

Quantum weights of dyons and of instantons with nontrivial holonomy

Dmitri Diakonov,^{1,2} Nikolay Gromov,³ Victor Petrov,² and Sergey Slizovskiy³

¹*NORDITA, Blegdamsvej 17, DK-2100 Copenhagen, Denmark*

²*St. Petersburg NPI, Gatchina, 188 300, St. Petersburg, Russia*

³*St. Petersburg State University, Faculty of Physics, Ulianovskaya 1, 198 904, St. Petersburg, Russia*

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We calculate exactly functional determinants for quantum oscillations about periodic instantons with a nontrivial value of the Polyakov line at spatial infinity. Hence, we find the weight or the probability with which calorons with nontrivial holonomy occur in the Yang-Mills partition function. The weight depends on the value of the holonomy, the temperature, Λ_{QCD} , and the separation between the BPS monopoles (or dyons) that constitute the periodic instanton. At large separation between constituent dyons, the quantum measure factorizes into a product of individual dyon measures, times a definite interaction energy. We present an argument that at temperatures below a critical one related to Λ_{QCD} , trivial holonomy is unstable, and that calorons “ionize” into separate dyons.

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I. MOTIVATION AND THE MAIN RESULT

There are two known generalizations of the standard self-dual instantons to nonzero temperatures. One is the periodic instanton of Harrington and Shepard [1] studied in detail by Gross, Pisarski, and Yaffe [2]. These periodic instantons, also called *calorons*, are said to have trivial holonomy at spatial infinity. It means that the Polyakov line

$$L = \text{P exp} \left(\int_0^{1/T} dt A_4 \right) \Big|_{|\vec{x}| \rightarrow \infty} \quad (1)$$

assumes values belonging to the group center $Z(N)$ for the $SU(N)$ gauge group [3]. The vacuum made of those instantons has been investigated, using the variational principle, in Ref. [4].

The other generalization has been constructed a few years ago by Kraan and van Baal [5] and Lee and Lu [6]; it has been called the *caloron with nontrivial holonomy* as the Polyakov line for this configuration does not belong to the group center. We shall call it for short the KvBLL caloron. It is also a periodic self-dual solution of the Yang-Mills equations of motion with an integer topological charge. In the limiting case when the KvBLL caloron is characterized by trivial holonomy, it is reduced to the Harrington-Shepard caloron. The fascinating feature of the KvBLL construction is that a caloron with a unit topological charge can be viewed as “made of” N Bogomolnyi-Prasad-Sommerfeld (BPS) monopoles or dyons [7,8].

Dyons are self-dual solutions of the Yang-Mills equations of motion with static (i.e., time-independent) action density, which have both the magnetic and electric field at infinity decaying as $1/r^2$. Therefore these objects carry both electric and magnetic charges (prompting their name). In the $(3+1)$ -dimensional $SU(2)$ gauge theory there are in fact two types of self-dual dyons [9]: M and L with (electric, magnetic) charges $(+, +)$ and $(-, -)$, and two types of anti-self-dual dyons \bar{M} and \bar{L} with charges $(+, -)$ and $(-, +)$, respectively. Their explicit fields can be found, e.g., in Ref.

[10]. In the $SU(N)$ theory there are $2N$ different dyons [9,11]: M_1, M_2, \dots, M_{N-1} ones with charges counted with respect to $N-1$ Cartan generators and one L dyon with charges compensating those of M_1, \dots, M_{N-1} to zero, and their anti-self-dual counterparts.

Speaking of dyons one implies that the Euclidean space-time is compactified in the “time” direction whose inverse circumference is temperature T , with the usual periodic boundary conditions for boson fields. However, the temperature may go to zero, in which case the 4D Euclidean invariance is restored.

The essence of the dyon is that the A_4 component of the dyon field tends to a constant value at spatial infinity. This constant A_4 can be eliminated by a time-dependent gauge transformation. However then the fields violate the periodic boundary conditions, unless A_4 has quantized values corresponding to trivial holonomy, i.e., unless the Polyakov line belongs to the group center. Therefore, in a general case one implies that dyons have a nonzero value of A_4 at spatial infinity and a nontrivial holonomy.

The KvBLL caloron of the $SU(2)$ gauge group (to which we restrict ourselves in this paper) with a unit topological charge is “made of” one L and one M dyon, with total zero electric and magnetic charges. Although the action density of isolated L and M dyons does not depend on time, their combination in the KvBLL solution is generally nonstatic: the L, M “constituents” show up not as 3D but rather as 4D lumps (see Fig. 1). When the temperature goes to zero while the separation between dyons remain fixed, these lumps merge, and the KvBLL caloron is reduced to the usual Belavin-Polyakov-Schwarz-Tyupkin instanton [12] (as is the standard Harrington-Shepard caloron), *plus* corrections of the order of T . However, the holonomy remains fixed and nontrivial at spatial infinity.

There is a strong argument against the presence of either dyons or KvBLL calorons in the Yang-Mills (YM) partition function at nonzero temperatures [2]. The point is that the 1-loop effective action obtained from integrating out fast varying fields where one keeps all powers of A_4 but expands in (covariant) derivatives of A_4 has the form [13]

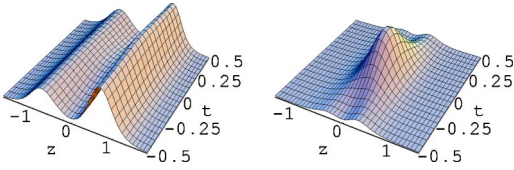


FIG. 1. The action density of the KvBLL caloron as function of z, t at fixed $x=y=0$, with the asymptotic value of A_4 at spatial infinity $v=0.9\pi T$, $\bar{v}=1.1\pi T$. It is periodic in the t direction. At large dyon separations the density becomes static (left, $r_{12}=1.5/T$). As the separation decreases the action density becomes more like a 4D lump (right, $r_{12}=0.6/T$). In both plots the L, M dyons are centered at $z_L = -vr_{12}/2\pi T$, $z_M = \bar{v}r_{12}/2\pi T$, $x_{L,M} = y_{L,M} = 0$. The axes are in units of temperature T .

$$S^1 \text{ loop} = \int d^4x [P(A_4) + E^2 f_E(A_4) + B^2 f_B(A_4) + \text{higher derivative terms}],$$

$$P(A_4) = \frac{1}{3T(2\pi)^2} v^2 (2\pi T - v)^2 \Big|_{\text{mod } 2\pi T},$$

$$v = \sqrt{A_4^a A_4^a} \quad [\text{for the } SU(2) \text{ group}], \quad (2)$$

where the perturbative potential energy term $P(A_4)$ has been known for a long time [2,14] (see Fig. 2). As follows from Eq. (1) the trace of the Polyakov line is related to v as

$$\frac{1}{2} \text{Tr } L = \cos \frac{v}{2T}. \quad (3)$$

The zeros of the potential energy correspond to $\frac{1}{2} \text{Tr } L = \pm 1$, i.e., to the trivial holonomy. If a dyon has $v \neq 2\pi T n$ at spatial infinity the potential energy is positive-definite and proportional to the 3D volume. Therefore, dyons and KvBLL calorons with nontrivial holonomy seem to be strictly forbidden: quantum fluctuations about them have an unacceptably large action.

Meanwhile, precisely these objects determine the physics of the supersymmetric YM theory where in addition to gluons there are gluinos, i.e., Majorana (or Weyl) fermions in

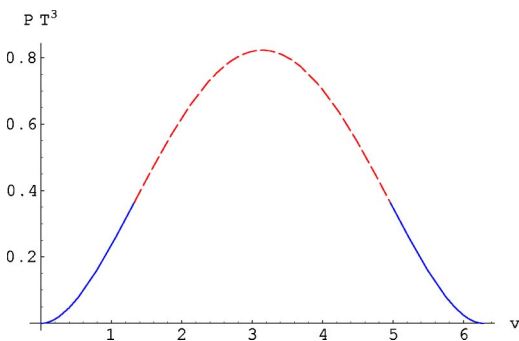


FIG. 2. Potential energy as function of v/T . Two minima correspond to $\frac{1}{2} \text{Tr } L = \pm 1$, the maximum corresponds to $\text{Tr } L = 0$. The range of the holonomy where dyons experience repulsion is shown in the dashed portion.

the adjoint representation. Because of supersymmetry, the boson and fermion determinants about L, M dyons cancel exactly, so that the perturbative potential energy (2) is identically zero for all temperatures, actually in all loops. Therefore, in the supersymmetric theory dyons are openly allowed. [To be more precise, the cancellation occurs when periodic conditions for gluinos are imposed, so it is the compactification in one (time) direction that is implied, rather than physical temperature which requires antiperiodic fermions.] Moreover, it turns out [11] that dyons generate a nonperturbative potential having a minimum at $v = \pi T$, i.e., where the perturbative potential would have the maximum. This value of A_4 corresponds to the holonomy $\text{Tr } L = 0$ at spatial infinity, which is the “most nontrivial”; as a matter of fact $\langle \text{Tr } L \rangle = 0$ is one of the confinement’s requirements. In the supersymmetric YM theory configurations having $\text{Tr } L = 0$ at infinity are not only allowed but dynamically preferred as compared to those with $\frac{1}{2} \text{Tr } L = \pm 1$. In nonsupersymmetric theory it looks as if it is the opposite.

Nevertheless, it has been argued in Ref. [15] that the perturbative potential energy (2) which forbids individual dyons in the pure YM theory might be overruled by nonperturbative contributions of an *ensemble* of dyons. For fixed dyon density, their number is proportional to the 3D volume and hence the nonperturbative dyon-induced potential as function of the holonomy (or of A_4 at spatial infinity) is also proportional to the volume. It may be that at temperatures below some critical one the nonperturbative potential wins over the perturbative one so that the system prefers $\langle \text{Tr } L \rangle = 0$. This scenario could then serve as a microscopic mechanism of the confinement-deconfinement phase transition [15]. It should be noted that the KvBLL calorons and dyons seem to be observed in lattice simulations below the phase transition temperature [16–18].

To study this possible scenario quantitatively, one first needs to find the quantum weight of dyons or the probability with which they appear in the Yang-Mills partition function. Unfortunately, the single-dyon measure is not well defined: it is too badly divergent in the infrared region owing to the weak (Coulomb-like) decrease of the fields. What makes sense and is finite, is the quantum determinant for small oscillations about the KvBLL caloron that possesses zero net electric and magnetic charges. To find this determinant is the primary objective of this study. The KvBLL measure depends on the asymptotic value of A_4 [or on the holonomy through Eq. (3)], on the temperature T , on Λ , the scale parameter obtained through the renormalization of the charge, and on the dyon separation r_{12} . At large separations between constituent L, M dyons of the caloron, one gets their weights and their interaction.

The problem of computing the effect of quantum fluctuations about a caloron with nontrivial holonomy is of the same kind as that for ordinary instantons (solved by ’t Hooft [19]) and for the standard Harrington-Shepard caloron (solved by Gross, Pisarski, and Yaffe [2]), being, however, technically much more difficult. The zero-temperature instanton is $O(4)$ symmetric, and the quantum weight depends on only one variable ρ , the instanton size. The Harrington-Shepard caloron is $O(3)$ symmetric, and the quantum weight

depends on two variables, ρ and T . The KvBLL caloron has only the axial $O(2)$ symmetry, and its quantum weight is a function of three variables, ρ (or r_{12}), T , and the holonomy. Nevertheless, we have managed to find the small-oscillation determinant *exactly*. It is possible because we are able to construct the exact propagator of spin-0, isospin-1 field in the KvBLL background, which by itself is some achievement.

As it is well known [2,19], the calculation of the quantum weight of a Euclidean pseudoparticle consists of three steps: (i) calculation of the metric of the moduli space or, in other words, computing the Jacobian composed of zero modes, needed to write down the pseudoparticle measure in terms of its collective coordinates, (ii) calculation of the functional determinant for nonzero modes of small fluctuations about a pseudoparticle, and (iii) calculation of the ghost determinant resulting from background gauge fixing in the previous step. Problem (i) has been actually solved already by Kraan and van Baal [5]. Problem (ii) is reduced to (iii) in the self-dual background field [20] since for such fields $\text{Det}(W_{\mu\nu}) = \text{Det}(-D^2)^4$, where $W_{\mu\nu}$ is the quadratic form for spin-1, isospin-1 quantum fluctuations and D^2 is the covariant Laplace operator for spin-0, isospin-1 ghost fields. Symbolically, one can write

KvBLL measure

$$= \int d(\text{collective coordinates}) \cdot \text{Jacobian} \cdot \text{Det}^{-1/2} \times (W_{\mu\nu}) \cdot \text{Det}(-D^2) \quad (4)$$

where the product of the last two factors is simply $\text{Det}^{-1}(-D^2)$ in the self-dual background. As usually, the functional determinants are normalized to free ones (with zero background fields) and UV regularized (we use the standard Pauli-Villars method). Thus, to find the quantum weight of the KvBLL caloron only the ghost determinant has to be computed.

To that end, we follow Zarembo [21] and find the derivative of this determinant with respect to the holonomy or, more precisely, to $v \equiv \sqrt{A_4^q A_4^a} |_{|\vec{x}| \rightarrow \infty}$. The derivative is expressed through the Green function of the ghost field in the caloron background. If a self-dual field is written in terms of the Atiyah-Drinfeld-Hitchin-Manin construction, and in the KvBLL case it basically is [5,6], the Green function is generally known [22–25] and we build it explicitly for the KvBLL case. Therefore, we are able to find the derivative $\partial \text{Det}(-D^2)/\partial v$. Next, we reconstruct the full determinant by integrating over v using the determinant for the trivial holonomy [2] as a boundary condition. This determinant at $v=0$ is still a nontrivial function of r_{12} and the fact that we match it from the $v \neq 0$ side is a serious check. Actually we need only one overall constant factor from Ref. [2] in order to restore the full determinant at $v \neq 0$, and we make a minor improvement of the Gross-Pisarski-Yaffe calculation as we have computed the needed constant analytically.

Although all the above steps can be performed explicitly, at some point the equations become extremely lengthy—

typical expressions are several Mbytes long and so far we have not managed to simplify them such that they would fit into a paper. However, we are able to obtain compact analytical expressions in the physically interesting case of large separation between dyons, $r_{12} \gg 1/T$. We have also used the exact formulas to check numerically some of the intermediate formulas, in particular at $v \rightarrow 0$.

If the separation is large in the temperature scale, $r_{12} \gg 1/T$, the final result for the quantum measure of the KvBLL caloron can be written down in terms of the 3D positions of the two constituent L, M dyons $\vec{z}_{1,2}$, their separation $r_{12} = |\vec{z}_1 - \vec{z}_2|$, the asymptotic of A_4 at spatial infinity denoted by $v \in [0, 2\pi T]$ and $\bar{v} = 2\pi T - v \in [0, 2\pi T]$ [see Eq. (80)]. We give here a simpler expression obtained in the limit when the separation between dyons is much larger than their core sizes:

$$\begin{aligned} \mathcal{Z}_{\text{KvBLL}} = & \int d^3 z_1 d^3 z_2 T^6 (2\pi)^{8/3} C \left(\frac{8\pi^2}{g^2} \right)^4 \left(\frac{\Lambda e^{\gamma_E}}{4\pi T} \right)^{22/3} \\ & \times \left(\frac{v}{2\pi T} \right)^{4v/3\pi T} \left(\frac{\bar{v}}{2\pi T} \right)^{4\bar{v}/3\pi T} \exp[-2\pi r_{12} P''(v)] \\ & \times \exp[-V^{(3)}P(v)], \end{aligned} \quad (5)$$

where the overall factor C is a combination of universal constants; numerically $C = 1.031419972084$. Λ is the scale parameter in the Pauli-Villars scheme; the factor g^{-8} is not renormalized at the one-loop level.

Since the caloron field has a constant A_4 component at spatial infinity, it is suppressed by the same perturbative potential $P(v)$ as given by Eq. (2). Its second derivative with respect to v is

$$P''(v) = \frac{1}{\pi^2 T} \left[v - \pi T \left(1 - \frac{1}{\sqrt{3}} \right) \right] \left[v - \pi T \left(1 + \frac{1}{\sqrt{3}} \right) \right].$$

If v is in the range between 0 and $\pi T(1 - 1/\sqrt{3})$ or between $\pi T(1 + 1/\sqrt{3})$ and $2\pi T$ (corresponding to the holonomy not too far from trivial, $0.787597 < \frac{1}{2} |\text{Tr} L| < 1$) the second derivative $P''(v)$ is positive, and the L and M dyons experience a linear attractive potential. Integration over the separation r_{12} of dyons inside a caloron converges. We perform this integration in Sec. VII, estimate the free energy of the caloron gas and conclude that trivial holonomy ($v=0, 2\pi T$) may be unstable, despite the perturbative potential energy $P(v)$. In the complementary range $\pi T(1 - 1/\sqrt{3}) < v < \pi T(1 + 1/\sqrt{3})$ (or $\frac{1}{2} |\text{Tr} L| < 0.787597$), $P''(v)$ is negative (see Fig. 2), and dyons experience a strong linear-rising repulsion. It means that for these values of v , integration over the dyon separations diverges: calorons with holonomy far from trivial “ionize” into separate dyons.

II. THE KVLL CALORON SOLUTION

Although the construction of the self-dual solution with nontrivial holonomy has been fully performed independently by Kraan and van Baal [5] and Lee and Lu [6] we have

found it more convenient for our purposes to use the gauge convention and the formalism of Kraan and van Baal (KvB) whose notations we follow in this paper.

The key quantities characterizing the KvBLL solution for a general $SU(N)$ gauge group are the $N-1$ gauge-invariant eigenvalues of the Polyakov line (1) at spatial infinity. For the $SU(2)$ gauge group to which we restrict ourselves in this paper, it is just one quantity, e.g. $\text{Tr} L$, Eq. (3). In a gauge where A_4 is static and diagonal at spatial infinity, i.e. $A_4|_{x \rightarrow \infty} = i\nu\tau_3/2$, it is this asymptotic value ν that characterizes the caloron solution in the first place. We shall also use the complementary quantity $\bar{\nu} \equiv 2\pi T - \nu$. Their relation to parameters $\omega, \bar{\omega}$ introduced by KvB [5] is $\omega = \nu/4\pi T$, $\bar{\omega} = \bar{\nu}/4\pi T = \frac{1}{2} - \omega$. Both ν and $\bar{\nu}$ vary from 0 to $2\pi T$. At $\nu = 0, 2\pi T$ the holonomy is said to be ‘‘trivial,’’ and the KvBLL caloron reduces to that of Harrington and Shepard [1].

There are, of course, many ways to parametrize the caloron solution. Keeping in mind that we shall be mostly interested in the case of widely separated dyon constituents, we shall parametrize the solution in terms of the coordinates of the dyons’ ‘‘centers’’ (we call constituent dyons L and M according to the classification in Ref. [10]):

$$L \text{ dyon: } \vec{z}_1 = -\frac{2\omega\vec{r}_{12}}{T},$$

$$M \text{ dyon: } \vec{z}_2 = \frac{2\bar{\omega}\vec{r}_{12}}{T},$$

$$\text{dyon separation: } \vec{z}_2 - \vec{z}_1 = \vec{r}_{12}, \quad |r_{12}| = \pi T \rho^2,$$

where ρ is the parameter used by KvB; it becomes the size of the instanton at $\nu \rightarrow 0$. We introduce the distances from the ‘‘observation point’’ \vec{x} to the dyon centers,

$$\begin{aligned} \vec{r} = \vec{x} - \vec{z}_1 &= \vec{x} + 2\omega\vec{r}_{12}, & r &= |\vec{r}|, \\ \vec{s} = \vec{x} - \vec{z}_2 &= \vec{x} - 2\bar{\omega}\vec{r}_{12}, & s &= |\vec{s}|. \end{aligned} \quad (6)$$

Henceforth we measure all dimensional quantities in units of temperature for brevity and restore T explicitly only in the final results.

The KvBLL caloron field in the fundamental representation is [5] (we choose the separation between dyons to be in the third spatial direction, $\vec{r}_{12} = r_{12}\vec{e}_3$):

$$\begin{aligned} A_\mu &= \delta_{\mu,4} i\nu \frac{\tau_3}{2} + \frac{i}{2} \bar{\eta}_{\mu\nu}^3 \tau_3 \partial_\nu \ln \Phi + \frac{i}{2} \Phi \text{Re}[(\bar{\eta}_{\mu\nu}^1 - i\bar{\eta}_{\mu\nu}^2) \\ &\times (\tau_1 + i\tau_2)(\partial_\nu + i\nu\delta_{\nu,4})\tilde{\chi}], \end{aligned} \quad (7)$$

where τ_i are Pauli matrices, $\bar{\eta}_{\mu\nu}^a$ are 't Hooft symbols [19] with $\bar{\eta}_{ij}^a = \epsilon_{aij}$ and $\bar{\eta}_{4\nu}^a = -\bar{\eta}_{\nu 4}^a = i\delta_{a\nu}$. ‘‘Re’’ means $2 \text{Re}(W) \equiv W + W^\dagger$ and the functions used are

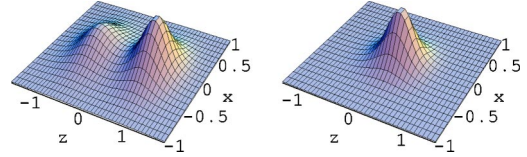


FIG. 3. The action density of the KvBLL caloron as function of z, x at fixed $t=y=0$. At large separations r_{12} the caloron is a superposition of two BPS dyon solutions (left, $r_{12}=1.5/T$). At small separations they merge (right, $r_{12}=0.6/T$). The caloron parameters are the same as in Fig. 1.

$$\begin{aligned} \hat{\psi} &= -\cos(2\pi x_4) + \overline{\text{ch}} \text{ch} + \frac{\vec{r}\vec{s}}{2rs} \overline{\text{sh}} \text{sh}, \\ \psi &= \hat{\psi} + \frac{r_{12}^2}{rs} \overline{\text{sh}} \text{sh} + \frac{r_{12}}{s} \overline{\text{sh}} \overline{\text{ch}} + \frac{r_{12}}{r} \overline{\text{sh}} \text{ch}, \\ \tilde{\chi} &= \frac{r_{12}}{\psi} \left(e^{-2\pi i x_4} \frac{\text{sh}}{s} + \frac{\overline{\text{sh}}}{r} \right), \quad \Phi = \frac{\psi}{\hat{\psi}}. \end{aligned} \quad (8)$$

We have introduced shorthand notations for hyperbolic functions:

$$\begin{aligned} \text{sh} &\equiv \sinh(sv), & \text{ch} &\equiv \cosh(sv), & \overline{\text{sh}} &\equiv \sinh(r\bar{\nu}), \\ \overline{\text{ch}} &\equiv \cosh(r\bar{\nu}). \end{aligned} \quad (9)$$

The first term in (7) corresponds to a constant A_4 component at spatial infinity ($A_4 \approx i\nu\tau_3/2$) and gives rise to the non-trivial holonomy. One can see that A_μ is periodic in time with period 1 (since we have chosen the temperature to be equal to unity). A useful formula for the field strength squared is [5]

$$\text{Tr} F_{\mu\nu} F_{\mu\nu} = \partial^2 \partial^2 \log \psi. \quad (10)$$

In the situation when the separation between dyons r_{12} is large compared to both their core sizes $1/\nu$ (M) and $1/\bar{\nu}$ (L), the caloron field can be approximated by the sum of individual BPS dyons [see Figs. 1 and 3 (left) and Fig. 4]. We give below the field inside the cores and far away from both cores.

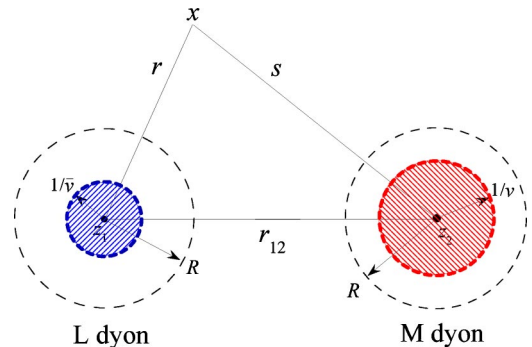


FIG. 4. Three regions of integration for well separated dyons.

A. Inside dyon cores

In the vicinity of the L dyon center \vec{z}_1 and far away from the M dyon ($sv \gg 1$) the field becomes that of the L dyon. It is instructive to write it in spherical coordinates centered at \vec{z}_1 . In the ‘‘stringy’’ gauge [10] in which the A_4 component is constant and diagonal at spatial infinity, the L dyon field is

$$\begin{aligned}
 A_4^L &= \frac{i\tau_3}{2} \left(\frac{1}{r} + 2\pi - \bar{v} \coth(\bar{v}r) \right), \quad A_r^L = 0, \\
 A_\theta^L &= i\bar{v} \frac{-\sin(2\pi x_4 - \phi)\tau_1 + \cos(2\pi x_4 - \phi)\tau_2}{2 \sinh(\bar{v}r)}, \\
 A_\phi^L &= i\bar{v} \frac{-\cos(2\pi x_4 - \phi)\tau_1 + \sin(2\pi x_4 - \phi)\tau_2}{2 \sinh(\bar{v}r)} \\
 &\quad - i\tau_3 \frac{\tan(\theta/2)}{2r}.
 \end{aligned} \tag{11}$$

Here A_θ , for example, is the projection of \vec{A} onto the direction $\vec{n}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$. The ϕ component has a string singularity along the z axis going in the positive direction. Notice that inside the core region ($\bar{v}r \leq 1$) the field is time dependent, although the action density is static. At large distances from the L dyon center, i.e., far outside the core one neglects exponentially small terms $\mathcal{O}(e^{-\bar{v}r})$ and the surviving components are

$$\begin{aligned}
 A_4^L &\rightarrow \left(v + \frac{1}{r} \right) \frac{i\tau^3}{2}, \\
 A_\phi^L &\rightarrow -\frac{\tan(\theta/2)}{r} \frac{i\tau^3}{2},
 \end{aligned}$$

corresponding to the radial electric and magnetic field components

$$E_r^L = B_r^L \rightarrow -\frac{1}{r^2} \frac{i\tau^3}{2}. \tag{12}$$

This Coulomb-type behavior of both the electric and magnetic fields prompts the name ‘‘dyon.’’

Similarly, in the vicinity of the M dyon and far away from the L dyon ($r\bar{v} \gg 1$) the field becomes that of the M dyon, which we write in spherical coordinates centered at \vec{z}_2 :

$$\begin{aligned}
 A_4^M &= \frac{i\tau_3}{2} \left(v \coth(vs) - \frac{1}{s} \right), \quad A_\theta^M = v \frac{\sin\phi\tau_1 - \cos\phi\tau_2}{2i \sinh(vs)}, \\
 A_r^M &= 0, \quad A_\phi^M = v \frac{\cos\phi\tau_1 + \sin\phi\tau_2}{2i \sinh(vs)} + i\tau_3 \frac{\tan(\theta/2)}{2s},
 \end{aligned} \tag{13}$$

the asymptotics of which is

$$\begin{aligned}
 A_4^M &\rightarrow \left(v - \frac{1}{s} \right) \frac{i\tau^3}{2}, \\
 A_\phi^M &\rightarrow \frac{\tan(\theta/2)}{s} \frac{i\tau^3}{2}, \\
 E_r^M &= B_r^M \rightarrow \frac{1}{s^2} \frac{i\tau^3}{2}.
 \end{aligned} \tag{14}$$

We see that in both cases the L, M fields become Abelian at large distances, corresponding to (electric, magnetic) charges $(-, -)$ and $(+, +)$, respectively. The corrections to the fields (11) and (13) are hence of the order of $1/r_{12}$ arising from the presence of the other dyon.

B. Far away from dyon cores

Far away from both dyon cores ($r\bar{v} \gg 1, sv \gg 1$; note that it does not necessarily imply large separations—the dyons may even be overlapping) one can neglect both types of exponentially small terms, $\mathcal{O}(e^{-r\bar{v}})$ and $\mathcal{O}(e^{-sv})$. With exponential accuracy the function $\tilde{\chi}$ in Eq. (8) is zero, and the KvBLL field (7) becomes Abelian [5]:

$$A_\mu^{\text{as}} = i \frac{\tau_3}{2} (\delta_{\mu 4} v + \bar{\eta}_{\mu\nu}^3 \partial_\nu \ln \Phi_{\text{as}}), \tag{15}$$

where Φ_{as} is the function Φ of Eq. (8) evaluated with the exponential precision:

$$\Phi_{\text{as}} = \frac{r+s+r_{12}}{r+s-r_{12}} = \frac{s-s_3}{r-r_3} \quad \text{if} \quad \vec{r}_{12} = r_{12} \vec{e}_3. \tag{16}$$

It is interesting that, despite being Abelian, the asymptotic field (15) retains its self-duality. This is because the third color component of the electric field is

$$E_i^3 = \partial_i A_4^3 = \partial_i \partial_3 \ln \Phi_{\text{as}}$$

while the magnetic field is

$$B_i^3 = \epsilon_{ijk} \partial_j A_k^3 = \partial_i \partial_3 \ln \Phi_{\text{as}} - \delta_{i3} \partial^2 \ln \Phi_{\text{as}},$$

where the last term is zero, except on the line connecting the dyon centers where it is singular; however, this singularity is an artifact of the exponential approximation used. Explicit evaluation of Eq. (15) gives the following nonzero components of the A_μ field far away from both dyon centers:

$$A_4^{\text{as}} = \frac{i\tau_3}{2} \left(v + \frac{1}{r} - \frac{1}{s} \right), \tag{17}$$

$$\begin{aligned}
 A_\phi^{\text{as}} &= -\frac{i\tau_3}{2} \left(\frac{1}{r} + \frac{1}{s} \right) \\
 &\quad \times \sqrt{\frac{(r_{12}-r+s)(r_{12}+r-s)}{(r_{12}+r+s)(r+s-r_{12})}}.
 \end{aligned} \tag{18}$$

In particular, far away from both dyons, A_4 is the Coulomb field of two opposite charges.

III. THE SCHEME FOR COMPUTING $\text{Det}(-D^2)$

As explained in Sec. I, to find the quantum weight of the KvBLL caloron, one needs to calculate the small oscillation determinant, $\text{Det}(-D^2)$, where $D_\mu = \partial_\mu + A_\mu$ and A_μ is the caloron field (7). Instead of computing the determinant directly, we first evaluate its derivative with respect to the holonomy v , and then integrate the derivative using the known determinant at $v=0$ [2] as a boundary condition.

If the background field A_μ depends on some parameter \mathcal{P} , a general formula for the derivative of the determinant with respect to such parameter is

$$\frac{\partial \log \text{Det}(-D^2[A])}{\partial \mathcal{P}} = - \int d^4x \text{Tr}(\partial_{\mathcal{P}} A_\mu J_\mu), \quad (19)$$

where J_μ is the vacuum current in the external background, determined by the Green function:

$$J_\mu^{ab} \equiv (\delta_c^a \delta_d^b \partial_x - \delta_c^a \delta_d^b \partial_y + A^{ac} \delta_d^b + A^{db} \delta_c^a) \mathcal{G}^{cd}(x, y)|_{y=x}$$

or simply

$$J_\mu \equiv \vec{D}_\mu \mathcal{G} + \mathcal{G} \vec{D}_\mu. \quad (20)$$

Here \mathcal{G} is the Green function or the propagator of spin-0, isospin-1 particle in the given background A_μ defined by

$$-D_x^2 G(x, y) = \delta^{(4)}(x - y) \quad (21)$$

and, in the case of nonzero temperatures, being periodic in time, meaning that

$$\mathcal{G}(x, y) = \sum_{n=-\infty}^{\infty} G(x_4, \vec{x}; y_4 + n, \vec{y}). \quad (22)$$

Equation (19) can be easily verified by differentiating the identity $\log \text{Det}(-D^2) = \text{Tr} \log(-D^2)$ [26]. The background field A_μ in Eq. (19) is taken in the adjoint representation, as is the trace. Hence, if the periodic propagator \mathcal{G} is known, Eq. (19) becomes a powerful tool for computing quantum determinants. Specifically, we take $\mathcal{P} = v$ as the parameter for differentiating the determinant, and there is no problem in finding $\partial_v A_\mu(r, s, r_{12}, x_0, v)$ for the caloron field (7). In this differentiation, we assume for convenience that the dyons' centers are fixed. [One could assume, for example, that the dyons' center of mass is fixed—it would lead to more complicated intermediate formulas but the same final result as the two frames differ by a global translation of which the determinant in question is invariant.]

The Green functions in self-dual backgrounds are generally known [25,27] and are built in terms of the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [28] for the given self-dual field. A subtlety appearing at nonzero temperatures is that the Green function is defined by Eq. (21) in the Euclidean \mathbb{R}^4 space where the topological charge is infinite because of the infinite number of repeated strips in the

compactified time direction, whereas one actually needs an explicitly periodic propagator (22). To overcome this nuisance, Nahm [25] suggested to pass on to the Fourier transforms of the infinite-range subscripts in the ADHM construction. We perform this program explicitly in Appendix A, first for the single dyon field and then for the KvBLL caloron. In this way, we get the finite-dimensional ADHM construction both for the dyon and the caloron, with very simple periodicity properties. Using it, we construct explicitly periodic propagators in Appendix B, also first for the dyon and then for the caloron case. For the KvBLL caloron it was not known previously. Using the obtained periodic propagators, in Appendix C we calculate the exact vacuum current (20) for the dyon, and in Appendix D we evaluate the vacuum current in the caloron background, with the help of the regularization carried out in Appendix E.

Although there is no principal difficulty in doing all calculations exactly for the whole caloron moduli space, at some point we loose the capacity of performing analytical calculations for the simple reason that expressions become too long, and so far we have not been able to put them into a manageable form in a general case. Therefore, we have to adopt a more subtle attitude. First of all we restrict ourselves to the part of the moduli space corresponding to large separations between dyons ($r_{12} \gg 1$). Physically, it seems to be the most interesting case (see Sec. I). Furthermore, at the first stage we take $r_{12}v, r_{12}\bar{v} \gg 1$, meaning that the dyons are well separated and do not overlap since the separation is then much bigger than the core sizes [see Figs. 1,3 (left)]. In this case, the vacuum current J_μ (20) becomes that of single dyons inside the spheres of some radius R surrounding the dyon centers, such that $1/v, 1/\bar{v} \ll R \ll r_{12}$, and outside these spheres it can be computed analytically with exponential precision, in correspondence with Sec. II B (see Fig. 4). Adding up the contributions of the regions near two dyons and of the far-away region, we get $d \text{Det}(-D^2)/dv$ for well-separated dyons. Integrating it over v we obtain the determinant itself up to a constant and possible $1/r_{12}$ terms.

This is already an interesting result by itself; however, we would like to compute the constant, which can be done by matching our calculation with that for the trivial caloron at $v=0$. It means that we have to extend the domain of applicability to $r_{12}v = \mathcal{O}(1)$ [or $r_{12}\bar{v} = \mathcal{O}(1)$], implying overlapping dyons, presented in Figs. 1 and 3 (right). To make this extension, we “guess” the analytical expression that would interpolate between $r_{12}v \gg 1$ where the determinant is already computed and $r_{12}v \ll 1$ where matching with the Gross-Pisarski-Yaffe (GPY) calculation [2] can be performed. At this point it becomes very helpful that we possess the exact vacuum current for the caloron, which, although too long to be put on paper, is nevertheless affordable for numerical evaluation (and can be provided on request). We check our analytical guess to an accuracy better than one-millionth. In this way we obtain the determinant up to an overall constant factor for any v, \bar{v} with the only restriction that $r_{12} \gg 1$. This constant factor is then read off from the GPY calculation [2]. Finally, we compute the $1/r_{12}$ and $\log r_{12}/r_{12}$ corrections in $\text{Det}(-D^2)$, which turn out to be quite nontrivial.

IV. $\text{Det}(-D^2)$ FOR WELL SEPARATED DYONS

The L, M dyon cores have sizes $1/\bar{v}$ and $1/v$, respectively, and in this section we consider the case of well-separated dyons, meaning that the distance between the two centers is much greater than the core sizes, $r_{12} \gg 1/v, 1/\bar{v}$. This situation is depicted in Figs. 1 and 3 (left). The two dyons are static in time, so that $\partial \log \text{Det}(-D^2)/\partial v$ (19) becomes an integral over 3D space, times $1/T$ set to unity. We divide the 3D volume into to three regions (Fig. 4): (i) a ball of radius R centered at the center of the M dyon, (ii) a ball of radius R centered at the L dyon, (iii) the rest of the space, with two balls removed. The radius R is chosen such that it is much larger than the dyon cores but much less than the separation: $r_{12} \gg R \gg 1/v, 1/\bar{v}$. Summing up the contributions from the three regions of space, we are satisfied to observe that the result does not depend on the intermediate radius R .

A. $\text{Det}(-D^2)$ for a single dyon

In region (i) the KvBLL caloron field can be approximated by the M dyon field (13), and the vacuum current by that inside a single dyon, both with the $\mathcal{O}(1/r_{12})$ accuracy. We make a more precise calculation, including the $\mathcal{O}(1/r_{12})$ terms, in Sec. V. The single-dyon vacuum current is calculated in Appendix C. Adding up the three parts of the vacuum current denoted there as $J_\mu^{s,r,m}$ we obtain the full isospin-1 vacuum current (in the stringy gauge)

$$\begin{aligned} J_r &= 0, \\ J_\phi &= -\frac{iv[\sin(\phi)T_2 + \cos(\phi)T_1][1 - sv \coth(sv)]^2}{24\pi^2 s^2 \sinh(sv)}, \\ J_\theta &= -\frac{iv[\sin(\phi)T_1 - \cos(\phi)T_2][1 - sv \coth(sv)]^2}{24\pi^2 s^2 \sinh(sv)}, \\ J_4 &= -iT_3 \left[\frac{[1 - sv \coth(sv)]^3}{6\pi^2 s^3} + \frac{1 - sv \coth(sv)}{3s} \right. \\ &\quad \left. + \frac{\coth(sv)[1 - sv \coth(sv)]^2}{2\pi s^2} \right], \end{aligned}$$

where $(T_c)^{ab} \equiv i\epsilon^{acb}$ are the isospin-1 generators. We contract J_μ (23) with dA_μ/dv from Eq. (13) according to Eq. (19). After taking the matrix trace, the integrand in Eq. (19) becomes spherically symmetric:

$$\begin{aligned} \text{Tr}[\partial_\nu A_\mu J_\mu] &= \frac{2}{3s} \left(\coth(sv) - sv + \frac{sv[sv \coth(sv) - 2]}{\sinh^2(sv)} \right) \\ &\quad - \frac{[sv \coth(sv) - 1]^3 [\sinh(2sv) - 3sv]}{6\pi^2 \sinh^2(sv) s^3} \\ &\quad + \frac{[sv \coth(sv) - 1]^2 [\sinh(2sv) - 2sv]}{2\pi \sinh^2(sv) \tanh(sv) s^2}. \end{aligned} \quad (23)$$

It has to be integrated over the spherical box of radius R . Fortunately, we are able to perform the integration analytically. The result for the M dyon is

$$\begin{aligned} &\frac{\partial \log \text{Det}(-D^2[\text{M dyon}])}{\partial v} \\ &= -\int_0^R ds \text{Tr}[\partial_\nu A_\mu J_\mu] 4\pi s^2 \\ &= -\frac{24(\gamma_E - \log \pi) + 53 + \frac{2}{3}\pi^2}{18\pi} + \frac{1}{v} + \frac{4\pi R^3}{3} \\ &\quad \times P'(v) - 2\pi R^2 P''(v) + 2\pi R P'''(v) - \frac{4}{3\pi} \log(Rv). \end{aligned} \quad (24)$$

As we see, it is badly infrared divergent, as it depends on the box radius R . Here $P(q)$ is the potential energy [see Eq. (2)]

$$\begin{aligned} P(q) &= \left[\frac{\pi^2}{12} \left(\frac{q}{\pi} - 2 \right)^2 \left(\frac{q}{\pi} \right)^2 \right], \\ P'(q) &= \frac{1}{3\pi^2} q(\pi - q)(2\pi - q), \\ P''(q) &= \frac{1}{3\pi^2} (3q^2 - 6\pi q + 2\pi^2), \\ P'''(q) &= \frac{2}{\pi^2} (q - \pi), \quad P^{IV}(q) = \frac{2}{\pi^2}. \end{aligned}$$

The IR-divergent terms arise from the asymptotics of the integrand. Neglecting exponentially small terms e^{-sv} in Eq. (23) we have

$$\begin{aligned} -\text{Tr}[\partial_\nu A_\mu J_\mu] &\simeq -4 \left[\frac{(1 - sv)^3}{12\pi^2 s^3} + \frac{(1 - sv)^2}{4\pi s^2} + \frac{1 - sv}{6s} \right] \\ &= P' \left(v - \frac{1}{s} \right) = P'(v) - P''(v) \frac{1}{s} \\ &\quad + \frac{1}{2} P'''(v) \frac{1}{s^2} - \frac{1}{6} P^{IV}(v) \frac{1}{s^3}. \end{aligned}$$

Integrating it over the sphere of radius R one gets the IR-divergent terms [the second line in Eq. (24)].

The fact that the IR-divergent part of $d \text{Det}(-D^2)/dv$ is directly related to the potential energy $P(A_4)$ is not accidental. At large distances the field of the dyon becomes a slowly varying Coulomb field [see Eq. (14)]. Therefore, the determinant can be generically expanded in the covariant derivatives of the background field [13,29] with the potential energy $P(A_4)$ being its leading zero-derivative term. The nontrivial fact, however, is that with exponential precision

the vacuum current is related to the variation of solely the leading term in the covariant derivative expansion of the effective action with no contribution from any of the sub-leading terms. This is a specific property of self-dual fields, and we observe it also in the following section.

B. Contribution from the far-away region

We now compute the contribution to $\partial \text{Det}(-D^2)/\partial v$ from the region of space far away from both dyon centers. With exponential accuracy (meaning neglecting terms of the order of $e^{-r\bar{v}}$ and e^{-sv}) the KvBLL caloron field is given by Eqs. (17) and (18), and only the A_4 component depends (trivially) on v . The caloron vacuum current with the same exponential accuracy is calculated in Appendix D. Combining the results given by Eqs. (D8) and (D9) and Eqs. (D11) and (D12) we see that $J_\varphi=0$ and for J_4 we have

$$J_4 = \frac{iT_3}{2} \left\{ \left[\frac{1}{3\pi^2} \left(\frac{1}{r} - \frac{1}{s} \right)^3 - \frac{1}{\pi} \left(\frac{1}{r} - \frac{1}{s} \right)^2 + \frac{2}{3} \left(\frac{1}{r} - \frac{1}{s} \right) \right] + \left[\frac{4}{\pi} \left(\frac{1}{r} - \frac{1}{s} \right)^2 - 8 \left(\frac{1}{r} - \frac{1}{s} \right) + \frac{8\pi}{3} \right] \omega + \left[16 \left(\frac{1}{r} - \frac{1}{s} \right) - 16\pi \right] \omega^2 + \frac{64\pi}{3} \omega^3 \right\}. \quad (25)$$

We remind the reader that r, s are distances from M, L dyon centers $\vec{z}_{1,2}$ and that $\omega = v/(4\pi)$. It is interesting that the separation $r_{12} = |z_1 - z_2|$ does not appear explicitly in the current. Moreover, it can be again written through the potential energy $P(A_4)$:

$$J_4 = \frac{1}{2} iT_3 P'(q) \Big|_{q=v+1/r-1/s}. \quad (26)$$

Therefore, in the far-away region one obtains

$$-\text{Tr}[\partial_\nu A_\mu J_\mu] = P' \left(v + \frac{1}{r} - \frac{1}{s} \right). \quad (27)$$

We have now to integrate Eq. (27) over the whole 3D space with two spheres of radius R surrounding the dyon centers removed:

$$\begin{aligned} - \int d^4x \text{Tr}[\partial_\nu A_\mu J_\mu] &= \int d^3x P' \left(v + \frac{1}{r} - \frac{1}{s} \right) \\ &= P'(v) \int d^3x + P'''(v) \int d^3x \\ &\quad \times \left(\frac{1}{r} - \frac{1}{s} \right) + \frac{1}{2} P'''(v) \int d^3x \left(\frac{1}{r} - \frac{1}{s} \right)^2 \\ &\quad + \frac{1}{6} P^{IV}(v) \int d^3x \left(\frac{1}{r} - \frac{1}{s} \right)^3. \end{aligned} \quad (28)$$

The first integral in Eq. (28) is the 3D volume V minus the volume of two spheres, $V - 2(4\pi/3)R^3$. The second integral

is zero by symmetry between the two centers, and so is the last one. The only nontrivial integral is

$$\int d^3x \left(\frac{1}{r} - \frac{1}{s} \right)^2 = 4\pi r_{12} - 8\pi R + \mathcal{O}\left(\frac{R^2}{r_{12}}\right). \quad (29)$$

Therefore, the contribution from the region far from both dyon centers is

$$\begin{aligned} \frac{\partial \log \text{Det}(-D^2)}{\partial v} \Big|_{\text{far}} &= P'(v) \left(V - 2\frac{4\pi}{3}R^3 \right) + \frac{1}{2} P'''(v) \\ &\quad \times (4\pi r_{12} - 8\pi R). \end{aligned} \quad (30)$$

C. Combining all three regions

We now add up the contributions to $\partial \log \text{Det}(-D^2)/\partial v$ from the regions surrounding the two dyons and from the far-away region. Since the contribution of the L dyon is the same as that of the M dyon with the replacement $v \rightarrow \bar{v}$ and since $\partial/\partial v = -\partial/\partial \bar{v}$, when adding up contributions of L, M core regions we have to antisymmetrize in $v \leftrightarrow \bar{v}$. It should be noted that $P(v)$ and $P''(v)$ are symmetric under this interchange, while $P'(v)$ and $P'''(v)$ are antisymmetric. Therefore, the combined contribution of both cores is, from Eq. (24),

$$\begin{aligned} \frac{\partial \log \text{Det}(-D^2)}{\partial v} \Big|_{\text{cores}} &= 2P'(v) \frac{4\pi}{3} R^3 + \frac{1}{2} P'''(v) 8\pi R + \frac{1}{v} \\ &\quad - \frac{1}{\bar{v}} - \frac{4}{3\pi} \ln \left(\frac{v}{\bar{v}} \right). \end{aligned} \quad (31)$$

Adding it up with the contribution from the far-away region, Eq. (30), we obtain the final result, which is independent of the intermediate radius R used to separate the regions:

$$\begin{aligned} \frac{\partial \log \text{Det}(-D^2)}{\partial v} &= P'(v) V + P'''(v) 2\pi r_{12} + \frac{1}{v} - \frac{1}{\bar{v}} \\ &\quad - \frac{4}{3\pi} \ln \left(\frac{v}{\bar{v}} \right). \end{aligned} \quad (32)$$

This equation can be easily integrated over v up to a constant which in fact can be a function of the separation r_{12} :

$$\begin{aligned} \log \text{Det}(-D^2) &= P(v) V + P''(v) 2\pi r_{12} + \left(1 - \frac{4v}{3\pi} \right) \log(v) \\ &\quad + \left(1 - \frac{4\bar{v}}{3\pi} \right) \log(\bar{v}) + f(r_{12}). \end{aligned} \quad (33)$$

Since in the above calculation of the determinant for well-separated dyons we have neglected the Coulomb field of one dyon inside the core region of the other, we expect that the unknown function $f(r_{12}) = \mathcal{O}(1/r_{12}) + c$, where c is the true integration constant. Our next aim will be to find it. The $\mathcal{O}(1/r_{12})$ corrections will be found later.

V. MATCHING THE DETERMINANT WITH TRIVIAL HOLONOMY

To find the integration constant, one needs to know the value of the determinant at $v=0$ (or $\bar{v}=0$) where the KvBLL caloron with nontrivial holonomy reduces to the Harrington-Shepard caloron with a trivial one and for which the determinant has been computed by GPY [2]. Before we match our determinant at $v \neq 0$ with that at $v=0$ let us recall the GPY result.

A. Det($-D^2$) at $v=0$

The $v=0$ periodic instanton is traditionally parametrized by the instanton size ρ . It is known [2,24] that the periodic instanton can be viewed as a mix of two BPS monopoles, one of which has an infinite size. It becomes especially clear in the KvBLL construction [5,6] where the size of one of the dyons becomes infinite as $v \rightarrow 0$ (see Sec. II). Despite one dyon being infinitely large, one can still continue to parametrize a caloron by the distance r_{12} between dyon centers, with $\rho = \sqrt{r_{12}/\pi}$. Since our determinant (33) is given in terms of r_{12} we have first of all to rewrite the GPY determinant in terms of r_{12} , too. Actually, GPY have interpolated the determinant in the whole range of ρ (hence r_{12}) but we shall be interested only in the limit $r_{12} \gg 1$. In this range the GPY result reads:

$$\begin{aligned} \log \text{Det}(-D^2)|_{v=0, T=1} &= \log \text{Det}(-D^2)|_{v=0, T=0} + \frac{4}{3} \pi r_{12} - \frac{4}{3} \log r_{12} + c_0 \\ &+ \mathcal{O}\left(\frac{1}{r_{12}}\right), \\ c_0 &= \frac{8}{9} - \frac{8\gamma_E}{3} - \frac{2\pi^2}{27} + \frac{4 \log \pi}{3}. \end{aligned} \quad (34)$$

We have made here a small improvement as compared to Ref. [2], namely (i) we have checked that the correction is of the order of $1/r_{12}$, based on an intermediate exact formula, and (ii) we have also managed to get an exact analytical expression for the constant.

The zero-temperature determinant is that for the standard BPST instanton [19,30]:

$$\begin{aligned} \log \text{Det}(-D^2)|_{v=0, T=0} &= \frac{2}{3} \log \mu + \frac{1}{3} \log\left(\frac{r_{12}}{\pi}\right) + \alpha(1), \\ \alpha(1) &= \frac{2\gamma_E}{3} - \frac{16}{9} + \frac{\log 2}{3} + \frac{2 \log(2\pi)}{3} - \frac{4\zeta'(2)}{\pi^2}, \end{aligned} \quad (35)$$

$$\alpha(1) = \frac{2\gamma_E}{3} - \frac{16}{9} + \frac{\log 2}{3} + \frac{2 \log(2\pi)}{3} - \frac{4\zeta'(2)}{\pi^2}, \quad (36)$$

where it is implied that the determinant is regularized by the Pauli-Villars method and μ is the Pauli-Villars mass (see Sec. VIA). Combining Eqs. (34) and (35) one obtains

$$\begin{aligned} \log \text{Det}(-D^2)|_{v=0, T=1} &= \frac{4}{3} \pi r_{12} - \log r_{12} + \frac{2}{3} \log \mu + c_1 \\ &+ \mathcal{O}\left(\frac{1}{r_{12}}\right), \end{aligned} \quad (37)$$

where

$$\begin{aligned} c_1 &= \log 2 + \frac{5}{3} \log \pi - \frac{8}{9} - 2\gamma_E - \frac{2\pi^2}{27} - \frac{4\zeta'(2)}{\pi^2} \\ &= 0.206602292859. \end{aligned} \quad (38)$$

We notice that $P(v) \rightarrow 0$ and $P''(v) \rightarrow \frac{2}{3}$ at $v \rightarrow 0$; therefore the first two terms in Eq. (33) become exactly equal to the first term in Eq. (37). At the same time, the last two terms in Eq. (33) become $\log v - \frac{5}{3} \log(2\pi) + c$ which is formally singular at $v \rightarrow 0$ and does not match the $-\log r_{12}$ in Eq. (37). The reason is that Eq. (33) has been derived assuming $r_{12} \gg 1/v, 1/\bar{v}$ and one cannot take the limit $v \rightarrow 0$ in that expression without taking simultaneously $r_{12} \rightarrow \infty$. In order to match the determinant at $v=0$ one needs to extend Eq. (33) to arbitrary values of $v r_{12}$. As we shall see, it will be important for the matching that $\log r_{12}$ has the coefficient -1 .

B. Extending the result to arbitrary values of $v r_{12}$

Let us take a fixed but large value of the dyon separation $r_{12} \gg 1$ such that both Eqs. (33) and (37) are valid. Actually, our aim is to integrate the exact expression for the derivative of the determinant

$$\begin{aligned} \partial_v \log \text{Det}(-D^2) &= \int \wp(x) d^4x, \\ \wp(x) &\equiv -\text{Tr}[\partial_v A_\mu J^\mu(x, x|A)], \end{aligned} \quad (39)$$

from $v=0$, where the determinant is given by Eq. (37), to some small value of $v \ll 1$ (but such that $v r_{12} \gg 1$) where Eq. (33) becomes valid. We shall parametrize this v as $v = k/r_{12} \ll 1$ with $k \gg 1$. The result of the integration over v must be equal to the difference between the right hand sides of Eqs. (33) and (37). We write it as

$$\begin{aligned} \int_0^{k/r_{12}} dv \int d^4x \wp(x) &= V \left[P\left(\frac{k}{r_{12}}\right) - P(0) \right] + 2\pi r_{12} \left[P''\left(\frac{k}{r_{12}}\right) - P''(0) \right] \\ &+ \log(k) + c - \frac{5}{3} \log(2\pi) \\ &- c_1 - \frac{2}{3} \log \mu + \mathcal{O}\left(\frac{k}{r_{12}}\right). \end{aligned} \quad (40)$$

Notice that $\log r_{12}$ has cancelled in the difference in the right-hand side (r.h.s.). We denote

$$c_2 \equiv c - c_1 - \frac{2}{3} \log \mu - \frac{5}{3} \log(2\pi). \quad (41)$$

We know that the first two terms in Eq. (40) come from far asymptotics. Denoting by $\bar{\varphi}$ our φ with subtracted asymptotic terms we have

$$\int_0^{k/r_{12}} dv \int d^4x \bar{\varphi}(x) = \log k + c_2 + \mathcal{O}\left(\frac{1}{r_{12}}\right). \quad (42)$$

In this integration we are in the domain $1/v \gg 1$ and $r_{12} \gg 1$ and we can simplify the integrand dropping terms, which are small in this domain. At this point it will be convenient to restore temporarily the temperature dependence. With $\beta \equiv 1/T$ our domain of interest is $1/v \gg \beta$ and $r_{12} \gg \beta$. Therefore we are in the small- β domain and can expand $\bar{\varphi}$ in series with respect to β :

$$\bar{\varphi} = \frac{1}{\beta^2} \varphi_0 + \frac{1}{\beta} \varphi_1 + \mathcal{O}(\beta^0). \quad (43)$$

As we shall see in a moment, only the first two terms are not small in this domain and we need to know only them to compute c_2 . It is a great simplification because $\varphi_{0,1}$ do not contain terms proportional to $e^{-\bar{v}r}$ since $\bar{v} = 2\pi T - v \rightarrow \infty$ at $\beta \rightarrow 0$, and what is left is time independent. Moreover, what is left after we neglect exponentially small terms are homogeneous functions of r, s, r_{12}, v and we can turn to the dimensionless variables:

$$\begin{aligned} \varphi_0(r, s, r_{12}, v) &= \frac{1}{r_{12}} \tilde{\varphi}_0\left(\frac{r}{r_{12}}, \frac{s}{r_{12}}, vr_{12}\right), \\ \varphi_1(r, s, r_{12}, v) &= \frac{1}{r_{12}^2} \tilde{\varphi}_1\left(\frac{r}{r_{12}}, \frac{s}{r_{12}}, vr_{12}\right). \end{aligned}$$

We rewrite the l.h.s. of Eq. (42) in terms of the new quantities:

$$\int d^4x \bar{\varphi}(x) = \frac{r_{12}^2}{\beta} \int d^3\tilde{x} \tilde{\varphi}_0 + r_{12} \int d^3\tilde{x} \tilde{\varphi}_1 + \mathcal{O}(\beta), \quad (44)$$

where $\tilde{x} = x/r_{12}$ is dimensionless. We see that it is indeed sufficient to take just the first two terms in the expansion (43) at $\beta \rightarrow 0$. The integration measure can be written in terms of the dimensionless variables $\tilde{r} = r/r_{12}, \tilde{s} = s/r_{12}$ as

$$d^3\tilde{x} = 2\pi \tilde{r} d\tilde{r} \tilde{s} d\tilde{s}, \quad (45)$$

where \tilde{r} and \tilde{s} are constrained by the triangle inequalities $\tilde{r} + \tilde{s} < 1$, $\tilde{r} + 1 < \tilde{s}$, and $\tilde{s} + 1 < \tilde{r}$, and we have integrated over the azimuth angle.

We have now to use the exact vacuum current in the caloron background to compute $\int \tilde{\varphi}_{0,1}$. First, it turns out that the first integral in Eq. (44) is zero. This is good news because had it been nonzero, Eq. (42) could not be right as its r.h.s. has no dependence on r_{12} other than possible $1/r_{12}$ terms. Second, we have noticed that the second integral in Eq. (44) is in fact

$$\int d^3\tilde{x} \tilde{\varphi}_1 = \frac{1}{vr_{12} + 1}. \quad (46)$$

Unfortunately, we were not able to verify it analytically but we checked numerically that it holds with the precision of a few units of 10^{-7} in the range of vr_{12} between 0 and 15. Combining Eqs. (44) and (46) we obtain for the l.h.s. of Eq. (42)

$$\begin{aligned} \int_0^{k/r_{12}} dv \int d^4x \bar{\varphi}(x) &= r_{12} \int_0^{k/r_{12}} \frac{dv}{vr_{12} + 1} = \log(k + 1) \\ &= \log k + \mathcal{O}\left(\frac{1}{k}\right). \end{aligned}$$

Therefore, we reproduce the r.h.s. of Eq. (42) and in addition find that $c_2 = 0$.

Equation (46) is sufficient to extend the result for the determinant (33) valid at $vr_{12} \gg 1$ to arbitrary values of vr_{12} , provided $r_{12} \gg 1$ (the extension to arbitrary values of $\bar{v}r_{12}$ is obtained by symmetry $v \leftrightarrow \bar{v}$). The final result for the determinant to the $1/r_{12}$ accuracy is

$$\begin{aligned} \log \text{Det}[-D^2] &= VP(v) + 2\pi P''(v)r_{12} + \left(1 - \frac{4v}{3\pi}\right) \\ &\quad \times \log(vr_{12} + 1) + \left(1 - \frac{4\bar{v}}{3\pi}\right) \log(\bar{v}r_{12} + 1) \\ &\quad + \frac{2}{3} \log(\mu r_{12}) + c_1 + \frac{5}{3} \log(2\pi) + \mathcal{O}\left(\frac{1}{r_{12}}\right), \end{aligned} \quad (47)$$

where μ is the UV cutoff and the numerical constant c_1 is given by Eq. (38). This expression is finite at $v \rightarrow 0, \bar{v} \rightarrow 0$ and coincides with the GPY result (37) in these limits. At $vr_{12} \gg 1$ we restore the previous result, Eq. (33), but now with the integration constant fixed: $c = \frac{2}{3} \log \mu + \frac{5}{3} \log(2\pi) + c_1$. Equation (47) is valid for any holonomy, i.e., for $v, \bar{v} \in [0, 2\pi]$, and the only restriction on its applicability is the condition that the dyon separation is large, $r_{12} \gg 1$.

C. $1/r_{12}$ corrections

Equation (47) can be expanded in inverse powers of $vr_{12}, \bar{v}r_{12}$, which gives $1/(vr_{12}), 1/(\bar{v}r_{12})$ (and higher) corrections; however, there are other $1/r_{12}$ corrections that are not accompanied by the $1/v, 1/\bar{v}$ factors: the aim of this section is to find them using the exact vacuum current.

To this end, we again consider the case $r_{12} \gg 1/v, 1/\bar{v}$ such that one can split the integration over 3D space into three regions shown in Fig. 4. In the far-away region one can use the same vacuum current (25) as it has an exponential precision with respect to the distances to both dyons. In the core regions, however, it is now insufficient to neglect completely the field of the other dyon, as we did in Sec. IV looking for the leading order. Since we are now after the $1/r_{12}$ correc-

tions, we have to use the exact field and the exact vacuum current of the caloron but we can neglect the exponentially small terms in their separation.

Another modification with respect to Sec. IV is that we find it more useful this time to choose r_{12} as the parameter \mathcal{P} in Eq. (19). We shall compute the $1/r_{12}^2$ terms in $\partial \text{Det}(-D^2)/\partial r_{12}$ and then restore the determinant itself since the limit of $r_{12} \rightarrow \infty$ is already known. Let us define how the KvBLL field depends on r_{12} . As seen from Eq. (7) the KvBLL field is a function of r, s, v, r_{12} only. We define the change in the separation $r_{12} \rightarrow r_{12} + dr_{12}$ as the symmetric displacement of each monopole center by $\pm dr_{12}/2$. It corresponds to

$$\frac{\partial r}{\partial r_{12}} = \frac{r_{12}^2 + r^2 - s^2}{4r_{12}r}, \quad \frac{\partial s}{\partial r_{12}} = \frac{r_{12}^2 + s^2 - r^2}{4r_{12}s}. \quad (48)$$

We shall use the definition (48) to compute the derivative of the caloron field (7) with respect to r_{12} .

Let us start from the M -monopole core region. To get the

$1/r_{12}$ correction to the determinant we need to compute its derivative in the $1/r_{12}^2$ order and expand correspondingly the caloron field and the vacuum current to this order. Wherever the distance r from the far-away L dyon appears in the equations, we replace it by $r = (r_{12}^2 + 2sr_{12}\cos\theta + s^2)^{1/2}$ where s is the distance from the M dyon and θ is the polar angle seen from the M -dyon center. Expanding in inverse powers of r_{12} we get coefficients that are functions of $s, \cos\theta$. One can easily integrate over θ as the integration measure in spherical coordinates is $2\pi s^2 ds d\cos\theta$. We leave out the intermediate equations and give only the end result for the integrand in Eq. (19). After integrating over $\cos\theta$ we obtain the following contribution from the core region of the M monopole:

$$\left. \frac{\partial \text{Det}(-D^2)}{\partial r_{12}} \right|_{M \text{ dyon core}} = -\frac{1}{r_{12}^2} \int_0^R I_{1/r_{12}^2} 16\pi s^2 ds + \mathcal{O}\left(\frac{1}{r_{12}^3}\right), \quad (49)$$

where I_{1/r_{12}^2} reads

$$\begin{aligned} I_{1/r_{12}^2} = & -\frac{\coth(sv)}{12\pi^2 s^3} - \frac{\coth(sv)}{9s} - \frac{sv^2 \coth(sv) \text{csch}(sv)^2}{36} - \frac{s^2 v^3 [2 + \cosh(2sv)] \text{csch}(sv)^4}{72} \\ & + \frac{v[-61 + 3 \cosh(2sv) + 4 \cosh(4sv)] \coth(sv) \text{csch}(sv)^4}{96\pi s} \\ & - \frac{v^2 [37 + 23 \cosh(2sv) + 4 \cosh(4sv)] \coth(sv) \text{csch}(sv)^4}{192\pi^2 s} + \frac{s^2 v^4 [4 + \cosh(2sv)] \coth(sv)^2 \text{csch}(sv)^4}{48\pi} \\ & - \frac{sv^3 [54 \cosh(sv) + 17 \cosh(3sv) + \cosh(5sv)] \text{csch}(sv)^7}{384\pi} \\ & + \frac{sv^4 [-406 \cosh(sv) - 81 \cosh(3sv) + 7 \cosh(5sv)] \text{csch}(sv)^7}{2304\pi^2} - \frac{s^2 v^5 [-24 - 33 \cosh(2sv) + \cosh(6sv)] \text{csch}(sv)^8}{1536\pi^2} \\ & + v \frac{6 + 13 \text{csch}(sv)^2}{72} + \frac{-10 - 5 \text{csch}(sv)^2 + 9 \text{csch}(sv)^4}{48\pi s^2} + v \frac{40 + 65 \text{csch}(sv)^2 + 19 \text{csch}(sv)^4}{192\pi^2 s^2} \\ & + v^2 \frac{-6 - 20 \text{csch}(sv)^2 + 9 \text{csch}(sv)^4 + 27 \text{csch}(sv)^6}{48\pi} + v^3 \frac{24 + 98 \text{csch}(sv)^2 + 285 \text{csch}(sv)^4 + 234 \text{csch}(sv)^6}{576\pi^2}. \quad (50) \end{aligned}$$

Fortunately we are able to integrate this function analytically:

$$\int_0^R I_{1/r_{12}^2} 16\pi s^2 ds = \frac{1}{v} - \frac{\pi^2 + 36\gamma_E + 69}{27\pi} + \frac{2v(v^2 - 3\pi v + 2\pi^2)}{9\pi} R^3 + \frac{4(6v\pi - 2\pi^2 - 3v^2)}{9\pi} R^2 + \frac{10(v - \pi)}{3\pi} R - \frac{4 \log(Rv/\pi)}{3\pi}. \quad (51)$$

For the L monopole core contribution one has to replace v by \bar{v} . Adding together contributions from L, M monopole cores we have

$$\left. \frac{\partial \text{Det}(-D^2)}{\partial r_{12}} \right|_{\text{cores}} = -\frac{1}{r_{12}^2} \left[\frac{1}{v} + \frac{1}{\bar{v}} - 2 \frac{\pi^2 + 36\gamma_E + 69}{27\pi} - \frac{8}{9\pi} (3v^2 - 6\pi v + 2\pi^2) R^2 - \frac{4 \log(R^2 v \bar{v} / \pi^2)}{3\pi} \right]. \quad (52)$$

Now let us turn to the far-away region. Recalling Eq. (26) we realize that the contribution of this region is determined by the potential energy:

$$\left. \frac{\partial \text{Det}(-D^2)}{\partial r_{12}} \right|_{\text{far}} = \int d^3x \partial_{r_{12}} P \left(v + \frac{1}{r} - \frac{1}{s} \right) = \frac{1}{2} P''(v) \int d^3x \partial_{r_{12}} \left(\frac{1}{r} - \frac{1}{s} \right)^2 + \frac{1}{24} P^{IV}(v) \int d^3x \partial_{r_{12}} \left(\frac{1}{r} - \frac{1}{s} \right)^4.$$

The integration region is the 3D volume with two balls of radius R removed. We use

$$\int d^3x \partial_{r_{12}} \left(\frac{1}{r} - \frac{1}{s} \right)^2 = 4\pi - \frac{16\pi R^2}{3r_{12}^2}, \quad \int d^3x \partial_{r_{12}} \left(\frac{1}{r} - \frac{1}{s} \right)^4 = \frac{2\pi}{3r_{12}^2} \left[48 \log \left(\frac{r_{12}}{R} \right) - 9\pi^2 + 8 \right]. \quad (53)$$

Adding up all three contributions we see that the region separation radius R gets cancelled (as it should), and we get

$$\partial_{r_{12}} \log \text{Det}(-D^2) = 2\pi P''(v) + \frac{1}{r_{12}^2} \left[\frac{4}{3\pi} \log \left(\frac{\sqrt{v} r_{12}^2}{\pi^2} \right) - \frac{1}{v} - \frac{1}{\sqrt{v}} + \frac{50}{9\pi} + \frac{8\gamma_E}{3\pi} - \frac{23\pi}{54} \right], \quad (54)$$

which can be easily integrated, with the result

$$\log \text{Det}(-D^2) = 2\pi P''(v) r_{12} + \frac{1}{r_{12}} \left[\frac{1}{v} + \frac{1}{\sqrt{v}} + \frac{23\pi}{54} - \frac{8\gamma_E}{3\pi} - \frac{74}{9\pi} - \frac{4}{3\pi} \log \left(\frac{\sqrt{v} r_{12}^2}{\pi^2} \right) \right] + \bar{c} + \mathcal{O} \left(\frac{1}{r_{12}^2} \right), \quad (55)$$

where \bar{c} is the integration constant that does not depend on r_{12} . Comparing Eq. (55) with Eq. (32) at $r_{12} \rightarrow \infty$ we conclude that

$$\begin{aligned} \bar{c} = & VP(v) + \frac{2}{3} \log \mu + \frac{3\pi - 4v}{3\pi} \log v + \frac{3\pi - 4\sqrt{v}}{3\pi} \log \bar{v} \\ & + \frac{5}{3} \log(2\pi) + c_1 \end{aligned} \quad (56)$$

and c_1 is given in Eq. (38). We note that the leading correction, $\log r_{12}/r_{12}$, arises from the far-away region and is related to the potential energy, similar to the leading r_{12} term. The terms proportional to $1/v$ and $1/\sqrt{v}$ can be extracted from expanding Eq. (47) [which is an additional independent check of Eq. (46)]. In fact, Eqs. (47) and (55) are complementary: Eq. (47) sums up all powers of $1/r_{12}v, 1/r_{12}\bar{v}$ but misses $(\log r_{12})/r_{12}$ and $1/r_{12}$ terms, whereas Eq. (55) collects all terms of that order but misses higher powers of $1/r_{12}v, 1/r_{12}\bar{v}$.

VI. QUANTUM WEIGHT OF THE KVBL CALORON

A. Quantum weight of a Euclidean pseudoparticle: generalities

If a field configuration \bar{A}_μ is a solution of the Yang-Mills Euclidean equation of motion, $D_\mu F_{\mu\nu} = 0$, its quantum weight is the contribution of the saddle point to the partition function

$$\mathcal{Z} = \int DA_\mu \exp(-S[A]), \quad S[A] = \frac{1}{4g^2} \int d^4x F_{\mu\nu}^a F_{\mu\nu}^a. \quad (57)$$

The general field over which one integrates in Eq. (57) can be written as

$$A_\mu = \bar{A}_\mu + a_\mu, \quad (58)$$

where \bar{A}_μ is the classical solution corresponding to the local minimum of the action and a_μ is the presumably small quantum oscillation about the solution. One expands the action around the minimum,

$$\begin{aligned} S[A] = & S[\bar{A}] - \frac{1}{g^2} \int d^4x a_\nu^a D_\mu^{ab}(\bar{A}) F_{\mu\nu}^b(\bar{A}) \\ & + \frac{1}{2g^2} \int d^4x a_\mu^a W_{\mu\nu}^{ab}(\bar{A}) a_\nu^b + \mathcal{O}(a^3), \end{aligned} \quad (59)$$

where the linear term is in fact absent since \bar{A} satisfies the equation of motion, and the quadratic form is

$$W_{\mu\nu}^{ab}(\bar{A}) = -D^2(\bar{A})^{ab} \delta_{\mu\nu} + (D_\mu D_\nu)^{ab}(\bar{A}) - 2f^{acb} F_{\mu\nu}^c(\bar{A}), \quad (60)$$

$$D_\mu^{ab}(\bar{A}) = \partial_\mu \delta^{ab} + f^{acb} \bar{A}_\mu^c. \quad (61)$$

We have written the covariant derivative in the adjoint representation; the relation with the fundamental representation is given by $a_\mu = -i a_\mu^a t^a$, $\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$ and similarly for $F_{\mu\nu}$, etc. The 1-loop approximation to the quantum weight corresponds to evaluating Eq. (57) in the Gaussian approximation in a_μ ; hence $\mathcal{O}(a^3)$ terms in Eq. (59) have been neglected.

The quadratic form (60) is highly degenerate since any fluctuation of the type $a_\mu^a = D_\mu^{ab}(\bar{A}) \Lambda^b(x)$ corresponding to an infinitesimal gauge transformation of the saddle-point

field \bar{A} nullifies it. Therefore, one has to impose a gauge-fixing condition on a_μ . The conventional choice is the background Lorenz gauge $D_\mu^{ab}(\bar{A})a_\mu^b=0$: with this condition imposed the operator W simplifies as the second term in Eq. (60) can be dropped. Fixing this gauge, however, brings in the Faddeev-Popov ghost determinant $\text{Det}[-D^2(\bar{A})]$.

To define the path integral, one decomposes the fluctuation field in the complete set of the eigenfunctions of the quadratic form,

$$a_\mu^a(x) = \sum_n c_n \psi_\mu^a(x), \quad W_{\mu\nu}^{ab} \psi_\nu^b = \lambda_n \psi_\mu^b, \quad D_\mu^{ab} \psi_\mu^b = 0, \quad (62)$$

and implies that the path integral is understood as the integral over Fourier coefficients in the decomposition:

$$DA_\mu(x) = \prod_n \frac{dc_n}{\sqrt{2\pi}}, \quad (63)$$

The quadratic form (60) has a finite number of zero modes related to the moduli space of the solution. Let the number of zero modes be p [for a self-dual solution with topological charge one $p=4N$ for the $SU(N)$ gauge group [27]]. Let ξ_i , $i=1, \dots, p$, be the set of collective coordinates characterizing the classical solution, of which the action $S[\bar{A}]$ is in fact independent. The zero modes are

$$\psi_{\mu i}^a(x) = \frac{\partial \bar{A}_\mu^a(x, \xi)}{\partial \xi_i} - D_\mu^{ab}(\bar{A}) \Lambda_i^b(x), \quad (64)$$

where the second term is subtracted in order for the zero modes to satisfy the background Lorenz condition, $D_\mu^{ab} \psi_\mu^b = 0$. The $p \times p$ metric tensor

$$g_{ij} = \int d^4x \psi_{\mu i}^a \psi_{\mu j}^a \quad (65)$$

defines the metric of the moduli space. Its determinant is actually the Jacobian for passing from integration over zero-mode Fourier coefficients c_i , $i=1, \dots, p$, in Eq. (63) to the integration over the collective coordinates ξ_i , $i=1, \dots, p$:

$$\prod_{i=1}^p \frac{dc_i}{\sqrt{2\pi}} = J \prod_{i=1}^p d\xi_i \left(\frac{1}{\sqrt{2\pi}} \right)^p, \quad J = \sqrt{\det g_{ij}}. \quad (66)$$

Finally, one has to normalize and regularize the ghost determinant $\text{Det}(-D^2)$ and the Gaussian integral of the quadratic form. One usually normalizes the contribution of a pseudoparticle to the partition function by dividing it by the free (i.e., zero background field) determinants, and regularizes it by dividing by the determinants of the $-D^2$ and $W_{\mu\nu}$ operators shifted by the Pauli-Villars mass μ [19,30]. It means that $\text{Det}(-D^2)$ is replaced by the ‘‘quadrupole’’ combination

$$\text{Det}(-D^2)_{n,r} = \frac{\text{Det}(-D^2)}{\text{Det}(-\partial^2)} \frac{\text{Det}(-\partial^2 + \mu^2)}{\text{Det}(-D^2 + \mu^2)} \quad (67)$$

and similarly for the determinant of the quadratic form,

$$\text{Det}'(W_{\mu\nu})_{n,r} = \frac{\text{Det}'(W_{\mu\nu})}{\text{Det}(-\partial^2 \delta_{\mu\nu})} \frac{\text{Det}(-\partial^2 \delta_{\mu\nu} + \mu^2)}{\text{Det}'(W_{\mu\nu} + \mu^2)}, \quad (68)$$

where the prime indicates that only the product of nonzero eigenvalues is taken. In the integration over Pauli-Villars fields, the zero eigenvalues are shifted by μ^2 . Hence the integration over the zero-mode Fourier coefficients in the Pauli-Villars fields produces the factor

$$\prod_{i=1}^p \int \frac{dc_i}{\sqrt{2\pi}} \exp\left[-\frac{1}{2g^2} c_i^2 (0 + \mu^2)\right] = \left(\frac{g}{\mu}\right)^p \quad (69)$$

which has to be taken in the minus first power. Finally, one obtains the following normalized and regularized expression for the 1-loop quantum weight of a Euclidean pseudoparticle:

$$\mathcal{Z} = \int \prod_{i=1}^p d\xi_i e^{-S[\bar{A}]} \left(\frac{\mu}{g\sqrt{2\pi}}\right)^p J [\text{Det}'(W_{\mu\nu})_{n,r}]^{-1/2} \times \text{Det}(-D^2)_{n,r}. \quad (70)$$

If the saddle-point field \bar{A}_μ is (anti)self-dual there is a remarkable relation between the two determinants [20]: $\text{Det}'(W_{\mu\nu})_{n,r} = \text{Det}^4(-D^2)_{n,r}$ which is satisfied if the background field is decaying fast enough at infinity and the Hilbert space of the eigenfunctions of the two operators is well defined. This is the case of the KvBLL caloron but not the case of a single BPS dyon having a Coulomb asymptotics. To define the dyon weight properly, one would need to consider it in a spherical box, which would violate most of the statements in this section. For this reason we prefer to consider the well-defined quantum weight of the KvBLL caloron in which case the product of two determinants in Eq. (70) becomes just $\text{Det}^{-1}(-D^2)$.

B. KvBLL caloron moduli space

The KvBLL moduli space has been studied in the original papers [5,6]; in particular in Ref. [5] the metric tensor g_{ij} (65) has been explicitly computed. We briefly review these results and adjust them to our needs.

The KvBLL classical solution has 8 parameters for the $SU(2)$ gauge group. These are the four center-of mass positions z_μ and the four quaternionic variables $\zeta = \rho U$ corresponding to the constituent monopoles relative position in space and one global gauge transformation (see Appendix A 3). The moduli space of the KvBLL caloron is a product of the base manifold $\mathbb{R}^3 \times S^1$ parameterized by the $\vec{z} \in \mathbb{R}^3$ and $z_4 \in [0,1]$, and the nontrivial part of the moduli space parameterized by the quaternion ζ . It should be noted that the change $\zeta \rightarrow -\zeta$, corresponding to the center of the $SU(2)$, leaves $\bar{A}_\mu(x)$ invariant, such that one has to factor out this symmetry.

The 8 zero modes $\psi_{\mu i}^a$ (64) satisfying the background Lorenz condition have been explicitly found in Ref. [5]. It one parametrizes the unitary matrix through Euler angles,

$$U = e^{-iY\tau_3/2} e^{i(\pi/2 - \theta)\tau_2/2} e^{-i\varphi\tau_1/2}, \quad 0 \leq Y \leq 4\pi, \\ 0 \leq \varphi \leq 2\pi, \quad 0 \leq \theta \leq \pi, \quad (71)$$

the metric is [5]

$$ds^2 = (2\pi)^2 [2dz_\mu dz_\mu + (1 + 8\pi^2 \omega \bar{\omega} \rho^2)(4d\rho^2 + \rho^2 d^2\Omega) \\ + \rho^2 (1 + 8\pi^2 \omega \bar{\omega} \rho^2)^{-1} d\Sigma_3^2] \quad (72)$$

where

$$d^2\Omega = \sin^2\theta d\varphi^2 + d\theta^2, \quad d\Sigma_3 = dY + \cos\theta d\varphi. \quad (73)$$

The first part describes the flat metric of the base manifold $\mathbb{R}^3 \times S^1$, and the remainder forms the nontrivial part of the metric. The variables are inside the ranges $\rho \in [0, \infty)$, $\theta \in [0, \pi)$, $\varphi \in [0, 2\pi)$, $Y \in [0, 4\pi)/Z_2 = [0, 2\pi)$ for the nontrivial part, and $z_4 \in [0, 1]$, $z_i \in \mathbb{R}$ for translational modes.

The collective coordinate Jacobian is immediately found from Eq. (72):

$$J = \sqrt{\det(g_{ij})} = 8(2\pi)^8 \rho^3 (1 + 8\pi^2 \rho^2 \omega \bar{\omega}) \sin\theta. \quad (74)$$

The factor $\sin\theta$ is needed to organize the orientation $SO(3)$ Haar measure normalized to unity,

$$\int d^3\mathcal{O} = \frac{1}{8\pi^2} \int_0^{2\pi} dY \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta = 1, \quad (75)$$

and the KvBLL measure written in terms of the caloron center, size, and orientation becomes

$$\int d^3z \int dz_4 \int d^3\mathcal{O} \int d\rho \rho^3 (1 + 8\pi^2 \omega \bar{\omega} \rho^2) 16(2\pi)^{10}. \quad (76)$$

This must be multiplied by the factors $(\mu/g\sqrt{2\pi})^8$ and $\exp(-S[\bar{A}]) = \exp(-8\pi^2/g^2)$ according to Eq. (70). As the result, the KvBLL measure is

$$\int d^3z \int dz_4 \int d^3\mathcal{O} \int \frac{d\rho}{\rho^5} (1 + 8\pi^2 \omega \bar{\omega} \rho^2) \\ \times (\mu\rho)^8 \frac{1}{4\pi^2} \left(\frac{8\pi^2}{g^2} \right)^4 e^{-8\pi^2/g^2}. \quad (77)$$

When the holonomy is trivial ($\omega=0$ or $\bar{\omega}=0$) it becomes the well-known measure of the BPST instanton [30] or that of the Harrington-Shepard caloron [2]. The difference between the two is that in the first case one integrates over any z_4 whereas in the second case the z_4 integration is restricted to $z_4 \in [0, 1/T]$. Equation (77) would have been the full result in the $\mathcal{N}=1$ supersymmetric theory where the determinant over nonzero modes is cancelled by the gluino determinant. In that case one would need to add the integral over Grassmann variables corresponding to the gluino zero modes.

C. Combining the Jacobian and the determinant over nonzero modes

According to the general equation (70), we have now to multiply Eq. (77) by the (regularized and normalized) determinant over nonzero modes, which has been calculated in Eq. (47). First of all, we notice that $\text{Det}^{-1}(-D^2)$ brings in an additional UV divergent factor $\mu^{-2/3}$. In combination with the classical action and the factor μ^8 coming from zero modes, it produces

$$\mu^{22/3} e^{-8\pi^2/g^2(\mu)} = \Lambda^{22/3}, \quad (78)$$

where Λ is the scale parameter obtained here through the ‘‘transmutation of dimensions.’’

We notice further that $\text{Det}^{-1}(-D^2)$ is independent of the $SU(2)$ orientation \mathcal{O} and of z_4 . Therefore, we integrate over these variables, which gives unity. Next, we introduce the centers of the constituent BPS dyons $\vec{z}_{1,2}$ such that $|z_1 - z_2| = r_{12} = \pi\rho^2$ and write

$$\int d^3\vec{z}_1 d^3\vec{z}_2 = \int d^3 \left(\frac{\vec{z}_1 + \vec{z}_2}{2} \right) d^3(\vec{z}_1 - \vec{z}_2) = 4\pi \int d^3z r_{12}^2 dr_{12} \\ = 8\pi^2 \int d^3z d\rho r_{12}^{5/2}. \quad (79)$$

Therefore, integration over $d^3z d\rho$ in Eq. (77) can be traded for integrating over the dyon positions in space, $\vec{z}_{1,2}$. Lastly, we restore the temperature from dimensional considerations and obtain our final result for the 1-loop quantum weight of the KvBLL caloron, written in terms of the coordinates of the dyon centers:

$$\mathcal{Z}_{\text{KvBLL}} = \int d^3z_1 d^3z_2 T^6 C \left(\frac{8\pi^2}{g^2} \right)^4 \left(\frac{\Lambda e^{\gamma_E}}{4\pi T} \right)^{22/3} \left(\frac{1}{Tr_{12}} \right)^{5/3} \\ \times \left(2\pi + \frac{\bar{v}}{T} r_{12} \right) (vr_{12} + 1)^{4v/3\pi T - 1} (\bar{v}r_{12} + 1)^{4\bar{v}/3\pi T - 1} \exp[-VP(v) - 2\pi r_{12} P''(v)], \quad (80)$$

where

$$C = \frac{64}{\pi^2} \exp \left[\frac{8}{9} - \frac{16\gamma_E}{3} + \frac{2\pi^2}{27} + \frac{4\zeta'(2)}{\pi^2} \right] = 1.031419972084 \quad (81)$$

and $P(v)$ is the potential energy

$$P(v) = \frac{1}{12\pi^2 T} v^2 \bar{v}^2, \\ P''(v) = \frac{1}{\pi^2 T} \left[\pi T \left(1 - \frac{1}{\sqrt{3}} \right) - v \right] \left[\bar{v} - \pi T \left(1 - \frac{1}{\sqrt{3}} \right) \right], \\ \bar{v} = 2\pi T - v. \quad (82)$$

We have collected the factor $4\pi e^{-\gamma_E T/\Lambda}$ because it is the natural argument of the running coupling constant at nonzero temperatures [13,31]. Here Λ is the scale parameter in the Pauli-Villars regularization scheme that we have used. It is related to scale parameters in other schemes: $\Lambda_{\text{PV}} = e^{1/22} \Lambda_{\overline{\text{MS}}} = 40.66 \cdot \exp(-3\pi^2/11N^2) \Lambda_{\text{lat}}$ [32]. The factor g^{-8} is not renormalized at the one-loop level: it starts to “run” at the 2-loop level (see below).

The KvBLL caloron weight (80) has been derived assuming the separation between constituent dyons is large in temperature units ($r_{12} \gg 1/T$) but the holonomy is arbitrary: $\frac{1}{2} \text{Tr} L \in [-1, 1]$ corresponding to $\nu, \bar{\nu} \in [0, 2\pi T]$. It means that Eq. (80) is valid not only for well-separated but also for overlapping dyons.

D. The limit of large dyon separation

In the limit when the separation of dyons is larger than their core sizes, $r_{12} \gg 1/\nu, 1/\bar{\nu}$, the caloron weight simplifies to

$$\begin{aligned} \mathcal{Z}_{\text{KvBLL}} = & \exp\left[-VT^3 \frac{4\pi^2}{3} \nu^2 (1-\nu)^2\right] \int d^3 z_1 d^3 z_2 T^6 \\ & \times (2\pi)^{(8/3)} C\left(\frac{8\pi^2}{g^2}\right)^4 \left(\frac{\Lambda e^{\gamma_E}}{4\pi T}\right)^{22/3} \nu^{(8/3)\nu} \\ & \times (1-\nu)^{(8/3)(1-\nu)} \exp\left[-2\pi r_{12} T\right. \\ & \left. \times \left(\frac{2}{3} - 4\nu(1-\nu)\right)\right], \end{aligned} \quad (83)$$

where we have introduced the dimensionless quantity $\nu = \nu/2\pi T \in [0, 1]$. In Sec. VC we have calculated the $1/r_{12} T$ correction to the determinant [see Eq. (55)]. Another correction arises from the Jacobian (74), which cancels the $1/\nu, 1/\bar{\nu}$ terms in Eq. (55). As a result, we get the following correction factor to Eq. (83):

$$\begin{aligned} & \exp\left[\frac{1}{r_{12} T} \left(\frac{4}{3\pi} \log[\nu(1-\nu)(2r_{12} T)^2] + c_{1/r_{12}}\right)\right. \\ & \left. + \mathcal{O}\left(\frac{1}{(r_{12} T)^2}\right)\right], \\ c_{1/r_{12}} = & \frac{74}{9\pi} + \frac{8\gamma_E}{3\pi} - \frac{23\pi}{54} = 1.946. \end{aligned} \quad (84)$$

One can define the interaction potential between L, M dyons as

$$\begin{aligned} V_{\text{LM}}(r_{12}) = & r_{12} T^2 2\pi \left(\frac{2}{3} - 4\nu(1-\nu)\right) - \frac{1}{r_{12}} \left(\frac{4}{3\pi} \log[\nu(1-\nu)(2r_{12} T)^2] + c_{1/r_{12}}\right) \\ & + \mathcal{O}\left(\frac{1}{r_{12}^2 T}\right). \end{aligned} \quad (85)$$

This interaction is a purely quantum effect: classically L, M dyons do not interact at all as the KvBLL caloron of which they are constituents has the same classical action for all L, M separations. Curiously, the interaction potential has the familiar “linear + Coulomb” form. Both terms depend seriously on the holonomy: the Polyakov line at spatial infinity is $\frac{1}{2} \text{Tr} L = \cos(\pi\nu)$. In the range $0.787597 < \frac{1}{2} |\text{Tr} L| < 1$ dyons experience asymptotically a constant attraction force; in the complementary range $\frac{1}{2} |\text{Tr} L| < 0.787597$ it is repulsive. It should be noted that in its domain of applicability $r_{12} \gg 1/\nu, 1/\bar{\nu} \gg 1/2\pi T$, the second term in Eq. (85) is a small correction as compared to the linear rising (or linear falling) interaction.

E. 2-loop improvement of the result

The factor $g(\mu)^{-8}$ in Eq. (80) is the bare coupling which is renormalized only at the 2-loop level. In the case of the zero-temperature instanton one can unambiguously determine the 2-loop instanton weight without explicit 2-loop calculations from the requirement that it should be invariant under the simultaneous change of the UV cutoff and of the bare coupling given at that cutoff, such that the scale parameter

$$\begin{aligned} \Lambda = & \mu \exp\left(-\frac{8\pi^2}{b_1 g^2(\mu)}\right) \left(\frac{16\pi^2}{b_1 g^2(\mu)}\right)^{b_2/2b_1^2} [1 + \mathcal{O}(g^2(\mu))], \\ b_1 = & \frac{11}{3} N, \quad b_2 = \frac{34}{3} N^2, \end{aligned} \quad (86)$$

remains invariant. The result [33] is that one has to replace the combination of the bare coupling constants

$$\begin{aligned} & \left(\frac{8\pi^2}{g^2(\mu)}\right)^{2N} \exp\left(-\frac{8\pi^2}{g^2(\mu)}\right) \\ & \rightarrow \beta(\tau)^{2N} \exp\left[-\beta^{\text{II}}(\tau) + \left(2N - \frac{b_2}{2b_1}\right) \frac{b_2}{2b_1} \frac{\ln \beta(\tau)}{\beta(\tau)}\right. \\ & \left. + \mathcal{O}\left(\frac{1}{\beta(\tau)}\right)\right], \end{aligned} \quad (87)$$

where

$$\beta(\tau) = b_1 \ln \frac{\tau}{\Lambda}, \quad \beta^{\text{II}}(\tau) = \beta(\tau) + \frac{b_2}{2b_1} \ln \frac{2\beta(\tau)}{b_1}, \quad (88)$$

and τ is the scale of the pseudoparticle, which is $1/\rho$ in the instanton case. In the case of the KvBLL caloron with widely separated constituents one has to take the temperature scale, $\tau = 4\pi T e^{-\gamma_E}$. Thus, the 2-loop recipe is to replace the factor $(8\pi^2/g^2)^4 (\Lambda e^{\gamma_E}/4\pi T)^{22/3}$ in Eqs. (80) and (83) by the r.h.s. of Eq. (87).

In contrast to the zero-temperature instanton, in the KvBLL caloron case this replacement is not the only effect of two loops. In particular, the potential energy $P(\nu)$ is modified in 2 loops [34]. Nevertheless, the above modifica-

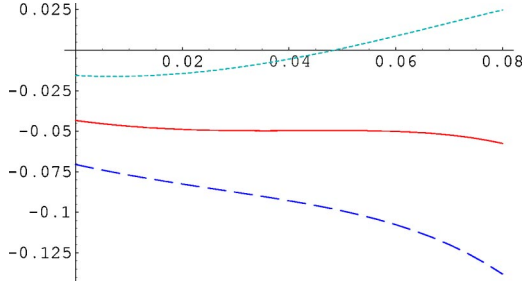


FIG. 5. Free energy of the caloron gas in units of $T^3 V$ at $T = 1.3\Lambda$ (dotted), $T = 1.125\Lambda$ (solid), and $T = 1.05\Lambda$ (dashed) as a function of the asymptotic value of A_4 in units of $2\pi T$.

tion is a very important effect of two loops, which needs to be taken into account if one wants to make a realistic estimate of the density of calorons with nontrivial holonomy at a given temperature. We remark that the additional large factor $4\pi e^{-\gamma_E} \approx 7.0551$ makes the running coupling numerically small even at $T \approx \Lambda$ [$1/\beta(\tau) \approx 0.07$], which may justify the use of semiclassical methods at temperatures around the phase transition. This numerically large scale is not accidental but originates from the fact that it is the Matsubara frequency $2\pi T$ rather than T itself which serves as a scale in all temperature-related problems. The additional order-of-unity factor $2e^{-\gamma_E}$ is specific for the Pauli-Villars regularization scheme used.

VII. CALORON DENSITY AND INSTABILITY OF THE TRIVIAL HOLONOMY

Since the caloron field has a constant A_4 component at spatial infinity, it is strongly suppressed by the potential energy $P(\nu)$, unless $\nu = 0, 2\pi T$ corresponding to trivial holonomy. Nevertheless, one may ask if the free energy of an ensemble of calorons can override this perturbative potential. We make below a crude estimate of the free energy of noninteracting KvBLL calorons. We shall consider only the case of small $\nu < \pi T(1 - 1/\sqrt{3})$. If ν exceeds this value the integral over dyon separations in Eq. (80) diverges, meaning that calorons with holonomy far from trivial “ionize” into separate dyons. We shall not consider this case here but restrict ourselves to small values of ν where the integral over the separation between dyon constituents converges, such that one can assume that KvBLL calorons are in the “atomic” phase. Integrating over the separation r_{12} in Eq. (80) gives the “fugacity” of calorons:

$$\zeta = T^3 f(T/\Lambda) I(\nu), \quad (89)$$

$$f(T/\Lambda) = 8\pi^2 C \beta^4 \exp\left[-\beta^{\text{II}} + \left(4 - \frac{34}{11}\right) \frac{34}{11} \frac{\ln \beta}{\beta}\right],$$

$$\beta = \frac{22}{3} \ln \frac{4\pi T}{\Lambda e^{\gamma_E}},$$

$$\beta^{\text{II}} = \beta + \frac{34}{11} \ln \frac{3\beta}{11}, \quad (90)$$

$$I(\nu) = \int_0^\infty dR R^{(1/3)} [1 + 2\pi\nu(1-\nu)R] (2\pi\nu R + 1)^{(8/3)\nu-1} [2\pi(1-\nu)R + 1]^{(8/3)(1-\nu)-1} \times \exp\left[-2\pi R \left(\frac{2}{3} - 4\nu + 4\nu^2\right)\right] \quad (91)$$

where we have introduced the dimensionless separation, $R = r_{12}T$, and the dimensionless $\nu = v/2\pi T$. One should be cautioned that Eq. (80) has been derived for $R \gg 1$, therefore the caloron fugacity is evaluated accurately if the integral (91) is saturated in the large- R region.

Assuming the Yang-Mills partition function is governed by a noninteracting gas of N_+ calorons and N_- anticalorons, one writes their grand canonical partition function as

$$\mathcal{Z}_{\text{cal}} = \exp\left[-VT^3 \frac{4\pi^2}{3} \nu^2 (1-\nu)^2\right] \sum_{N_+, N_-} \frac{1}{N_+! N_-!} \times \left(\int d^3 z \zeta\right)^{N_+ + N_-} = \exp[-VT^3 \mathcal{F}(\nu, T)], \quad (92)$$

where $\mathcal{F}(\nu, T)$ is the free energy of the caloron gas, including the perturbative potential energy:

$$\mathcal{F}(\nu, T) = \frac{4\pi^2}{3} \nu^2 (1-\nu)^2 - 2f(T/\Lambda) I(\nu). \quad (93)$$

We plot the free energy as function of ν in Fig. 5 at several temperatures. The function $f(T/\Lambda)$ rapidly drops with increasing temperature. Therefore, at high temperatures the perturbative potential energy prevails, and the minimal free energy corresponds to trivial holonomy. However, at $T \approx \Lambda$ the caloron fugacity becomes sizable, and an opposite trend is observed. In this model, $T_c = 1.125\Lambda$ is the critical temperature where the trivial holonomy becomes an unstable point, and the system rolls towards large values of ν where the present approach fails since at large ν calorons anyhow have to “ionize” into separate dyons.

Although several simplifying assumptions have been made in this derivation, it may indicate the instability of the trivial holonomy at temperatures below some critical one related to Λ .

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APPENDIX A: ADHM CONSTRUCTION FOR THE BPS DYON AND THE KVBLL CALORON

1. General ADHM construction

The basic object in the ADHM construction [28] is the $k \times (k+1)$ quaternionic-valued matrix Δ which is taken to be linear in the space-time variable x :

$$\Delta(x) = \mathcal{A} + \mathcal{B}x, \quad x \equiv x_\mu \sigma_\mu, \quad \sigma_\mu = (i\vec{\tau}, 1_2). \quad (\text{A1})$$

The ADHM gauge potential is given by

$$A_\mu(x) = v^\dagger(x) \partial_\mu v(x), \quad (\text{A2})$$

where $v(x)$ is a $(k+1)$ -dimensional quaternionic vector, the normalized solution to

$$\Delta^\dagger(x) v(x) = 0, \quad (\text{A3})$$

and k is the topological charge of the gauge field. An important property of the ADHM construction is that the operator $\Delta^\dagger(x) \Delta(x)$ is a real-valued matrix:

$$f = [\Delta(x)^\dagger \Delta(x)]^{-1} \in \mathbb{R}^{k \times k}. \quad (\text{A4})$$

In what follows we shall use the equation

$$\Delta f \Delta^\dagger = 1 - v v^\dagger. \quad (\text{A5})$$

It becomes obvious when one notes that both sides are projectors onto the space orthogonal to the vector v , which follows from $v^\dagger v = 1$, $\Delta^\dagger v = 0$.

In the case of finite temperatures, because of the infinite number of copies of space in the compact direction, the topological charge $k = \infty$, and it is convenient to make a discrete Fourier transformation with respect to the infinite range indices. The Fourier transformed $v(x)$ are 2×2 matrix-valued functions $v(x_\mu, z)$ of a new variable $z \in [-1/2, 1/2]$ and Δ becomes a differential operator in z .

2. ADHM construction for the BPS dyon

As stated above, at nonzero temperatures the essence of the ADHM construction is the introduction of 2×2 matrix-valued functions $v(x_\mu, z)$. The scalar product is defined as

$$\langle v_1 | v_2 \rangle = \int_{-1/2}^{1/2} v_1^\dagger(x_\mu, z) v_2(x_\mu, z) dz. \quad (\text{A6})$$

For the BPS dyon solution v has been found by Nahm [25]:

$$v(x_\mu, z) = \sqrt{\frac{vr}{\sinh(vr)}} \exp(izvx^\dagger) \quad (\text{A7})$$

where $\sigma_\mu^\dagger = (1_2, -i\vec{\tau})$, $x^\dagger = x_\mu \sigma_\mu^\dagger$ and $r = |\vec{x}|$. The matrix-valued function v is the solution of the equation

$$\Delta^\dagger(x) v(x, z) = 0, \quad \Delta^\dagger(x) = i\partial_z + vx^\dagger \quad (\text{A8})$$

normalized to unity,

$$\langle v | v \rangle = 1. \quad (\text{A9})$$

The gauge field is expressed through v as

$$A_\mu = \langle v | \partial_\mu v \rangle. \quad (\text{A10})$$

We use anti-Hermitian fields such that the covariant derivative is $D_\mu = \partial_\mu + A_\mu$. Comparing Eq. (A8) with the general Eq. (A1) we conclude that in this case

$$\mathcal{A} = -i\partial_z, \quad \mathcal{B} = v. \quad (\text{A11})$$

Equation (A7) corresponds to the ‘‘hedgehog’’ gauge. However we find it more convenient to work in the stringy gauge where A_μ has a pure gauge stringlike singularity. One proceeds from the hedgehog gauge to the stringy gauge using the singular gauge transformation (see, e.g., Ref. [10])

$$v \rightarrow v^s = v S_-^\dagger, \quad A_\mu \rightarrow A_\mu^s = S_- A_\mu S_-^\dagger + S_- \partial_\mu S_-^\dagger \quad (\text{A12})$$

with

$$S_- = e^{-i(\phi/2)\tau^3} e^{i(\pi-\theta)/2\tau_2} e^{-i(\phi/2)\tau^3} \quad (\text{A13})$$

having the property that it ‘‘gauge-combs’’ A_4 at spatial infinity to a fixed (third) direction:

$$S_- n^a \tau^a S_-^\dagger = \tau^3. \quad (\text{A14})$$

In the stringy gauge

$$v^s = S_-^\dagger \sqrt{\frac{vr}{\sinh(vr)}} \exp[zv(ix_4 + r\tau^3)]. \quad (\text{A15})$$

One can check that $A_\mu = \langle v^s | \partial_\mu v^s \rangle$ gives the M dyon field in the stringy gauge as in Eq. (13). We note that in the stringy gauge v^s has a remarkable property

$$v^s(x_4 + n, \vec{x}) = e^{invz} v^s(x_4, \vec{x}). \quad (\text{A16})$$

3. ADHM construction for the KvBLL caloron

Unfortunately, the original paper [5] does not present an explicit expression for v , the main ingredient of the ADHM construction. We could have used Ref. [6] but it seems that Ref. [5] is more informative in some other respects. Therefore, we have to calculate v ourselves.

From the point of view of the original ADHM construction v is a quaternionic vector of infinite length since finite-temperature field configuration can be viewed as an infinite set of equal strips, the total topological charge in \mathbb{R}^4 being infinite. The bracket is formally defined as a contraction along this infinite-dimension side:

$$\langle v | v \rangle \equiv v^\dagger \tilde{v}. \quad (\text{A17})$$

The gauge potential results from

$$A_\mu(x) = v^\dagger(x) \partial_\mu v(x), \quad D_\mu = \partial_\mu + A_\mu. \quad (\text{A18})$$

The vector $v(x)$ is the normalized solution of the equation

$$\Delta^\dagger(x)v(x)=0, \quad \Delta(x)=\begin{pmatrix} \lambda \\ B-x \end{pmatrix}, \quad (\text{A19})$$

where B is a square quaternionic matrix, λ is an (infinite) quaternionic vector, $x \equiv x_\mu \sigma_\mu$, $\sigma_\mu = (i\vec{\tau}, 1_2)$. Introducing the notation

$$v(x) = \Phi^{-(1/2)}(x) \begin{pmatrix} -1 \\ u(x) \end{pmatrix}, \quad u(x) = (B^\dagger - x^\dagger)^{-1} \lambda^\dagger, \quad (\text{A20})$$

Eq. (A19) becomes

$$(B^\dagger - x^\dagger)u(x) = \lambda^\dagger. \quad (\text{A21})$$

The inverse of the matrix $(B^\dagger - x^\dagger)$ exists almost in all points. The points where it does not exist are monopole positions. We are interested in those singular points that lie in the interval $0 < x_4 < 1$ (we have rescaled the units to set temperature $T=1$). The unknown function $\Phi(x)$ is determined from the normalization of v :

$$\Phi(x) = 1 + u^\dagger u. \quad (\text{A22})$$

The formalism of infinite-dimensional matrices is not convenient. Following Nahm [25] we pass to the Fourier transforms in the discrete but infinite-range indices and get instead a continuous variable $z \in [-\frac{1}{2}, \frac{1}{2}]$. In the notation of Ref. [5]:

$$(B^\dagger - x^\dagger)_{nm} = - \int_{-1/2}^{1/2} \frac{dz}{2\pi i} e^{-2\pi i z n} \hat{D}_x^\dagger(z) e^{2\pi i z m},$$

$$\hat{D}_x^\dagger(z) = -\partial_z + \hat{A}^\dagger(z) + 2\pi i x^\dagger \equiv -\partial_z + 2\pi i r^\dagger(z),$$

where

$$\hat{A}(z) = 2\pi i [\xi + \vec{r}_{12} \cdot \vec{\sigma} \Theta_\omega(z)]. \quad (\text{A23})$$

Here the function $\Theta_\omega(z) = 2\bar{\omega}$ when $z \in [-\omega, \omega]$ and -2ω otherwise; $r^\dagger(z) \equiv r_\mu^\dagger(z) \sigma_\mu^\dagger$, $x^\dagger \equiv x_\mu \sigma_\mu^\dagger$. As it can be seen from Eq. (A23), the quaternion ξ simply represents the center of mass position of the whole system and can be set to zero, $\xi=0$. We define $r_\mu(z) = r_\mu$ when $z \in [\omega, 1-\omega]$, and $r_\mu(z) = s_\mu$ otherwise, where

$$\vec{s} = \vec{x} - 2\bar{\omega} \vec{r}_{12}, \quad \vec{r} = \vec{x} + 2\omega \vec{r}_{12}, \quad s_4 = r_4 = x_4. \quad (\text{A24})$$

Here $\bar{\omega} \equiv \frac{1}{2} - \omega$, and r_μ and s_μ have the meaning of the vectors from the dyon centers to the ‘‘observation’’ point, $\vec{r}_{12} = \vec{r} - \vec{s}$ has the meaning of the separation dyon. We choose separation between dyon in the 3D direction: $\vec{r}_{12} = r_{12} \vec{e}_3$. As for λ ,

$$\lambda_n^\dagger = \rho U^\dagger e^{-2\pi i n \bar{\omega} \cdot \vec{\tau}}, \quad (\text{A25})$$

where U is a unitary matrix, and $\rho > 0$. We have an additional constraint [5]

$$\vec{r}_{12} \cdot \vec{\sigma} = \pi \rho^2 (U^\dagger \bar{\omega} \cdot \vec{\sigma} U) / \omega. \quad (\text{A26})$$

Here $2\pi i \bar{\omega} \cdot \vec{\tau}$ is the value of A_4 at spatial infinity, $\omega = |\bar{\omega}|$. It can be seen that $\pi \rho^2 = r_{12}$. We choose to rotate the A_4 direction in color space instead of rotating monopole positions, so we do not lose the generality of the solution. We connect the vector $\bar{\omega}$ and U by

$$U^\dagger \bar{\omega} \cdot \vec{\tau} U = \omega \tau_3. \quad (\text{A27})$$

Writing down the m th component of the (infinite) quaternionic vector as a Fourier transform

$$u_m(x) = \int_{-1/2}^{1/2} u(x, z) e^{-2\pi i m z} dz, \quad (\text{A28})$$

Eq. (A19), which we have to solve, can be rewritten as

$$\begin{aligned} & [\partial_z - 2\pi i r^\dagger(z)]u(x, z) \\ &= 2\pi i \rho U^\dagger \sum_n e^{-2\pi i n \bar{\omega} \cdot \vec{\tau}} e^{2\pi i z n} \\ &= 2\pi i \rho U^\dagger [P_+ \delta(z - \omega) + P_- \delta(z + \omega)], \end{aligned} \quad (\text{A29})$$

where

$$P_\pm = \frac{1}{2} (1 \pm \bar{\omega} \cdot \vec{\tau} / \omega). \quad (\text{A30})$$

Equation (A29) is piecewise homogeneous; therefore we present its solution in the form

$$u(x, z) = \begin{cases} \exp(2\pi i s^\dagger z) B_1, & -\omega < z < \omega \\ \exp[2\pi i r^\dagger(z - 1/2)] B_2, & \omega < z < 1 - \omega \end{cases} \quad (\text{A31})$$

and match the values and the derivatives of u at the end points of the pieces,

$$\begin{aligned} e^{-2\pi i r^\dagger \bar{\omega}} B_2 - e^{2\pi i s^\dagger \omega} B_1 &= f_1, \\ e^{-2\pi i s^\dagger \omega} B_1 - e^{2\pi i r^\dagger \bar{\omega}} B_2 &= f_2, \end{aligned} \quad (\text{A32})$$

where

$$f_1 = 2\pi i \rho U^\dagger P_+, \quad f_2 = 2\pi i \rho U^\dagger P_-, \quad \bar{\omega} = \frac{1}{2} - \omega.$$

Note that $B_{1,2}$ are matrices that generally do not commute:

$$\begin{aligned} B_2 &= (e^{-2\pi i s^\dagger \omega} e^{-2\pi i r^\dagger \bar{\omega}} - e^{2\pi i s^\dagger \omega} e^{2\pi i r^\dagger \bar{\omega}})^{-1} (e^{-2\pi i s^\dagger \omega} f_1 \\ &\quad + e^{2\pi i s^\dagger \omega} f_2) \equiv b_{22} b_{21} e^{-2\pi i x_4 \omega \tau_3} U^\dagger / \hat{\psi}, \\ B_1 &= (e^{-2\pi i r^\dagger \bar{\omega}} e^{-2\pi i s^\dagger \omega} - e^{2\pi i r^\dagger \bar{\omega}} e^{2\pi i s^\dagger \omega})^{-1} (e^{-2\pi i r^\dagger \bar{\omega}} f_2 \\ &\quad + e^{2\pi i r^\dagger \bar{\omega}} f_1) \equiv b_{12} b_{11} e^{-2\pi i x_4 \omega \tau_3} U^\dagger / \hat{\psi}, \end{aligned} \quad (\text{A33})$$

where

$$\begin{aligned}
 b_{22} &= [-\cos(\pi x_4)(\overline{\text{ch}_{1/2}\text{sh}_{1/2}\hat{r}} + \overline{\text{ch}_{1/2}\text{sh}_{1/2}\hat{s}}) + i \sin(\pi x_4) \\
 &\quad \times (\overline{\text{ch}_{1/2}\text{ch}_{1/2} + \hat{r}\hat{s}\overline{\text{sh}_{1/2}\text{sh}_{1/2}})], \\
 b_{12} &= [-\cos(\pi x_4)(\overline{\text{ch}_{1/2}\text{sh}_{1/2}\hat{r}} + \overline{\text{ch}_{1/2}\text{sh}_{1/2}\hat{s}}) + i \sin(\pi x_4) \\
 &\quad \times (\overline{\text{ch}_{1/2}\text{ch}_{1/2} + \hat{s}\hat{r}\overline{\text{sh}_{1/2}\text{sh}_{1/2}})], \\
 b_{21} &= 2\pi i \rho (\overline{\text{ch}_{1/2} - \hat{s}\tau_3\text{sh}_{1/2}}), \\
 b_{11} &= 2\pi i \rho (\overline{\text{ch}_{1/2} + \hat{r}\tau_3\text{sh}_{1/2}}) e^{\pi i x_4 \tau_3}, \tag{A34}
 \end{aligned}$$

$$\hat{\psi} \equiv -\cos(2\pi x_4) + \overline{\text{chch}} + \frac{\vec{s} \cdot \vec{r}}{sr} \overline{\text{shsh}}.$$

Hat over the variable (notation found also in Ref. [5]) means contraction of the corresponding normalized vector with Pauli matrices, e.g., $\hat{\omega} \equiv \vec{\omega} \cdot \vec{\tau} / \omega$. We denote for brevity

$$\begin{aligned}
 \text{sh} &\equiv \sinh(4\pi s \omega), \quad \text{ch} \equiv \cosh(4\pi s \omega), \\
 \overline{\text{sh}} &\equiv \sinh(4\pi r \overline{\omega}), \quad \overline{\text{ch}} \equiv \cosh(4\pi r \overline{\omega}), \tag{A35}
 \end{aligned}$$

and the hyperbolic functions with subscript 1/2 are the corresponding functions of half the same arguments. Combining Eqs. (A31), (A33), and (A34) back into Eq. (A20) one gets the two-dimensional quaternionic vector $v(x, z)$ which is the base for the construction of the Green's function (see Appendix B). Note that we have made a Fourier transform of u (A28) and got a continuous index z , so that scalar products of infinite-dimensional vectors become z integrations, see Eq. (A38).

We note that U is actually a gauge transformation of v . Therefore, the gauge potential A_μ^U is obtained by a global gauge transformation of $A_\mu^{U=1}$. We conclude that the determinant does not depend on the relative ‘‘color orientation’’ of the Polyakov line or holonomy, and of the vector \vec{r}_{12} connecting monopole centers. Thus, we set $U=1$ and $\vec{\omega} = \omega \vec{e}_3$.

We notice further that $v(x, z)$ built above gives A_μ that is not periodic in time direction and zero A_4 at spatial infinity. It is a peculiar feature of the ‘‘algebraic’’ gauge used in Ref. [5]. It is more convenient to use the gauge in which the fields are periodic. To that end we make a nonperiodic gauge transformation $g = e^{2\pi i x_4 \omega \tau_3}$ and obtain

$$v(x, z)^{\text{per}} = \Phi^{-1/2}(x) \begin{pmatrix} -g \\ w(x, z) \end{pmatrix}, \quad w = ug, \tag{A36}$$

meaning

$$w(x, z) = u(x, z) e^{2\pi i x_4 \omega \tau_3}. \tag{A37}$$

In terms of the Fourier-transformed v the bracket takes the form

$$\langle v | v \rangle \equiv v_1^\dagger \tilde{v}_1 + \int_{-1/2}^{1/2} v_2^\dagger \tilde{v}_2 dz, \tag{A38}$$

where v_1 is an upper element and v_2 is a lower one.

Now let us determine $\Phi(x)$. We use the following identities:

$$\begin{aligned}
 b_{12}^\dagger b_{12} &= b_{22}^\dagger b_{22} = \hat{\psi}/2, \quad b_{21}^\dagger b_{21} = 4\pi^2 \rho^2 \left(\overline{\text{ch}_{1/2} - \frac{s_3}{s} \text{sh}_{1/2}} \right), \\
 b_{11}^\dagger b_{11} &= 4\pi^2 \rho^2 \left(\overline{\text{ch}_{1/2} + \frac{r_3}{r} \text{sh}_{1/2}} \right). \tag{A39}
 \end{aligned}$$

Note that the right-hand sides of Eq. (A39) are proportional to the unity 2×2 matrix. Now we can easily calculate the normalization:

$$\begin{aligned}
 \langle w | w \rangle &= \hat{\psi}^{-2} b_{11}^\dagger b_{11} b_{12}^\dagger b_{12} \int_{-\omega}^{\omega} dz e^{-4\pi \vec{s} \cdot \vec{\tau} z} \\
 &\quad + \hat{\psi}^{-2} b_{21}^\dagger b_{21} b_{22}^\dagger b_{22} \int_{-\overline{\omega}}^{\overline{\omega}} dz e^{-4\pi \vec{r} \cdot \vec{\tau} z} \\
 &= \frac{\pi \rho^2}{\hat{\psi}} \left[\left(\frac{\overline{\text{ch}_{1/2}\text{sh}_{1/2}}}{s} + \frac{\overline{\text{ch}_{1/2}\text{sh}_{1/2}}}{r} \right) \right. \\
 &\quad \left. + \frac{r_{12}}{sr} \overline{\text{sh}_{1/2}\text{sh}_{1/2}} \right] \equiv \frac{\psi - \hat{\psi}}{\hat{\psi}}.
 \end{aligned}$$

We used the identity $\vec{r} - \vec{s} = \vec{r}_{12} = r_{12} \vec{e}_3$. Thus for Φ we get

$$\Phi = \frac{\psi}{\hat{\psi}}, \quad \psi = \hat{\psi} + r_{12} \left(\frac{\overline{\text{ch}} \overline{\text{sh}}}{s} + \frac{\overline{\text{chsh}}}{r} \right) + \frac{r_{12}^2}{sr} \overline{\text{shsh}}. \tag{A40}$$

We have checked the A_μ of the KvBLL caloron (7) by calculating $\langle v^{\text{per}} | \partial_\mu v^{\text{per}} \rangle$. Note that v^{per} has the following periodicity property (for integer n):

$$v^{\text{per}}(z, x_4 + n) = e^{2\pi i n z} v^{\text{per}}(z, x_4). \tag{A41}$$

APPENDIX B: SPIN-0 ISOSPIN-1 PROPAGATOR

1. General construction of the Green function

Once the self-dual field is found in terms of the ADHM construction, such that the gauge field is written as $A_\mu = \langle v | \partial_\mu v \rangle$ where the scalar product is defined in Eq. (A38), it is possible to construct explicitly the Green function of spin-0 isospin-1 field in the background of the self-dual field [5,6,25]. The solution of the equation

$$-(D_\mu^2)^{ca}(x) G^{ab}(x, y) = \delta^{cb} \delta^{(4)}(x - y) \tag{B1}$$

is given by

$$\begin{aligned}
 G^{ab}(x,y) &= \frac{\frac{1}{2} \text{Tr} \tau^a \langle v(x)|v(y) \rangle \tau^b \langle v(y)|v(x) \rangle}{4\pi^2(x-y)^2} \\
 &+ \frac{1}{4\pi^2} \int_{-1/2}^{1/2} dz_1 dz_2 dz_3 dz_4 M(z_1, z_2, z_3, z_4) \\
 &\times \frac{1}{2} \text{Tr} [\mathcal{V}^\dagger(x, z_1) \mathcal{V}(x, z_2) \tau^a] \\
 &\times \text{Tr} [\mathcal{V}^\dagger(y, z_4) \mathcal{V}(y, z_3) \tau^b], \quad (\text{B2})
 \end{aligned}$$

where $\mathcal{V}(x, z) \equiv \mathcal{B}^\dagger v(x, z)$.

We denote the first term by G_1 and the second term (the M part) by G_2 . The only new object is the function $M(z_1, z_2, z_3, z_4)$ which we determine below. As we shall see, we do not need M with arbitrary arguments, but only at $z_3 = z_4$. For coincident arguments we obtain

$$M(z_1, z_2, z, z) = \delta(z_1 - z_2) M(z_1, z), \quad (\text{B3})$$

see below.

The propagator (B2) is written for the \mathbb{R}^4 space and does not obey the periodicity condition. The periodic propagator, however, can be easily obtained from it by a standard procedure:

$$\mathcal{G}(x, y) = \sum_{n=-\infty}^{+\infty} G(x_4, \vec{x}; y_4 + n, \vec{y}). \quad (\text{B4})$$

In what follows it will be convenient to split it into three parts:

$$\mathcal{G}(x, y) = \mathcal{G}^r(x, y) + \mathcal{G}^s(x, y) + \mathcal{G}^m(x, y),$$

$$\mathcal{G}^s \equiv G_1|_{n=0}, \quad \mathcal{G}^r \equiv \sum_{n \neq 0} G_1, \quad \mathcal{G}^m \equiv \sum_n G_2. \quad (\text{B5})$$

The vacuum current (20) will be also split into three parts, in accordance to which part of the periodic propagator (B5) is used to calculate it:

$$J_\mu = J_\mu^r + J_\mu^s + J_\mu^m. \quad (\text{B6})$$

2. Propagator in the BPS dyon background

In Appendix A2 we have found the needed periodic quaternion $v(x, z)$ for the single BPS monopole [see Eq. (A15)]. The 4-argument function M for the BPS monopole was computed in Ref. [25]. The result with the two last arguments taken equal is

$$\begin{aligned}
 M(z_3, z_4, z, z) &= \delta(z_3 - z_4) M(z_3, z), \\
 M(z, z') &= -\frac{1}{4v^2} (2|z - z'| - 1 + 4zz'). \quad (\text{B7})
 \end{aligned}$$

Equations (A15) and (B7) completely determine the periodic propagator defined in Eqs. (B2) and (B4) in the BPS dyon background. The use of this propagator is demonstrated in Appendix C.

3. Propagator in the KvBLL caloron background

In Appendix A3 we have found the needed quaternion $v(x, z)$ for the KvBLL caloron. In this Appendix we derive the M function for the KvBLL caloron. The propagator (B2) will be then completely determined in the caloron background.

In the notations of Ref. [25] M is an infinite-dimensional rank-4 tensor, with indices running from 1 to k , the topological charge in \mathbb{R}^4 . As in the case of v , it is convenient to make the Fourier transformation with respect to the indices:

$$\begin{aligned}
 M_{pqnm} &= \int_{-1/2}^{1/2} M(z_1, z_2, z_3, z_4) e^{2\pi i(-pz_1 + qz_2 + nz_3 - mz_4)} \\
 &\times dz_1 dz_2 dz_3 dz_4. \quad (\text{B8})
 \end{aligned}$$

The tensor M_{pqmn} is defined by the equation [27]

$$\begin{aligned}
 \frac{1}{2} \text{Tr} [(\mathcal{A}^\dagger \mathcal{A})_{il} (\mathcal{B}^\dagger \mathcal{B})_{mj} + (\mathcal{B}^\dagger \mathcal{B})_{il} (\mathcal{A}^\dagger \mathcal{A})_{mj} \\
 - 2(\mathcal{A}^\dagger \mathcal{B})_{il} (\mathcal{B}^\dagger \mathcal{A})_{mj}] M_{rsij} &= \delta_{rl} \delta_{sm}, \quad (\text{B9})
 \end{aligned}$$

All indices here run from 1 to k as rectangular $k \times (k+1)$ matrices \mathcal{A} and \mathcal{B} are contracted along the longer side. Here \mathcal{A} and \mathcal{B} are

$$\Delta(x) \equiv \mathcal{A} + \mathcal{B}x, \quad \mathcal{A} = \Delta(0), \quad \mathcal{B} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \quad (\text{B10})$$

Equation (B9) can be rewritten as

$$\begin{aligned}
 \frac{1}{2} \text{Tr} [(\Delta^\dagger \Delta(0))_{il} \delta_{mj} + (\Delta^\dagger \Delta(0))_{mj} \delta_{il} - 2B_{il}^\dagger B_{mj}] M_{rsij} \\
 = \delta_{rl} \delta_{sm}, \quad (\text{B11})
 \end{aligned}$$

where $\Delta^\dagger \Delta(0) = \lambda^\dagger \lambda + B^\dagger B$, B and λ are found in Eqs. (A23) and (A25), respectively. In our case k is infinite and we rewrite Eq. (B11) in the Fourier basis:

$$\begin{aligned}
 \frac{1}{2} \text{Tr} \left[\tilde{\Delta}^\dagger \tilde{\Delta}(0, z_3) + \tilde{\Delta}^\dagger \tilde{\Delta}(0, z_4) + 2 \left(\frac{\partial_{z_3}}{2\pi i} + r^\dagger(z_3) \right) \left(\frac{\partial_{z_4}}{2\pi i} \right. \right. \\
 \left. \left. - r(z_4) \right) \right] M(z_1, z_2, z_3, z_4) = \delta(z_1 - z_3) \delta(z_2 - z_4),
 \end{aligned}$$

where $r(z) = r_i \sigma_i$ when $z \in [\omega, 1 - \omega]$, and $r(z) = s_i \sigma_i$ otherwise; $\sigma_i = i \tau_i$. Zero components of r_μ , s_μ are absent because $x_\mu = 0$. We use

$$\tilde{\Delta}^\dagger \tilde{\Delta}(0, z) = -\frac{\partial_z^2}{4\pi^2} + r^2(z) + \frac{\rho^2}{2} [\delta(z - \omega) + \delta(z + \omega)]. \quad (\text{B12})$$

Here the first two terms come from the Fourier transformation of $B^\dagger B$ (A23) and the last one comes from the Fourier transformation of $\lambda^\dagger \lambda$. We obtain the explicit equation for M :

$$\left(-\frac{(\partial_{z_3} + \partial_{z_4})^2}{4\pi^2} + |\vec{r}(z_3) - \vec{r}(z_4)|^2 \right) M(z_1, z_2, z_3, z_4) + \frac{\rho^2}{2} [\delta(z_3 - \omega) + \delta(z_3 + \omega) + \delta(z_4 - \omega) + \delta(z_4 + \omega)] M(z_1, z_2, z_3, z_4) = \delta(z_1 - z_3) \delta(z_2 - z_4). \tag{B13}$$

In the case $z_3 = z_4$, which is the only one we need as we shall see in a moment, we look for the solution in the form

$$M(z_1, z_2, z, z) = \delta(z_1 - z_2) M(z_1, z). \tag{B14}$$

The equation for the two-argument function simplifies to

$$\left(-\frac{\partial_z^2}{4\pi^2} + \frac{r_{12}}{\pi} \delta(z - \omega) + \frac{r_{12}}{\pi} \delta(z + \omega) \right) M(z', z) = \delta(z - z'), \tag{B15}$$

where $r_{12} = \pi \rho^2$. We see that the solution has to be piecewise linear in its arguments. The solution is symmetric in its two arguments and for $z < z'$ is given by

$$M(z, z') = \begin{cases} \frac{32r_{12}^2 \pi^3 \omega(z - \omega)(1 - z' - \omega) - 8r_{12} \pi^2 (\omega^2 + z(z' - 1)) + \pi}{2d(8r_{12} \pi \omega \bar{\omega} + 1)}, & z, z' \in [\omega, 1 - \omega] \\ \frac{1}{2} \pi \left(\frac{4\pi z(1 - 2z')}{8r_{12} \pi \omega \bar{\omega} + 1} + \frac{1}{r_{12}} \right), & z \in [-\omega, \omega], z' \in [\omega, 1 - \omega] \\ \frac{4r_{12} \pi^2 [z - z' - 2z'z - 2(\omega - 1)\omega - 8r_{12} \pi (z' - \omega)(z + \omega)\bar{\omega}] + \pi}{2r_{12}(8r_{12} \pi \omega \bar{\omega} + 1)}, & z, z' \in [-\omega, \omega]. \end{cases} \tag{B16}$$

Outside this range M is defined by periodicity: $M(z + n, z' + m) = M(z, z')$, where n, m are integers.

Now let us demonstrate that actually only the two-argument function $M(z, z')$ is needed to construct the propagator satisfying the periodicity. It turns out that making the Green function periodic simplifies G_2 (Sec. III). One has from the definitions (B2)–(B5):

$$\mathcal{G}^m \equiv \sum_n \frac{1}{8\pi^2} \int_{-1/2}^{1/2} dz_1 \cdots dz_4 M(z_1, \dots, z_4) \times \text{Tr}[\mathcal{V}^\dagger(x, z_1) \mathcal{V}(x, z_2) \tau^a] \times \text{Tr}[\mathcal{V}^\dagger(y^n, z_4) \mathcal{V}(y^n, z_3) \tau^b], \tag{B17}$$

where $y_4^n = y_4 + n$, $\vec{y}^n = \vec{y}$. Using Eq. (A41) we set

$$\mathcal{V}(y^n, z) = e^{2\pi i n m z} \mathcal{V}(y, z). \tag{B18}$$

Further on, we note that for $|\eta| \leq 1$ one has

$$\sum_n \text{Tr}[\mathcal{V}^\dagger(y^n, z_4) \mathcal{V}(y^n, z_3) \tau^b] = \text{Tr}[\mathcal{V}^\dagger(y, z_4) \mathcal{V}(y, z_3) \tau^b] \delta(z_3 - z_4) \frac{1}{|\eta|}.$$

Now we can see that making the Green's function periodic results in the substitution

$$M(z_1, z_2, z_3, z_4) \rightarrow \frac{1}{|\eta|} M(z_1, z_2, z_3, z_3) \delta(z_3 - z_4) = \frac{1}{|\eta|} M(z_1, z_3) \delta(z_1 - z_2) \delta(z_3 - z_4).$$

It follows from Eqs. (A16) and (A41) that for the monopole one has to take $\eta = v/(2\pi) < 1$ and for the KvBLL caloron $\eta = 1$. In both cases the M part of the periodic propagator is given by

$$\mathcal{G}^m = \frac{1}{8\pi^2 |\eta|} \int_{-1/2}^{1/2} dz dz' M(z, z') \times \text{Tr}[\mathcal{V}^\dagger(x, z) \mathcal{V}(x, z) \tau^a] \text{Tr}[\mathcal{V}^\dagger(y, z') \mathcal{V}(y, z') \tau^b], \tag{B19}$$

where the two-argument M functions are found in Eqs. (B7) and (B16), respectively.

APPENDIX C: VACUUM CURRENT IN THE BPS MONOPOLE BACKGROUND

We compute the vacuum current (20) in the BPS monopole background in this Appendix. We assume $0 < v < 2\pi$ and work in the stringy gauge (13) dropping the index s in v^s given by Eq. (A15).

1. Singular part of the monopole current J_μ^s

This part of the current corresponds to the second term \mathcal{G}^s in Eq. (B5). At $x \rightarrow y$ this part of the propagator is singular. The regularization is presented in Appendix E. Equations (E2) and (A11) state:

$$J_\mu^{s\ ab} = i\varepsilon^{adb} \text{tr}(\tau^d j_\mu), \quad j_\mu = \frac{v^2}{12\pi^2} \langle v | f \sigma_\mu \Delta^\dagger f | v \rangle - \text{H.c.},$$

$$\Delta^\dagger(x) = i\partial_z + vx^\dagger. \quad (\text{C1})$$

The function $f(z, z', x)$ for the BPS monopole is known [25]:

$$f(x; z, z') = -\frac{e^{ivx_4(z-z')}}{2vs} \left(\sinh vs |z-z'| + \coth \frac{vs}{2} \sinh vsz \sinh vsz' - \tanh \frac{vs}{2} \cosh vsz \cosh vsz' \right). \quad (\text{C2})$$

Here we denoted by s the distance to the M -monopole center. It is helpful to calculate the action of the Green function on v . Since the monopole is a static configuration, we can take $x_4=0$, moreover f is a scalar function and we can move S_-^\dagger matrix to the left:

$$\begin{aligned} |v\rangle &\equiv S_- f |v\rangle|_{x_4=0} \\ &= \frac{\cosh(svz) \tanh(sv/2) - 2z \sinh(svz)}{4\sqrt{sv} \sinh(sv)} \mathbf{1}_2 \\ &\quad + \frac{\sinh(svz) \coth(sv/2) - 2z \cosh(svz)}{4\sqrt{sv} \sinh(sv)} \tau_3. \end{aligned}$$

We use the following identities

$$\begin{aligned} S_- \vec{n}_r \vec{\tau} S_-^\dagger &= \tau_3, \\ S_- \vec{n}_\theta \vec{\tau} S_-^\dagger &= -\cos(\phi) \tau_1 - \sin(\phi) \tau_2, \\ S_- \vec{n}_\phi \vec{\tau} S_-^\dagger &= \sin(\phi) \tau_1 - \cos(\phi) \tau_2 \end{aligned} \quad (\text{C3})$$

and arrive, after simple algebra, at

$$\begin{aligned} \{j_4, j_r, j_\theta, j_\phi\} &= \frac{v^3}{12\pi^2} \langle v | \{i, -\tau_3, \cos \phi \tau_1 + \sin \phi \tau_2, \\ &\quad -\sin \phi \tau_1 + \cos \phi \tau_2\} (\partial_z - v\tau_3 s) | v \rangle + \text{H.c.} \end{aligned}$$

Finally we obtain the singular part of the vacuum current:

$$\begin{aligned} J_r^s &= 0, \\ J_\phi^s &= -\frac{iv[s^2v^2\text{csch}^2(sv) + sv \coth(sv) - 2]}{24\pi^2s^2\sinh(sv)} \\ &\quad \times [T_1 \cos(\phi) + T_2 \sin(\phi)], \\ J_\theta^s &= -\frac{iv[s^2v^2\text{csch}^2(sv) + sv \coth(sv) - 2]}{24\pi^2s^2\sinh(sv)} \\ &\quad \times [T_1 \sin(\phi) - T_2 \cos(\phi)], \\ J_4^s &= -\frac{i[1 - s^3v^3 \coth(sv) \text{csch}^2(sv)]}{24\pi^2s^3} T_3, \end{aligned} \quad (\text{C4})$$

where $(T_c)^{ab} \equiv i\varepsilon^{acb}$.

2. Regular part of the monopole current J_μ^r

We are going to calculate the part of the current that corresponds to

$$\begin{aligned} (\mathcal{G}^r)^{ab}(x, y) &\equiv \sum_{n \neq 0} \frac{1}{8\pi^2(x-y_n)^2} \text{Tr}[\tau^a \langle v(x) | v(y_n) \rangle \\ &\quad \times \tau^b \langle v(y_n) | v(x) \rangle], \\ y_n &= \vec{y}, \quad y_{n4} = y_4 + n, \end{aligned} \quad (\text{C5})$$

namely

$$J_\mu^r = J_\mu^{r1} + J_\mu^{r2}, \quad J_\mu^{r1} = A_\mu \mathcal{G}^r + \mathcal{G}^r A_\mu, \quad J_\mu^{r2} = (\partial_\mu^x - \partial_\mu^y) \mathcal{G}^r.$$

At first consider J_μ^{r1} . We have to compute \mathcal{G}^r with equal arguments. Substituting (A15) into (C5) and calculating the trace one has

$$(\mathcal{G}^r)^{ab}(x, x) \equiv \sum_{n \neq 0} \int dz dz' \frac{\text{Tr}}{8\pi^2 n^2}, \quad (\text{C6})$$

where

$$\begin{aligned} \text{Tr} &= 2 \frac{s^2 v^2 e^{inv(z-z')}}{\sinh^2(sv)} \{ \cosh[2sv(z-z')] (\delta^{ab} - \delta^{a3} \delta^{b3}) \\ &\quad + \cosh[2sv(z+z')] \delta_3^a \delta_3^b + \sinh[2sv(z-z')] T_3^{ab} \}, \end{aligned}$$

with $T^c = i\varepsilon^{acb}$. To compute the sum in this expression we use the summation formula (note that $v < 2\pi$)

$$\sum_{n \neq 0} \frac{e^{izn}}{4\pi^2 n^2} = \frac{z^2}{8\pi^2} - \frac{|z|}{4\pi} + \frac{1}{12}, \quad -2\pi < z < 2\pi. \quad (\text{C7})$$

It remains now to calculate integrals over z and z' . The result is

$$\mathcal{G}^r(x,x) = \left[\frac{3 \coth(sv) - sv[3 \operatorname{csch}^2(sv) + 2]}{8\pi s} + \frac{[sv \coth(sv) - 1]^2}{8\pi^2 s^2} (\delta^{ab} - \delta_3^a \delta_3^b) + \left[\frac{sv \operatorname{csch}^2(sv) - \coth(sv)}{8\pi s} + \frac{1 - s^2 v^2 \operatorname{csch}^2(sv)}{16\pi^2 s^2} + \frac{1}{12} \right] \delta^{ab} \right]$$

Now we turn to the J_μ^r part of the current where we have to sum over n a derivative of the propagator. First of all we consider derivatives of the trace in (C5). One finds for $x=y$:

$$\begin{aligned} (\partial_\theta^x - \partial_\theta^y) \operatorname{Tr} &= \frac{2s^2 v^2 i}{\sinh(sv)^2} [\cosh(2svz) + \cosh(2svz')] \\ &\quad \times (T_1 \sin \phi - T_2 \cos \phi) e^{ivn(z-z')}, \\ (\partial_\phi^x - \partial_\phi^y) \operatorname{Tr} &= \left(\sin \theta \frac{4s^2 v^2 i}{\sinh(sv)^2} \cosh[sv(z+z')] \right. \\ &\quad \times \cosh[sv(z-z')] (T_1 \cos \phi + T_2 \sin \phi) \\ &\quad \left. - \frac{4s^2 v^2 i}{\sinh(sv)^2} \cosh[2sv(z-z')] \right. \\ &\quad \left. \times (1 - \cos \theta) T_3 \right) e^{ivn(z-z')}, \\ (\partial_4^x - \partial_4^y) \operatorname{Tr} &= -\frac{4s^2 v^3 i}{\sinh^2(sv)} (z-z') \sinh[2sv(z-z')] \\ &\quad \times T_3 e^{ivn(z-z')}, \end{aligned} \quad (\text{C8})$$

$$(\partial_s^x - \partial_s^y) \operatorname{Tr} = 0.$$

Here only terms even in $z-z'$ were left. The last two equations are especially clear as we can drop out the matrices S in Eq. (A15).

A derivative of the denominator of (C5) is equal to zero for $x=y$ except for the derivative with respect to x_4 , but in this case we have the expression of the form (C6) with $n^3/4$ instead of n^2 in the denominator. Now we can sum over n . We use the summation formula

$$\sum_{n \neq 0} \frac{e^{izn}}{i\pi^2 n^3} = \frac{z^3}{6\pi^2} - \frac{z|z|}{2\pi} + \frac{z}{3}, \quad -2\pi < z < 2\pi. \quad (\text{C9})$$

Next one has to integrate over z, z' . Combining all pieces we obtain:

$$J_r^r = 0,$$

$$J_\phi^r = i \frac{\cos(\phi) T_1 + \sin(\phi) T_2}{48 \sinh^3(sv) \pi^2 s^2} \wp_1,$$

$$J_\theta^r = i \frac{\sin(\phi) T_1 - \cos(\phi) T_2}{48 \sinh^3(sv) \pi^2 s^2} \wp_1, \quad (\text{C10})$$

$$J_4^r = \frac{i T_3}{24 \pi^2 s^3} \wp_2,$$

where we denote

$$\begin{aligned} \wp_1 &\equiv [s^2 v^3 + 6\pi s^2 v^2 + 3v + 3s(v + \pi) \sinh(2sv)v \\ &\quad - (s^2 v^3 + 3v + 6\pi) \cosh(2sv) + 6\pi], \\ \wp_2 &\equiv 8\pi^2 s^2 [-1 + sv \coth(sv)] - 12\pi s \coth(sv) \\ &\quad \times [-1 + sv \coth(sv)]^2 + (-3(1 + 4s^2 v^2) \\ &\quad + sv\{4(3 + s^2 v^2) \coth(sv) \\ &\quad + 3sv[-4 + sv \coth(sv)] \operatorname{csch}^2(sv)\}). \end{aligned}$$

We have used spherical coordinates. For example, a projection of \vec{J} onto the direction $\vec{n}_\theta = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta)$ is denoted by J_θ .

3. M part of the monopole current J_μ^m

Combining together Eqs. (B7) and (B19) we have for the M part of the periodic Green's function:

$$\mathcal{G}^m = -\frac{v}{16\pi} \int_{-1/2}^{1/2} dz dz' (2|z-z'| - 1 + 4zz') \times \operatorname{Tr}[v^\dagger(x,z)v(x,z)\tau^a] \operatorname{Tr}[v^\dagger(y,z')v(y,z')\tau^b] \quad (\text{C11})$$

Note that we can drop out S_- in (A15). In the stringy gauge one has

$$v^\dagger(x_\mu, z)v(x_\mu, z) = \frac{vs}{\sinh(vs)} \exp[-2vs\tau^2 z]. \quad (\text{C12})$$

It means that \mathcal{G}^m has only the 33 component. Taking the trace we get

$$\operatorname{Tr}[v^\dagger(x_\mu, z)v(x_\mu, z)\tau^3] = -2 \frac{vs}{\sinh(vs)} \sinh(2vsz). \quad (\text{C13})$$

Therefore the only nonzero component of \mathcal{G}^m is

$$\begin{aligned} \mathcal{G}_m^{33}(x,y) &= -\frac{v^3 r_x r_y}{4\pi} \int_{-1/2}^{1/2} dz dz' (2|z-z'| - 1 + 4zz') \\ &\quad \times \frac{\sinh(2vr_y z') \sinh(2vr_x z)}{\sinh(vr_x) \sinh(vr_y)} \end{aligned}$$

$(r_x = |\vec{x}|, r_y = |\vec{y}|)$. Performing the integrations we get

$$\mathcal{G}_m^{33}(x,y) = -\frac{1}{4\pi v} \left(\frac{1}{r_x r_y} + \frac{v r_x \coth v r_y - v r_y \coth v r_x}{r_y^2 - r_x^2} \right). \quad (\text{C14})$$

Note that (C14) is symmetric in its arguments. For that reason the contribution to the current coming from (ordinary) derivatives of \mathcal{G}^m is zero,

$$\partial_x \mathcal{G}^m(x,y) - \partial_y \mathcal{G}^m(x,y)|_{x=y} = 0,$$

and only the anticommutator $\{\mathcal{G}^m(x,x), A_\mu\}$ remains. Taking the limit $x \rightarrow y$, we get for \mathcal{G}^m

$$\mathcal{G}_m^{33}(x,x) = \frac{1}{8\pi s} \left[\coth(vs) - \frac{2}{vs} + \frac{vs}{\sinh^2(vs)} \right], \quad (\text{C15})$$

and for the contribution to the current $J_\mu^m = A_\mu \mathcal{G}^m + \mathcal{G}^m A_\mu$, we obtain in spherical coordinates

$$J_r^m = 0,$$

$$J_\phi^m = -\frac{i[\sin(\phi)T_2 + \cos(\phi)T_1][sv \coth(sv) + s^2 v^2 \text{csch}^2(sv) - 2]}{16\pi \sinh(sv)s^2},$$

$$J_\theta^m = -\frac{i[\sin(\phi)T_1 - \cos(\phi)T_2][sv \coth(sv) + s^2 v^2 \text{csch}^2(sv) - 2]}{16\pi \sinh(sv)s^2}, \quad (\text{C16})$$

$$J_4^m = 0.$$

Adding up (C4), (C10), and (C16) we obtain the full vacuum current in the BPS background [see Eq. (23)].

APPENDIX D: VACUUM CURRENT IN THE KVBLL CALORON BACKGROUND

There are no principal problems to make the calculation of the caloron Green's function and the ensuing vacuum currents exactly. One can consider this Appendix as an instruction how to perform the exact calculation. In fact, we have done it but unfortunately the exact result for the current is about 200 pages long and thus too large to be printed. However, in certain limits the expressions drastically simplify. In particular, assuming the case when the dyons inside the caloron are widely separated such that their cores do not overlap, it is relatively easy to find the KvBLL caloron current with the exponential precision (i.e., dropping out term of the order $e^{-r\vec{v}}, e^{-s\vec{v}}$). This will be sufficient to find the determinant of the KvBLL caloron for large r_{12} up to some constant.

With the exponential precision, the only nonzero components of the KvBLL caloron's gauge potential in fundamental representation are (see Sec. II)

$$A_4 \approx \frac{i\tau_3}{2} \left(4\pi\omega + \frac{1}{r} - \frac{1}{s} \right),$$

$$A_\varphi \approx -\frac{i\tau_3}{2} \left(\frac{1}{r} + \frac{1}{s} \right) \sqrt{\frac{(r_{12}-r+s)(r_{12}+r-s)}{(r_{12}+r+s)(r+s-r_{12})}}. \quad (\text{D1})$$

We are using the coordinates x_4, r, s, φ , where r, s are defined in (A24) and φ is defined by

$$\vec{x} = x_e (\cos \varphi \vec{e}_2 + \sin \varphi \vec{e}_1) + \left(\frac{r_{12}^2 + r^2 - s^2}{2r_{12}} - 2r_{12}\omega \right) \vec{e}_3,$$

$$x_e \equiv \frac{\sqrt{(r_{12}+r-s)(r_{12}+s-r)(r+s-r_{12})(r_{12}+r+s)}}{2r_{12}}. \quad (\text{D2})$$

One can easily check the consistency of this definition, i.e., that

$$s = |\vec{s}|, \quad r = |\vec{r}|, \quad \text{where} \quad \vec{s} = \vec{x} - 2\omega \vec{r}_{12},$$

$$\vec{r} = \vec{x} + 2\omega \vec{r}_{12}, \quad \vec{r}_{12} = r_{12} \vec{e}_3.$$

Since $A_r = 0$ and $A_s = 0$ we have to calculate only the J_4 and J_φ components.

We shall use the ADHM construction. The main steps of the calculation are similar to that for the monopole. Dropping out exponentially small terms in Eq. (A36) one has in the periodic gauge

$$v(x, z) \simeq \sqrt{\frac{r+s-r_{12}}{r+s+r_{12}}} \begin{pmatrix} -e^{2\pi i x_4 \omega \tau_3} \\ w(x, z) \end{pmatrix}, \quad (\text{D3})$$

$$w(x, z) \simeq \begin{cases} -i\pi\rho e^{2\pi(i s^\dagger z - \omega s)} \left[\frac{r_{12}+r-s}{r_{12}} \tau_3 + \frac{2x_\rho}{r+s-r_{12}} (\tau_1 \sin \varphi + \tau_2 \cos \varphi) \right], & -\omega < z < \omega \\ i\pi\rho e^{2\pi(ir^\dagger z - ir^\dagger/2 - r\bar{\omega})} \left[\frac{r_{12}+s-r}{r_{12}} (i1_2 \sin \pi x_4 + \tau_3 \cos \pi x_4) + \frac{2x_\rho}{r+s-r_{12}} [\sin(\pi x_4 - \varphi) \tau_1 - \cos(\pi x_4 - \varphi) \tau_2] \right], & \omega < z < 1 - \omega \end{cases} \quad (\text{D4})$$

where $s^\dagger \equiv s_\mu \sigma_\mu^\dagger$, $s \equiv |s^\dagger|$, $s_4 = x_4$. We shall use the following formulas to pass to the cylindrical coordinates x_3, x_ρ, φ :

$$\frac{\partial r}{\partial x_3} = \frac{r_{12}^2 + r^2 - s^2}{2r_{12}r}, \quad \frac{\partial r}{\partial x_\rho} = \frac{x_\rho}{r}, \quad \frac{\partial r}{\partial \varphi} = 0$$

$$\frac{\partial s}{\partial x_3} = -\frac{r_{12}^2 + s^2 - r^2}{2r_{12}s}, \quad \frac{\partial s}{\partial x_\rho} = \frac{x_\rho}{s}, \quad \frac{\partial s}{\partial \varphi} = 0. \quad (\text{D5})$$

1. Singular part of the caloron current J_μ^s

Let us calculate the singular part of the vacuum current with exponential precision. It is related to the zero Matsubara frequency. Similar to the monopole case, we could use Eq. (E2), where the Green's function (A4) for the case of KvBLL caloron was found in Ref. [5]. However it is more convenient to use Eq. (E10) because then we have only to take derivatives of the simple expression (D3) and no integrations arise. Equation (E2) would have been more suitable for the exact calculation.

It is straightforward to calculate the quantity Γ from Eq. (E9). It is sufficient to calculate the second time derivative:

$$\Gamma \simeq \langle \partial_4 \partial_4 v | v \rangle - A_4^2. \quad (\text{D6})$$

Bearing in mind that Γ is a vector under gauge transformations, we can perform calculations in any gauge. Up to the exponentially small terms we have

$$\Gamma^{ab} = -\frac{(r+s)[(r-s)^2 + r_{12}(r+s)] \delta^{ab}}{4r^2 s^2 (r_{12} + r + s)}. \quad (\text{D7})$$

One can observe from Eq. (D1) that all terms with derivatives in the right-hand side of Eq. (E10) are zero. Writing the Laplace operator in the cylindrical coordinates we find

$$j_4^s \simeq \frac{1}{48\pi^2} \left[\left\langle \partial_4 \left(\partial_4^2 + \partial_3^2 + \frac{1}{x_\rho} \partial_\rho x_\rho \partial_\rho + \frac{1}{x_\rho^2} \partial_\varphi^2 \right) v \middle| v \right\rangle - \text{H.c.} \right]$$

$$+ \frac{1}{24\pi^2} (A_4^3 + A_\varphi A_4 A_\varphi + 6A_4 \Gamma),$$

$$j_\varphi^s \simeq \frac{1}{48\pi^2} \left[\left\langle \frac{\partial_\varphi}{x_\rho} \left(\partial_4^2 + \partial_3^2 + \frac{1}{x_\rho} \partial_\rho x_\rho \partial_\rho + \frac{1}{x_\rho^2} \partial_\varphi^2 \right) v \middle| v \right\rangle - \text{H.c.} \right] + \frac{1}{24\pi^2} (A_\varphi^3 + A_4 A_\varphi A_4 + 6A_\varphi \Gamma).$$

Taking the derivatives we obtain simple expressions:

$$j_4^s = \frac{i\tau_3}{48\pi^2} \left(\frac{1}{r^3} - \frac{1}{s^3} \right), \quad (\text{D8})$$

$$j_\varphi^s = -\left(\frac{1}{r} + \frac{1}{s} \right) \frac{i\tau_3 x_\rho r_{12}}{8\pi^2 r s (r_{12} + r + s)^2}. \quad (\text{D9})$$

2. Regular part of the caloron current J_μ^r

Next we calculate the temperature-dependent part of the KvBLL caloron vacuum current. As in the monopole case (Appendix C 2) we divide the current into two parts,

$$J_\mu^r = J_\mu^{r2} + J_\mu^{r1}, \quad (\text{D10})$$

where

$$J_\mu^{r2} = \sum_{n \neq 0} \frac{1}{8\pi^2} (\partial_\mu^x - \partial_\mu^y) \frac{\text{Tr}[\tau^a v^\dagger(x) v(y) \tau^b v^\dagger(y) v(x)]}{(x-y)^2},$$

$$J_\mu^{r1} = \sum_{n \neq 0} \frac{1}{8\pi^2} \left\{ A_\mu, \frac{\text{Tr}[\tau^a v^\dagger(x) v(y) \tau^b v^\dagger(y) v(x)]}{n^2} \right\}$$

and $y_4 = x_4 + n$. The quaternion function $v(x, z)$ has been constructed in Appendix A 3 [actually called $v^{\text{per}}(x, z)$ there]. It is important that $v(x, z)$ has the remarkable periodicity property (A41).

In evaluating the above currents the tactic is to factor the matrix part out of the integrals over z . We use the following notation for the integrals over z :

$$I_+^n \equiv \int_{-\omega}^{\omega} e^{2\pi i n z} \cosh(4\pi s z) dz,$$

$$\bar{I}_+^n \equiv \int_{-\bar{\omega}}^{\bar{\omega}} e^{2\pi i n(z+1/2)} \cosh(4\pi r z) dz,$$

$$I_-^n \equiv \int_{-\omega}^{\omega} e^{2\pi i n z} \sinh(4\pi s z) dz,$$

$$\bar{I}_-^n \equiv \int_{-\bar{\omega}}^{\bar{\omega}} e^{2\pi i n(z+1/2)} \sinh(4\pi r z) dz.$$

We obtain the following relations for the matrix structures:

$$\hat{\psi}^2 w^\dagger(x_4) w(x_4+n) = (I_+^n \beta + I_-^n \beta_s) + (\bar{I}_+^n \bar{\beta} + \bar{I}_-^n \bar{\beta}_r),$$

$$\hat{\psi}^2 w^\dagger(x_4+n) w(x_4) = (I_+^n \beta - I_-^n \beta_s) + (\bar{I}_+^n \bar{\beta} - \bar{I}_-^n \bar{\beta}_r),$$

$$\begin{aligned} \hat{\psi}^2 w^\dagger(x_4) \bar{\partial}_4 w(x_4+n) \\ = (I_+^n \beta_0 + I_-^n \beta_s^0 + i \partial_s I_-^n \beta + i \partial_s I_+^n \beta_s) + \dots, \end{aligned}$$

$$\begin{aligned} \hat{\psi}^2 w^\dagger(x_4+n) \bar{\partial}_4 w(x_4) \\ = (I_+^n \beta_0 - I_-^n \beta_s^0 - i \partial_s I_-^n \beta + i \partial_s I_+^n \beta_s) + \dots, \end{aligned}$$

where ... means the same expression but with bar over each quantity and r instead of s . The notation $\bar{\partial}_4$ means derivative from the right minus derivative from the left. The definition and the evaluation of the matrix structures with the exponential precision is

$$\beta_s^0 \equiv b_{11}^\dagger b_{12}^\dagger \hat{\delta}_4 b_{12} b_{11} \approx o(e^{4\pi s \omega} e^{8\pi r \bar{\omega}}),$$

$$\begin{aligned} \bar{\beta}_r^0 &\equiv b_{21}^\dagger b_{22}^\dagger \hat{r} \bar{\delta}_4 b_{22} b_{21} \\ &\approx \frac{i\pi^2 r_{12}}{2} (\vartheta+1)(s_3-1) e^{8\pi s \omega} e^{4\pi r \bar{\omega}}, \end{aligned}$$

$$\beta_0 \equiv b_{11}^\dagger b_{12}^\dagger \bar{\delta}_4 b_{12} b_{11} \approx o(e^{4\pi s \omega} e^{8\pi r \bar{\omega}}),$$

$$\begin{aligned} \bar{\beta}_0 &\equiv b_{21}^\dagger b_{22}^\dagger \bar{\delta}_4 b_{22} b_{21} \\ &\approx \frac{\pi^2 r_{12}}{2i} (\vartheta+1)(s_3-1) \hat{\omega} e^{8\pi s \omega} e^{4\pi r \bar{\omega}}, \end{aligned}$$

$$\begin{aligned} \beta_s &\equiv b_{11}^\dagger b_{12}^\dagger \hat{s} b_{12} b_{11} \\ &\approx \frac{\pi r_{12}}{4} (\vartheta+1)(r_3+1) \hat{\omega} e^{4\pi s \omega} e^{8\pi r \bar{\omega}}, \end{aligned}$$

$$\begin{aligned} \bar{\beta}_r &\equiv b_{21}^\dagger b_{22}^\dagger \hat{r} b_{22} b_{21} \\ &\approx \frac{\pi r_{12}}{4} (\vartheta+1)(s_3-1) \hat{\omega} e^{8\pi s \omega} e^{4\pi r \bar{\omega}}, \end{aligned}$$

$$\begin{aligned} \beta &\equiv b_{11}^\dagger b_{12}^\dagger b_{12} b_{11} \\ &\approx \frac{r_{12} \pi}{4} (\vartheta+1)(r_3+1) e^{4\pi s \omega} e^{8\pi r \bar{\omega}}, \end{aligned}$$

$$\begin{aligned} \bar{\beta} &\equiv b_{21}^\dagger b_{22}^\dagger b_{22} b_{21} \\ &\approx \frac{r_{12} \pi}{4} (\vartheta+1)(1-s_3) e^{8\pi s \omega} e^{4\pi r \bar{\omega}}, \end{aligned}$$

where

$$\begin{aligned} \hat{\psi} &\approx \frac{1}{4} (\vartheta+1) e^{4\pi s \omega} e^{4\pi r \bar{\omega}}, \quad \vartheta \equiv \frac{\vec{r}\vec{s}}{sr} = \frac{r^2+s^2-r_{12}^2}{2sr}, \\ r r_3 &= s s_3 = \frac{r_{12}^2+r^2-s^2}{2r_{12}}. \end{aligned}$$

Substituting this into in the currents $J_\mu^{r_1, r_2}$ we obtain certain sums, which are of the form

$$\begin{aligned} \sum_{n \neq 0} I_+^n I_+^n / (4\pi^2 n^2), \quad \sum_{n \neq 0} \cos(2\pi n \omega) g I_+^n / (4\pi^2 n^2), \\ \sum_{n \neq 0} \sin(4\pi n \omega) / (\pi^2 n^3). \end{aligned}$$

All such sums can be calculated using the summation formulas

$$\begin{aligned} \sum_{n \neq 0} \frac{e^{2\pi i n(z-1/2)}}{4\pi^2 n^2} = \frac{z^2}{2} - \frac{1}{24} \equiv c_2(z), \quad -\frac{1}{2} \leq z \leq \frac{1}{2}, \\ \sum_{n \neq 0} \frac{e^{2\pi i n(z-1/2)}}{\pi^2 n^3} = 8\pi i \left(\frac{z^3}{6} - \frac{z}{24} \right) \equiv c_3(z), \\ -\frac{1}{2} \leq z \leq \frac{1}{2}. \end{aligned}$$

For example,

$$\begin{aligned} \sum_{n \neq 0} I_+^n I_+^n / (4\pi^2 n^2) = \int_{-\omega}^{\omega} \int_{-\omega}^{\omega} c_2(z+z'-1/2) \cosh(4\pi s z) \\ \times \cosh(4\pi s z') dz dz' \end{aligned}$$

and so on. With some help from MATHEMATICA we come to the final result

$$\begin{aligned} J_4^r = & \left[\frac{1}{4\pi^2 r^3} - \frac{1}{\pi^2 r^2 s} + \frac{1}{\pi^2 r s^2} - \frac{1}{4\pi^2 s^3} - \frac{1}{\pi r^2} + \frac{2}{\pi r s} \right. \\ & - \frac{1}{\pi s^2} + \frac{2}{3r} - \frac{2}{3s} + \left(\frac{4}{\pi r^2} - \frac{8}{\pi r s} + \frac{4}{\pi s^2} - \frac{8}{r} + \frac{8}{s} \right. \\ & \left. \left. + \frac{8\pi}{3} \right) \omega + \left(\frac{16}{r} - \frac{16}{s} - 16\pi \right) \omega^2 + \frac{64\pi \omega^3}{3} \right] \frac{i T_3}{2}, \quad (D11) \end{aligned}$$

$$J_\varphi^r = \left(\frac{1}{r} + \frac{1}{s} \right) \frac{i T_3 x_\varrho r_{12}}{4\pi^2 r s (r_{12} + r + s)^2}. \quad (D12)$$

3. M part of the caloron current J_μ^m

This part of the current is especially simple: with exponential precision it is zero. The main steps are the same as in the case of a single monopole. The starting formula is our Eq. (B19). First of all we note that only the lower components of v are left and only the $a=3$ component is nonzero:

$$\text{Tr}[\mathcal{V}^+(x,z)\mathcal{V}(x,z)\tau^a] = \frac{1}{\Phi(x)} \text{Tr}[w^+(x,z)w(x,z)\tau^a] \propto \delta^{a3}.$$

Inspecting the definition of the M part of the propagator (B19) we observe that

$$G^{m\ ab}(x,y) \propto \delta^{a3} \delta^{b3}, \quad G^{m\ ab}(x,y) = G^{m\ ab}(y,x). \quad (\text{D13})$$

The second equation means that the terms with derivatives in the expression for the current (20) cancel each other. It follows from the first one that the product of G^m and $A_\mu^{ab} \propto \varepsilon^{3ab}$ is equal to zero, too. Therefore we conclude that

$$J_\mu^m \approx 0. \quad (\text{D14})$$

APPENDIX E: REGULARIZATION OF THE CURRENT

Here we consider in more detail J_μ^s , the contribution to the current from the singular (as $x \rightarrow y$) part of the propagator $\mathcal{G}^s(x,y)$ defined by Eq. (B5). This part is obviously temperature-independent, so the zero-temperature results are applicable. We regularize the current by setting $x-y = \epsilon$ and inserting a parallel transporter to support gauge invariance (see, e.g., Ref. [20]):

$$J_\mu^s = J_\mu^{s1} + J_\mu^{s2},$$

$$J_\mu^{s1} \equiv [A_\mu(z - \epsilon/2)G^s(z - \epsilon/2, z + \epsilon/2) + G^s(z - \epsilon/2, z + \epsilon/2)A_\mu(z + \epsilon/2)] \text{P exp} \left(- \int_x^y A_\mu dz_\mu \right), \quad (\text{E1})$$

$$J_\mu^{s2} \equiv [(\partial_\mu^x - \partial_\mu^y)G^s(x,y)] \text{P exp} \left(- \int_x^y A_\mu dz_\mu \right),$$

where $x = z - \epsilon/2$, $y = z + \epsilon/2$ and we imply averaging over all directions of ϵ in the 4D space. This regularization method was proved to be equivalent to the ζ -function regularization approach [35].

For a background field written in terms of the ADHM construction, a useful expression for the vacuum current was derived in Refs. [20,35]. In the $SU(2)$ case it acquires the form:

$$J_\mu^{s\ ab} = i \varepsilon^{adb} \text{tr}(\tau^d j_\mu),$$

$$j_\mu = \frac{1}{12\pi^2} \langle v | \mathcal{B} f (\sigma_\mu \Delta^\dagger \mathcal{B} - \mathcal{B}^\dagger \Delta \sigma_\mu^\dagger) f \mathcal{B}^\dagger | v \rangle$$

(see Appendix A for notations of the ADHM construction elements).

We would like to derive another expression for this part of the current—in terms of derivatives. In some cases it is more useful. We start from writing our result:

$$j_\mu = \frac{1}{48\pi^2} [(D_\mu D^2 \langle v |) | v \rangle - \text{H.c.}]. \quad (\text{E2})$$

Let us prove it. First of all we consider the action of one derivative

$$D_\mu \langle v(x) | = \partial_\mu \langle v | - \partial_\mu \langle v | v \rangle \langle v | = \partial_\mu \langle v | (1 - |v\rangle \langle v|) \\ = \partial_\mu \langle v | \Delta f \Delta^\dagger = - \langle v | \partial_\mu \Delta f \Delta^\dagger \quad (\text{E3})$$

$$= - \langle v | \mathcal{B} \sigma_\mu f \Delta^\dagger. \quad (\text{E4})$$

At the end of the first line we have used Eq. (A5). The first equation in the second line comes from differentiating the ADHM equation

$$0 = \partial_\mu (\langle v | \Delta) = \partial_\mu \langle v | \Delta + \langle v | \partial_\mu \Delta.$$

The last equation follows from the definition (B10). Therefore we obtain

$$D_\mu \langle v(x) | v(y) \rangle = - \langle v(x) | \mathcal{B} \sigma_\mu f_x \Delta_x^\dagger | v(y) \rangle \\ = - \langle v(x) | \mathcal{B} \sigma_\mu f_x (x-y)^\dagger \mathcal{B}^\dagger | v(y) \rangle,$$

where in the last line we have used the ADHM equation (A3). We next consider two derivatives. It is important here that f is proportional to the unity 2×2 matrix. We have

$$D_x^2 \langle v(x) | v(y) \rangle = - D_\mu^x [\langle v(x) | \mathcal{B} \sigma_\mu f_x (x-y)^\dagger \mathcal{B}^\dagger | v(y) \rangle] \\ = \langle v(x) | \mathcal{B} \sigma_\mu f_x \Delta_x^\dagger \mathcal{B} \sigma_\mu f_x (x-y)^\dagger \mathcal{B}^\dagger | v(y) \rangle \\ - \langle v(x) | \mathcal{B} \sigma_\mu \partial_\mu f_x (x-y)^\dagger \mathcal{B}^\dagger | v(y) \rangle \\ - \langle v(x) | \mathcal{B} \sigma_\mu f_x \sigma_\mu^\dagger \mathcal{B}^\dagger | v(y) \rangle \\ = -4 \langle v(x) | \mathcal{B} f_x \mathcal{B}^\dagger | v(y) \rangle.$$

We have used here

$$\sigma_\mu \partial_\mu f_x = - f_x \partial_\mu (\sigma_\mu \Delta^\dagger \Delta) f_x = - f_x \sigma_\mu (\sigma_\mu^\dagger \mathcal{B}^\dagger \Delta + \Delta^\dagger \mathcal{B} \sigma_\mu) f_x \\ = -2 f_x \mathcal{B}^\dagger \Delta f_x = f_x \sigma_\mu \Delta^\dagger \mathcal{B} \sigma_\mu f_x.$$

We have also used that the derivative of the inverse operator is $\partial(O^{-1}) = -O^{-1}(\partial O)O^{-1}$, as well as the relations

$$\sigma_\mu \sigma_\mu^\dagger = 4, \quad \sigma_\mu c \sigma_\mu = -2c^\dagger, \quad (\text{E5})$$

where c is an arbitrary quaternion.

Finally, let us consider three derivatives:

$$\begin{aligned}
 D_\mu^x D_x^2 \langle v(x) | v(y) \rangle &= -4D_\mu^x [\langle v(x) | \mathcal{B} f_x \mathcal{B}^\dagger | v(y) \rangle] \\
 &= 4 \langle v(x) | \mathcal{B} f_x \sigma_\mu \Delta_x^\dagger \mathcal{B} f_x \mathcal{B}^\dagger | v(y) \rangle \\
 &\quad + 4 \langle v(x) | \mathcal{B} \partial_\mu f_x \mathcal{B}^\dagger | v(y) \rangle.
 \end{aligned}$$

Notice that the last term is hermitian at $x=y$. Thus we have proven that the current written in form of Eq. (E2) is equivalent to that of Eq. (E2):

$$\begin{aligned}
 &\frac{1}{48\pi^2} [(D_\mu D^2 \langle v | v \rangle) - \text{H.c.}] \\
 &= \frac{1}{12\pi^2} \langle v | \mathcal{B} f (\sigma_\mu \Delta^\dagger \mathcal{B} - \mathcal{B}^\dagger \Delta \sigma_\mu^\dagger) f \mathcal{B}^\dagger | v \rangle.
 \end{aligned}$$

In fact it is more useful to rewrite everything in terms of ordinary rather than covariant derivatives:

$$\begin{aligned}
 (D_\mu D^2 \langle v | v \rangle) &= \langle \partial_\mu \partial^2 v | v \rangle + A_\nu A_\mu A_\nu - \partial_\nu A_\nu A_\mu - A_\nu \partial_\nu A_\mu \\
 &\quad - \partial_\mu A_\nu A_\nu + \partial_\mu \partial_\nu A_\nu + 6A_\mu \Gamma, \quad (\text{E6})
 \end{aligned}$$

where A_μ is in the fundamental representation and

$$\frac{1}{2} (D_\mu D_\nu + D_\nu D_\mu \langle v | v \rangle) = \delta_{\mu\nu} \Gamma. \quad (\text{E7})$$

We have to prove that the left-hand side of Eq. (E7) is a Lorentz scalar as is the right-hand side. Note that Eq. (E5) is

proportional to $x-y$. The only way to obtain a nonzero result at $x-y \rightarrow 0$ is to differentiate this factor:

$$\begin{aligned}
 &\frac{1}{2} (D_\mu D_\nu + D_\nu D_\mu \langle v | v \rangle) \\
 &= -\frac{1}{2} \langle v | \mathcal{B} (\sigma_\mu \sigma_\nu^\dagger + \sigma_\nu \sigma_\mu^\dagger) f \mathcal{B}^\dagger | v \rangle \\
 &= -\delta_{\mu\nu} \langle v | \mathcal{B} f \mathcal{B}^\dagger | v \rangle. \quad (\text{E8})
 \end{aligned}$$

It follows from Eq. (E8) that Γ is Hermitian. We can write Γ as follows:

$$\begin{aligned}
 \Gamma \delta_{\mu\nu} &= \langle \partial_\mu \partial_\nu v | v \rangle + \frac{1}{2} (\partial_\mu A_\nu + \partial_\nu A_\mu) \\
 &\quad - \frac{1}{2} (A_\mu A_\nu + A_\nu A_\mu). \quad (\text{E9})
 \end{aligned}$$

Finally, the regularized singular part of the current can be written as

$$\begin{aligned}
 j_\mu &= \frac{1}{48\pi^2} (\langle \partial_\mu \partial^2 v | v \rangle - \text{H.c.}) + \frac{1}{24\pi^2} \\
 &\quad \times (A_\nu A_\mu A_\nu + \partial_\mu \partial_\nu A_\nu + 3A_\mu \Gamma + 3\Gamma A_\mu). \quad (\text{E10})
 \end{aligned}$$

Equations (E9) and (E10) are used for the calculation of the singular part of the vacuum current in Appendix D 1.

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- be always regularized by putting them in a finite-size box, the theorem on the relation between spin-1 and spin-0 determinants (Sec. I) is generally violated. This is one of the reasons we consider well-behaved calorons rather than individual dyons, although they are more “elementary.”
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