 (Nova Science, Commack, 1987)].
- [21] J. Schwinger, Phys. Rev. **82**, 664 (1951).
- [22] C.N. Leung, Y.J. Ng, and A.W. Ackley, Phys. Rev. D **54**, 4181 (1996); D.-S. Lee, C.N. Leung, and Y.J. Ng, *ibid.* **55**, 6504 (1997); E.J. Ferrer and V. de la Incera, *ibid.* **58**, 065008 (1998); Int. J. Mod. Phys. A **14**, 3963 (1999); Phys. Lett. B **481**, 287 (2000); E. Elizalde, E.J. Ferrer, and V. de la Incera, Phys. Rev. D **68**, 096004 (2003).
- [23] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1964).
- [24] M.J. Duff and M. Ramon-Medrano, Phys. Rev. D **12**, 3357 (1975); G.K. Savvidy, Phys. Lett. **71B**, 133 (1977); S.G. Matinyan and G.K. Savvidy, Nucl. Phys. **B134**, 539 (1978); H.B. Nielsen and P. Olesen, *ibid.* **B144**, 376 (1978); V.V. Skalozub, Sov. J. Nucl. Phys. **43**, 665 (1986).
- [25] V.V. Skalozub, Sov. J. Nucl. Phys. **45**, 1058 (1987); J. Ambjorn and P. Olesen, Nucl. Phys. **B315**, 606 (1989); **B330**, 193 (1990).
- [26] L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978); S.P. Mkheyev and A.Yu. Smirnov, Sov. J. Nucl. Phys. **42**, 913 (1985).
- [27] W. Dittrich and M. Reuter, *Effective Lagrangians in Quantum Electrodynamics*, Lecture Notes in Physics Vol. 220 (Springer-Verlag, Berlin, 1985).
- [28] A.D. Linde, Phys. Lett. **86B**, 39 (1979); E.J. Ferrer, V. Incera, and A.E. Shabad, Phys. Lett. B **185**, 407 (1987); Nucl. Phys. **B309**, 120 (1988).
- [29] A. Gynther, Phys. Rev. D **68**, 016001 (2003); F. Sannino and K. Tuominen, *ibid.* **68**, 016007 (2003); V.P. Gusynin, V.A. Miransky, and I. Shovkovy, Phys. Lett. **58B**, 82 (2004).
- [30] B.W. Lee and R. Shrock, Phys. Rev. D **16**, 1444 (1977); K. Fujikawa and R.E. Shrock, Phys. Rev. Lett. **45**, 963 (1980).
- [31] P.B. Pal and T.N. Pham, Phys. Rev. D **40**, 259 (1989); J.F. Nieves, *ibid.* **40**, 866 (1989).
- [32] A. Kusenko and G. Segre, Phys. Rev. Lett. **77**, 4872 (1996).
- [33] For a discussion on the relation between the index of refraction and the different velocity notions, see L. Brillouin, *Wave Propagation and Group Velocity* (Academic Press, New York, 1960); R.L. Smith, Am. J. Phys. **38**, 978 (1970); A. Kuzmich *et al.*, Phys. Rev. Lett. **86**, 3925 (2001).
- [34] I.J. Drummond and S.J. Harthrell, Phys. Rev. D **22**, 343 (1980).
- [35] K. Scharnhorst, Phys. Lett. B **236**, 354 (1990).
- [36] E.J. Ferrer, V. de la Incera, and A. Romero, Phys. Lett. B **515**, 341 (2001).
- [37] J.L. Latorre, P. Pascual, and R. Tarrach, Nucl. Phys. **B437**, 60 (1995); G.M. Shore, *ibid.* **B460**, 379 (1996); K. Scharnhorst, Ann. Phys. (N.Y.) **7**, 700 (1998); W. Dittrich and H. Gies, Phys. Rev. D **58**, 025004 (1998); A.D. Dolgov and I.D. Novikov, Phys. Lett. B **442**, 82 (1998).
- [38] *Higher Transcendental Functions*, Vol. II, *California Inst. of Technology, Bateman Manuscript Project* (McGraw-Hill Book Company, Inc., New York, 1953).
- [39] E. D. Rainville, *Special Functions* (The MacMillan Company, New York, 1960).
- [40] K. Enqvist, Int. J. Mod. Phys. D **7**, 331 (1998).
- [41] K.A. Olive, D.N. Schramm, G. Steigman, and T. Walker, Phys. Lett. B **236**, 454 (1990); L.M. Krauss and P. Romanelli, Astrophys. J. **358**, 47 (1990).
- [42] K. Enqvist, V. Semikoz, A. Shukurov, and D. Sokoloff, Phys. Rev. D **48**, 4557 (1993).
- [43] J. Cannellos, E.J. Ferrer, and V. de la Incera, Phys. Lett. B **542**, 123 (2002); J. Cannellos, E.J. Ferrer, and V. de la Incera, *Self Energy of Chiral Leptons in a Background Hypermagnetic Field, Proceedings of MRST-2002, Perimeter Institute, Waterloo, Canada, 2002*, edited by V. Elias, R. J. Epp, and R. C.

- Myers, AIP Conf. Proc. No. 646 (AIP, Melville, NY, 2003), p. 117.
- [44] A. Ayala, A. Bashir, and S. Sahu, Phys. Rev. D **69**, 045008 (2004).
- [45] A.D. Linde, Rep. Prog. Phys. **42**, 389 (1979); Phys. Lett. **96B**, 178 (1980); K. Kajantie, M. Laine, K. Rummukainen, and M.E. Shaposhnikov, Nucl. Phys. **B493**, 413 (1997).
- [46] C. Thompson and R.C. Duncan, Astrophys. J. **408**, 194 (1993).