Fermion masses and coupling unification in E_6: Life in the desert

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We present an E_6 grand unified model with a realistic pattern of fermion masses. All standard model fermions are unified in three fundamental 27-plets (i.e., supersymmetry is not invoked), which involve in addition right-handed neutrinos and three families of vectorlike heavy quarks and leptons. The lightest of those can lie in the low-TeV range, being accessible to future collider experiments. As a result of the high symmetry, the masses and mixings of all fermions are closely related. The new heavy fermions play a crucial role for the quark and lepton mass matrices and the bilarge neutrino oscillations. In all channels generation mixing and $\mathcal{CP}$ violation arise from a single antisymmetric matrix. The E_6 breaking proceeds via an intermediate-energy region with $SU(3)_L\times SU(3)_R\times SU(3)_C$ gauge symmetry and a discrete left-right symmetry. This breaking pattern leads in a straightforward way to the unification of the three gauge coupling constants at high scales, providing for a long proton lifetime. The model also provides for the unification of the top, bottom, and tau Yukawa couplings and for new interesting relations in flavor and generation space.

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I. INTRODUCTION

The exceptional group E_6 [1,2] is the preferred group for grand unification. All standard model (SM) fermions are in the lowest **27** representation. Its maximal subgroup $SU(3)\times SU(3)\times SU(3)$ can be viewed as an extension of the Weinberg-Salam group $SU(2)_L\times U(1)_Y\times SU(3)_C \to SU(3)_L\times SU(3)_R\times SU(3)_C\equiv G_{333}$. The fermions can be described by singlet and triplet representations of the $SU(3)$ groups only. Using for all fermion fields left-handed two component Weyl spinor fields, the quantum number assignments are (for each generation) [3]

$$\text{quarks:}\quad Q_L(x)=(3,1,\bar{3}),$$

$$\text{leptons:}\quad L(x)=(\bar{3},3,1),$$

$$\text{antiquarks:}\quad Q_R(x)=(1,\bar{3},3). \tag{1.1}$$

The 78 generators of E_6 consist of the three $SU(3)$ adjoint octet generators and the generators $F(3,3,3)$ and $\overline{F}(\bar{3},\bar{3},\bar{3})$ of coset E_6/G_{333}.

The beautiful cyclic symmetry of E_6 is apparent from Eqs. (1.1) and from the fact that F takes a quark field into a lepton field, a lepton field into an antiquark field, and an antiquark field into a quark field. An additional argument favoring E_6 is its appearance through compactification of the ten-dimensional $E_8\times E_8$ heterotic superstring theory on a Calabi-Yau manifold. The compactification process can lead either to four-dimensional E_6 gauge symmetry (which is anomaly free and left-right symmetric) or to some of E_6's maximal subgroups [4]. The phenomenology of the E_6 grand unified theory (GUT) attracted attention earlier [1,2,5], and its active study has been continued until recently [6]. The phenomenology and properties of G_{333} triunification models are also interesting [7].

According to Eqs. (1.1) one has besides the SM fermions additional quark and antiquark fields with the same charges as the corresponding down quarks, two $SU(2)_L$ doublet leptons (containing additional "active" neutrinos), and two SM singlets—"right-handed" neutrinos for each generation.

$SO(10)$ and E_6 grand unified theories in old times usually predicted small neutrino mixings since in straightforward applications the large symmetry obtained from these groups connects the neutrino mixings with the small mixings observed in the quark sector. After the observation of large mixings in neutrino oscillations one had to return to the smaller $SU(5)$ group (the minimal version of it does not involve right-handed neutrinos) or needed several Higgs bosons of the same representation or special composite operators and fine-tuning procedures. In this paper we will show, however, that the consequent use of the fermion and scalar particle interactions and spectra of E_6 allows us to construct a realistic GUT model.

We consider at first the Yukawa sector of E_6 with its symmetric and antisymmetric matrices in flavor and generation space. After defining the model, we can calculate from it the mass spectrum of ordinary and new fermions and their mixings in terms of a few parameters only. An interesting feature is that the mass matrices of quarks and leptons are strongly influenced by the flavor mixing of the SM particles with heavy fermions as was suggested by Bjorken, Pakvasa, and Tuan [8]. Earlier suggestions for the mixing of the SM particles with new heavy fermions can be found in [9]. Our work is done in the spirit of Ref. [8]. As in this reference, our scenario favors a relatively light mass scale for some of the new particles [10-plets of $SO(10)$]. The lightest can lie in the low-TeV region or even below. A major difference to [8] is the full use of the discrete left-right symmetry of E_6, valid at the intermediate symmetry G_{333}. It is broken solely by the Majorana property of very heavy neutral leptons (the right-handed heavy neutrinos). The use of an antisymmetric Higgs

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representation proposed many years ago [5] plays a decisive role. The corresponding antisymmetric matrix determines the generation mixing and the $\mathcal{C}P$ violation in all heavy and light channels (in the basis in which the up-quark mass matrix is diagonal). The inclusion of all neutral leptons of E_6 allows us to connect the mass matrix of the heavy neutrinos with the diagonal up-quark mass matrix and the antisymmetric generation matrix. It leads to bimaximal mixings of the light neutrinos which then change to a bilarge mixing pattern at the weak scale by renormalization effects. All mass ratios and mixing angles of light and heavy fermions are simply related to each other.

We then study the gauge coupling and top-bottom-tau unification in E_6. It is achieved by an unbroken G_{333} subgroup as an intermediate symmetry. The discrete left-right $\mathcal{D}_{LR}$ symmetry, which is unbroken at these intermediate energies, plays also an important role. The breaking scale M_I of the intermediate symmetry is not a free parameter, but uniquely fixed by the standard model couplings: $M_{\mathrm{I}}\simeq 1.5\times 10^{13}$ GeV. M_I also determines the scale of light and heavy neutrinos in agreement with experiment. The unification of the couplings occurs above 10^{16} GeV in our specific model at 2×10^{17} GeV and thus suppresses proton decay. The renormalization of mass ratios, various Yukawa matrices, and the scaling of the neutrino mass matrix are studied in detail.

Our model is nonsupersymmetric as the one in [8]. The hierarchy problem persists but it is hoped that its eventual solution would not change the basic features of our approach.

II. PARTICLE ASSIGNMENTS IN E_6 AND THE YUKAWA SECTOR

Let us first consider the lowest particle generation

$$(Q_L)_i^a=\begin{pmatrix} u^a\\ d^a\\ D^a\end{pmatrix},\quad L_k^i=\begin{pmatrix} L_1^1 & E^- & e^-\\ E^+ & L_2^2 & \nu\\ e^+ & \hat{\nu} & L_3^3\end{pmatrix},$$
$$(Q_R)_a^k=(\hat{u}_a,\ \hat{d}_a,\ \hat{D}_a), \tag{2.1}$$

where $i,k,a=1,2,3$; a is a color index. In this description $SU(3)_L$ acts vertically and $SU(3)_R$ horizontally. The charges are obtained from the operator

$$Q=\left(I_3+\frac{1}{2}Y\right)_L+\left(I_3+\frac{1}{2}Y\right)_R, \tag{2.2}$$

with I_3, Y defined as usual. Before symmetry breaking equivalent forms of Eqs. (2.1) can be obtained by applying left and right $\mathcal{U}$-spin rotations.

The charge conjugation operator interchanges left with right-handed indices:

$$\mathcal{C}(Q_L)_i^a\mathcal{C}^{-1}=(Q_R)_a^i,\quad \mathcal{C}L_k^i\ \mathcal{C}^{-1}=L_i^k,$$
$$\mathcal{C}(Q_R)_a^i\mathcal{C}^{-1}=(Q_L)_i^a. \tag{2.3}$$

It leaves the commutation relations for the E_6 generators unchanged and is often called $\mathcal{D}_{LR}$ parity. The new lepton fields L_1^1, L_2^2, L_3^3 are identical with their own antiparticle fields. Nevertheless, if two of these fields—say, L_1^1 and L_2^2—are connected by a single mass term, a four-component Dirac field can be formed. The two fields then behave like a (vector like) particle-antiparticle pair. The parity and $\mathcal{C}P$ operations change the left handed two component fields into right handed ones:

$$\mathcal{P}(Q_L)_i^a(t,x)\mathcal{P}^{-1}=\sigma_2(Q_R)_a^{i*}(t,-x),$$
$$\mathcal{P}L_k^i(t,x)\mathcal{P}^{-1}=\sigma_2L_i^{k*}(t,-x),$$
$$\mathcal{P}(Q_R)_a^i(t,x)\mathcal{P}^{-1}=\sigma_2(Q_L)_i^{a*}(t,-x),$$
$$\mathcal{C}P(Q_L)_i^a(t,x)\mathcal{C}P^{-1}=\sigma_2(Q_L)_i^{a*}(t,-x),$$
$$\mathcal{C}PL_k^i(t,x)\mathcal{C}P^{-1}=\sigma_2L_k^{i*}(t,-x),$$
$$\mathcal{C}P(Q_R)_a^i(t,x)\mathcal{C}P^{-1}=\sigma_2(Q_R)_a^{i*}(t,-x). \tag{2.4}$$

By including the generation quantum number α $(=1,2,3)$ all basic fermions are now classified by the left-handed Weyl fields Ψ_r^α with the E_6 flavor index r running from 1 to 27.

The product of two **27**'s of E_6 decomposes into a symmetric $\overline{\mathbf{27}}$, an antisymmetric $\overline{\mathbf{351}}_A$, and a symmetric $\overline{\mathbf{351}}_S$ representation:

$$\mathbf{27}\times\mathbf{27}=\overline{\mathbf{27}}+\overline{\mathbf{351}}_A+\overline{\mathbf{351}}_S. \tag{2.5}$$

Consequently, the Yukawa interactions in the E_6 Lagrangian contain in general the three Higgs fields

$$H=H(27),\quad H_A=H(351_A),\quad H_S=H(351_S). \tag{2.6}$$

Each of the three Higgs fields couples to the fermions together with a 3×3 matrix G acting on the generation space:

$$G_{\alpha\beta}=[G(27)]_{\alpha\beta},\quad A_{\alpha\beta}=[G(351_A)]_{\alpha\beta},$$
$$S_{\alpha\beta}=[G(351_S)]_{\alpha\beta}. \tag{2.7}$$

The E_6 invariant Yukawa interaction reads

$$\mathcal{L}_Y=((\Psi_r^\alpha)^T\mathrm{i}\sigma_2\Psi_s^\beta)[G_{\alpha\beta}H_{rs}+A_{\alpha\beta}(H_A)_{rs}+S_{\alpha\beta}(H_S)_{rs}]$$
$$+\mathrm{H.c.} \tag{2.8}$$

G and S are symmetric matrices in generation space, while A is an antisymmetric matrix. $\mathcal{L}_Y$ is invariant with respect to $\mathcal{C}$, the right$\leftrightarrow$left operation. In case of real vacuum expectation values (VEVs) of the Higgs fields, the part of $\mathcal{L}_Y$ obtained from the real part of these matrices is formally even under the $\mathcal{C}P$ and $\mathcal{P}$ operations, while the term arising from their imaginary parts is formally odd under $\mathcal{C}P$ and $\mathcal{P}$.

The decomposition of the Higgs fields with respect to the G_{333} subgroup reads

$$H=(\bar{3},3,1)+(1,\bar{3},3)+(3,1,\bar{3}), \tag{2.9}$$

$$H_A=(\overline{3},3,1)+(1,\overline{3},3)+(3,1,\overline{3})+(\overline{3},\overline{6},1)+(1,\overline{3},\overline{6})+(\overline{6},1,\overline{3})+(6,3,1)+(1,6,3)+(3,1,6)+(3,8,\overline{3})+(\overline{3},3,8)+(8,\overline{3},3), \tag{2.10}$$

$$H_S=(\overline{3},3,1)+(1,\overline{3},3)+(3,1,\overline{3})+(6,\overline{6},1)+(1,6,\overline{6})+(\overline{6},1,6)+(3,8,\overline{3})+(\overline{3},3,8)+(8,\overline{3},3). \tag{2.11}$$

The color singlet parts whose neutral members can develop VEVs are

$$H\rightarrow(\overline{3},3,1),$$
$$H_A\rightarrow(\overline{3},3,1)+(\overline{3},\overline{6},1)+(6,3,1),$$
$$H_S\rightarrow(\overline{3},3,1)+(6,\overline{6},1). \tag{2.12}$$

We note that the parts containing a sextet or antisextet representation can only couple to leptons.

III. MODEL

The vacuum expectation values of the three Higgs fields determine the particle spectrum. To be in accord with the SM the masses of the new particles of E_6 have to get heavy (at least of order TeV). Thus, Higgs components which are $SU(2)_L$ singlets can have large VEVs. The members of $SU(2)_L$ doublets, on the other hand, should be of the order of the weak scale, while the VEVs of $SU(2)_L$ triplets are expected to vanish.

In order to define our model to be predictive and to have very few unknown parameters, we need some specific assumptions concerning the three Higgs fields, about the generation matrices G, A, S and the symmetry breaking pattern. We do not consider Higgs field components which carry color. They are supposed to acquire masses of the order of the GUT scale from appropriate Higgs potentials.

We allow VEVs for all color singlet and neutral components of H:

$$\langle H_1^1\rangle=e_1^1,\quad \langle H_k^i\rangle=e_k^i,\quad \text{for } i,k=2,3. \tag{3.1}$$

However, by a biunitary left and right $\mathcal{U}$-spin transformation in flavor space [i.e., on the $SU(3)_L$ and $SU(3)_R$ indices 2 and 3] we can choose a proper basis for which

$$e_3^2=e_2^3=0. \tag{3.2}$$

Our first assumption concerns the VEVs of H_A and H_S. $\langle H_A\rangle$ can mix the standard model particles with the new heavy D and L states [the 10-plet of $SO(10)$]: $d\leftrightarrow D$, $e\leftrightarrow E$, $\nu=L_3^2\leftrightarrow L_2^2$. This is achieved by components of H_A which involve left and right $\mathcal{U}$-spin 1/2 indices. For the $(\overline{3},3,1)$ sector of H_A we take, therefore,

$$\langle (H_A)_k^i\rangle=f_k^i,\quad i,k=2,3. \tag{3.3}$$

For the $(\overline{3},\overline{6},1)$ sector of H_A one has, correspondingly,

$$\langle H_A^{i\{1,k\}}\rangle=f^{i\{1,k\}},\quad i,k=2,3, \tag{3.4}$$

and for the sector $(6,3,1)$,

$$\langle H_{A\{1,k\}i}\rangle=f_{\{1,k\}i},\quad i,k=2,3. \tag{3.5}$$

In our numerical treatment we will restrict the VEVs in Eqs. (3.4), (3.5) to those with $i=3$, $k=2,3$ and $i=2$, $k=3$ which should be the dominant ones. With respect to $\mathcal{U}$ spin, $f^{3\{1,3\}}$ is the analogue of f_2^3. While f_2^3 mixes d with D, $f^{3\{1,3\}}$ mixes e^- with E^- and ν with L_2^2.

The VEVs of the symmetric Higgs field H_S can provide large Majorana masses for the heavy leptons L_2^3 and L_3^3. They arise from the $H_S(6,\overline{6},1)$ sector. Here we have to take the $SU(2)_L$ singlets and left- and right-handed $\mathcal{U}$-spin triplets

$$\langle (H_S)_{\{3,3\}}\{2,2\}\rangle=F^{\{2,2\}},\quad \langle (H_S)_{\{3,3\}}\{3,3\}\rangle=F^{\{3,3\}}. \tag{3.6}$$

All other components of $\langle H_A\rangle$ and $\langle H_S\rangle$ are taken to be zero or negligible in our calculations.

Of particular interest is the question of the breaking of the left-right symmetry of E_6. $F^{\{22\}}$ as obtained from $\langle H_S\rangle$ breaks this symmetry strongly. It could be the dominant manifestation of $\mathcal{D}_{LR}$ symmetry breaking. $\langle H\rangle$ and $\langle H_A\rangle$ on the other hand need not break this symmetry significantly. A strict left-right symmetry in this sector would imply the relations

$$f_k^i=-f_i^k,\quad f^{i\{1,k\}}=f_{\{1,k\}i},\quad i,k=2,3. \tag{3.7}$$

The signs follow by taking the H_A part of the Yukawa interaction to be even under $\mathcal{D}_{LR}$ when H_A is replaced by $\langle H_A\rangle$. As a consequence of Eqs. (3.7) the f's are of the order of the weak scale even though some are standard model singlets and thus only protected by the discrete $\mathcal{D}_{LR}$ symmetry itself. If this is indeed the case, it implies new particles in the few TeV region as we will see.

The next assumption concerns the generation matrices G, A, and S. The symmetric matrix $G_{\alpha\beta}$ can be diagonalized by an orthogonal transformation, which leaves the symmetry properties of $A_{\alpha\beta}$ and $S_{\alpha\beta}$ unchanged. By choosing this basis, the up-quark mass matrix is diagonal because, according to the above-assumed properties of $\langle H_A\rangle$ and $\langle H_S\rangle$, only $\langle H\rangle$ contributes to it:

$$(m_U)_{\alpha\beta}=G_{\alpha\beta}e_1^1=g_\alpha\delta_{\alpha\beta}e_1^1. \tag{3.8}$$

As a consequence, the quark mixing angles and the $\mathcal{CP}$ violating phase must entirely come from the inclusion of the Higgs H_A with its antisymmetric generation matrix $A_{\alpha\beta}$ as proposed in Ref. [5]. Thus, $A_{\alpha\beta}$ has to contain imaginary parts which cannot be rotated away using quark phase redefinitions. This leads us to assume that the matrix A is—in our phase convention—purely imaginary—i.e., a Hermitian matrix. The normalized matrix contains then only two parameters, in fact only one when utilizing a discrete generation exchange symmetry for A as shown later.

We suggest that the generation matrices G, A, and S are not independent of each other. In particular, the coupling matrix S for the heaviest leptons should have an intimate relation with the generation matrices of the charged fermions [10]. S may then be expanded in terms of G and A. Speculatively we assume that the generation mixing matrix S is a combination of the bilinear product G^2 and the commutator $[G,A]$. The generation mixing in this sector is then due to the same matrix A which causes the mixing of the charged fermions. As it turns out this structure for S is crucial for bilarge neutrino mixings. In fact, it leads to bimaximal mixing which is then changed to bilarge mixing by renormalization group effects.

The last assumption concerns the breaking pattern of E_6, which is presumably the origin of the breakings seen in the Yukawa sector. We suppose the following symmetry breaking chain:

$$E_6 \xrightarrow{M_{\rm GUT}} SU(3)_L\times SU(3)_R\times SU(3)_C\times \mathcal{D}_{LR}$$
$$\xrightarrow{M_I} SU(2)_L\times U(1)_Y\times SU(3)_C. \tag{3.9}$$

Here $M_{\rm GUT}$ is the GUT scale and $\mathcal{D}_{LR}$ denotes the discrete left$\leftrightarrow$right symmetry operation. As we will show below, the breaking chain (3.9) leads in a straightforward way to the unification of the gauge coupling constants. The first breaking step to the intermediate symmetry can be caused by a scalar 650-plet which contains two G_{333} singlets. One of them $\mathcal{S}_+$ is even under $\mathcal{D}_{LR}$ $(\mathcal{S}_+\to\mathcal{S}_+)$, while the second one $\mathcal{S}_-$ is odd $(\mathcal{S}_-\to-\mathcal{S}_-)$. It then follows from the symmetries at M_I and above M_I that $\mathcal{S}_+$ has the nonzero VEV and $\langle\mathcal{S}_-\rangle=0$. This ensures that in the $(M_I,\ M_{\rm GUT})$ interval $L\leftrightarrow R$ symmetry is precise and the equality of the coupling constants g_L and g_R is protected also at the quantum level.

We take two Higgs $SU(2)_L$ doublets of H—namely, $H_1^{1,2}$ and $H_2^{1,2}$—to be relatively light. The remaining Higgs boson masses of the color neutral components of H and H_A are taken to be of order M_I or higher. The only exception is the $SU(2)_L$ doublet $(H_A)_2^{1,2}$ which can be much lighter than M_I because of the left-right symmetry $(f_2^2\approx 0)$ in the H_A sector mentioned above. But it must be heavier than ≈ 500 TeV not to induce flavor changing processes above presently known limits. All Higgs bosons not mentioned are assumed to have masses at the order of the GUT scale.

Before starting our investigation, let us state the quark and lepton masses at the scale $\mu=M_Z$ [11,12]:

$$\begin{aligned}
&m_u=(1.8\pm 0.4)\ \text{MeV},\quad m_d=(3.3\pm 0.7)\ \text{MeV},\\
&m_s=(62\pm 12)\ \text{MeV},\quad m_c=(0.64\pm 0.04)\ \text{GeV},\\
&m_b=(2.89\pm 0.03)\ \text{GeV},\quad m_t=(173\pm 5)\ \text{GeV},\\
&m_e=0.487\ \text{MeV},\quad m_\mu=102.8\ \text{MeV},\\
&m_\tau=1.747\ \text{GeV},
\end{aligned}\tag{3.10}$$

as obtained from the analysis of experimental data. The general hierarchical structure of the SM masses and of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements will be used in the following. Some of the masses—in particular m_t, m_b, and m_τ—are taken as input parameters.

IV. QUARK MASS MATRIX

Because of the hierarchical structure of the quark masses and mixing angles, it is convenient to express them in terms of powers of a small dimensionless parameter. We introduce the parameter σ [10] with the value

$$\sigma=0.058, \tag{4.1}$$

for which

$$|V_{cb}|\simeq\frac{\sigma}{\sqrt{2}},\qquad \left|\frac{m_u}{m_c}\right|\simeq\left|\frac{m_c}{m_t}\right|\simeq|V_{ub}|\simeq\sigma^2 \tag{4.2}$$

hold within experimental uncertainties. One also has $(m_s/m_b)|V_{us}|\approx\sigma^2$.

According to our assumption (3.1), (3.2) the up-quark mass matrix is

$$(m_U)_{\alpha\beta}\equiv G_{\alpha\beta}\langle H_1^1\rangle=g_\alpha\delta_{\alpha\beta}e_1^1=\mathrm{Diag}\left(\frac{m_u}{m_t},\ \pm\frac{m_c}{m_t},\ 1\right)m_t. \tag{4.3}$$

At the scale $\mu\simeq M_Z$ we can write

$$m_U\simeq\mathrm{Diag}(\sigma^4,\ \pm\sigma^2,\ 1)173\ \text{GeV}. \tag{4.4}$$

The signs of the mass parameters are in general of no relevance because of the freedom to change phases. But since we keep G and A to be Hermitian matrices, the Jarlskog determinant obtained from the commutator of mass matrices depends on the sign chosen in Eq. (4.3) giving two solutions for the area of the unitarity triangle.

Because of the existence of the D quarks, the down-quark (big) mass matrix is a 6×6 matrix, which contains the antisymmetric generation matrix A:

$$M_{d,D}=\begin{array}{c} \\ d \\ D\end{array}\!\!\begin{array}{c}\qquad\hat{d}\qquad\qquad\hat{D}\\ \begin{pmatrix} e_2^2G+f_2^2A, & f_3^2A\\ f_2^3A, & e_3^3G+f_3^3A\end{pmatrix}\end{array}. \tag{4.5}$$

Here e_2^2, f_2^2, and f_3^2 are mass scales of order of the weak scale, while at least e_3^3 should describe a heavy mass scale. In accordance with our model assumptions we have $f_2^3,f_3^3\ll e_3^3$ which allows us to integrate out the $D,\hat{D}$ states and to write down the seesaw formula

$$m_D\simeq e_2^2G+f_2^2A-\frac{f_2^3f_3^2}{e_3^3}(AG^{-1}A). \tag{4.6}$$

The D-quark mass matrix is simply proportional to the up-quark mass matrix:

$$M_D \simeq e_3^3 G = \frac{e_3^3}{e_1^1} m_U . \tag{4.7}$$

Although Eq. (4.5) should only be valid at the unification scale and has to be carefully scaled down for a determination of m_D at $\mu \simeq M_Z$, we will use Eq. (4.6) at M_Z for a first orientation.

The first entry in Eq. (4.6) is responsible for the mass of the bottom quark, while the second term must provide for the small mixing angles and the large $\mathcal{CP}$ violating phase. The third term gives a correction to the symmetric part of the mass matrix which is important for the strange quark mass. We expect, therefore,

$$e_2^2 \simeq m_b , \quad f_2^2 |A_{23}| \simeq |V_{cb}| m_b ,$$

$$f_2^2 |A_{12}| \simeq m_s |V_{us}|, \quad \text{and} \quad f_2^2 |A_{13}| \simeq |V_{ub}| m_b . \tag{4.8}$$

From Eq. (4.2) one then gets, for f_2^2 and the generation matrix A,

$$f_2^2 \simeq \frac{\sigma m_b}{\sqrt{2}\lambda_A}, \quad A \simeq \begin{pmatrix} 0, & \mathrm{i}\sigma, & -\mathrm{i}\sigma \\ -\mathrm{i}\sigma, & 0, & \frac{\mathrm{i}}{\sqrt{2}} \\ \mathrm{i}\sigma, & -\frac{\mathrm{i}}{\sqrt{2}}, & 0 \end{pmatrix} \lambda_A \sqrt{2}. \tag{4.9}$$

We introduced a scaling factor $\sqrt{2}\lambda_A$ (as discussed in Sec. VIII) such that $f_2^2 = \langle H_2^2 \rangle$ and the scale-dependent matrix A is normalized according to $\mathrm{Tr}(A^2) \simeq 2\lambda_A^2$. We remark that the antisymmetric matrix A taken here is also antisymmetric with respect to the (discrete) interchange of the second generation with the third one. We know, of course, that the matrix A can have its strictly antisymmetric form only above M_I, the breaking point of the left-right symmetry. Thus, in our renormalization group treatment we take the matrix A as given in Eq. (4.9) to be strictly valid at $\mu = M_{\mathrm{GUT}}$, even though we anticipated its form at a low scale. By going down from $\mu = M_{\mathrm{GUT}}$ to M_I, the matrix A "splits" into a matrix A^Q for the quarks and a matrix A^L for the leptons. By going further down to M_Z, A^Q as well as A^L each splits into three matrices relevant for the sectors indicated by the superscripts:

$$A^Q \to (A^{d\hat{d}},\ A^{d\hat{D}},\ A^{D\hat{d}}),$$

$$A^L \to (A^{e^- e^+},\ A^{e^- E^+},\ A^{E^- e^+}). \tag{4.10}$$

These matrices are no more strictly antisymmetric. Obviously, also the matrix G splits into more matrices. Between M_{GUT} and M_I we have $G \to (G_Q, G_L)$ for the quarks and leptons. Below M_I, one gets

$$G_Q \to (G^{u\hat{u}},\ G^{d\hat{d}},\ G^{D\hat{D}}),$$

$$G_L \to (G^{e^+e^-},\ G^{L\bar{L}},\ G^{\nu\hat{\nu}}). \tag{4.11}$$

We calculated these matrices at $\mu = M_Z$ in a model specified in Sec. VIII, which has a unification scale of 2×10^{17} GeV. The matrix $e_2^2 G$ in Eq. (4.6) becomes $e_2^2 G^{d\hat{d}}$. It is obtained from $m_U = e_1^1 G^{u\hat{u}}$ at M_Z [Eq. (4.4)] scaled up to the GUT scale, where $G^{u\hat{u}} = G^{d\hat{d}}$ holds and then scaled down to M_Z. In our approximation, it is still a diagonal matrix. In the same way, one obtains the matrix replacing G^{-1} in the $D\hat{D}$ channel of Eq. (4.6). It is denoted by $(G^{D\hat{D}})^{-1}$. The matrices $G^{D\hat{D}}$ and $G^{d\hat{d}}$ at $\mu = M_Z$ are given in [13].

With these changes the mass matrix for the down quarks becomes now

$$m_D(M_Z) = e_2^2 G^{d\hat{d}} + f_2^2 A^{d\hat{d}} - \frac{f_2^3 f_3^2}{e_3^3} A^{d\hat{D}} (G^{D\hat{D}})^{-1} A^{D\hat{d}},$$

$$e_2^2 (G^{d\hat{d}})_{33} \simeq m_b^0, \quad f_2^2 (A^{d\hat{d}})_{23} \simeq \frac{\mathrm{i}\sigma m_b^0}{\sqrt{2}}. \tag{4.12}$$

m_b^0 will slightly differ from the mass of the bottom quark because of the mixing occurring in m_D.

After having found the renormalization group effects on the matrices G and A, the only parameter for calculating the d-quark masses and the CKM matrix is $f_2^3 f_3^2 / [e_3^3 (G^{D\hat{D}})_{33}]$. We use this parameter for a fit of the Cabibbo angle $|V_{us}|$. Because of our expectation of an approximate left right symmetry [see Eq. (3.7)] we look for a negative value of this parameter and find

$$\frac{f_3^2 f_2^3}{e_3^3 (G^{D\hat{D}})_{33}} \simeq -4.75\times 10^{-5}\ \mathrm{GeV}. \tag{4.13}$$

Upon diagonalization of the down-quark mass matrix (4.12), with the negative sign taken in Eq. (4.4), one obtains

$$m_d(M_Z) \simeq 2.66\ \ \mathrm{MeV}, \quad m_s(M_Z) \simeq 49.7\ \ \mathrm{MeV},$$

$$m_b(M_Z) \simeq 2.89\ \ \mathrm{GeV},$$

$$|V_{us}| \simeq 0.217, \quad |V_{cb}| \simeq 0.045, \quad |V_{ub}| \simeq 0.0034, \tag{4.14}$$

and for the angles of the unitarity triangle,

$$\alpha \simeq 84°, \quad \beta \simeq 20°, \quad \gamma \simeq 76°. \tag{4.15}$$

To obtain the correct value for $m_b(M_Z)$ we took for the (3,3) element of $e_2^2 G^{d\hat{d}}$, $m_b^0 = 2.859$ GeV. A similar good fit is obtained if in Eq. (4.4) the positive sign is chosen. The number given in Eq. (4.13) then changes to -3.26×10^{-5} GeV and the angles of the unitarity triangle become

$$\alpha \simeq 95°, \quad \beta \simeq 21°, \quad \gamma \simeq 64°. \tag{4.16}$$

In the following we will use the negative sign in Eq. (4.4).

The results (4.14)–(4.16) are in good agreement with present experimental data. The mass of the strange quark is a bit low but still within the bounds of Eq. (3.10). We also see that Weinberg's suggestion [13]

$$|V_{us}| \approx \sqrt{\frac{m_d}{m_s}} \tag{4.17}$$

is valid. It follows from the smallness of the (1,1) entry $e_2^2\sigma^4$ in Eq. (4.12) due to the small first generation up quark mass. We further note that the term in m_D, which arises from the mixing with the heavy D quarks, reduced the angle γ from the originally obtained value $\simeq 90°$ [5] to a lower value.

Besides Eq. (4.13) there is no restriction on the value of e_3^3 except that f_2^3/e_3^3 has to be sufficiently small to justify the seesaw formula and thereby the near unitarity of the CKM mixing matrix. However, as mentioned in Sec. III, the VEVs $\langle H\rangle$ and $\langle H_A\rangle$ may approximately respect the left-right symmetry of E_6 and of the intermediate symmetry in contrast to the large VEV of H_S. This idea is supported by the small value found for f_2^2 in Eq. (4.9). It would be zero for a strict left-right symmetry in this channel and is indeed small ($f_2^2 \simeq 0.093$ GeV) compared to the weak interaction scale. One can then expect that the product $-f_3^2 f_2^3$ is not of order $M_Z M_I$ but not much higher than $(M_Z)^2$. This gives us a rough estimate for e_3^3 and thus for the masses of the D quarks:

$$e_3^3(G^{D\hat{D}})_{33} \approx \frac{2.1\times 10^4}{\mathrm{GeV}} M_Z^2 \quad \text{or} \quad \approx \frac{3.07\times 10^4}{\mathrm{GeV}} M_Z^2. \tag{4.18}$$

From these relations, which are of course sensitive to the value taken for the weak scale input, we expect $e_3^3(G^{D\hat{D}})_{33}$ to be of order $(10^7 – 10^8)$ GeV. Taking $e_3^3(G^{D\hat{D}})_{33} = 4\times 10^7$ GeV as an example (and scaling effects into account), one obtains

$$M_{D1} \simeq 557 \ \mathrm{GeV}, \quad M_{D2} \simeq 129 \ \mathrm{TeV},$$

$$M_{D3} \simeq 4\times 10^4 \ \mathrm{TeV}. \tag{4.19}$$

A more detailed discussion of the heavy fermions and their masses is presented in Sec. VII.

V. CHARGED LEPTON MASS MATRIX

The charged lepton mass matrix has the same structure as the down-quark mass matrix. By going from quarks to leptons E_6 Clebsch-Gordon coefficients have to be taken into account. Quarks and leptons couple to $H(\bar{3},3,1)$ according to the combination

$$q_i \hat{q}^k + \frac{1}{2}\varepsilon_{ii'i''}\varepsilon^{kk'k''} L_{k'}^{i'} L_{k''}^{i''}. \tag{5.1}$$

The $(\bar{3},3,1)$ sector of the Higgs field H_A couples only to quarks, the sectors $(\bar{3},\bar{6},1)$ and $(6,3,1)$ only to leptons. Thus, the relevant 6×6 matrix at the GUT scale is

$$M_{e,E} = \begin{array}{c c} & \begin{array}{cc} e^+ & E^+ \end{array} \\ \begin{array}{c} e^- \\ E^- \end{array} & \begin{pmatrix} -e_2^2 G - (f^{2\{1,3\}} - f_{\{1,3\}2})A, & f^{3\{1,3\}}A \\ -f_{\{1,3\}3}A, & -e_3^3 G + (f^{3\{1,2\}} - f_{\{1,2\}}3)A \end{pmatrix} \end{array}. \tag{5.2}$$

Using the same arguments as for the down-quark mass matrix the f's in the diagonal elements are small compared to the main terms. After integrating out the E-type states, the mass matrix for the charged leptons of the SM is generated and has at $\mu \simeq M_Z$ the form

$$m_E \simeq -e_2^2 G^{e^-e^+} - (f^{2\{13\}} - f_{\{13\}}2)A^{e^-e^+} - \frac{f_{\{1,3\}3} f^{3\{1,3\}}}{e_3^3} A^{e^-E^+} (G^{L\bar{L}})^{-1} A^{E^-e^+}. \tag{5.3}$$

The first term is constructed like $e_2^2 G^{d\hat{d}}$, but for leptons. The contribution of VEVs in the second term should be as small as the corresponding term f_2^2 in the quark mass matrix. Diagonalizing Eq. (5.3) one gets, with

$$f^{2\{13\}} - f_{\{13\}2} \simeq 0.042 \ \mathrm{GeV},$$

$$\frac{f_{\{1,3\}3} f^{3\{1,3\}}}{e_3^3 (G^{L\bar{L}})_{33}} \simeq 12.6\times 10^{-5} \ \mathrm{GeV}, \tag{5.4}$$

the charged lepton masses

$$m_e = 0.488 \ \mathrm{MeV}, \quad m_\mu = 102.8 \ \mathrm{MeV}, \quad m_\tau = 1.748 \ \mathrm{GeV}. \tag{5.5}$$

For obtaining the correct value of the tau lepton mass we took m_τ^0 [the (3,3) element of $e_2^2 G^{e^- e^+}$] to be 1.689 GeV. The contributions from the first term in Eq. (5.3) for the light generations are proportional to σ^4 and σ^2, respectively, and thus negligibly small. The muon mass receives its essential contribution from the third term in Eq. (5.3)—i.e., from the mixing with the heavy leptons. The contributions from the second and third terms to the electron mass are comparable. There is some $\mathcal{CP}$ violation due to the second term in Eq. (5.3). The corresponding unitarity triangle, for charged leptons, has the angles $\alpha \simeq 43°$, $\beta \simeq 62°$, $\gamma \simeq 75°$. After diagonalization of the charged lepton matrix, this $\mathcal{CP}$ violation will affect the neutrino mixings. The charged lepton mixing angles turn out to be small: $|V_{e\mu}| \simeq 0.034$, $|V_{e\tau}| \simeq 0.003$, $|V_{\mu\tau}| \simeq 0.068$. Therefore, the large neutrino mixings are not due to the mixings in the charged lepton sector but should come from the neutral lepton sector where large Majorana masses appear. In the next section it will be shown that this is indeed the case. Comparing now Eq. (4.13) with Eq. (5.4) we get $f_{\{1,3\}3} f^{3\{1,3\}} / - f_3^2 f_2^3 \simeq (1.6)^2$. Considering the analogy of f_3^2 with $f_{\{1,3\}3}$ and f_2^3 with $f^{3\{1,3\}}$ this appears to be a reasonable value. In Secs. VII, VIII we will use $|f_{\{1,3\}3}| \simeq |f^{3\{1,3\}}|$, $|f_3^2| \simeq |f_2^3|$—i.e., appropriate left-right symmetry in these channels.

VI. NEUTRAL LEPTON MASS MATRIX

The fundamental fermion representation of E_6 contains five neutral two-component fields. Thus, for three generations, the mass matrix for these neutral leptons is a 15×15 matrix. According to the assumption stated in Sec. III, it is given by

$$M_L = \begin{array}{c} \\ L_3^2 \\ L_2^3 \\ L_3^3 \\ L_1^1 \\ L_2^2 \end{array} \begin{array}{c} \begin{array}{ccccc} L_3^2 & L_2^3 & L_3^3 & L_1^1 & L_2^2 \end{array} \\ \begin{pmatrix} 0 & -e_1^1 G & 0 & -f^{3\{1,3\}} A & 0 \\ -e_1^1 G & F^{\{2,2\}} S & 0 & -f_{\{1,3\}3} A & 0 \\ 0 & 0 & F^{\{3,3\}} S & e_2^2 G + h_2^2 A & e_1^1 G \\ -f^{3\{1,3\}} A^T & -f_{\{1,3\}3} A^T & e_2^2 G + h_2^2 A^T & 0 & e_3^3 G + h_3^3 A \\ 0 & 0 & e_1^1 G & e_3^3 G + h_3^3 A^T & 0 \end{pmatrix} \end{array}, \tag{6.1}$$

where

$$h_2^2 = f^{2\{1,3\}} + f_{\{1,3\}2}, \quad h_3^3 = f^{3\{1,2\}} + f_{\{1,2\}3}, \tag{6.2}$$

and L_3^2 stands for the standard light neutrino fields. All ingredients in this matrix arising from the Higgs fields H and H_A are defined in the previous sections. We notice, however, that in the $L_1^1 L_2^2$ block the contribution of the f's is additive. The new elements are the ones containing the symmetric generation matrix S. They give rise to genuine Majorana mass terms and are of particular significance in the diagonalization process. The strength of the H_S Higgs contribution to M_L is governed by the constants $F^{\{2,2\}}$ and $F^{\{3,3\}}$ carrying right-handed $\mathcal{U}$-spin quantum numbers. $F^{\{3,3\}}$ essentially fixes the Majorana mass for the heavy L_3^3 leptons which are expected to be of the order of the $SU(3)_L \times SU(3)_R$ breaking scale. The constant $F^{\{2,2\}}$ of similar strengths breaks the left-right symmetry and thus is responsible for the dominant breaking of this symmetry.

We can reduce the matrix M_L to a 9×9 matrix by knowing that e_3^3 is much larger than the other elements in the same row and column—in particular, if $f^{3\{1,3\}}$ and $f^{3\{1,2\}}$ are indeed of the order of the weak scale. This allows us to integrate out the L_1^1, L_2^2 states. With the abbreviations

$$f = f^{3\{1,3\}} e_1^1 / e_3^3, \quad \overline{f} = f_{\{1,3\}3} e_1^1 / e_3^3, \tag{6.3}$$

one finds

$$M_L{}' = \begin{array}{c} \\ L_3^2 \\ L_2^3 \\ L_3^3 \end{array} \begin{array}{c} \begin{array}{ccc} L_3^2 & L_2^3 & L_3^3 \end{array} \\ \begin{pmatrix} 0 & -e_1^1 G & fA \\ -e_1^1 G & F^{\{2,2\}} S & \overline{f} A \\ fA^T & \overline{f} A^T & F^{\{3,3\}} S \end{pmatrix} \end{array} \tag{6.4}$$

and

$$M_{L_1^1 L_2^2} \approx e_3^3 G. \tag{6.5}$$

We neglected in Eq. (6.4) a correction to the (3-3) block—namely, $-(e_1^1/e_3^3) e_2^2 G$. It is small compared to the large eigenvalues of $F^{\{3,3\}} S$.

For values of $f^{3\{1,3\}}$ of the order of the weak scale and $F^{\{(2,2)\}}, F^{\{(3,3)\}}$ near $M_I \approx 10^{13}$ GeV, we can again apply the seesaw mechanism and finally arrive at the 3×3 Majorana matrix for the light neutrinos,

$$m_\nu \simeq -\frac{(e_1^1)^2}{F^{\{2,2\}}} G S^{-1} G - \frac{f^2}{F^{\{3,3\}}} A^T S^{-1} A, \tag{6.6}$$

and for the mass matrices of the heavy Majorana neutrinos,

$$M_{L_2^3} \simeq F^{\{2,2\}} S, \quad M_{L_3^3} \simeq F^{\{3,3\}} S. \tag{6.7}$$

Only the first term in Eq. (6.6) need to be considered, since the remaining one can safely be neglected. Therefore, the neutrino mass matrix (6.6) is mainly due to the decoupling of $L_2^3 = \hat{\nu}$ states. It scales with the masses of these heavy lepton states.

We expect (see Sec. III) S to be related to the two other generation matrices G and A. The main term for S should be G^2, which leads to a diagonal nondegenerate mass matrix [see Eq. (6.6)]. We then add a term linear in A, the commutator $[G,A]$, with a tiny coefficient. It implies that also in this sector generation mixing is solely due to the antisymmetric matrix A. We take for S, divided by the overall coupling strength λ_S to the Higgs field H_S, the real and bilinear construct

$$S/\lambda_S = G^2 + \mathrm{i} x \sigma^3 [G,A], \tag{6.8}$$

with the single parameter x. The G^2 term with its dominant element $\simeq 1$ for the third generation serves for generation hierarchy and for the normalization of S/λ_S [for which the σ^3 term in Eq. (6.8) can be neglected]. With no renormalization effects included, the matrix S, as defined in Eq. (6.8) reads

$$S/\lambda_S \simeq \begin{pmatrix} \sigma^8, & -\sigma^6 x, & -\sigma^4 x \\ -\sigma^6 x, & \sigma^4, & \dfrac{\sigma^3}{\sqrt{2}} x \\ -\sigma^4 x, & \dfrac{\sigma^3}{\sqrt{2}} x, & 1 \end{pmatrix}. \tag{6.9}$$

In each element of Eq. (6.9) only the leading powers of σ are shown.

By inverting the matrix S defined in Eq. (6.8) and using Eq. (6.6), one finds, for m_ν,

$$m_\nu \simeq \begin{pmatrix} -1, & \left(1 - \dfrac{1}{\sqrt{2}}\sigma x\right) x, & -\left(1 - \dfrac{1}{\sqrt{2}}\sigma x\right) x \\ \left(1 - \dfrac{1}{\sqrt{2}}\sigma x\right) x, & x^2 - 1, & \left(x - \dfrac{1}{\sqrt{2}}\sigma\right) x \\ -\left(1 - \dfrac{1}{\sqrt{2}}\sigma x\right) x, & \left(x - \dfrac{1}{\sqrt{2}}\sigma\right) x, & x^2 - 1 \end{pmatrix} \frac{(e_1^1 \lambda_\tau)^2}{(1 - 2x^2 + \sqrt{2} x^3 \sigma) \lambda_S F^{\{2,2\}}}. \tag{6.10}$$

For the simplicity of representation Eq. (6.10) contains only the zeroth and first powers in σ. Taking the full expression makes numerically little difference. The interesting feature of m_ν is the fact that it produces for any value of $x \gtrsim 1.5$ automatically an almost perfect bimaximal neutrino mixing pattern with a normal (not inverted) neutrino spectrum. By changing x, solely the ratio of mass square differences

$$R = \frac{m_2^2 - m_1^2}{m_3^2 - m_2^2}, \tag{6.11}$$

changes (m_i denotes the three ν eigenstates mass ordered according to $m_1 < m_2 < m_3$). The experimentally observed ratio $R \approx 0.03$ is obtained for $x \simeq 3.5$.

However, for a proper calculation of the neutrino mass matrix at $\mu = M_I$ and $\mu = M_Z$, renormalization effects have to be taken into account. This is particularly necessary because of the large generation splitting of the heavy neutrino states $L_2^3 = \hat{\nu}$ caused by the G^2 term in the matrix S. We have to integrate out these states in steps and to redefine m_ν in each step. We start by using Eq. (6.8) at the scale M_I with $G = G_L$ and $A = A_L$ and proceed according to the rules given in the Appendix. It turns out that renormalization effects strongly influence the neutrino mass matrix and thus also the mixing pattern. The bimaximal mixing is changed to a bi-large mixing. The calculation is again performed for the gauge and Yukawa unification at 2×10^{17} GeV described in Sec. VIII. As in the examples given in [15] we find that the renormalization coefficients strongly reduce the mixing angle θ_{12} observed in solar neutrino experiments while the angle θ_{23} observed in atmospheric neutrino experiments is less affected. The renormalization coefficients also increase the value of the ratio R.

A good description of the known neutrino data is obtained by changing the value of our parameter $x = 3.5$ to

$$x \simeq 2.8. \tag{6.12}$$

With this value we obtain $R \simeq 0.055$. Larger values of x reduce R. However, this would lead to a too strong reduction of the solar neutrino oscillation probability.

With $x \simeq 2.8$ one obtains for the mass matrix of the light neutrinos at $\mu = M_Z$:

$$m_\nu = -\begin{pmatrix} -0.135, & 0.67, & -0.62 \\ 0.67, & 3.75, & 4.61 \\ -0.62, & 4.61, & 2.81 \end{pmatrix} \frac{M_0}{10}, \tag{6.13}$$

with

$$M_0=\frac{(\lambda_\tau(M_I)e_1^1(M_I))^2}{\lambda_S F^{\{2,2\}}}\simeq\frac{(102.17\ \text{GeV})^2}{M_{\hat{\nu}_3\hat{\nu}_3}}. \quad (6.14)$$

Here λ_τ is the coupling of the third generation lepton to $H_2^{1,2}$ and e_1^1 is the VEV of the Higgs field H_1^1 as used before.

We obtain the mass-squared difference observed in atmospheric neutrino experiments $\Delta m^2_{\text{atm}}\simeq 2.5\times 10^{-3}$ eV^2 when setting $M_{\hat{\nu}_3\hat{\nu}_3}\simeq 1.6\times 10^{14}$ GeV. It is very satisfying that this scale is of the same order of magnitude as expected from the value of M_I, the breaking point of the left-right symmetry [14]. With this value of $M_{\hat{\nu}_3\hat{\nu}_3}$ the neutrino mass eigenvalues turn out to be

$$m_1=0.0023\ \text{eV},\quad m_2=0.0120\ \text{eV},\quad m_3=0.0516\ \text{eV},$$
$$m_2^2-m_1^2\simeq 1.4\times 10^{-4}\ (\text{eV})^2,$$
$$m_3^2-m_2^2\simeq 2.5\times 10^{-3}\ (\text{eV})^2. \quad (6.15)$$

To obtain the neutrino mixing matrix, one has to go to a basis in which the charged lepton mass matrix is diagonal. Diagonalizing Eq. (5.3) and denoting by ν_α the weak eigenstates ($\alpha=e,\mu,\tau$), we find, from Eq. (6.13),

$$\nu_\alpha=U^\nu_{\alpha i}\nu_i, \quad (6.16)$$

$$U^\nu_{\alpha i}\simeq\begin{pmatrix}-0.2+0.86\text{i}, & -0.46-0.08\text{i}, & 0.026\\ -0.29-0.015\text{i}, & 0.044+0.55\text{i}, & 0.78\\ 0.37-0.017\text{i}, & -0.036-0.69\text{i}, & 0.62\end{pmatrix}_{\alpha i}. \quad (6.17)$$

Here we took a special phase choice for the neutrino flavor eigenstates such that the third column has only real and positive elements.

Our results for the three mixing angles relevant in neutrino oscillation experiments as obtained from Eq. (6.17) are

$$\sin^2\theta_{13}\simeq 6.8\times 10^{-4},\quad \sin^2\theta_{12}\simeq 0.22,\quad \sin^2\theta_{23}\simeq 0.61. \quad (6.18)$$

These values and the ratio $R\simeq 0.055$ of mass-squared differences are quite close to the results [16] of SuperKamiokande [17,18], SNO [19], and KamLAND [20], the CHOOZ limit [21], and the observations of the disappearance of solar neutrinos [22].

We can also get from Eq. (6.17) the neutrino unitarity triangle, defined in analogy with the quark unitarity triangle. It turns out to be

$$\alpha_\nu\simeq 74°,\quad \beta_\nu\simeq 6°,\quad \gamma_\nu\simeq 100°. \quad (6.19)$$

The phases of the elements of the first row of U^ν are "Majorana phases" relevant for neutrinoless double-β-decay experiments. With the convention used in Eq. (6.17) we get

$$\delta_{11}\simeq 103°,\quad \delta_{12}\simeq -170°,\quad \delta_{13}=0. \quad (6.20)$$

For the quantity $\langle m_\nu\rangle_{ee}$ which determines the decay rate we find

$$\langle m_\nu\rangle_{ee}=|m_1(U^\nu_{e1})^2+m_2(U^\nu_{e2})^2+m_3(U^\nu_{e3})^2|$$
$$\simeq 9.5\times 10^{-4}\ \text{eV}. \quad (6.21)$$

The matrix U^ν, in particular its deviation from bimaximal mixing, depends via the renormalization parameters to some extent on the way unification is obtained. But a bilarge mixing with near maximal mixing in the μ-τ sector will always result from the basic assumptions of our E_6 model outlined in Sec. III.

VII. DESERT IS BLOOMING

In our model the masses of the heavy down quarks D and the corresponding leptons L, which form 10-plets of $SO(10)$, have a generation splitting similar to the up quarks. The absolute values of these masses cannot be given. However, if $\langle H_A\rangle$ still respects to some extent the left-right symmetry of E_6 as discussed above, the lightest D and L states lie in the TeV region. In Sec. VIII, we present a numerical solution of the problem of the gauge and Yukawa coupling unification for $M_{\text{GUT}}=2\times 10^{17}$ GeV and $e_3^3(G^{D\hat{D}})_{33}=3.98\times 10^7$ GeV. This solution also fixes the so-far undetermined VEVs of H and H_A. With the values quoted there [Eq. (8.36)], we can now directly diagonalize the 6×6 matrices (4.5) and (5.2) which determine the mixing of the SM particles with the D and L states. This mixing, although important for the mass matrices, does not seriously violate the unitarity relations for the SM particles. For example, the sum of the squares of the second row of the CKM matrix differs from one only by 1.2×10^{-4}. In the charged lepton sector, the corresponding deviation amounts to 3.7×10^{-4}.

We can list now the mass values of the new particles by using again $e_3^3(G^{D\hat{D}})_{33}=4\times 10^7$ GeV and setting $\lambda_S F^{\{2,2\}}=\lambda_S F^{\{3,3\}}=1.6\times 10^{14}$ GeV in accordance with the neutrino results:

$$M_{D_1}\simeq 557\ \text{GeV},\quad M_{D_2}\simeq 129\ \text{TeV},$$
$$M_{D_3}\simeq 4\times 10^4\ \text{TeV},$$
$$M_{L_1}\simeq 355\ \text{GeV},\quad M_{L_2}\simeq 103\ \text{TeV},$$
$$M_{L_3}\simeq 3.83\times 10^4\ \text{TeV},$$
$$M(L_2^3)_1\simeq 5.8\times 10^5\ \text{GeV},\quad M(L_2^3)_2\simeq 9.6\times 10^8\ \text{GeV},$$
$$M(L_2^3)_3\simeq 1.6\times 10^{14}\ \text{GeV},$$
$$M(L_3^3)_1\simeq 5.8\times 10^5\ \text{GeV},\quad M(L_3^3)_2\simeq 9.6\times 10^8\ \text{GeV},$$
$$M(L_3^3)_3\simeq 1.6\times 10^{14}\ \text{GeV}. \quad (7.1)$$

In the evaluation we took the most important renormalization effects into account (see Sec. VIII and the Appendix). As we see, the desert is populated between the mass scales M_Z and M_{GUT}. The mass ratios for different generations of the standard model singlet neutrinos are even more drastic than the corresponding ratios for the D quarks and the $SU(2)_L$ doublet heavy leptons.

Our specific unification model allows us to calculate numerous properties of the old and new particles, in particular those related to their decay properties. We will present here a few examples only.

From the 6×6 mass matrix (4.5), for the quarks, one can calculate the coupling matrices in generation space for the couplings of the light and heavy mass eigenstates to the appropriate light Higgs field components:

$$d^T \mathrm{i}\sigma_2 \mathcal{C}^{d\hat{d}}\ \hat{d}\ H_2^2, \quad d^T \mathrm{i}\sigma_2 \mathcal{C}^{d\hat{D}}\ \hat{D}\ H_2^2, \quad D^T \mathrm{i}\sigma_2 \mathcal{C}^{D\hat{d}}\ \hat{d}\ H_2^2. \tag{7.2}$$

We find, without using the remaining freedom of changing phases,

$$\mathcal{C}^{d\hat{d}} \simeq \begin{pmatrix} (-0.18+1.6\mathrm{i})\times 10^{-4}, & (-0.25+7.3\mathrm{i})\times 10^{-4}, & -(3.2+0.9\mathrm{i})\times 10^{-5} \\ (7.4+0.28\mathrm{i})\times 10^{-4}, & (3.3+1.3\mathrm{i})\times 10^{-3}, & (-0.42+1.5\mathrm{i})\times 10^{-4} \\ -(9.9+0.4\mathrm{i})\times 10^{-3}, & -0.043+0.014\mathrm{i}, & 0.069+\mathrm{i} \end{pmatrix},$$

$$\mathcal{C}^{d\hat{D}} \simeq \begin{pmatrix} 0, & -4.4\times 10^{-7}, & 0 \\ -(0.3+3.4\mathrm{i})\times 10^{-2}, & 0, & 4.1\mathrm{i}\times 10^{-6} \\ -0.77-8.9\mathrm{i}, & 0.39, & 0 \end{pmatrix} \times 10^{-3},$$

$$\mathcal{C}^{D\hat{d}} \simeq \begin{pmatrix} 0, & (0.33-3.8\mathrm{i})\times 10^{-2}, & 0.84-9.7\mathrm{i} \\ -4.8\times 10^{-7}, & 0, & 0.43 \\ 0, & -4.6\times 10^{-6}, & 0 \end{pmatrix} \times 10^{-3}. \tag{7.3}$$

Of course, similar results can be derived for the lepton couplings to the Higgs fields.

For the weak interaction process $D \to u W_L$ one can introduce the matrix $\mathbf{V}^{uD}$ as an extension of the CKM matrix

$$\overline{u}\gamma^\mu (1-\gamma_5) \mathbf{V}^{uD} D (W_L)^+_\mu . \tag{7.4}$$

From Eq. (4.5) one gets

$$\mathbf{V}^{uD} \simeq \begin{pmatrix} -3.5\times 10^{-4}, & -0.04, & -1.1\times 10^{-4} \\ -0.95+11\mathrm{i}, & 0, & 1.3\times 10^{-3} \\ 0.81-9.4\mathrm{i}, & 0.42, & 0 \end{pmatrix} \times 10^{-3}. \tag{7.5}$$

There are also right-handed current interactions of the standard model particles with the heavy $SU(2)_R$ vector bosons $W_R^{\pm}$:

$$\overline{u}\gamma^\mu (1+\gamma_5) \mathbf{V}^{\hat{u}\hat{d}} d (W_R)^+_\mu , \tag{7.6}$$

where $\mathbf{V}^{\hat{u}\hat{d}}$ is slightly different but has the same structure as the CKM matrix.

Of particular interest for the decay properties of the mass eigenstates of $\hat{\nu}$ neutrinos are the Dirac masses connecting the flavor eigenstates of the light neutrinos (in a basis in which the charged lepton matrix is diagonal) with the heavy neutrinos. Using Eq. (5.3), $e_1^1 G_L(M_I)$ from Eqs. (A22), (A23), and diagonalizing $S(M_I)$ we obtain

$$m^{\mathrm{Dirac}} \simeq \begin{array}{c} \\ \nu_1 \\ \nu_2 \\ \nu_3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \quad\hat{\nu}_1 \quad\quad & \quad\quad \hat{\nu}_2 \quad\quad & \quad\quad \hat{\nu}_3 \end{array} \\ \begin{pmatrix} (1.7-10\mathrm{i})\times 10^{-4}, & (-1.2-11\mathrm{i})\times 10^{-3}, & -0.41-0.0099\mathrm{i} \\ (-4.6+0.3\mathrm{i})\times 10^{-3}, & 0.033+0.32\mathrm{i}, & 3.8+7.7\mathrm{i} \\ (3.3+3.9\mathrm{i})\times 10^{-3}, & -0.034+0.041\mathrm{i}, & -93.7+83.6\mathrm{i} \end{pmatrix} \end{array} \mathrm{GeV}. \tag{7.7}$$

VIII. UNIFICATION OF COUPLINGS

A. Gauge coupling unification with intermediate $SU(3)_L \times SU(3)_R \times SU(3)_C$ symmetry

As is known, the SM does not lead to the unification of the gauge coupling constants. In our scenario, there are the additional Dirac fermions D and L below the GUT scale $M_{\rm GUT}$. However, these do not alter the unification picture of the standard model significantly. We still need to introduce an intermediate breaking scale M_I.

A large group like E_6 with high-dimensional representations should first be broken by a step which lowers the symmetry considerably. It is natural to break E_6 to the maximal subgroup $SU(3)_L\times SU(3)_R\times SU(3)_C$. As we will see, this has the advantage that the corresponding intermediate scale is not an arbitrary parameter but fixed. The breaking at the GUT scale can be achieved in the scalar sector by a Higgs $H(650)$, which contains two G_{333} singlets (1,1,1), $\mathcal{S}_+$ and $\mathcal{S}_-$. $\mathcal{S}_+$ is even under $\mathcal{D}_{LR}$ and thus keeps the left-right symmetry, while $\mathcal{S}_-$ is odd. We have to take $\langle \mathcal{S}_-\rangle=0$ and $\langle \mathcal{S}_+\rangle$ to be different from zero for the breaking. It keeps $g_L = g_R[g_L=g_{SU(3)_L},\ g_R=g_{SU(3)_R}]$ for $\mu \geqslant M_I$. The reason is that at the intermediate scale M_I the $SU(2)_L$ gauge coupling $g_2(\mu)$ and the hypercharge coupling $g_1(\mu)$ have to respect the $SU(3)_L\times SU(3)_R$ symmetry. Since the $U(1)_Y$ hypercharge is a combination of Y_L, I_{3R}, and Y_R, according to Eq. (2.2), the intermediate symmetry automatically requires the matching

$$g_2(M_I)=g_1(M_I)=g_L(M_I)=g_R(M_I),$$
$$g_L(\mu)=g_R(\mu) \quad \text{for} \quad \mu\geqslant M_I. \tag{8.1}$$

The relation $g_L(\mu)=g_R(\mu)$ for $\mu\geqslant M_I$ holds even at the quantum level since it is protected by $\mathcal{D}_{LR}$ parity. As a consequence, M_I is fixed by the meeting point of g_2 and g_1. From thereon the two curves continue as a single one up to $M_{\rm GUT}$ where $g_L=g_R$ unifies with $g_C=g_{SU(3)_C}$. For this to happen the states $H_A(6,3,1)$ and $H_A(\overline{3},\overline{6},1)$ will play a central role as we will see shortly.

The details are as follows: Below M_I, the field content consists of the fermionic generations of the standard model together with two light Higgs doublets and the three Dirac particles $D(D,\hat{D})$, $L^0(L_1^1,L_2^2)$, $L^-(L_2^1,L_1^2)$. The two Higgs doublets are $H_1^{1,2}$, $H_2^{1,2}$ with $\langle H_1^1\rangle^2+\langle H_2^2\rangle^2=v_0^2\leqslant v^2=(174\ \text{GeV})^2$. v_0 will be smaller than v in case an additional Higgs meson with standard model quantum numbers has a nonzero VEV.

The corresponding b factors for the evolution of the couplings are

$$(b_1,b_2,b_3)=\left(\frac{21}{5},-3,-7\right), \tag{8.2}$$

as obtained from the standard model fermions and two Higgs doublets. The additional b factors for the D's and L's for each generation are

$$(b_1,b_2,b_3)^D=\left(\frac{4}{15},0,\frac{2}{3}\right),\quad (b_1,b_2,b_3)^L=\left(\frac{2}{5},\frac{2}{3},0\right). \tag{8.3}$$

Apart from these additional states, there are more scalar doublets $H_d^m(m=1,2,3,4)$, which are involved in the construction of the fermion sector. One of them comes from $H(\overline{3},3,1)$ and the other three from $H_A(\overline{3},3,1)$. H and H_A also contain two isosinglet fields H_+^n $(n=1,2)$ carrying the same $U(1)_Y$ charge as e^+. If some of the corresponding states with masses $\mu(H_d^m)$ and $\mu(H_+^m)$ lie below M_I, each of them will contribute to the b factors according to

$$(b_1,b_2,b_3)^{H_d}=\left(\frac{1}{10},\frac{1}{6},0\right),\quad (b_1,b_2,b_3)^{H_+}=\left(\frac{1}{5},0,0\right). \tag{8.4}$$

Four more Higgs components which are SM singlets could also be relatively light, but they do not contribute to the running of the gauge couplings. Thus, the solution of the renormalization group (RG) equation (at the one-loop level) for the gauge couplings at M_I reads

$$\alpha_a^{-1}(M_I)=\alpha_a^{-1}(M_Z)-\frac{b_a}{2\pi}\ln\frac{M_I}{M_Z}-\frac{b_a^{H_d}}{2\pi}\sum_m \ln\frac{M_I}{\mu(H_d^m)} -\frac{b_a^{H_+}}{2\pi}\ln\frac{M_I^2}{\mu(H_+^1)\mu(H_+^2)}-\frac{b_a^D}{2\pi}\ln\frac{M_I^3}{M_{D_1}M_{D_2}M_{D_3}} -\frac{b_a^L}{2\pi}\ln\frac{M_I^3}{M_{L_1}M_{L_2}M_{L_3}}. \tag{8.5}$$

Here b_a^D, M_D, and b_a^L, M_L denote masses and b factors of D and L states, respectively. The matching of g_1 and g_2 at M_I gives

$$\ln\frac{M_I}{M_Z}=\frac{5\pi}{18}[\alpha_1^{-1}(M_Z)-\alpha_2^{-1}(M_Z)]+\frac{1}{108}\sum_m \ln\frac{M_I}{\mu(H_d^m)} -\frac{1}{36}\ln\frac{M_I^2}{\mu(H_+^1)\mu(H_+^2)}-\frac{1}{27}\ln\frac{M_{L_1}M_{L_2}M_{L_3}}{M_{D_1}M_{D_2}M_{D_3}}. \tag{8.6}$$

At the GUT scale we should have $M_{L_i}=M_{D_i}$. According to Secs. IV–VI, $M_{L_i}\simeq M_{D_i}$ should hold approximately also at lower scales, since they are determined by $\langle H_3^3\rangle$. Thus, for the determination of M_I we can safely neglect the last term in Eq. (8.6). Taking the masses $\mu(H_d^m)\simeq\mu(H_+^n)\simeq M_I$, also the second term can be neglected. With $\alpha_1^{-1}(M_Z)=59$ and $\alpha_2^{-1}(M_Z)=29.6$ we then obtain, for M_I, the breaking point of the intermediate symmetry, $M_I\simeq 1.3\times 10^{13}$ GeV. According to our model, however, one extra Higgs $SU(2)_L$ doublet—namely, $(H_A)_2^{1,2}$—should have a mass much below M_I, as was discussed in Sec. III. The small VEV found for it, in Sec. IV, supported this view. Let us thus take its mass $\nu(H_{A2}^{1,2})=M_A=M_{D_3}\approx 4\times 10^4$ TeV, which is far above the lowest allowed value ($\sim$500 TeV) and does not lead to flavor changing neutral currents. With this value, the second term in Eq. (8.6) leads only to a slight increase of M_I: $M_I \simeq 1.5\times 10^{13}$ GeV. In general, the value of M_I is rather stable with respect to modifications of our model concerning the Higgs sectors $H(\overline{3},3,1)$ and $H_A(\overline{3},3,1)$. It is highly interesting that the value obtained for M_I is close (see [14]) to the phenomenologically obtained mass scale $(\lambda_S F^{\{2,2\}})$ necessary to describe the mass-squared difference observed in atmospheric neutrino oscillations. Moreover, the same scale also describes the breaking point of the left-right symmetry.

For the precise calculation of $\alpha_L(M_I)=\alpha_R(M_I) =\alpha_{1,2}(M_I)$ from Eq. (8.5), we need input masses for the D quarks and the leptons L. The mass of the third generation D quark we take is based on the discussion about an approximate left-right symmetry in the H and H_A sectors. We use

$$M_{D_3}\simeq e_3^3(G^{D\hat{D}})_{33}\simeq 4\times 10^4 \ \ \text{TeV}. \tag{8.7}$$

Before renormalization, the lepton L_3 has the same mass. The ratios for the generation splitting of these quarks and leptons are $\sigma^4{:}\sigma^2{:}1$. The corresponding input in Eq. (8.5) allows us now to calculate the values $\alpha_3(M_I)$ and $\alpha_1(M_I) =\alpha_2(M_I)$, which can then be used as initial conditions to go up to $M_{\rm GUT}$. After the study of the Yukawa coupling unification at $M_{\rm GUT}$, one can go back to the scales of the D and L states to find renormalized values for their masses (see the next section). The corresponding change of Eq. (8.5) will little affect the values of α_3 and $\alpha_1=\alpha_2$ at M_I, from which one can start again. The result is

$$\alpha_3^{-1}(M_I)=\alpha_C^{-1}(M_I)\simeq 31.43,$$

$$\alpha_1^{-1}(M_I)=\alpha_2^{-1}(M_I)=\alpha_{L,R}^{-1}(M_I)=35.63. \tag{8.8}$$

The D and L masses, found this way, are quoted in Sec. VII and have already been used in form of the mass matrices $M_D=e_3^3G^{D\hat{D}}$ and $M_E=e_3^3G^{L\overline{L}}$ in Secs. IV and V.

Above the scale of M_I G_{333} is unbroken and the quark-lepton states are unified together with the D states in $Q_L(3,1,\overline{3})$, $Q_R(1,\overline{3},3)$ and the leptons L in $L(\overline{3},3,1)$ multiplets. For the fermion masses we needed besides the VEVs from $H(\overline{3},3,1)$ also those from $H_A(\overline{3},3,1)$. We take the masses of these Higgs bosons to be negligible for scales above M_I [similar to the mass of $H(\overline{3},3,1)$]. In fact, we have to do that because some members lie below M_I and the full $(SU(3))^2$ symmetry must hold above M_I. The corresponding b factors for $\mu \geqslant M_I$ are, therefore,

$$(b_L,b_R,b_C)^{M_I}=(-4,-4,-5). \tag{8.9}$$

With these values the meeting point $g_L=g_R=g_C$ would be above the Planck scale because $b_L=b_R$ is not much different from b_C. We know, however, from our treatment of the charged lepton sector, that the vacuum expectation values of $H_A(6,3,1)$ and $H_A(\overline{3},\overline{6},1)$ play an important role. Since lying above the M_I scale, the masses of these two Higgs bosons are equal due to the left-right $\mathcal{D}_{LR}$ symmetry: $M(6,3,1)=M(\overline{3},\overline{6},1)\equiv M_6$. They contribute to the renormalization with the b factors

$$(b_L,b_R,b_C)^6=\left(\frac{7}{2},\frac{7}{2},0\right). \tag{8.10}$$

We now have, for $\mu \geqslant M_I$,

$$\alpha_C^{-1}(\mu)=\alpha_3^{-1}(M_I)-\frac{b_C^{M_I}}{2\pi}\ln\frac{\mu}{M_I} \tag{8.11}$$

and

$$\alpha_{L,R}^{-1}(\mu)=\alpha_{L,R}^{-1}(M_I)-\frac{b_{L,R}^{M_I}}{2\pi}\ln\frac{\mu}{M_I}-\theta(\mu-M_6)\frac{b_{L,R}^{6}}{2\pi}\ln\frac{\mu}{M_6}. \tag{8.12}$$

The grand unification energy $M_{\rm GUT}$ can now be obtained by setting $\mu=M_{\rm GUT}$ and equating Eqs. (8.11) and (8.12). $M_{\rm GUT}$ depends on M_6 and increases with increasing M_6. It is interesting that even for low values of M_6 close to M_I we get a large values for $M_{\rm GUT}$. For instance for $M_6\simeq 3M_I$ we have $M_{\rm GUT}\simeq 10^{16}$ GeV. Already for $M_6\simeq 5\times 10^{16}$ GeV we get $M_{\rm GUT}\simeq 3\times 10^{18}$ GeV. Therefore, in our model we have $M_{\rm GUT}\gtrsim 10^{16}$ GeV, which thus ensures proton stability compatible with present experimental limits. But we still have to see which restrictions are forced on us by top-bottom-tau unification.

B. Top-bottom-tau unification

In this section we study the running of the Yukawa couplings and their unification. We concentrate on the unification of the third-generation couplings λ_t, λ_b, λ_τ for the top, bottom, and tau fermions, respectively. In the SM, because of the small mixings in the quark sector, their evolution is little affected by the other couplings. In the considered model, the situation is different. Apart from the fermion couplings to $H(\overline{3},3,1)$ [first coupling in Eq. (2.8)] also couplings with H_A are important. In particular, the Higgs fields $H_A^6(6,3,1)$, $H_A^{\overline{6}}(\overline{3},\overline{6},1)$ with common mass $M_6<M_{\rm GUT}$ are important for gauge coupling unification. Therefore, above the scale M_I, the following Yukawa couplings are relevant for renormalization:

$$Q_LG_QQ_RH+\frac{1}{2}LG_LLH+Q_LA^QQ_RH_A+\frac{1}{2}LA^LLH_A^6 + \frac{1}{2}L\overline{A}^LLH_A^{\overline{6}}. \tag{8.13}$$

We have to distinguish the coupling matrices G_Q, G_L, A^Q, A^L, but have $A^L=\overline{A}^L$ due to the left-right $\mathcal{D}_{LR}$ symmetry which holds above $\mu=M_I$. The elements of the diagonal matrices G_Q, G_L determine the masses M_{D_i}, M_{L_i}, respectively.

As a consequence of the first term in Eq. (8.13) one has already at the G_{333} level top-bottom unification: $\lambda_t(\mu)$ and $\lambda_b(\mu)$ must unify at M_I and evolve then further as a single coupling $\lambda_{Q_3}(\mu)$. This coupling should then unify with $\lambda_\tau(\mu)=\lambda_{L_3}(\mu)$ at $\mu=M_{\rm GUT}$.

Below the scale M_I the coupling matrices G_Q, G_L, A^Q, and $A^L=\overline{A^L}$ split into more matrices depending on the Higgs field components they are attached to. In an obvious notation we have

$$G_Q\to(G^{u\hat{u}},\quad G^{d\hat{d}},\quad G^{D\hat{D}}),$$

$$G_L\to(G^{e^-e^+},\quad G^{L\overline{L}},\quad G^{\nu\hat{\nu}}),$$

$$A^Q\to(A^{d\hat{d}},\quad A^{d\hat{D}},\quad A^{D\hat{d}}),$$

$$A^L\to(A^{e^-e^+},\quad A^{E^-e^+},\quad A^{e^-E^+}). \tag{8.14}$$

We left out the matrices $A^{D\hat{D}}$, $A^{E^-E^+}$ and additional matrices from the neutral lepton sector. They are multiplied with VEVs which are—in our model—small compared to competing terms in the same channel. In the approximations we use for the renormalization the G matrices remain diagonal and the diagonal elements of the matrices A remain zero. Furthermore, the matrices connected to $\overline{A^L}$ are the same as the ones from A^L. But the matrices derived from $A^L=\overline{A^L}$ are no longer strictly antisymmetric.

The most important elements of the matrices (8.14) are the (3,3) elements of the G's and the (2,3) and (3,2) elements of the A's: $(G^{u\hat{u}})_{33}=\lambda_t(\mu)$, $(G^{e^-e^+})_{33}=\lambda_\tau(\mu)$, etc. For the matrix elements of $A^{d\hat{d}}$ we define

$$(A^{d\hat{d}})_{23}={\rm i}\overline{\lambda}_A(\mu),\quad (A^{d\hat{d}})_{32}=-{\rm i}\hat{\lambda}_A(\mu). \tag{8.15}$$

Clearly, we have $\overline{\lambda}_A(\mu)=\hat{\lambda}_A(\mu)$ for $\mu\geqslant M_I$.

There is a restriction from the mass of the vector boson W for a combination of the VEVs multiplying the coupling matrices. With the notation

$$e_1^1=v_0\sin\beta,\quad e_2^2=v_0\cos\beta, \tag{8.16}$$

the condition is

$$\begin{aligned}v_0^2+(f_2^2)^2+(f_3^2)^2+(f_{\{1,3\}}3)^2+(f^{2\{13\}}-f_{\{13\}}2)^2\\=(174\ \ {\rm GeV})^2.\end{aligned} \tag{8.17}$$

Since e_1^1 and e_2^2 contribute to the masses of the third generation, the v_0^2 term should be the dominant one. At the scale $\mu=M_Z$ one has

$$\lambda_t(M_Z)=\frac{m_t}{v_0\sin\beta},\quad \lambda_b(M_Z)=\frac{m_b^0}{v_0\cos\beta},$$

$$\lambda_\tau(M_Z)=\frac{m_\tau^0}{v_0\cos\beta}. \tag{8.18}$$

Here m_b^0 and m_τ^0 are a little smaller than m_b and m_τ, respectively, since they refer to the diagonal parts of the down quark and charged lepton mass matrices. In Secs. IV and V, we found $m_b^0/m_b\simeq0.989$, $m_\tau^0/m_\tau\simeq0.966$.

We can now set up the renormalization group equations for λ_t, λ_b, λ_τ, $\overline{\lambda}_A$, and $\hat{\lambda}_A$. They are connected with each other and—as a result of the $SU(3)_L\times SU(3)_R$ symmetry at $\mu\geqslant M_I$—no other coupling intervenes. Below M_I we have, for $\eta_t=\lambda_t^2/4\pi$, $\eta_b=\lambda_b^2/4\pi$, $\eta_\tau=\lambda_\tau^2/4\pi$, $\hat{\eta}_A=(\hat{\lambda}_A)^2/4\pi$, and $\overline{\eta}_A=(\overline{\lambda}_A)^2/4\pi$,

$$\begin{aligned}2\pi\eta_t'=\frac{9}{2}\eta_t^2+\frac{1}{2}\eta_t\eta_b-\eta_t\left(\frac{17}{20}\alpha_1+\frac{9}{4}\alpha_2+8\alpha_3\right)\\+\theta(\mu-M_A)\frac{1}{2}\eta_t\hat{\eta}_A,\end{aligned} \tag{8.19}$$

$$\begin{aligned}2\pi\eta_b'=\frac{9}{2}\eta_b^2+\frac{1}{2}\eta_b\eta_t+\eta_b\eta_\tau-\eta_b\left(\frac{1}{4}\alpha_1+\frac{9}{4}\alpha_2\right.\\\left.+8\alpha_3\right)+\theta(\mu-M_A)\eta_b\left(\frac{1}{2}\hat{\eta}_A+\overline{\eta}_A\right),\end{aligned} \tag{8.20}$$

$$2\pi\eta_\tau'=\frac{5}{2}\eta_\tau^2+3\eta_\tau\eta_b-\eta_\tau\left(\frac{9}{4}\alpha_1+\frac{9}{4}\alpha_2\right), \tag{8.21}$$

$$\begin{aligned}2\pi\hat{\eta}_A'=\frac{1}{2}\hat{\eta}_A\left(\eta_b+\eta_t-\frac{1}{5}\alpha_1-3\alpha_2-16\alpha_3\right)+\theta(\mu\\-M_A)\\\times\hat{\eta}_A\left(\frac{9}{2}\hat{\eta}_A+3\overline{\eta}_A-\frac{3}{20}\alpha_1-\frac{3}{4}\alpha_2\right),\end{aligned} \tag{8.22}$$

$$\begin{aligned}2\pi\overline{\eta}_A{}'=\overline{\eta}_A\left(\eta_b-\frac{1}{10}\alpha_1-\frac{3}{2}\alpha_2-8\alpha_3\right)+\theta(\mu-M_A)\\\times\overline{\eta}_A\left(\frac{9}{2}\overline{\eta}_A+3\hat{\eta}_A-\frac{3}{20}\alpha_1-\frac{3}{4}\alpha_2\right).\end{aligned} \tag{8.23}$$

At $\mu=M_I$ the matching

$$\eta_t=\eta_b\equiv\eta_{Q_3},\quad \eta_\tau\equiv\eta_{L_3},\quad \overline{\eta}_A=\hat{\eta}_A\equiv\eta_A \tag{8.24}$$

is required.

Above M_I we have for η_{Q_3}, η_{L_3}, η_A, and $\eta_{AL}=(\lambda_A^L)^2/4\pi$ the equations

$$2\pi\eta_{Q_3}'=6\eta_{Q_3}^2+\eta_{Q_3}\eta_{L_3}+3\eta_{Q_3}\eta_A-\eta_{Q_3}8(\alpha_{L,R}+\alpha_C), \tag{8.25}$$

$$\begin{aligned}2\pi\eta_{L_3}'=2\eta_{L_3}^2+3\eta_{L_3}\eta_{Q_3}-\eta_{L_3}\frac{56}{3}\alpha_{L,R}\\+\theta(\mu-M_6)3\eta_{L_3}\eta_{AL},\end{aligned} \tag{8.26}$$

$$2\pi\eta_A'=9\eta_A^2+\frac{3}{2}\eta_A\eta_{Q_3}-\eta_A8(\alpha_{L,R}+\alpha_C), \tag{8.27}$$

$$2\pi\eta'_{AL}=\eta_{AL}\left(\frac{1}{2}\eta_{L_3}-16\alpha_{L,R}\right)$$
$$+\theta(\mu-M_6)\,\eta_{AL}\left(4\eta_{AL}-\frac{14}{3}\alpha_{L,R}\right). \quad (8.28)$$

$i\lambda_A^L$ is the (2,3) element of $A^L=\overline{A}^L$ and is only needed above M_I. The matching condition at $M_{\rm GUT}$ for the final unification of the couplings reads

$$\eta_{Q_3}(M_{\rm GUT})=\eta_{L_3}(M_{\rm GUT}),\quad \eta_A(M_{\rm GUT})=\eta_{AL}(M_{\rm GUT}). \quad (8.29)$$

The procedure of finding a solution with gauge and top-bottom-tau unification is the following: A given value of $M_{\rm GUT}\gtrsim 10^{16}$ GeV (otherwise no solution is possible) fixes M_6. Taking then trial values for $\eta_{Q_3}(M_{\rm GUT})$ and $\eta_A(M_{\rm GUT})$ and solving Eqs. (8.25)–(8.28) gives their values at M_I. These values determine $\eta_t(M_I)=\eta_b(M_I)$, $\eta_\tau(M_I)$, and $\overline{\eta}_A(M_I)=\hat{\eta}_A(M_I)$. The renormalization group equations (8.19)–(8.23) allow us then to calculate $\lambda_t(M_Z)$, $\lambda_b(M_Z)$, $\lambda_\tau(M_Z)$. Clearly, the input values $\eta_{Q_3}(M_{\rm GUT})$ and $\eta_A(M_{\rm GUT})$ have now to be changed such that λ_τ/λ_b becomes equal to m_τ^0/m_b^0 and λ_t, λ_b are in the perturbative region—i.e., $\lesssim 3$. If this can be achieved, one can calculate from Eq. (8.18) v_0^2 and $\tan\beta$:

$$v_0^2=\frac{m_t^2}{\lambda_t^2}+\frac{(m_b^0)^2}{\lambda_b^2},\quad \tan\beta=\frac{m_t}{m_b^0}\frac{\lambda_b}{\lambda_t}. \quad (8.30)$$

Of course, only solutions with $v_0<v=174$ GeV are acceptable.

C. Numerical solution for $M_{\rm GUT}=2\times10^{17}$ GeV, $M_{D_3}=M_A=4\times10^4$ TeV

Here we present a numerical solution of the problem of gauge and Yukawa coupling unification in E_6, which satisfies all above-mentioned requirements. We choose the unification scale to be 2×10^{17} GeV, the masses of the heaviest D state and the Higgs field $(H_A)_2^{1,2}$ both equal to 3.98×10^7 GeV.

Further input values are the third-generation masses

$$m_t(M_Z)=173\ {\rm GeV},\quad m_b^0(M_Z)=2.859\ {\rm GeV},$$
$$m_\tau^0(M_Z)=1.689\ {\rm GeV}, \quad (8.31)$$

the three gauge coupling constants at $\mu=M_Z$ and a suitable value for $\eta_{t,b,\tau}$ at $M_{\rm GUT}$:

$$\eta_{t,b,\tau}(M_{\rm GUT})\simeq 0.0381. \quad (8.32)$$

For this latter value all couplings remain in the perturbative region and $v_0<v=174$ GeV. As a result we find the solution

$$M_6\simeq 1.6\times10^{16}\ {\rm GeV},\quad M_I\simeq 1.5\times10^{13}\ {\rm GeV},$$
$$\eta_A(M_{\rm GUT})=\eta_{AL}(M_{\rm GUT})\simeq 0.0336, \quad (8.33)$$

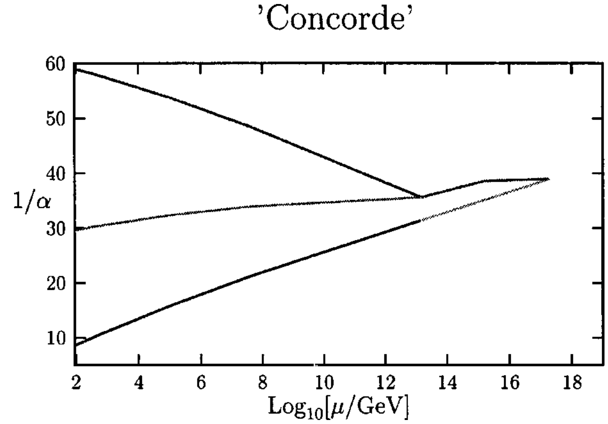

FIG. 1. Unification of gauge couplings. $M_I\simeq 1.5\times10^{13}$ GeV, $M_6\simeq 1.6\times10^{16}$ GeV, $M_{\rm GUT}\simeq 2\times10^{17}$ GeV, and $\alpha_G^{-1}\simeq 39$.

with the consequences

$$\alpha_G^{-1}(M_{\rm GUT})\simeq 38.99,$$
$$\eta_{t,b}(M_I)=\eta_{Q_3}(M_I)\simeq 0.0412,$$
$$\eta_\tau(M_I)=\eta_{L_3}(M_I)\simeq 0.0536,$$
$$\hat{\eta}_A(M_I)=\overline{\eta}_A(M_I)=\eta_A(M_I)\simeq 0.0368,$$
$$\eta_{AL}(M_I)\simeq 0.0598, \quad (8.34)$$

$$\lambda_\tau(M_Z)=\lambda_b(M_Z)\frac{m_\tau^0}{m_b^0}\simeq 0.612,\quad \lambda_t(M_Z)\simeq 1.127,$$
$$v_0\simeq 153.48\ {\rm GeV},\quad \tan\beta\simeq 55.59. \quad (8.35)$$

From the value found for v_0, we can now determine $(f_3^2)^2+(f_{\{13\}}3)^2$ from Eq. (8.17). Using then Eqs. (4.13), (5.4) together with $e_3^3(G^{D\hat{D}})_{33}=3.98\times10^7$ GeV and $|f_3^2|=|f_2^3|$, $|f_{\{13\}3}|=|f^{3\{13\}}|$, we finally get

$$f_3^2=\pm 43.504\ {\rm GeV},\quad f_2^3=\mp 43.504\ {\rm GeV},$$
$$f_{\{13\}3}=f^{3\{13\}}=69.484\ {\rm GeV}. \quad (8.36)$$

The solution for the gauge coupling and Yukawa coupling unification given here has been applied in the previous sections, in particular for the evaluation of the renormalization parameters for all the different mass matrices.

In Fig. 1—"Concorde"—we show the evolution of the gauge couplings and their unification. Figure 2—"Bermuda triangle"—exhibits the running of the Yukawa couplings η_t, η_b, η_τ, and their unification. In Fig. 3—"desert spider"—the running of the (2,3) and (3,2) elements of the A matrices and their unification is presented. In these evaluations the splittings between the masses M_{D_i} and M_{L_i} have been taken into account.

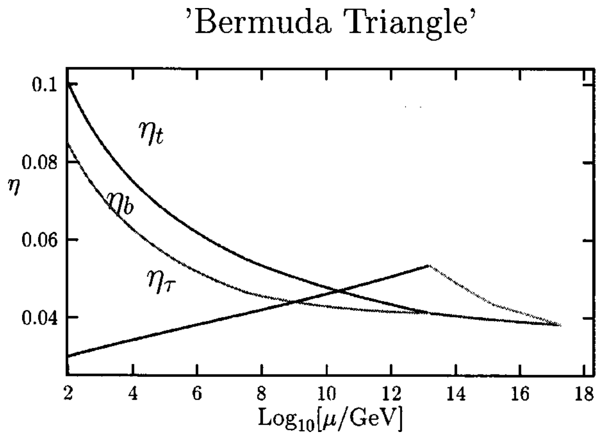

FIG. 2. t-b-τ unification: $\lambda_t(M_{\mathrm{GUT}})=\lambda_b(M_{\mathrm{GUT}})=\lambda_\tau(M_{\mathrm{GUT}})\simeq 0.692$.

IX. CONCLUSIONS

The E_6 model presented has many attractive features. Only few input data are sufficient to obtain a realistic picture of the fermion masses and their mixings. The presence of new heavy fermions in the ''desert'' plays an important role even for the mass matrices of the SM particles. All generation mixings and $\mathcal{C}P$ violations arise from a single antisymmetric matrix A, which mixes the light fermions but also the light with the heavy fermions. The latter effect also contributes in an important way to the eigenvalues of the quark and lepton mass matrices. For instance, the main part of the μ meson mass and of the strange quark mass is generated by virtual transitions to heavy fermions. As a side remark we note that the antisymmetric generation mixing matrix found here could lead to significant effects in rare weak decay processes with fixed phases of the new contributions. The matrix A, in combination with G, is also responsible for the bilarge mixing of the light neutrinos and their oscillation pattern. In the limit of no renormalization effects, the neutrino mixing is bimaximal.

Those heavy new particles, which form 10-plets with respect to $SO(10)$, have a hierarchical spectrum similar to the spectrum of the up quarks. The lightest ones are expected to lie in the low-TeV region.

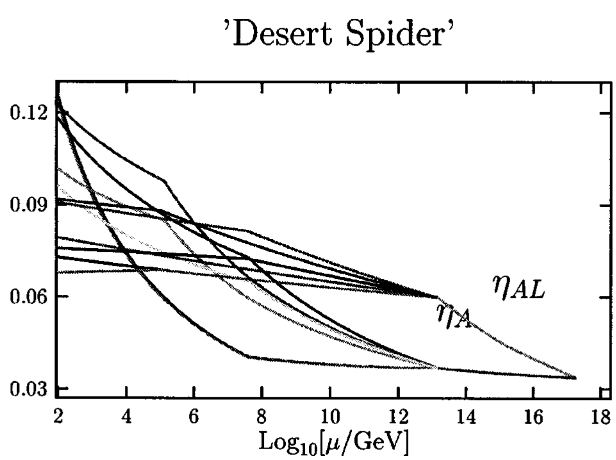

FIG. 3. Unification of (2,3), (3,2) elements of A^Q and A^L matrices. $A^Q_{23}(M_{\mathrm{GUT}})=A^L_{23}(M_{\mathrm{GUT}})\simeq 0.65i$.

The group E_6 provides new insights into the unification of the three gauge couplings and into the unification of the Yukawa couplings of top, bottom, and tau. The intermediate symmetry $SU(3)_L\times SU(3)_R\times SU(3)_C$ with a discrete left-right symmetry plays a decisive role. The breaking point of this intermediate symmetry is fixed by the known gauge couplings g_1 and g_2. Simultaneously it determines the mass scales for the light and heavy neutrinos. We achieved a solution of the gauge and Yukawa coupling unification with strongly constraint parameters. It describes the evolution and the final convergence of many coupling matrices which differ significantly at low energies. The solution allows us to calculate quite a number of properties such as transition matrices from heavy to light fermions, Majorana phases, and the double-β-decay matrix element. As a result of the high unification scale ($>10^{16}$ GeV), the model adequately suppresses dimension-6 operators which induce nucleon decays. The proton lifetime is above the presently accessible range.

The presented E_6 model can be supersymmetrized without changing the construction of the Yukawa sector. A supersymmetric version would, however, affect the coupling unification picture given here.

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APPENDIX: NEUTRINO MASS MATRIX RENORMALIZATION

Here we will perform the renormalization analysis for the neutrino sector.

Dimension-5 operators, responsible for neutrino masses, are generated by integrating out the ''right-handed'' states $(L_2^3)_\alpha=\hat{\nu}_\alpha$. The $\hat{\nu}_\alpha$ masses are determined by the matrix $F^{\{2,2\}}S$. The matrix S describes the generation-dependent Majorana couplings of these states to the symmetric sextet component of the Higgs field H_S. In our model S is postulated to be the bilinear matrix product in generation space (6.8). We take this form to be valid at M_I with $G\to G_L$ and $A\to A^L$. With the appropriate scaling factors, at $\mu=M_I$ the matrix S has the form

$$S/\lambda_S\simeq\begin{pmatrix} (\kappa_1^L)^2\sigma^8, & -\kappa_2^L\kappa_a^L\sigma^6 x, & -\sigma^4 x \\ -\kappa_2^L\kappa_a^L\sigma^6 x, & (\kappa_2^L)^2\sigma^4, & \dfrac{\sigma^3}{\sqrt{2}}x \\ -\sigma^4 x, & \dfrac{\sigma^3}{\sqrt{2}}x, & 1 \end{pmatrix}. \tag{A1}$$

Again, as in Eq. (6.9), only the leading terms in σ are exhibited here. The renormalization coefficients κ are shown in Eq. (A23). At the scale M_I, the mass matrix for the states $\hat{\nu}_\alpha$ is

$$M_{\nu\nu}^{\alpha\beta}(M_I)=F^{\{2,2\}}S^{\alpha\beta}. \tag{A2}$$

The eigenvalues are $\mu_1\simeq\sigma^8[(\kappa_1^L)^2-(\kappa_a^L)^2x^2-x^2+(\kappa_a^L/\kappa_2^L)\sqrt{2}x^3\sigma]\lambda_S F^{\{2,2\}}$, $\mu_2\simeq\sigma^4(\kappa_2^L)^2\lambda_S F^{\{2,2\}}$, and $\lambda_S F^{\{2,2\}}$. As we can see, these scales are separated by large distances. Because of this fact, strong renormalization effects occur. They are the cause of the difference between the neutrino mass matrices (6.10) and (6.13). The decoupling of the three $\hat{\nu}_\alpha$ states occurs step by step. Thus the renormalization has to be performed separately in each energy interval [23,15]. By generalizing the results of Ref. [15] we present model-independent formula for the running of the neutrino mass matrix m_ν and then apply them to our model.

The couplings in the neutrino sector involve Dirac and Majorana mass terms. One can choose a basis in which the mass matrix for the heavy neutrinos is diagonal. Thus, without loss of generality, one can write the coupling terms in the form

$$\nu_\alpha(\lambda^{\rm Dirac})^{\alpha\beta}\hat{\nu}_\beta+\frac{1}{2}\mu_\alpha\hat{\nu}_\alpha\hat{\nu}_\alpha. \tag{A3}$$

The matrix $\lambda^{\rm Dirac}$ is related to the original Dirac matrix $\lambda_0^{\rm Dirac}=G^{\nu\hat{\nu}}e_1^1=G_L e_1^1$ (for $\mu\geqslant M_I$) via

$$\lambda^{\rm Dirac}=\lambda_0^{\rm Dirac}U_S^T. \tag{A4}$$

Here the unitary matrix U_S diagonalizes the matrix $M_{\hat{\nu}\hat{\nu}}$:

$$(U_S M_{\hat{\nu}\hat{\nu}}U_S^T)^{\alpha\beta}=\delta^{\alpha\beta}\mu_\alpha. \tag{A5}$$

By integrating out the state $\hat{\nu}_\alpha$, the light neutrino mass matrix gets a contribution at the scale μ_α. Without renormalization effects, m_ν would have the form

$$m_\nu^{\alpha\beta}=-(Y_1+Y_2+Y_3)^{\alpha\beta}, \tag{A6}$$

where

$$Y_1^{\alpha\beta}=\frac{1}{\mu_1}(\lambda^{\rm Dirac})^{\alpha 1}(\lambda^{\rm Dirac})^{\beta 1},$$

$$Y_2^{\alpha\beta}=\frac{1}{\mu_2}(\lambda^{\rm Dirac})^{\alpha 2}(\lambda^{\rm Dirac})^{\beta 2},$$

$$Y_3^{\alpha\beta}=\frac{1}{\mu_3}(\lambda^{\rm Dirac})^{\alpha 3}(\lambda^{\rm Dirac})^{\beta 3}. \tag{A7}$$

The division of the mass matrix in three parts is convenient in order to see the contributions coming from each integrated state $\hat{\nu}_\alpha$. Each $d=5$ operator (Y_i), generated on the scale μ_i, runs from this scale down to M_Z according to the RG equations

$$4\pi\frac{d}{dt}Y_i^{\alpha\beta}=Y_i^{\alpha\beta}(6\eta_t-3\alpha_2),\quad \text{with}\quad (\alpha,\beta)\neq(3,3), \tag{A8}$$

$$4\pi\frac{d}{dt}Y_i^{3\alpha}=Y_i^{3\alpha}\left(6\eta_t+\frac{1}{2}\eta_\tau-3\alpha_2\right),\quad \text{with}\quad \alpha\neq 3, \tag{A9}$$

$$4\pi\frac{d}{dt}Y_i^{33}=Y_i^{33}(6\eta_t+\eta_\tau-3\alpha_2). \tag{A10}$$

These equations have the solutions

$$Y_i^{\alpha\beta}(\mu')=Y_i^{\alpha\beta}(\mu)r_g(\mu')r_g^{-1}(\mu), \tag{A11}$$

$$Y_i^{3\alpha}(\mu')=Y_i^{3\alpha}(\mu)r_g(\mu')r_\tau(\mu')r_g^{-1}(\mu)r_\tau^{-1}(\mu). \tag{A12}$$

$$Y_i^{33}(\mu')=Y_i^{33}(\mu)r_g(\mu')r_\tau^2(\mu')r_g^{-1}(\mu)r_\tau^{-2}(\mu), \tag{A13}$$

where

$$r_g(\mu)=\rho_t^{-6}(\mu)\rho_{\alpha_2}^3(\mu),\quad r_\tau(\mu)=\rho_\tau^{-1/2}(\mu). \tag{A14}$$

Before the emergence of the $d=5$ operators, only the Dirac couplings run. They obey the equations [24]

$$4\pi\frac{d}{dt}(\lambda^{\rm Dirac})^{\alpha\beta}=0,\quad (\alpha,\beta)\neq(3,3), \tag{A15}$$

$$4\pi\frac{d}{dt}(\lambda^{\rm Dirac})^{3\alpha}=\lambda^{3\alpha}\frac{1}{2}\eta_\tau,\quad \alpha\neq 3. \tag{A16}$$

According to these equations, we have, for $\mu<M_I$,

$$(\lambda^{\rm Dirac})^{\alpha\beta}(\mu)=(\lambda^{\rm Dirac})^{\alpha\beta}(M_I),$$
$$(\lambda^{\rm Dirac})^{3\alpha}(\mu)=(\lambda^{\rm Dirac})^{3\alpha}(M_I)r_\tau(\mu). \tag{A17}$$

With all these results, one can write down the light neutrino mass matrix at the scale $\mu=M_Z$:

$$m_\nu^{\alpha\beta}(M_Z)=-\mathcal{Y}^{\alpha\beta}r_g(M_Z), \tag{A18}$$

where

$$\begin{aligned}\mathcal{Y}^{\alpha\beta}=&\sum_{i,j=1,2}\delta^{\alpha i}\delta^{\beta j}[Y_3^{ij}+r_g^{-1}(\mu_2)Y_2^{ij}+r_g^{-1}(\mu_1)Y_1^{ij}]\\&+\sum_{i=1,2}(\delta^{\alpha 3}\delta^{\beta i}+\delta^{\beta 3}\delta^{\alpha i})r_\tau(M_Z)[Y_3^{i3}+r_g^{-1}(\mu_2)Y_2^{i3}\\&+r_g^{-1}(\mu_1)Y_1^{i3}]\delta^{\alpha 3}\delta^{\beta 3}r_\tau^2(M_Z)[Y_3^{33}+r_g^{-1}(\mu_2)Y_2^{33}\\&+r_g^{-1}(\mu_1)Y_1^{33}].\end{aligned} \tag{A19}$$

The quantities Y_i are given in Eq. (A7) and the renormalization factors in Eq. (A14).

We now apply this result to our model. The matrix $(\lambda^{\rm Dirac})$ is built according to Eq. (A4) taking

$$\lambda_0^{\mathrm{Dirac}}(M_I)=\mathrm{Diag}(\kappa_1^L\sigma^4,\ -\kappa_2^L\sigma^2,\ 1)\lambda_\tau(M_I)e_1^1(M_I). \tag{A20}$$

The value of e_1^1 at the scale M_I can be calculated from the RG equation

$$4\pi\frac{d}{dt}e_1^1=e_1^1\left(-3\eta_t+\frac{9}{20}\alpha_1+\frac{9}{4}\alpha_2\right), \tag{A21}$$

knowing its value at the scale $\mu=M_Z$:

$$e_1^1(M_I)=e_1^1(M_Z)\rho_t^{-3}(M_Z)\rho_{\alpha_1}^{9/20}(M_Z)\rho_{\alpha_2}^{9/4}(M_Z). \tag{A22}$$

In our model, it turns out that $e_1^1(M_I)\simeq 124.52$ GeV.

The values of the factors appearing in Eq. (A1) are

$$\kappa_1^L\simeq 0.707,\ \ \kappa_2^L\simeq 0.746,\ \ \kappa_a^L\simeq 1.017. \tag{A23}$$

After constructing the matrix λ^{Dirac}, one can calculate all elements of Y_i according to Eq. (A7). The numerical values of the renormalization factors, appearing in Eqs. (A18), (A19), are

$$r_g(M_Z)\simeq 0.58,\quad r_g(\mu_1)\simeq 0.75,$$
$$r_g(\mu_2)\simeq 0.87,\quad r_\tau(M_Z)\simeq 0.96. \tag{A24}$$

The diagonalization of the neutrino mass matrix obtained in this way allows us to calculate the neutrino mixing and the ratio $R=\Delta m^2_{\mathrm{sol}}/\Delta m^2_{\mathrm{atm}}$ from the single parameter x. To obtain values for R and the mixing angles, which lie within experimental bounds, we have to use $x\simeq 2.8$ which is bit smaller than the value $x=3.5$ found without renormalization corrections [25]. The mixing is now no more bimaximal, but still bilarge. The results are discussed in the text (Sec. VI).

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