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Border of spacetime

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It is still uncertain whether the cosmic censorship conjecture is true or not. To get a new insight into this issue, we propose the concept of the border of spacetime as a generalization of the spacetime singularity and discuss its visibility. The visible border, corresponding to the naked singularity, is not only relevant to mathematical completeness of general relativity but also a window into new physics in strongly curved spacetimes, which is in principle observable.

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In general relativity, the singularity theorems tell us that spacetime singularities exist in generic gravitational collapse spacetime (see, e.g., Ref. [1]). For singularities formed in gravitational collapse, Penrose [2,3] proposed the so-called cosmic censorship conjecture, which has two versions. For spacetimes which contain physically reasonable matter fields and develop from generic nonsingular initial data, the weak one claims that there is no singularity which is visible from infinity, while the strong one claims that there is no singularity which is visible to any observer. A singularity censored by the strong version is called a naked singularity, while a singularity censored by the weak version is called a globally naked singularity. There is no general proof for the conjecture at present. Recent development on critical behavior ([4,5], see also Ref. [6]) and self-similar attractor ([7], see also Ref. [8]) has shown that there are naked-singular solutions which result from nonsingular initial data and contain physically reasonable matter fields. The critical solution has one unstable mode while the self-similar attractor has no unstable mode against spherical perturbation, although it is still uncertain whether these examples are stable against all other possible perturbations. If all examples of naked singularities were shown to be unstable, would it mean that they are all rubbish?

We have already known that general relativity will have the limitation of its applicable scale in a high-energy side. A simple and natural discussion on quantum effects of gravity yields the Planck energy $E_{\rm Pl}\sim 10^{19}$ GeV as a cut-off scale Λ. Some theories with large extra dimensions may have much lower cut-off scale, which could be TeV scale [9,10]. The energy scale of the curved spacetime can be measured by the curvature through Einstein's field equations. Then if the above expectation is true, general relativity is not applicable to the spacetime region whose curvature strength exceeds $\Lambda^4/E_{\rm Pl}^2$. This consideration naturally leads to the notion of the border of spacetime as follows.

Let $(\mathcal{M},g)$ be a spacetime manifold $\mathcal{M}$ with a metric g. We call a spacetime region $\mathcal{A}\subset\mathcal{M}$ a *border* if and only if the following inequality is satisfied:

$$\inf_{\mathcal{A}} F \geqslant \frac{\Lambda^4}{E_{\rm Pl}^2}, \qquad (1)$$

where the curvature strength F is given, for instance, by

$$F := \max(|R^a{}_a|, |R^{ab}R_{ab}|^{1/2}, |R^{abcd}R_{abcd}|^{1/2}). \qquad (2)$$

We denote the union of all borders in $\mathcal{M}$ by $\mathcal{U}_{\rm B}$. We call a border $\mathcal{A}$ a *visible border* if and only if $J^+(\mathcal{A},\mathcal{M})\cap(\mathcal{M}-\mathcal{U}_{\rm B})$ is not empty, where $J^+(\mathcal{A},\mathcal{M})$ is the causal future of $\mathcal{A}$ in $\mathcal{M}$ [1].

We can also naturally define a *globally visible border*. To make the definition precise, we assume that the spacetime $\mathcal{M}$ is asymptotically flat and thus $(\mathcal{M},g)$ is conformally embedded into a space $(\tilde{\mathcal{M}},\tilde{g})$ as an unphysical spacetime manifold with boundary $\bar{\mathcal{M}}=\mathcal{M}\cup\partial\mathcal{M}$, where the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ in $\tilde{\mathcal{M}}$ consists of the future and past null infinities $\mathcal{I}^+$ and $\mathcal{I}^-$ [1]. We call a border $\mathcal{A}$ a *globally visible border* if and only if $J^+(\mathcal{A},\bar{\mathcal{M}})\cap\mathcal{I}^+$ is not empty. The asymptotic flatness implies that a globally visible border is a visible border since $\mathcal{I}^+$ is attached to a nonborder region of $\mathcal{M}$ in $\tilde{\mathcal{M}}$. According to these definitions, naked-singular spacetimes do not necessarily involve visible borders. See Fig. 1 for an example in which there is no visible border but naked singularity.

The presence of visible border implies the incompleteness of future predictability of general relativity. Even already known examples of naked singularity formation will show

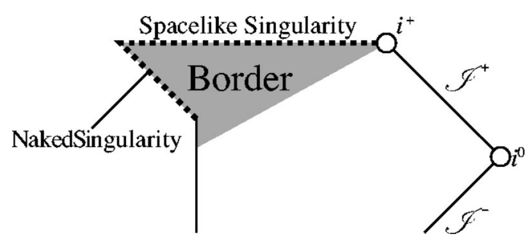

FIG. 1. Penrose diagram of a possible spacetime in which there is no visible border even if a naked singularity exists. The shaded region is a border region. The causal future of the border contains no nonborder region.

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that visible borders appear with nonzero probability in the collapse of physically reasonable matter fields. These visible borders will be stable against perturbations even if these perturbations prevent the formation of naked singularities. Even in the case of black hole formation, globally visible borders can appear. If the mass of the black hole $M_{\rm bh}$ is smaller than $E_{\rm Pl}^3/\Lambda^2$, the curvature strength in its exterior can be larger than the cut-off scale $\Lambda^4/E_{\rm Pl}^2$ and hence it is necessarily a globally visible border. The critical behavior shows a possibility that such small black holes form in our universe within the framework of general relativity, and thus strongly suggests that the cosmic censorship conjecture is violated in this physical sense. It is noted that general relativity and other conventional physics are still applicable within the maximum Cauchy development in $(\mathcal{M}-\mathcal{U}_{\rm VB})$, where $\mathcal{U}_{\rm VB}$ is the union of all visible borders. Just on $\mathcal{U}_B$, the effects of new physics will affect the spacetime metric directly or might invalidate the notion of spacetime manifold itself, where the new physics may be quantum gravity or possibly the classical theories other than general relativity, e.g., dilatonic gravity and higher-dimensional theory. In $(\mathcal{M}-\mathcal{U}_{\rm B})$, the effects of new physics will enter as boundary conditions.

However, if extremely severe fine tuning of initial conditions is required for the formation of visible border, we might say that visible borders are censored practically. We focus on this issue below. First, if a naked-singular collapse solution is stable against all possible perturbations and an attractor, no fine tuning is needed for the formation of visible borders.

Next, suppose a naked-singular collapse solution is a spherically symmetric self-similar solution with one unstable mode. According to the renormalization group scenario [11], this solution will be identified with a critical solution. Let t and r be appropriate time and radial coordinates, respectively, and also $\tau\equiv -\ln(-Et)$ and $x\equiv\ln[Er/(-Et)]$, where $E(\ll\Lambda)$ is the characteristic energy scale at the initial moment. A general solution $h(\tau,x)$ is expressed by a trajectory in the space of initial data sets (SIDS), which is the space of functions of x, while a self-similar solution $h_0(x)$ corresponds to a fixed point. If a fixed point has stable perturbations, there is a family of solutions asymptotically approaching the fixed point. This family will form a manifold embedded in the SIDS, which is called the stable manifold S of the fixed point. For the case of the fixed point with one unstable mode, its stable manifold S is codimension one in SIDS. Here let us consider a one-parameter family of initial data sets parametrized by p. This family has generically intersections with S by virtue of the codimension of S in SIDS. A value $p=p^*$ at an intersection corresponds to the critical value. The one-parameter family in the SIDS induces a one-parameter family of trajectories $h=h_p(\tau,x)$. Then only the trajectory $h=h_{p^*}(\tau,x)$ asymptotically approaches the fixed point $h=h_0(x)$ with one unstable mode, which corresponds to the critical solution.

For an initial data set with $p\approx p^*$, the solution after a long time will be described by

$$h(\tau,x)\approx h_0(x)+(p-p^*)e^{\kappa\tau}f_{\rm rel}(x), \tag{3}$$

where κ (>0) and $f_{\rm rel}$ are the eigenvalue and eigenfunction of the unstable mode, respectively. The first term $h_0(x)$ on the right-hand side in Eq. (3) is dominant at first, and thus the solution initially shows almost self-similar behavior. When the second term becomes comparable to the first one, the self-similar behavior is lost and the collapsing mass will start to form a black hole or bounce to disperse away. Let the mass within $x\leqslant x_{\rm bh}$ collapse to a black hole. Then the formation time $\tau=\tau_{\rm bh}$ of the black hole is estimated as $e^{\tau_{\rm bh}}\sim|(p-p^*)f_{\rm rel}(x_{\rm bh})/h_0(x_{\rm bh})|^{-\kappa^{-1}}=O(|p-p^*|^{-\kappa^{-1}})$, where $h_0(x_{\rm bh})$ and $f_{\rm rel}(x_{\rm bh})$ are of order unity. Using this result, we obtain the length scale $r=r_{\rm bh}$ of the collapsing mass at $\tau=\tau_{\rm bh}$, i.e., the gravitational radius $M_{\rm bh}/E_{\rm Pl}^2$ as

$$\frac{M_{\rm bh}}{E_{\rm Pl}^2}\sim r_{\rm bh}=E^{-1}e^{-\tau_{\rm bh}+x_{\rm bh}}=O(E^{-1}|p-p^*|^{\kappa^{-1}}). \tag{4}$$

Therefore the black hole mass satisfies the above power-law behavior, where the index $\gamma\equiv\kappa^{-1}$ is called the critical exponent. The curvature strength F is then estimated to be

$$F\sim\frac{E_{\rm Pl}^4}{M_{\rm bh}^2}=O(E^2|p-p^*|^{-2\kappa^{-1}}). \tag{5}$$

From the above equation, the width of fine tuning for the appearance of visible borders is estimated to be

$$|p-p^*|=O\left(\left(\frac{\Lambda^2}{E_{\rm Pl}E}\right)^{-\kappa}\right). \tag{6}$$

The above estimate also applies to the subcritical case, where the curvature strength reaches a maximum at the bounce and the matter field eventually disperses away. If κ is very small or equivalently γ is very large, the width of fine-tuning is not too small even for $E\ll\Lambda^2/E_{\rm Pl}$.

For a naked-singular solution with n unstable modes, an n-parameter family of initial data sets has generically intersections with the stable manifold of codimension n. Then the fine tuning should be considered in the n-dimensional parameter space. Clearly, naked-singular solutions with fewer unstable modes of small eigenvalues are physically more important. Moreover, although in the above discussion we have supposed spherically symmetric perturbations, it is reasonably expected that the discussion goes similarly even for nonspherical perturbations [12]. We can also infer that the discussion also applies even to nonspherical naked-singular solutions.

Finally we discuss the detectability of visible borders in practice. Let the mass M be distributed nearly spherically and homogeneously within the length scale L. Then the curvature strength F is typically estimated as $M/E_{\rm Pl}^2L^3$. If this mass is visible to an observer at infinity, $M/E_{\rm Pl}^2\lesssim L$ should be satisfied. This means that the curvature strength produced by the spherical visible mass satisfies $F\lesssim E_{\rm Pl}^4/M^2$. The definition (1) of visible border implies $E_{\rm Pl}^4/M^2\gtrsim\Lambda^4/E_{\rm Pl}^2$ so that the mass M can form a spherically symmetric visible border. This means that the mass M of the spherically symmetric visible border should satisfy

$$M \lesssim M_{\Lambda} := \frac{E_{\rm Pl}^3}{\Lambda^2}. \tag{7}$$

The upper bound M_{Λ} is equal to or smaller than the lunar mass $\sim 10^{27}$ g$\sim 10^{48}$ erg from the experimental constraint $\Lambda \gtrsim 1$ TeV. If the cut-off scale is much higher than TeV scale and if the energy conservation law in a usual sense holds and the mass of the visible border is not going to be negative, the effect of one almost spherically symmetric visible border will be rather small as an energy source in astrophysical situations so that it may be difficult to astronomically observe the direct signal from inside the visible border. In order that visible borders may be observable in a practical astronomical sense, the mass should be distributed in very specific manners, e.g., a highly elongated visible border and a loosely bound cluster of almost spherical visible borders, in an accordance with the hoop conjecture [13]. On the other hand, if recently proposed TeV scale gravity describes real gravitational physics at TeV scale, visible borders will be observed by the planned high-energy collider experiments and/or as astrophysical high-energy phenomena.

The appearance of visible borders with nonzero probability implies not only the limitation of general relativity but also a new window into extremely high-curvature spacetime physics in principle observable.

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[1] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1972).

[2] R. Penrose, Riv. Nuovo Cimento **I**, 252 (1969); reprinted in Gen. Relativ. Gravit. **34**, 1141 (2002).

[3] R. Penrose, in *General Relativity, an Einstein Centenary Survey*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979), p. 581.

[4] M.W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).

[5] J.M. MartínGarcía and C. Gundlach, Phys. Rev. D **68**, 024011 (2003).

[6] C. Gundlach, Phys. Rep. **376**, 339 (2003).

[7] T. Harada and H. Maeda, Phys. Rev. D **63**, 084022 (2001).

[8] T. Harada, to appear in the Proceedings of the International Conference on Gravitation and Cosmology (ICGC-2004), edited by B.R. Iyer, V. Kuriakose, and C.V. Vishveshwara, gr-qc/0407109.

[9] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, *ibid.* **436**, 257 (1998).

[10] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); **83**, 4690 (1999).

[11] T. Koike, T. Hara, and S. Adachi, Phys. Rev. Lett. **74**, 5170 (1995).

[12] C. Gundlach, Phys. Rev. D **65**, 084021 (2002).

[13] K. S. Thorne, in *Magic Without Magic*, edited by J. R. Klauder (Freeman, San Francisco, 1972), p. 231.