Chiral perturbation theory at $\mathcal{O}(a^2)$ for lattice QCD

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We construct chiral effective Lagrangian for two lattice theories: one with Wilson fermions and the other with Wilson sea fermions and Ginsparg-Wilson valence fermions. For each of these theories we construct the Symanzik action through $\mathcal{O}(a^2)$. The chiral Lagrangian is then derived, including terms of $\mathcal{O}(a^2)$, which have not been calculated before. We find that there are only few new terms at this order. Corrections to existing coefficients in the continuum chiral Lagrangian are proportional to a^2 and appear in the Lagrangian at $\mathcal{O}(a^2p^2)$ or higher. Similarly, $O(4)$ symmetry-breaking terms enter the Symanzik action at $\mathcal{O}(a^2)$, but contribute to the chiral Lagrangian at $\mathcal{O}(a^2p^4)$ or higher. We calculate the light meson masses in chiral perturbation theory for both lattice theories. At next-to-leading order, we find that there are no $\mathcal{O}(a^2)$ corrections to the valence-valence meson mass in the mixed theory due to the enhanced chiral symmetry of the valence sector.

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I. INTRODUCTION

Chiral perturbation theory (χPT) [1,2] plays an important role in the analysis of current lattice QCD data. Simulations with the quark masses as light as realized in nature are not feasible on present-day computers. Instead one simulates with heavier quark masses and performs a chiral extrapolation to the physical quark masses using the analytic predictions of χPT. To perform the chiral extrapolation one must first take the continuum limit of the lattice data, since χPT describes continuum QCD and is not valid for nonzero lattice spacing. However, it is common practice not to perform the continuum extrapolation and nevertheless fit the lattice data to continuum χPT, assuming that the lattice artifacts are small.

A strategy to reduce this systematic uncertainty was proposed in Refs. [3–7] (a different approach was taken in Refs. [8,9] in the strong coupling limit). There it was shown how the discretization effects stemming from a nonzero lattice spacing can be included in χPT. The basic idea is that lattice QCD is, close to the continuum limit, described by Symanzik's effective theory, which is QCD with additional higher dimensional terms [10–14]. The derivation of χPTfrom QCD can then be extended to this effective theory with additional symmetry-breaking parameters. The result is a chiral expansion in which the leading dependence on the lattice spacing is explicit. This idea was numerically examined in Ref. [15] for a theory with two dynamical sea quarks on a coarse lattice using the results of Ref. [7]. The characteristic chiral log behavior in the pseudo scalar meson mass and decay constant was observed.

A similar approach was taken in Ref. [16] for analyzing lattice theories with two types of lattice fermions—Wilson fermions for the sea quarks and Ginsparg-Wilson fermions for the valence quarks. The latter can be implemented using domain wall [17–19], overlap [20–24], perfect action [25,26], and chirally improved fermions [27,28]. There are several advantages in using different lattice fermions in numerical simulations. Since massless Ginsparg-Wilson fermions exhibit an exact chiral symmetry even at nonzero lattice spacing [29], it is possible to simulate such valence fermions with masses much smaller than the valence quark masses accessible using Wilson fermions [30,31]. This allows a wider numerical sampling of points in the chiral regime of QCD. In addition, the valence sector exhibits all the benefits stemming from the Ginsparg-Wilson relation [32], such as the absence of additive mass renormalization, of operator mixing among different chiral multiplets, and of lattice artifacts linear in the lattice spacing a [23–26,33,34].

In this paper we extend the results of both Refs. [7] and [16] by calculating the chiral Lagrangian including the $\mathcal{O}(a^2)$ lattice effects. There are various reasons for doing this. First, the lattice spacings in current unquenched simulations are not very small, so that neglecting the $\mathcal{O}(a^2)$ contributions might not be justified. Second, the use of nonperturbatively improved Wilson fermions in lattice simulations is becoming more common. The leading corrections for these fermions are of $\mathcal{O}(a^2)$ and hence need to be computed in order to know how the continuum limit is approached.

At $\mathcal{O}(a^2)$ many operators enter the Symanzik action and need to be taken into account for constructing the chiral Lagrangian. Four-fermion operators appear for the first time and operators that explicitly break Euclidean rotational symmetry are encountered. Nevertheless, the number of new op-

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erators in the chiral Lagrangian is rather small (three for the Wilson action and four for the mixed fermion theory). This is important in practical applications, since every new operator comes with an undetermined low-energy constant. These constants enter the analytic expressions for physical observables and too many free parameters limit the predictability of the chiral extrapolations.

The paper is organized as follows: χPT for the Wilson action is discussed in Sec. II, including the partially quenched case in Sec. II E. The mixed theory with Wilson sea and Ginsparg-Wilson valence quarks is treated in Sec. III. In Sec. IV, we discuss the chiral power counting and compute the pseudoscalar meson mass including the $\mathcal{O}(a^2)$ contributions for both cases. We end with some general comments in Sec. V.

II. WILSON ACTION

In this section we formulate the chiral effective theory for the Wilson lattice action. First, the Wilson action and its symmetries are briefly reviewed, then the local Symanzik action through $\mathcal{O}(a^2)$ is presented. Based on the symmetry properties of the Symanzik action we construct the chiral effective theory. Finally, we consider the extension to the partially quenched case.

A. Lattice action

We consider an infinite hypercubic lattice with lattice spacing a. The quark and antiquark fields are represented by ψ and $\bar{\psi}$, respectively. Wilson's fermion action [35] is given by

$$S_W = a^4 \sum_x \bar{\psi}(D_W + m_0)\psi(x),$$
$$D_W = \frac{1}{2}\{\gamma_\mu(\nabla_\mu^* + \nabla_\mu) - ar\nabla_\mu^*\nabla_\mu\}, \tag{1}$$

where m_0 denotes the $N_f \times N_f$ bare quark mass matrix and r the Wilson parameter. ∇_μ^*, ∇_μ are the usual covariant, nearest-neighbor backward and forward difference operators.

The Wilson action in Eq. (1) possesses several discrete symmetries—charge conjugation, parity—as well as an $SU(N_c)$ color gauge symmetry. The introduction of a discrete space-time lattice reduces the rotation symmetry group O(4) to the discrete hypercubic group.

Next, we consider the group of chiral flavor transformations,

$$G = SU(N_f)_L \otimes SU(N_f)_R. \tag{2}$$

Introducing the usual projection operators $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ the left- and right-handed fermion fields are defined by

$$\psi_{L,R} = P_\mp \psi, \quad \bar{\psi}_{L,R} = \bar{\psi} P_\pm. \tag{3}$$

Under a transformation $L \otimes R \in G$ these chiral components transform according to

$$\psi_L \to L\psi_L, \quad \bar{\psi}_L \to \bar{\psi}_L L^\dagger,$$
$$\psi_R \to R\psi_R, \quad \bar{\psi}_R \to \bar{\psi}_R R^\dagger. \tag{4}$$

For small quark masses and lattice spacing, G is an approximate symmetry group of the theory, broken only by the mass and the Wilson terms. If all the quark masses are nonzero but equal the vector subgroup with $L = R$ is a symmetry of the action S_W.

To complete the definition of the lattice theory one should also define a gauge action S_{YM}. However, the precise choice of the gauge action is irrelevant for the purpose of our analysis, so we leave it unspecified.

B. Symanzik action

The Symanzik action for the Wilson lattice action, up to and including $\mathcal{O}(a^2)$, has been calculated first in Ref. [13]. The analysis to $\mathcal{O}(a)$ has been later elaborated on in Ref. [14]. We restate these results in a slightly different form. The explicit breaking of chiral symmetry by the Wilson term leads to an additive renormalization of the quark mass. This causes the pion to become massless along a critical line $m_0 = m_c(a) \sim 1/a$, and a physical quark mass can be defined as the distance from this line, $m_q = m_0 - m_c$. The operators in the Symanzik action are constructed from the quark and gauge fields and their derivatives and powers of m_q. We list all terms in the action through $\mathcal{O}(a^2)$ that are allowed by the symmetries, organized in powers of a (again, we only focus on the fermion action). We use the notation

$$S_S = S_0 + aS_1 + a^2 S_2 \ldots,$$
$$S_k = \sum_i c_i^{(k+4)} O_i^{(k+4)}, \tag{5}$$

where $O_i^{(n)}$ are local operators of dimension n and the constants $c_i^{(n)}$ are unknown coefficients.

Some allowed operators in the Symanzik action are obtained by multiplying lower-dimensional operators with powers of the quark mass m_q. For example, $a\,\mathrm{tr}(m_q)\bar{\psi}\not{D}\psi$, $a\bar{\psi}m_q\not{D}\psi$ and four similar operators at $\mathcal{O}(a^2)$ contribute to the wave function renormalization of the quark fields. Performing a (flavor-dependent) field redefinition one can eliminate these operators while keeping the kinetic term trivial in flavor space. Similarly, the operators $a\,\mathrm{tr}(m_q^2)\bar{\psi}\psi$, $a\,\mathrm{tr}^2(m_q)\bar{\psi}\psi$, $a\,\mathrm{tr}(m_q)\bar{\psi}m_q\psi$, and $a\bar{\psi}m_q^2\psi$ [and seven more operators at $\mathcal{O}(a^2)$] renormalize the mass matrix m_q. These operators can be effectively accounted for by replacing m_q with the renormalized mass m; we do not list them here explicitly. At $\mathcal{O}(a^2)$ in the Symanzik action, differences between inserting m and m_q are at least $\mathcal{O}(a^3)$ and can be neglected. With these caveats we find the following list of operators (we use the same notation as in Ref. [13]):

$$S_0: \quad O_1^{(4)} = \bar{\psi} \not{D} \psi, \qquad O_2^{(4)} = \bar{\psi} m \psi. \tag{6}$$

$$S_1: \quad O_1^{(5)} = \bar{\psi} D_\mu D_\mu \psi, \qquad O_2^{(5)} = \bar{\psi} i \sigma_{\mu\nu} F_{\mu\nu} \psi. \tag{7}$$

$$\begin{aligned} S_2,\ \text{bilinears:} \quad & O_1^{(6)} = \bar{\psi} \not{D}^3 \psi, && O_5^{(6)} = \bar{\psi} m D_\mu D_\mu \psi, \\ & O_2^{(6)} = \bar{\psi}(D_\mu D_\mu \not{D} + \not{D} D_\mu D_\mu)\psi, && O_6^{(6)} = \mathrm{tr}(m) \quad \bar{\psi} D_\mu D_\mu \psi, \\ & O_3^{(6)} = \bar{\psi} D_\mu \not{D} D_\mu \psi, && O_7^{(6)} = \bar{\psi} m i \sigma_{\mu\nu} F_{\mu\nu} \psi, \\ & O_4^{(6)} = \bar{\psi} \gamma_\mu D_\mu D_\mu D_\mu \psi, && O_8^{(6)} = \mathrm{tr}(m) \quad \bar{\psi} i \sigma_{\mu\nu} F_{\mu\nu} \psi. \end{aligned} \tag{8}$$

$$\begin{aligned} S_2,\ \text{four-quark operators:} \quad & O_9^{(6)} = (\bar{\psi}\psi)^2, && O_{14}^{(6)} = (\bar{\psi} t^a \psi)^2, \\ & O_{10}^{(6)} = (\bar{\psi}\gamma_5\psi)^2, && O_{15}^{(6)} = (\bar{\psi} t^a \gamma_5 \psi)^2, \\ & O_{11}^{(6)} = (\bar{\psi}\gamma_\mu\psi)^2, && O_{16}^{(6)} = (\bar{\psi} t^a \gamma_\mu \psi)^2, \\ & O_{12}^{(6)} = (\bar{\psi}\gamma_\mu\gamma_5\psi)^2, && O_{17}^{(6)} = (\bar{\psi} t^a \gamma_\mu \gamma_5 \psi)^2, \\ & O_{13}^{(6)} = (\bar{\psi}\sigma_{\mu\nu}\psi)^2, && O_{18}^{(6)} = (\bar{\psi} t^a \sigma_{\mu\nu} \psi)^2, \end{aligned} \tag{9}$$

where t^a are the $SU(N_c)$ generators. This list of four-quark operators is slightly different from the one in Ref. [13]. Sheikholeslami and Wohlert's list contains operators with flavor group generators. However, both lists are equivalent and are related by Fierz identities (see Appendix A). Our choice of operators is guided by the fact that for the study of chiral transformation properties, it is more convenient to consider four-quark operators with a trivial flavor structure.

In the context of on-shell improvement, equations of motion have been used to reduce the number of operators at $\mathcal{O}(a)$ in the Symanzik action [13,14]. This involves a redefinition of the effective fields, which are matched to their lattice counterparts [14]. Only the Pauli term $O_2^{(5)}$ is left at this order and can be subsequently canceled by adding the clover term to the lattice action with a properly adjusted coefficient. The generalization of the arguments in Ref. [14] to $\mathcal{O}(a^2)$ has not been carried out yet. We therefore continue with the formulation of the chiral effective theory without making use of equations of motion.

We distinguish two types of operators in the Symanzik action: those that break chiral symmetry and those that do not. At $\mathcal{O}(a)$ all operators break chiral symmetry. At $\mathcal{O}(a^2)$ there are ten symmetry-breaking operators: $O_5^{(6)} - O_{10}^{(6)}$, $O_{13}^{(6)} - O_{15}^{(6)}$, and $O_{18}^{(6)}$. Fermionic operators that do not break the chiral symmetry first appear at $\mathcal{O}(a^2)$. Purely gluonic operators (which we have not listed above) also belong to the second type of operators as they are trivially invariant under chiral transformations. They too enter at $\mathcal{O}(a^2)$.

The operator $O_4^{(6)}$ deserves special attention. While respecting the chiral symmetries it is not invariant under O(4) rotations. This means that it does affect the structure of the chiral Lagrangian by inducing O(4) symmetry breaking terms in it. The analysis leading to the Symanzik action reveals that such terms must be at least of $\mathcal{O}(a^2)$.

C. Spurion analysis

At $\mathcal{O}(a^0)$, the Symanzik action is QCD-like. For small a and m we assume the lattice theory to exhibit the same spontaneous symmetry-breaking pattern $SU(N_f)_L \otimes SU(N_f)_R \rightarrow SU(N_f)_V$ as continuum QCD.[1] Consequently, the low-energy physics is dominated by Nambu-Goldstone bosons, which acquire small masses due to the soft explicit symmetry breaking by the small quark masses and discretization effects. The low-energy chiral effective field theory is written in terms of these light bosons.

To construct the chiral Lagrangian we follow the standard procedure of spurion analysis. We write a term in the Symanzik Lagrangian as $C_0 O$ where O contains the fields and their derivatives and C_0 is the remaining constant factor. For symmetry breaking terms, O changes under a chiral transformation of the fermionic fields $O \rightarrow O'$. We then promote C_0 to the status of a spurion C, with the transformation $C \rightarrow C'$ such that $CO = C'O'$. The chiral effective theory is constructed from the Nambu-Goldstone fields and the spurions with the requirement that the action is invariant under chiral transformations if the spurions are transformed as well. Once the terms in the chiral Lagrangian are obtained, each spurion is set to its original constant value $C = C_0$. This procedure guarantees that the chiral effective theory explicitly breaks chiral symmetry in the same manner as the underlying theory

[1]We assume to be outside of the Aoki phase [36–38].

defined by the Symanzik action and reproduces the same Ward identities.

It might appear that one needs many spurion fields to accommodate all the symmetry-breaking operators in the Symanzik action. However, this is not the case. Two spurions that transform in the same way will lead to the same terms in the chiral Lagrangian; therefore, it is enough to consider only one of them. This is discussed in Appendix B. Since we organize the chiral perturbation theory as an expansion in m and a, we do distinguish between spurions that transform the same way but have different m or a dependence.

In the following we list the representative spurions. Shown are the transformation rules for the different spurions under chiral transformations and the constant values to which the spurions are assigned in the end.

$$\begin{aligned}\mathcal{O}(a^0):\quad & M\to LMR^\dagger,\quad M^\dagger\to RM^\dagger L^\dagger,\\ & M_0=M_0^\dagger=m=\mathrm{diag}(m_1,\ldots,m_{N_f}).\end{aligned}\tag{10}$$

This makes the mass term $\overline{\psi}_L M\psi_R+\overline{\psi}_R M^\dagger\psi_L$ invariant under the chiral transformations of Eq. (4).

$$\begin{aligned}\mathcal{O}(a):\quad & A\to LAR^\dagger,\quad A^\dagger\to RA^\dagger L^\dagger,\\ & A_0=A_0^\dagger=aI,\end{aligned}\tag{11}$$

where I is the flavor identity matrix. The spurion A renders the operators $O_1^{(5)}$ and $O_2^{(5)}$ in Eq. (7) invariant.

$$\begin{aligned}\mathcal{O}(a^2):\quad & B\equiv B_1\otimes B_2\to LB_1R^\dagger\otimes LB_2R^\dagger,\\ & B^\dagger\equiv B_1^\dagger\otimes B_2^\dagger\to RB_1^\dagger L^\dagger\otimes RB_2^\dagger L^\dagger,\\ & C\equiv C_1\otimes C_2\to RC_1L^\dagger\otimes LC_2R^\dagger,\\ & C^\dagger\equiv C_1^\dagger\otimes C_2^\dagger\to LC_1^\dagger R^\dagger\otimes RC_2^\dagger L^\dagger,\\ & B_0=B_0^\dagger=C_0=C_0^\dagger=a^2I\otimes I.\end{aligned}\tag{12}$$

These spurions are introduced to make the symmetry-breaking four-quark operators invariant and therefore carry four flavor indices (see Ref. [39] and references therein). Consider, for example, the operator

$$\begin{aligned}(\overline{\psi}\psi)(\overline{\psi}\psi)=&(\overline{\psi}_L\psi_R)(\overline{\psi}_L\psi_R)+(\overline{\psi}_R\psi_L)(\overline{\psi}_L\psi_R)\\ &+(\overline{\psi}_R\psi_L)(\overline{\psi}_R\psi_L)+(\overline{\psi}_L\psi_R)(\overline{\psi}_R\psi_L).\end{aligned}\tag{13}$$

The first term on the right-hand side can be made invariant with the spurion B as can be seen from

$$\begin{aligned}B\overline{\psi}_L\psi_R\overline{\psi}_L\psi_R&=B_{ijkl}(\overline{\psi}_L)_i(\psi_R)_j(\overline{\psi}_L)_k(\psi_R)_l\\ &=\overline{\psi}_LB_1\psi_R\overline{\psi}_LB_2\psi_R.\end{aligned}\tag{14}$$

Similarly, all the other symmetry-breaking four-quark operators can be made invariant using the spurions B, C, and their Hermitian conjugates.

No additional spurion fields need to be introduced to make the symmetry-breaking bilinears at $\mathcal{O}(a^2)$ invariant. The combinations $AA^\dagger M$ and $\mathrm{tr}(AM^\dagger)A$ already transform in the right way to make $O_5^{(6)}-O_8^{(6)}$ invariant, and their constant values have the right powers in a and m. Another source of potentially new spurions at $\mathcal{O}(a^2)$ are squares of $\mathcal{O}(a)$ spurions. However, note that A^2, $A^\dagger A$, $(A^\dagger)^2$, and $AA^\dagger$ transform exactly like B, C, $B^\dagger$, and $C^\dagger$, respectively, and therefore need not be treated separately.

D. Chiral Lagrangian

The chiral Lagrangian is expanded in powers of p^2, m, and a. Generalizing the standard chiral power counting, the leading-order Lagrangian contains the terms of $\mathcal{O}(p^2,m,a)$, while the terms of $\mathcal{O}(p^4,p^2m,p^2a,m^2,ma,a^2)$ are of next-to-leading order. In terms of the dimensionless expansion parameters m/Λ_χ and $a\Lambda_\chi$, where $\Lambda_\chi\approx 1$ GeV is the typical chiral symmetry-breaking scale, this power counting assumes that the size of the chiral symmetry breaking due to the mass and the discretization effects are of comparable size.[2]

For the Wilson action, all next-to-leading order terms have already been computed in Ref. [7], except for the $\mathcal{O}(a^2)$ terms. We are now in the position to calculate these contributions, which are the ones associated with the spurions B and C. We find the following three new terms (and their Hermitian conjugates)

$$\langle B_1\Sigma^\dagger\rangle\langle B_2\Sigma^\dagger\rangle\to a^2\langle\Sigma^\dagger\rangle^2,\tag{15}$$

$$\langle B_1\Sigma^\dagger B_2\Sigma^\dagger\rangle\to a^2\langle\Sigma^\dagger\Sigma^\dagger\rangle,\tag{16}$$

$$\langle C_1\Sigma\rangle\langle C_2\Sigma^\dagger\rangle\to a^2\langle\Sigma\rangle\langle\Sigma^\dagger\rangle.\tag{17}$$

Here $\Sigma=\exp(2i\Pi/f)$, with Π being the matrix of Nambu-Goldstone fields. Σ transforms under the chiral transformations in Eq. (4) as $\Sigma\to L\Sigma R^\dagger$. The angled brackets are traces over flavor indices, and the arrows indicate assigning $B=B_0$, $C=C_0$, according to Eq. (12).

So far we only considered the operators in the Symanzik action that explicitly break chiral symmetry. Operators that do not break chiral symmetry also contribute at $\mathcal{O}(a^2)$. These operators do not add any new terms to the chiral Lagrangian, but simply modify the coefficients in front of already existing operators. At leading order, for example, the kinetic term is $f^2/4\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\rangle$. There are corrections to f^2 due to the symmetry-conserving terms in the Symanzik action: $f^2\to f^2+a^2K$ (K is another unknown low-energy constant.) This leads to the correction $a^2K\langle\partial_\mu\Sigma\partial\Sigma_\mu^\dagger\rangle$ for the kinetic term. Thus given a term of $\mathcal{O}(p^2)$ there is another term of $\mathcal{O}(a^2p^2)$. In general, we can rewrite the coefficient of any allowed operator in the chiral Lagrangian to obtain a new allowed operator which is $\mathcal{O}(a^2)$ *higher*. These terms are beyond next-to-leading order and are not included in the present work.

[2]A more detailed discussion of the power-counting scheme is given in Sec. IV.

As already mentioned, the operator $O_4^{(6)}$ breaks the O(4) symmetry in the Symanzik action. However, in order to break the O(4) symmetry, while still preserving the discrete hypercubic symmetry, an operator must carry at least four space-time indices. In the chiral Lagrangian, these are provided by the partial derivative ∂_μ, hence the operator is at least of $\mathcal{O}(p^4)$. Adding the fact that it is also an $\mathcal{O}(a^2)$ effect, we see that the leading O(4) symmetry-breaking terms in the chiral Lagrangian are of $\mathcal{O}(p^4a^2)$ (an example is the operator $a^2\Sigma_\mu\langle\partial_\mu\partial_\mu\Sigma\partial_\mu\partial_\mu\Sigma^\dagger\rangle$). Hence, up to the order considered here, O(4) breaking terms can be excluded from the analysis.

Finally we can write down the terms of $\mathcal{O}(a^2)$ which enter the next-to-leading order chiral Lagrangian. In terms of the two parameters

$$\hat{m}\equiv 2B_0 m = 2B_0\mathrm{diag}(m_1,\ldots,m_{N_f}), \quad \hat{a}\equiv 2W_0 a, \tag{18}$$

which have been introduced in Ref. [16],[3] these terms are

$$\mathcal{L}[a^2] = -\hat{a}^2 W_6'\langle\Sigma^\dagger+\Sigma\rangle^2 - \hat{a}^2 W_7'\langle\Sigma^\dagger-\Sigma\rangle^2 - \hat{a}^2 W_8'\langle\Sigma^\dagger\Sigma^\dagger+\Sigma\Sigma\rangle. \tag{19}$$

The coefficients W_i' are new unknown low-energy constants. Putting it all together, also quoting the terms in the Lagrangian of $\mathcal{O}(a)$ from Ref. [7],[4] we find

$$\begin{aligned}\mathcal{L}_\chi =& \frac{f^2}{4}\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\rangle - \frac{f^2}{4}\langle\hat{m}\Sigma^\dagger+\Sigma\hat{m}\rangle - \hat{a}\frac{f^2}{4}\langle\Sigma^\dagger+\Sigma\rangle - L_1\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\rangle^2 - L_2\langle\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\rangle\langle\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\rangle - L_3\langle(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger)^2\rangle \\ &+ L_4\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\rangle\langle\hat{m}\Sigma^\dagger+\Sigma\hat{m}\rangle + \hat{a}W_4\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\rangle\langle\Sigma^\dagger+\Sigma\rangle + L_5\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger(\hat{m}\Sigma^\dagger+\Sigma\hat{m})\rangle + \hat{a}W_5\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger(\Sigma^\dagger+\Sigma)\rangle \\ &- L_6\langle\hat{m}\Sigma^\dagger+\Sigma\hat{m}\rangle^2 - \hat{a}W_6\langle\hat{m}\Sigma^\dagger+\Sigma\hat{m}\rangle\langle\Sigma^\dagger+\Sigma\rangle - L_7\langle\hat{m}\Sigma^\dagger-\Sigma\hat{m}\rangle^2 - \hat{a}W_7\langle\hat{m}\Sigma^\dagger-\Sigma\hat{m}\rangle\langle\Sigma^\dagger-\Sigma\rangle \\ &- L_8\langle\hat{m}\Sigma^\dagger\hat{m}\Sigma^\dagger+\Sigma\hat{m}\Sigma\hat{m}\rangle - \hat{a}W_8\langle\hat{m}\Sigma^\dagger\Sigma^\dagger+\Sigma\Sigma\hat{m}\rangle + \mathcal{L}[a^2] + \text{higher order terms.}\end{aligned} \tag{20}$$

Here, the parameters L_i are the usual Gasser-Leutwyler coefficients of continuum χPT.

E. Partially quenched QCD

Partially quenched QCD is formally represented by an action with sea, valence, and ghost quarks [40]. We collect the quark fields in $\Psi=(\psi_S,\psi_V)$, where ψ_S describes the sea quarks, and ψ_V contains both the anticommuting valence quarks and commuting ghost fields. The same is done for the antiquark fields. The mass matrix is given by $m=\mathrm{diag}(m_S,m_V')$, with m_S being the $N_f\times N_f$ mass matrix for the sea quarks and $m_V'=\mathrm{diag}(m_V,m_V)$ is the $2N_V\times 2N_V$ mass matrix for the valence quarks and valence ghosts.

We consider partially quenched lattice QCD with Wilson's fermion action Eq. (1) for all three types of fields. The discrete symmetries and the color-gauge symmetry is as in the unquenched case. The group of chiral flavor transformations, however, is different. If all the masses and the Wilson parameter r are set to zero, the action is invariant under transformations in the graded group[5]

$$G_{PQ} = SU(N_f+N_V|N_V)_L\otimes SU(N_f+N_V|N_V)_R. \tag{21}$$

Based on the symmetries of the lattice theory the Symanzik action for partially quenched lattice QCD is obtained as before. The result is easily quoted: One can simply replace ψ and $\bar{\psi}$ in the Symanzik action for the unquenched theory with the extended fields Ψ and $\bar{\Psi}$ because the only two- and four-quark operators that are invariant under the extended, graded flavor group are still $\bar{\Psi}\Psi$ and its square.

The leading term in the Symanzik action is partially quenched QCD, for which the construction of the chiral Lagrangian (first introduced in Ref. [42]) is essentially the same as for the unquenched case [41]. This remains true when higher dimensional operators in the Symanzik action are included, and the analysis of Sec. II D is readily extended to the partially quenched case. In particular, the form of the chiral Lagrangian for partially quenched lattice QCD with Wilson fermions is *exactly* the same as in Eq. (20). The difference is in the definition of the angled brackets, which now denote supertraces, and the interpretation of Σ and m. These need to be appropriately redefined to reflect the larger flavor content of partially quenched χPT.

III. MIXED ACTION

In this section we consider a lattice theory with Wilson sea quarks and Ginsparg-Wilson valence quarks. As before

[3]Unlike in Ref. [16], here we define $\hat{a}$ without the factor of c_{SW}. The coefficient c_{SW} is not kept explicit as we do not use equations of motion, and S_1 contains $ac_1^{(5)}O_1^{(5)}$ besides the Pauli term $ac_{SW}O_2^{(5)}$. Note that c_{SW} does not refer to the coefficient of the clover-leaf term of improved lattice actions.

[4]There are some typos in the Lagrangian [Eq. (2.10)] in Ref. [7]. The Lagrangian in Eq. (20) is the correct one.

[5]See Ref. [41] for a more honest discussion of the symmetry group of partially quenched QCD.

we first construct Symanzik's effective action through $\mathcal{O}(a^2)$. We then derive the chiral Lagrangian for this theory.

A. Lattice action

The use of different lattice fermions for sea and valence quarks is a generalization of partially quenched lattice QCD. Theoretically it too is formulated by an action with sea and valence quarks and valence ghosts. However, in addition to allowing different quark masses ($m_S \neq m_V$), the Dirac operator in the sea sector is chosen to be different from the one for the valence quarks and ghosts. For this reason we will refer to this type of lattice theory as a "mixed action" theory.

The mixed action theory with Wilson sea quarks and Ginsparg-Wilson valence quarks is defined in Ref. [16]. We refer the reader to this reference for details and notation. Here we just quote that the flavor symmetry group of the mixed lattice action is

$$G_M = G_{Sea} \otimes G_{Val},$$

$$G_{Sea} = SU(N_f)_L \otimes SU(N_f)_R,$$

$$G_{Val} = SU(N_V|N_V)_L \otimes SU(N_V|N_V)_R. \tag{22}$$

The quark mass term in the mixed action breaks both G_{Sea} and G_{Val}. However, in the massless case G_{Val} becomes an exact symmetry [29] while G_{Sea} is still broken by the Wilson term. Because of the different Dirac operators there is no symmetry transformation that mixes the valence and sea sectors, in contrast to the partially quenched case [cf. Eq. (21)].

B. Symanzik action

The Symanzik action for the mixed theory can be derived using the results of the previous section. It is convenient to separately discuss three types of terms—those that contain only sea quark fields, those that contain only valence fields, and those that contain both.

For the first type of terms the analogy with the previous section is evident: the relevant symmetry group is $G_{Sea} = G$, and the explicit symmetry-breaking structure is the same. Thus, all bilinear operators $O_i^{(n)}(\psi)$ and four-quark operators $O_i^{(n)}(\psi,\psi)$, listed in Sec. II B, appear in Symanzik's action, once ψ is replaced by ψ_S.[6]

The construction of the purely valence terms is also analogous to the one for the Wilson action in Sec. II B. However, there are stricter symmetry constraints for Ginsparg-Wilson quarks and ghosts because the Ginsparg-Wilson action possesses an exact chiral symmetry when the quark mass is set to zero. All operators without any insertions of the quark mass must therefore be chirally invariant. Further, operators with insertions of the quark mass m must become chirally invariant when m is transformed like a spurion field. In particular, all dimension-3 and dimension-5 operators are forbidden. Several dimension-6 operators at $\mathcal{O}(a^2)$ are also excluded. Only the bilinears $O_1^{(6)} - O_5^{(6)}$, $O_7^{(6)}$ of Eq. (8) and the four-quark operators $O_i^{(6)}(\psi_V,\psi_V)$, $i=11$, 12, 16, and 17, of Eq. (9), are G_{Val} invariant and are therefore allowed.

For terms of the third type, note that the symmetry group G_M forbids bilinears that mix valence and sea quarks. Thus, the only terms containing both sea and valence fields are four-quark operators that are products of two bilinears—one from each sector. Again, only the four terms $O_i^{(6)}(\psi_S,\psi_V)$, $i=11$, 12, 16, and 17, are allowed. All the others break the chiral symmetry in the valence sector when $m_V=0$.

From these considerations it follows that the Symanzik action for the mixed lattice action up to and including $\mathcal{O}(a^2)$ contains the following terms:

$$S_0: \quad O_i^{(4)}(\psi_S), \quad O_i^{(4)}(\psi_V), \quad i=1,2. \tag{23}$$

$$S_1: \quad O_i^{(5)}(\psi_S), \quad i=1,2. \tag{24}$$

$$\begin{aligned} S_2,\ \text{bilinears:} \quad & O_i^{(6)}(\psi_S), \quad i=1-8, \\ & O_i^{(6)}(\psi_V), \quad i=1-5,7. \end{aligned} \tag{25}$$

$$\begin{aligned} S_2,\ \text{four-quark operators:} \quad & O_i^{(6)}(\psi_S,\psi_S), \quad i=9-18, \\ & O_i^{(6)}(\psi_V,\psi_V), \quad O_i^{(6)}(\psi_S,\psi_V), \quad i=11,12,16,17. \end{aligned} \tag{26}$$

[6]We make the dependence of bilinear operators on the fields explicit by writing $O(\psi)$. All the four-quark operators that we consider have the structure $O(\psi_1,\psi_2) = \overline{\psi_1}\Omega^J\psi_1\overline{\psi_2}\Omega^J\psi_2$. Here Ω denotes any combination of Clifford algebra elements and color group generators with a combined index J, which is contracted.

C. Spurion analysis

S_0, the leading term in the Symanzik action, is just the continuum action of partially quenched QCD. In the $m\to 0$ limit it is invariant under the flavor symmetry group G_{PQ} inEq. (21), which is larger than G_M [Eq. (22)], the symmetry group of the underlying lattice action. For a sufficiently small a (and m), S_0 determines the spontaneous symmetry-breaking pattern and the symmetry properties of the Nambu-Goldstone particles in the theory. It follows that the mixed theory contains the same set of light particles as partially quenched QCD.

For the construction of the chiral effective theory, we introduce spurion fields that make the entire Symanzik action invariant under G_{PQ}. Notice that all the operators proportional to a and a^2 break G_{PQ}, the flavor symmetry of the leading term. This is obvious for operators that appear with sea quark fields only, such as the dimension 5 operators. However, even if an operator appears "symmetrically" in ψ_S and ψ_V, as in Eq. (25), it still breaks G_{PQ}. To illustrate this point let us consider any of the bilinear terms, suppressing all γ matrices and color-group generators. Any bilinear that is invariant under all rotations of G_{PQ} must have the flavor structure $\bar{\Psi}\Psi=\bar{\psi}_S\psi_S+\bar{\psi}_V\psi_V$. In general, though, $\bar{\psi}_S\psi_S$ and $\bar{\psi}_V\psi_V$ will not appear in the Symanzik action with equal coefficients, and therefore will not be invariant under transformations in G_{PQ} that mix the sea and valence sectors.

As before we begin the construction of the chiral Lagrangian by listing the representative spurions required at each order in a to make the Symanzik action invariant. Shown are the transformation properties of the spurions under chiral transformations in G_{PQ} and the constant structures to which the spurion fields are assigned in the end. Since different operators appear in the sea and valence sector, it is convenient to introduce the projection operators

$$P_S=\mathrm{diag}(I_S,0), \quad P_V=\mathrm{diag}(0,I_V), \tag{27}$$

where I_S denotes the $N_f\times N_f$ identity matrix in the sea sector, and I_V the $2N_V\times 2N_V$ identity matrix in the space of valence quarks and ghosts (recall that ψ_V includes both valence quarks and ghosts).

$$\mathcal{O}(a^0):\quad M\to LMR^\dagger,\quad M^\dagger\to RM^\dagger L^\dagger,$$
$$M_0=M_0^\dagger=m=\mathrm{diag}(m_S,m_V'). \tag{28}$$
$$\mathcal{O}(a):\quad A\to LAR^\dagger,\quad A^\dagger\to RA^\dagger L^\dagger,$$
$$A_0=A_0^\dagger=aP_S. \tag{29}$$

The last spurion arises from the sea sector symmetry breaking terms at $\mathcal{O}(a)$.

The quark bilinears $O_1^{(6)}-O_4^{(6)}$ at $\mathcal{O}(a^2)$ couple fields with the same chirality. Since there are bilinears for both sea and valence fields we obtain the following spurions:

$$\mathcal{O}(a^2),\ \text{bilinears:}\quad B\to LBL^\dagger,\quad C\to RCR^\dagger,$$
$$B_0,\ C_0\in\{a^2P_S,a^2P_V\}. \tag{30}$$

No *additional* spurion fields need to be introduced to make the remaining bilinears $O_5^{(6)}-O_8^{(6)}$ invariant. Appropriate combinations of the spurion fields M and A (and their complex conjugates) have already the required transformation behavior and the correct constant structure.

We can distinguish two types of four-quark operators. The first type is made of bilinears that only couple fields of the same chirality. These operators appear with only sea or valence fields as well as in the "mixed" form $O(\psi_S,\psi_V)$. The remaining four-quark operators, which couple fields with opposite chirality, appear only with sea quarks. We therefore introduce the following spurions:

$$\mathcal{O}(a^2),\ \text{four-quark operators:}$$
$$D\equiv D_1\otimes D_2\to LD_1L^\dagger\otimes LD_2L^\dagger,$$
$$E\equiv E_1\otimes E_2\to RE_1R^\dagger\otimes RE_2R^\dagger,$$
$$F\equiv F_1\otimes F_2\to LF_1L^\dagger\otimes RF_2R^\dagger,$$
$$G\equiv G_1\otimes G_2\to RG_1R^\dagger\otimes LG_2L^\dagger,$$
$$D_0,E_0,F_0,G_0\in\{a^2P_S\otimes P_S,a^2P_S\otimes P_V,a^2P_V\otimes P_V\}, \tag{31}$$
$$H\equiv H_1\otimes H_2\to LH_1R^\dagger\otimes LH_2R^\dagger,$$
$$H^\dagger\equiv H_1^\dagger\otimes H_2^\dagger\to RH_1^\dagger L^\dagger\otimes RH_2^\dagger L^\dagger,$$
$$J\equiv J_1\otimes J_2^\dagger\to LJ_1R^\dagger\otimes RJ_2^\dagger L^\dagger,$$
$$J^\dagger\equiv J_1^\dagger\otimes J_2\to RJ_1^\dagger L^\dagger\otimes LJ_2R^\dagger,$$
$$H_0=H_0^\dagger=J_0=J_0^\dagger=a^2P_S\otimes P_S. \tag{32}$$

Squaring the spurions of $\mathcal{O}(a)$ does not lead to any new spurions.

D. Chiral Lagrangian

The chiral Lagrangian for the mixed action theory including the cutoff effects linear in a is derived in Ref. [16]. Terms of $\mathcal{O}(a^2)$ are constructed from the spurions in Eqs. (30)–(32). It is easily checked that the spurions B, C, D, and E lead necessarily to operators higher than $\mathcal{O}(a^2)$ [at least $\mathcal{O}(p^2a^2,ma^2)$], so we can ignore them. From the other spurions we obtain the following independent operators (and their Hermitian conjugates):

$$\langle F_1\Sigma F_2\Sigma^\dagger\rangle\to a^2\langle\tau_3\Sigma\tau_3\Sigma^\dagger\rangle, \tag{33}$$
$$\langle H_1\Sigma^\dagger H_2\Sigma^\dagger\rangle\to a^2\langle P_S\Sigma^\dagger P_S\Sigma^\dagger\rangle, \tag{34}$$
$$\langle H_1\Sigma^\dagger\rangle\langle H_2\Sigma^\dagger\rangle\to a^2\langle P_S\Sigma^\dagger\rangle\langle P_S\Sigma^\dagger\rangle, \tag{35}$$

$$\langle J_1\Sigma^\dagger\rangle\langle J_2^\dagger\Sigma\rangle\to a^2\langle P_S\Sigma^\dagger\rangle\langle P_S\Sigma\rangle. \tag{36}$$

For Eq. (33) we use the fact that $P_S=\frac{1}{2}(I+\tau_3)$ and $P_V=\frac{1}{2}(I-\tau_3)$, with $\tau_3=\mathrm{diag}(I_S,-I_V)$. When assigning $F_{1,2}=(I\pm\tau_3)$ and expanding, the fields Σ and $\Sigma^\dagger$ are next to each other and cancel whenever the identity matrix is inserted, so the only nontrivial operator is the one shown in Eq. (33).

We conclude that for the mixed action theory with Wilson sea and Ginsparg-Wilson valence quarks the terms of $\mathcal{O}(a^2)$ in the chiral Lagrangian are

$$\begin{aligned}\mathcal{L}[a^2]=&-\hat{a}^2W_M\langle\tau_3\Sigma\tau_3\Sigma^\dagger\rangle-\hat{a}^2W_6'\langle P_S\Sigma^\dagger+\Sigma P_S\rangle^2\\&-\hat{a}^2W_7'\langle P_S\Sigma^\dagger-\Sigma P_S\rangle^2\\&-\hat{a}^2W_8'\langle P_S\Sigma^\dagger P_S\Sigma^\dagger+\Sigma P_S\Sigma P_S\rangle.\end{aligned} \tag{37}$$

The parameters $\hat{m}$ and $\hat{a}$ are defined as in the unquenched case in Eq. (18). Note that the projector P_S in the last three terms implies that these operators involve only the sea-sea block of Σ.

The final result, including the terms from Ref. [16], reads

$$\begin{aligned}\mathcal{L}_\chi=&\frac{f^2}{4}\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\rangle-\frac{f^2}{4}\langle\hat{m}\Sigma^\dagger+\Sigma\hat{m}\rangle-\hat{a}\frac{f^2}{4}\langle P_S\Sigma^\dagger+P_S\Sigma\rangle-L_1\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\rangle^2-L_2\langle\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\rangle\langle\partial_\mu\Sigma\partial_\nu\Sigma^\dagger\rangle\\&-L_3\langle(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger)^2\rangle+L_4\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\rangle\langle\hat{m}\Sigma^\dagger+\Sigma\hat{m}\rangle+\hat{a}W_4\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\rangle\langle P_S\Sigma^\dagger+\Sigma P_S\rangle+L_5\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger(\hat{m}\Sigma^\dagger+\Sigma\hat{m})\rangle\\&+\hat{a}W_5\langle\partial_\mu\Sigma\partial_\mu\Sigma^\dagger(P_S\Sigma^\dagger+\Sigma P_S)\rangle-L_6\langle\hat{m}\Sigma^\dagger+\Sigma\hat{m}\rangle^2-\hat{a}W_6\langle\hat{m}\Sigma^\dagger+\Sigma\hat{m}\rangle\langle P_S\Sigma^\dagger+\Sigma P_S\rangle-L_7\langle\hat{m}\Sigma^\dagger-\Sigma\hat{m}\rangle^2\\&-\hat{a}W_7\langle\hat{m}\Sigma^\dagger-\Sigma\hat{m}\rangle\langle P_S\Sigma^\dagger-\Sigma P_S\rangle-L_8\langle\hat{m}\Sigma^\dagger\hat{m}\Sigma^\dagger+\Sigma\hat{m}\Sigma\hat{m}\rangle-\hat{a}W_8\langle\hat{m}\Sigma^\dagger P_S\Sigma^\dagger+\Sigma P_S\Sigma\hat{m}\rangle\\&+\mathcal{L}[a^2]+\text{higher order terms.}\end{aligned} \tag{38}$$

The chiral Lagrangian for the mixed action theory at $\mathcal{O}(a^2)$ has four terms while there are only three terms at this order in the chiral Lagrangian for the Wilson action. The reason that the mixed theory has an additional operator (and consequently an additional unknown low-energy constant multiplying it) is its reduced symmetry group, G_M in Eq. (22), compared to G_{PQ} in Eq. (21). The use of different Dirac operators for sea and valence quarks forbids transformations between the sea and valence sectors and allows the additional term $\langle\tau_3\Sigma\tau_3\Sigma^\dagger\rangle$ in Eq. (37).

The presence of more terms in the Lagrangian does not entail that chiral expressions for all observables in the mixed theory depend on more free parameters than in χPT for the Wilson action. By definition, the correlation functions measured in numerical simulations involve operators that are made of valence quarks only, and the enhanced chiral symmetry of the Ginsparg-Wilson fields plays an important role in that sector. The chiral symmetry leads to constraints on operators in the Symanzik action that contain valence fields, and ultimately it restricts and simplifies the form of chiral expressions for valence quark observables. This can already be seen by considering the terms in $\mathcal{L}[a^2]$ with coefficients W_6', W_7', and W_8' [see Eq. (37)]. These terms depend only on the sea-sea block of Σ. This entails that all the multi-pion interaction vertices obtained from these terms necessarily contain some mesons with at least a single sea quark in them. Consequently, these terms cannot contribute at tree level to any expectation value of operators made entirely out of valence fields. This is easily understood: the W' terms arise from the breaking of chiral symmetry in the sea sector by the Wilson term, and this breaking is communicated to the valence sector only through loop effects. A more concrete demonstration of this point is provided by the calculation of the pseudo scalar valence-valence meson mass in the next section.

IV. APPLICATION

We conclude our analysis of the chiral effective theories for the Wilson action and the mixed action theory with an explicit calculation of the light meson masses. Before presenting the calculations, however, a discussion of the chiral power counting is appropriate.

A. Power counting

χPT reproduces low-momentum correlation functions of the underlying theory, provided that the typical momentum p and the mass of the Nambu-Goldstone boson M_{NGB} are sufficiently small, $p\ll\Lambda_\chi$ and $M_{NGB}\ll\Lambda_\chi$. The standard convention is to consider p and M_{NGB} as formally of the same order, and take a single expansion parameter $\epsilon\sim M_{NGB}^2/\Lambda_\chi^2\sim p^2/\Lambda_\chi^2$. Thus, a typical next-to-leading order (one-loop) expression for a correlation function in χPT has the structure

$$C=C_{\mathrm{LO}}+C_{\mathrm{NLO}}+\cdots,$$

$$C_{\mathrm{LO}}=\mathcal{O}(\epsilon)=\mathcal{O}\left(\frac{M_{NGB}^2}{\Lambda_\chi^2},\ \frac{p^2}{\Lambda_\chi^2}\right),$$

$$C_{\mathrm{NLO}}=\mathcal{O}(\epsilon^2)=\mathcal{O}\left(\frac{M_{NGB}^4}{\Lambda_\chi^4},\ \frac{p^4}{\Lambda_\chi^4},\ \frac{M_{NGB}^2p^2}{\Lambda_\chi^4}\right). \tag{39}$$

In some cases of interest the momentum scale and the Nambu-Goldstone boson mass are significantly different, $p \ll M_{NGB}$ for instance. In such a case one could treat the two dimensionless parameters separately and introduce another expansion parameter p/M_{NGB}. However, as long as both M_{NGB}^2/Λ_χ^2 and p^2/Λ_χ^2 are sufficiently small, Eq. (39) still holds. Consequently, a reasonable approach in the case that p and M_{NGB} are very different is to take Eq. (39) and to ignore (or not calculate) terms that are smaller than the error associated with the larger expansion parameter.

In the case of χPT for lattice theories there are two possible sources of explicit chiral symmetry breaking: the quark masses and the lattice spacing. Consequently, the mass of the pseudo-Nambu-Goldstone boson is given by $M_{NGB}^2/\Lambda_\chi^2 \sim m/\Lambda_\chi + a\Lambda_\chi$. The discussion of the previous paragraph applies here as well: we can take $\varepsilon \sim p^2/\Lambda_\chi^2 \sim m/\Lambda_\chi \sim a\Lambda_\chi$ and Eq. (39) (properly extended) still holds. As long as the largest of these parameters is sufficiently small, this is a consistent power-counting scheme, and Eq. (39) is applicable even when some of the dimensionless parameters are significantly smaller than the others. This is the power-counting that is used in organizing the terms in the Lagrangians in Eqs. (20) and (38).

A different power-counting scheme does need to be employed in some cases. To illustrate this we consider a realistic example: for some fermion actions there are no discretization effects at $\mathcal{O}(a)$. This is the case, for example, for nonperturbatively $\mathcal{O}(a)$ improved Wilson fermions. If, in addition, the lattice spacing in a simulation is large such that $a^2\Lambda_\chi^2 \sim m/\Lambda_\chi$, an expansion in two parameters may be required, and the leading-order contributions are $\mathcal{O}(p^2/\Lambda_\chi^2, m/\Lambda_\chi, a^2\Lambda_\chi^2)$.

B. Pseudoscalar-meson masses

We now turn to the calculation of the pseudoscalar-meson masses. As in Refs. [7] and [16], we only consider mesons with different valence flavor indices ($A \neq B$). In addition, we also take the sea quark masses and the valence quark masses to be separately degenerate. For the partially quenched Wilson action we find

$$\begin{aligned}
M_{AB}^2 = {} & (\hat{m}_{\mathrm{Val}} + \hat{a}) + \frac{1}{16 N_f f^2 \pi^2} (\hat{m}_{\mathrm{Val}} + \hat{a}) [\hat{m}_{\mathrm{Val}} - \hat{m}_{\mathrm{Sea}} \\
& + (2\hat{m}_{\mathrm{Val}} - \hat{m}_{\mathrm{Sea}} + \hat{a}) \ln(\hat{m}_{\mathrm{Val}} + \hat{a})] - \frac{8}{f^2} (\hat{m}_{\mathrm{Val}} + \hat{a}) \\
& \times [N_f (L_4 \hat{m}_{\mathrm{Sea}} + W_4 \hat{a}) + L_5 \hat{m}_{\mathrm{Val}} + W_5 \hat{a}] \\
& + \frac{8 N_f}{f^2} [2 L_6 \hat{m}_{\mathrm{Val}} \hat{m}_{\mathrm{Sea}} + W_6 (\hat{m}_{\mathrm{Val}} + \hat{m}_{\mathrm{Sea}}) \hat{a} + 2 W_6' \hat{a}^2] \\
& + \frac{16}{f^2} [L_8 \hat{m}_{\mathrm{Val}}^2 + W_8 \hat{m}_{\mathrm{Val}} \hat{a} + W_8' \hat{a}^2] + \mathcal{O}(\epsilon^3), \qquad (40)
\end{aligned}$$

where N_f is the number of sea quark flavors.

Next, we consider the mixed action theory. A direct calculation shows that *there are no $\mathcal{O}(a^2)$ corrections to the pseudoscalar-meson mass*. The expression for M_{AB} is, therefore, the same as in Ref. [16], which we quote here for completeness:[7]

$$\begin{aligned}
M_{AB}^2 = {} & \hat{m}_{\mathrm{Val}} + \frac{1}{16 N_f f^2 \pi^2} \hat{m}_{\mathrm{Val}} [\hat{m}_{\mathrm{Val}} - \hat{m}_{\mathrm{Sea}} \\
& - \hat{a} + (2\hat{m}_{\mathrm{Val}} - \hat{m}_{\mathrm{Sea}} - \hat{a}) \ln(\hat{m}_{\mathrm{Val}})] \\
& - \frac{8}{f^2} \hat{m}_{\mathrm{Val}} [(L_5 - 2L_8) \hat{m}_{\mathrm{Val}} \\
& + N_f (L_4 - 2L_6) \hat{m}_{\mathrm{Sea}} + N_f (W_4 - W_6) \hat{a}] + \mathcal{O}(\epsilon^3). \qquad (41)
\end{aligned}$$

The fact that there are no $\mathcal{O}(a^2)$ contributions at next-to-leading order is not as surprising as one might think at first. Only the valence quark mass term breaks the chiral symmetry for Ginsparg-Wilson fermions. Hence the pseudoscalar-meson mass is proportional to the quark mass and vanishes in the limit $m_{Val} \to 0$. It follows that any lattice contribution to M_{AB}^2 is suppressed by at least one factor of m_{Val}, and the largest lattice correction quadratic in the lattice spacing is of $\mathcal{O}(m_{Val} a^2)$. Note that this higher order term becomes the leading discretization effect in the meson mass if an $\mathcal{O}(a)$ improved Wilson action is used for the sea quarks. This example illustrates the beneficial properties of Ginsparg-Wilson fermions, which are preserved even in the presence of a "non-Ginsparg-Wilson" sea sector.

V. SUMMARY

In the previous sections we presented chiral Lagrangian for two lattice theories: one with Wilson fermions and the other with Wilson sea fermions and Ginsparg-Wilson valence fermions. One consequence of the analysis is that corrections to the low-energy constants of continuum χPT (coming from symmetry-conserving discretization effects) are of $\mathcal{O}(a^2)$. Since the coefficients in the chiral Lagrangian themselves multiply terms of $\mathcal{O}(p^2)$ (B_0 and f) and $\mathcal{O}(p^4)$ (Gasser-Leutwyler coefficients), such effects can only be detected by measuring observables at the accuracy of $\mathcal{O}(a^2p^2)$ and $\mathcal{O}(a^2p^4)$, respectively. Another important discretization effect that enters the Symanzik action at $\mathcal{O}(a^2)$ is the breaking of O(4) rotational invariance. An O(4) breaking term in the chiral Lagrangian, however, must contain at least four derivatives, so it is a higher-order term as well [at least $\mathcal{O}(a^2p^4)$].

The main purpose of constructing chiral effective theories for lattice actions is to capture discretization effects analytically and to guide the chiral extrapolations of numerical lattice data. This is achieved by the explicit a dependence of observables that can be calculated in these effective theories.

[7]In Ref. [16] the number of flavors N_f was set to 3.

In particular, the chiral Lagrangian is sufficient for the determination of the pseudoscalar-meson masses. For the calculation of matrix elements, such as f_π, an additional a dependence coming from the effective continuum operators needs to be taken into account, but no conceptual difficulties are expected to arise in this step.

There is a more subtle cutoff dependence that is not explicit in the Symanzik action. All the unknown coefficients in the Symanzik action, including c_{SW}, implicitly include short-distance effects that make them a dependent. For the chiral Lagrangian this results in an implicit a dependence of the low-energy constants [7]. The existence of a well-defined continuum limit implies that all the parameters of continuum χPT, such as the Gasser-Leutwyler coefficients, have a leading a-independent part. The other coefficients in the Lagrangian, loosely referred to as the W's, are expected to show a weak, logarithmic a dependence.

From a practical point of view there are several ways to approach this issue. One option is not to vary a. For a given lattice spacing a, one fits the chiral forms by only varying the quark masses. Note that even if a is not varied, the inclusion of the discretization effects in the chiral expressions, particularly in the chiral logarithms, is more accurate than simply using the continuum expressions. From the fits, one extracts values for the coefficients in the Lagrangian, including the W's. Applying this procedure again, independently and for lattice data with different lattice spacings, these parameters are allowed to vary with a. It should be verified that the values obtained in this way for the continuum low-energy constants do not exhibit an a dependence beyond the error expected at the order of the calculation. It might be the case that the a dependence of the W's is so slow that they do not change much over the range of lattice spacings simulated. In that case a simultaneous fit in a and m might be appropriate. More generally, a simultaneous fit can be used when the a dependence of the W's is known. In particular, provided that the equations of motion can be consistently applied through $\mathcal{O}(a^2)$ to eliminate all but the Pauli term in the Symanzik action at $\mathcal{O}(a)$, one can treat the W's that enter the chiral Lagrangian at $\mathcal{O}(a)$ as being proportional to a single parameter c_{SW}. If the a dependence of this parameter is numerically known, one has control over all the a dependence in the chiral Lagrangian at $\mathcal{O}(a)$, pushing the unknown a dependence to $\mathcal{O}(a^2)$. This is "automatically" done in $\mathcal{O}(a)$-improved lattice simulations.

All the qualifications of the previous paragraphs notwithstanding, chiral perturbation theory for lattice actions provides a better understanding of the relation between lattice observables and their continuum counterparts. It is encouraging that at $\mathcal{O}(a^2)$ only a few new low-energy constants are needed. Thus χPT is still predictive at this order and it is likely to play an important role in the extraction of quantitative predictions of QCD from numerical simulations.

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APPENDIX A: FLAVOR, COLOR, AND DIRAC STRUCTURE OF FOUR-QUARK OPERATORS IN THE SYMANZIK ACTION

In this section we discuss four-quark operators in the Symanzik action that are invariant under the vector flavor symmetry group $SU(N_f)_V$, the color-gauge group $SU(N_c)$, the hypercubic transformations, parity, and charge conjugation.

It is convenient to label the quark fields $\overline{\psi}^{(1)}$, $\psi^{(2)}$, $\overline{\psi}^{(3)}$, $\psi^{(4)}$. Considering first the flavor group, we write the most general term (summation over repeated indices is assumed)

$$C_{i_1 i_2 i_3 i_4}\overline{\psi}^{(1)}_{i_1}\psi^{(2)}_{i_2}\overline{\psi}^{(3)}_{i_3}\psi^{(4)}_{i_4}. \tag{A1}$$

There are only two possibilities for C (up to a multiplicative constant), which make this term invariant:

$$C_{i_1 i_2 i_3 i_4}=\delta_{i_1 i_2}\delta_{i_3 i_4}, \quad \text{and} \quad C_{i_1 i_2 i_3 i_4}=\delta_{i_1 i_4}\delta_{i_3 i_2}. \tag{A2}$$

These correspond (up to a sign from the interchange of the Grassman fields) to

$$\overline{\psi}^{(1)}_i\psi^{(2)}_i\ \overline{\psi}^{(3)}_j\psi^{(4)}_j \quad \text{and} \quad \overline{\psi}^{(1)}_i\psi^{(4)}_i\ \overline{\psi}^{(3)}_j\psi^{(2)}_j. \tag{A3}$$

At this point we are free to redefine the labels on the quark fields in the second term by exchanging the second and fourth indices. In this way we only need to consider the first invariant in the last equation. From this point on, the order of the fields will remain fixed, so the labels of the fields can be dropped, and the trivial flavor contractions will be suppressed.

The same analysis holds for the color structure. The difference is that now we have already exhausted the freedom to reshuffle and relabel the fields—they are distinguishable by their flavor indices—and so there are two genuinely different invariant operators:

$$\overline{\psi}_a\psi_a\ \ \overline{\psi}_b\psi_b, \quad \text{and} \quad \overline{\psi}_a\psi_b\ \ \overline{\psi}_b\psi_a. \tag{A4}$$

We find it convenient to "untwist" the color indices in the second term using the Fierz rule

$$\delta_{ac}\delta_{bd}=\frac{1}{N_c}\delta_{ab}\delta_{cd}+2t^e_{ab}t^e_{cd}, \tag{A5}$$

where the t^e are the generators of the color group in the fundamental representation. The possible terms can now be written as

$$\overline{\psi}\psi\ \ \overline{\psi}\psi, \quad \text{and} \quad \overline{\psi}t^a\psi\ \ \overline{\psi}t^a\psi, \tag{A6}$$

where the contraction of color indices is straightforward and can also be suppressed.[8]

Finally, we take into account the Dirac structure. To maintain the hypercubic symmetry and parity invariance, the space-time indices must be contracted in pairs and γ_5 matrices must appear in pairs. One set of invariant terms can be obtained by adding a Dirac structure to the terms in Eq. (A6) in the following way:

$$\bar{\psi}\Gamma^A\psi\;\;\bar{\psi}\Gamma^A\psi \quad \text{and} \quad \bar{\psi}\Gamma^A t^a\psi\;\;\bar{\psi}\Gamma^A t^a\psi, \tag{A7}$$

where $\bar{\psi}\Gamma^A\psi$ can be a scalar, pseudoscalar, vector, pseudovector, or a tensor, with A denoting the appropriate space-time indices. In addition, as in the cases of color and flavor, it is also possible to have the Dirac matrices connect the first and fourth fields, and the second and the third. These operators, however, are linearly dependent on the previous terms, because of the identity

$$\Gamma^A_{\alpha\delta}\Gamma^B_{\gamma\beta}=\sum_{C,D} K^{AB}_{CD}\Gamma^C_{\alpha\beta}\Gamma^D_{\gamma\delta},$$

$$K^{AB}_{CD}=\frac{1}{16}\mathrm{Tr}[\Gamma^A\Gamma^D\Gamma^B\Gamma^C]. \tag{A8}$$

This identity holds for any pair of Clifford algebra elements and not only for the case $A=B$ in which we are interested.

This completes the derivation of the list of four-quark operators in Eq. (9). There are several equivalent sets of operators. A different path leads to the list of operators that appear in Ref. [13]: starting with the color structure, one considers the invariants of Eq. (A3), but with color indices instead of the flavor ones. Fierz rules can be used to replace the identity matrices with color generators that are either ''straight'' (connecting the first and second fields, and the third and fourth) or ''crossed.'' As was done above with respect to flavor, it is possible to choose a convention in which all the color generators are straight (reorder the fields). Thus the only invariant is $\bar{\psi}t^a\psi\bar{\psi}t^a\psi$. Once that convention is fixed, one again faces the possibility of crossed Dirac and flavor indices. The Dirac matrices are straightened in the same manner as above. Finally, using Fierz rules for the flavor group, one can also eliminate the crossed Kronecker deltas at the price of introducing terms with flavor group generators β^i. The final set of invariants is $\bar{\psi}\Gamma^A t^a\psi\bar{\psi}\Gamma^A t^a\psi$ and $\bar{\psi}\Gamma^A t^a\beta^i\psi\bar{\psi}\Gamma^A t^a\beta^i\psi$.

[8]In fact, the color structure is completely inconsequential in the construction of the chiral Lagrangian.

APPENDIX B: REDUNDANCY OF SPURIONS

We note the following fact: If A and B are two spurions, which are of the same order in the (m,a) power counting, transform in the same manner, and have a similar ''original'' structure, $B_0=kA_0$ where k carries no indices, then one can use only one of them to construct the chiral action. The reason is the following. If $f(A)$ is an operator in the chiral Lagrangian, which contains A, then $f(B)$ is also an allowed term because of the assumption that both spurions transform in the same way. Since the spurions transform linearly, the relation between the constants A_0 and B_0 also holds for the spurions. Assuming a power expansion in the spurions, this leads to $f(B)=k^n f(A)$. Recalling that each operator in the chiral Lagrangian appears with an unknown coefficient, we have in the Lagrangian

$$K_1 f(A)+K_2 f(B)=(K_1+K_2k^n)f(A). \tag{B1}$$

Since neither K_1 nor K_2 are known (and in most cases neither is k), this is equivalent to considering only a single term in the Lagrangian, $Kf(A)$, which we would have written anyway if we had considered only the first spurion.

Example: At $\mathcal{O}(a)$ the Symanzik Lagrangian contains the terms

$$ac_1\bar{\psi}_L D_\mu D_\mu\psi_R+ac_2\bar{\psi}_L i\sigma_{\mu\nu}F_{\mu\nu}\psi_R. \tag{B2}$$

To make these terms invariant one can introduce two spurions A and B, which are flavor matrices that transform as $A\to LAR^\dagger$, $B\to LBR^\dagger$. Both are $\mathcal{O}(a)$, and their constant values are $A_0=ac_1I$, $B_0=ac_2I$ (here I is the flavor identity matrix). With these spurions it is possible to construct the following invariant terms in the chiral Lagrangian [at $\mathcal{O}(a)$]:

$$K_1\langle A\Sigma^\dagger\rangle+K_2\langle B\Sigma^\dagger\rangle \tag{B3}$$

but after setting the spurions to their constant values we obtain only a single term

$$K_1ac_1\langle\Sigma^\dagger\rangle+K_2ac_2\langle\Sigma^\dagger\rangle=aK'\langle\Sigma^\dagger\rangle, \tag{B4}$$

which we would have writing down even if we had kept only A in the analysis.

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