Low-energy chiral Lagrangian from the spectral quark model

E. Megías,* E. Ruiz Arriola,† and L. L. Salcedo‡
Departamento de Física Moderna, Universidad de Granada, E-18071 Granada, Spain

W. Broniowski§
The H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, PL-31342 Cracow, Poland
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We analyze the recently proposed spectral quark model in the light of chiral perturbation theory in curved space-time. In particular, we calculate the chiral coefficients $L_1, \ldots, L_{10}$, as well as the coefficients L_{11}, L_{12}, and L_{13}, appearing when the model is coupled to gravity. The analysis is carried for the SU(3) case. We analyze the pattern of chiral symmetry breaking as well as elaborate on the satisfaction of anomalies. Matching the model results to resonance meson exchange yields the relation between the masses of the scalar, tensor, and vector mesons, $M_{f_0}=M_{f_2}=\sqrt{2}M_V=4\sqrt{3/N_c}\pi f_\pi$. Finally, the large-$N_c$ limit suggests the dual relations in the vector and scalar channels, $M_V=M_S=2\sqrt{6/N_c}\pi f_\pi$ and $\langle r^2\rangle_S^{1/2}=\langle r^2\rangle_V^{1/2}=1\sqrt{N_c}/2\pi f_\pi=0.59$ fm.

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I. INTRODUCTION

The low-energy structure of QCD in the presence of external electroweak and gravitational sources is best described by chiral perturbation theory (ChPT) [1–5] (for review see, e.g., Ref. [6]). In the meson sector, the spontaneous breaking of chiral symmetry dominates at low energies and systematic calculations of the corresponding low-energy constants (LECs) have been carried out in the recent past up to two-loop accuracy [7–10] or by using the Roy equations [11] (see also [12,13]). For strong and electroweak processes involving pseudoscalar mesons the bulk of the LECs is saturated in terms of resonance exchanges [14], which can be justified in the large-N_c limit in a certain low-energy approximation [15] by imposing the QCD short-distance constraints. In the case of gravitational processes similar ideas apply [5], although less information is known [16]. Nowadays, ChPT can be used as a qualitative and quantitative test to any model of low-energy hadron structure.

In the quest to understand the microscopic dynamics underlying the LECs, their calculation in chiral quark models has been undertaken many times [17–30]. The effort has been made to compute $L_1, \ldots, L_{10}$, which correspond to the flat-space-time case. The calculation of L_{11}, L_{12}, and L_{13}, encoding the coupling to gravitational sources, has seldomly been considered (see, however, [31]). Roughly speaking, these calculations are generally described in terms of some long-wavelength expansion of the fermion determinant associated with the constituent-quark degrees of freedom. A detailed scrutiny shows, however, that the implementation of the necessary regularization is not always satisfactory from several viewpoints. The regularization of a low-energy chiral quark model corresponds to a physical suppression of the high-energy quark states. This can be achieved in a number of different ways—e.g., by cutoffs, form factors, or momentum-dependent masses, provided they do not break symmetries such as the gauge invariance and chiral symmetry. Thus, the regularization should not be removed in the end. In such a situation, where the high-energy quark states are suppressed above a certain scale Λ, one should expect a powerlike behavior Λ^n/Q^n for any large-momentum external leg of the quark loop in the high-momentum limit. In the language of the parton model this high-energy behavior corresponds to the onset of scaling.

As a matter of fact, one of the questions which could not be answered by low-energy calculations concerns the low-energy resolution scale where these models are supposedly defined. Actually, in order to properly answer this question one should look instead into *high-energy processes* and demand parton-model relations on the constituent quarks. As pointed out in Ref. [32], a sensible scheme is obtained by demanding that the momentum fraction carried by the valence quarks in a hadron saturate the energy-momentum sum rule. Once this initial scale is defined one can use the QCD evolution to compute an observable at a higher scale. This way the QCD radiative corrections are incorporated. In fact, using the analysis of the Durham group carried out a decade ago [33] for the case of the pion, one obtains the result that the valence quarks saturate the energy-momentum sum rule at $\mu_0=313$ MeV if the leading-order (LO) Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) QCD perturbative analysis is carried out. Although this scale looks quite low, the impressive agreement obtained for the parton distribution functions of the pion after the DGLAP evolution in LO [34,35] and next-to-leading order (NLO) [35] (see also Ref. [32], and Ref. [36] where the comparison to the E615 data [37] is made) supports this interpretation of the low resolution scale. Moreover, using that scale, the pion distribution amplitude [38] and the off-forward generalized parton functions [36] agree well also with the recent transverse lattice calculations [39,40], which presumably incorporate the nonperturbative evolution.

A proper identification of the low-energy matrix elements entering the high-energy processes is grounded on the ab-

*Electronic address: emegias@ugr.es
†Electronic address: earriola@ugr.es
‡Electronic address: salcedo@ugr.es
§Electronic address: Wojciech.Broniowski@ifj.edu.pl

sence of logarithmic corrections in the low-energy model in the high-energy limit, since the proper QCD radiative logarithmic corrections are automatically and completely incorporated by the QCD evolution. Not surprisingly, this condition imposes severe constraints on the kind of admissible regularization schemes. In a recent work the spectral quark model (SQM) has been proposed [41,42], implementing the so-called spectral regularization (see below) complying to these powerlike short-distance constraints.

In the present paper we extend the SQM to the SU(3) flavor group and include finite current quark mass. Instead of using the construction of vertices based on the Ward-Takahashi identities, employed in Refs. [41,42], it is by far more convenient to define the effective action depending on the nonlinear pseudoscalar meson fields in the presence of external scalar, pseudoscalar, vector, axial, and gravitational sources. The latter have never been considered in chiral quark model calculations. This effective action is defined in Sec. II. We also show in Sec. III how one can explicitly eliminate the spectral function in terms of the quark momentum-dependent mass and wave-function renormalization. Following the standard procedure we perform the gradient expansion of the spectral-regularized fermion determinant for both the anomalous (Sec. IV) and the nonanomalous sectors in curved space-time (Sec. V). As a consequence the structure of the energy-momentum tensor may be analyzed. Remarkably, our spectral regularization method complies with the QCD anomaly without removing the regularization. Therefore the standard Wess-Zumino-Witten [43,44] term is generated for a finite regularization. In the nonanomalous sector we find, through the comparison to the standard chiral Lagrangian [3,4], that the low-energy constants at $\mathcal{O}(p^4)$ associated with terms which are nonvanishing in the chiral limit are completely independent of the regularization details. The LECs associated with terms carrying the current quark mass coefficients do depend on the particular ansatz for the spectral regularization, and we evaluate them using the regularization based on the meson dominance of form factors [42]. Such a model has provided a satisfactory description of the quark self-energy of the recent lattice data [45]. Finally (Sec. VII), we also confront the large-N_c relations [15] and discuss the consequences of extending the present model to include these constraints. The Appendix contains details of the formalism in the curved space-time.

II. EFFECTIVE ACTION OF THE SPECTRAL QUARK MODEL

In a recent work the spectral quark model has been introduced [41,42]. The approach is similar in spirit to the model of Efimov and Ivanov [46], proposed many years ago. It is based on the formal introduction of the generalized Lehmann representation for the quark propagator:

$$S(p)=\int_C d\omega \frac{\rho(\omega)}{\not p-\omega} \equiv \frac{Z(p^2)}{\not p-M(p^2)}, \tag{1}$$

where $\rho(\omega)$ is a (generally complex) quark spectral function and C denotes a suitable contour in the complex ω plane. The function $M(p^2)$ is the quark self-energy, while $Z(p^2)$ is the quark wave-function renormalization. In the case of *analytic confinement*—i.e., when the propagator does not have poles, a sensible definition of a constituent quark mass is (from now on we drop the index C from the ω integral, which is implicitly understood to run along the contour C)

$$M_Q=M(0)=\int d\omega \frac{\rho(\omega)}{\omega} \Big/ \int d\omega \frac{\rho(\omega)}{\omega^2}. \tag{2}$$

As discussed at length in Ref. [42], the proper normalization and the conditions of finiteness of hadronic observables are achieved by requesting an infinite set of *spectral conditions* for the moments of the quark spectral function $\rho(\omega)$—namely,

$$\rho_0 \equiv \int d\omega \rho(\omega)=1, \tag{3}$$

$$\rho_n \equiv \int d\omega \omega^n \rho(\omega)=0, \quad \text{for } n=1,2,3,\ldots. \tag{4}$$

Physical observables are proportional to the zeroth and the inverse moments,

$$\rho_{-k} \equiv \int d\omega \omega^{-k} \rho(\omega), \quad \text{for } k=0,1,2,3,\ldots, \tag{5}$$

as well as to the "*log moments*,"

$$\rho_n' \equiv \int d\omega \log(\omega^2/\mu^2) \omega^n \rho(\omega) = \int d\omega \log(\omega^2) \omega^n \rho(\omega), \quad \text{for } n=1,2,3,4,\ldots. \tag{6}$$

Obviously, when an observable is proportional to the dimensionless zeroth moment, $\rho_0=1$, the result does not depend explicitly on the regularization. The spectral conditions (4) remove the dependence on the scale μ in Eq. (6), thus guaranteeing the absence of any dimensional transmutation. The only exception is the zeroth-log moment

$$\rho_0'(\mu^2)=\int d\omega \log(\omega^2/\mu^2)\rho(\omega), \tag{7}$$

which does depend on a scale μ and is *not* regularized by the spectral method (see the discussion below). No standard requirement of positivity for the spectral strength, $\rho(\omega)$, is made. Unlike other regularizations, such as the dimensional regularization or ζ-function regularization, the spectral regularization is physical in the sense that it provides a high-energy suppression in one-quark-loop amplitudes and is not removed at the end of the calculation. It also improves on a Pauli-Villars regularization, because it complies with the factorization property of correlation functions, form factors, etc., in the high-energy limit; i.e., it guarantees the absence of logarithmic corrections to form factors. The phenomenological success of the SQM in describing structure functions of the pion, generalized parton distributions [36,47], and the pion light-cone wave function [38,48] suggests that the whole scheme deserves to be thoroughly pursued further.

In Ref. [42] it was argued that there are a number of terms in the one-quark-loop effective low-energy chiral Lagrangian which correspond to taking the infinite-cutoff limit. The terms with explicit chiral-symmetry breaking do not correspond to this class. The purpose of this paper is to analyze these terms, which are specific both to the regularization and the choice of couplings in the spectral quark Lagrangian. For completeness we also consider the gauge couplings and gravitational couplings, which allows us a determination of all low-energy constants in the SU(3) sector in the SQM approach.

The effective action complying to the solution of the Ward-Takahashi identities via the gauge technique of Delbourgo and West [49] corresponds in our case to the minimum substitution prescription for the spectral quark. It yields a quark fermionic determinant of the form

$$\Gamma[U,s,p,v,a,g] = -\mathrm{i}N_c \int d\omega \rho(\omega) \mathrm{Tr}\log(i\mathbf{D}), \qquad (8)$$

where the Dirac operator is given by

$$\begin{aligned} i\mathbf{D} &= i\not{d} - \omega U^5 - \hat{m}_0 + (\not{v} + \not{a}\gamma_5 - s - i\gamma^5 p) \\ &= iD - \omega U^5. \end{aligned} \qquad (9)$$

The derivative d_μ is frame (local Lorentz) and general-coordinate covariant and it includes the spin connection (see the Appendix for notation). The symbols s, p, v_μ, and a_μ denote the external scalar, pseudoscalar, vector, and axial flavor sources, respectively, given in terms of the generator of the flavor SU(3) group,

$$s = \sum_{a=0}^{N_F^2-1} s_a \frac{\lambda_a}{2}, \ldots, \qquad (10)$$

with λ_a representing the Gell-Mann matrices. The tensor $g_{\mu\nu}$ is the metric external source representing the coupling to a gravitational field. The matrix $U^5 = U^{\gamma_5}$, and $U = u^2 = e^{i\sqrt{2}\Phi/f}$ is the flavor matrix representing the pseudoscalar octet of mesons in the nonlinear representation:

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}. \qquad (11)$$

The matrix $\hat{m}_0 = \mathrm{Diag}(m_u, m_d, m_s)$ is the current quark mass matrix and f denotes the pion weak-decay constant in the chiral limit, to be determined later on from the proper normalization condition of the pseudoscalar fields. For a bilocal (Dirac- and flavor-matrix-valued) operator $A(x,x')$ one has

$$\mathrm{Tr}\, A = \int d^4x \sqrt{-g}\, \mathrm{tr}\langle A(x,x)\rangle, \qquad (12)$$

with tr denoting the Dirac trace and $\langle\ \rangle$ the flavor trace. Moreover, $g = \det g_{\mu\nu}$ is the determinant of the curved space-time metric. Finally, in the second line of Eq. (9) we have introduced the Dirac operator D corresponding to the external fields only. The $U_A(1)$ is taken into account by extending the matrix to the U(3) sector, $U \to \overline{U} = U e^{\mathrm{i}\eta_8/(3f)}$ with $\det U = 1$, adding the customary term

$$\mathcal{L} = -\frac{f^2}{4} m_{\eta_1}^2 \left\{ \theta - \frac{i}{2}[\log\det U - \log\det U^\dagger] \right\}^2. \qquad (13)$$

The Dirac operator given by Eq. (9) transforms covariantly under local chiral transformations (see the Appendix).

Formally, in the flat space-time the effective action (8) looks quite familiar and we should point out here that the main difference with similar actions, such as, e.g., the one of Ref. [21], is related to the regularization procedure. Actually, the method of Ref. [21] consists of taking $\rho(\omega) = \delta(\omega - M_Q)$ with M_Q being the constituent-quark mass. This choice satisfies the normalization condition $\rho_0 = 1$, but does not comply to the $\rho_n = 0$ spectral requirements. The problem can be avoided if one uses suitable regularization methods, such as the dimensional or ζ-function regularization, but then *logarithmic corrections* to form factors are generated and the well-known Landau instability found long ago in Refs. [50,51] sets in.

The pion form factor obtained from the ζ-function regularization used, for instance, in Ref. [21] for $t = -Q^2$ becomes, in the chiral limit,

$$F(Q^2) = -\frac{4N_c M_Q^2}{(4\pi)^2 f_\pi^2} \int_0^1 dx \log\left[\frac{x(1-x)Q^2 + M_Q^2}{\mu^2}\right],$$

where the pion weak-decay constant is given by $f_\pi^2 = 4N_c M_Q^2 \log(\mu^2/M_Q^2)/(4\pi)^2$. While the proper normalization $F(0) = 1$ is obtained, at large momenta one has a logarithmic behavior $F(Q^2) \to \log(Q^2)$, instead of the powerlike behavior, which poses a problem. On the other hand, the spectral regularization method yields [48] $F(Q^2) \to N_c/4\pi^2 f_\pi^2 (2\rho_4'/Q^2 - 2\rho_6'/Q^4 + \cdots)$, with no logarithms present. This twist expansion property allows us to extract in a clean way the low-energy matrix elements relevant for high-energy processes [32].

Our effective action looks also similar to Nambu–Jona-Lasinio (NJL) bosonized models [52] (for reviews see, e.g., Refs. [53–57]). Again, the main and important difference has to do with the interpretation of the regularization method. As discussed in Refs. [32] in NJL models one can only regularize quark loops—i.e., closed quark lines—so a direct comparison for the Lehmann representation, where quark lines are open is somewhat misleading. The fact that in the SQM the Lehmann ‘‘regularization’’ carries over to open quark lines has important consequences as regards the consistency of high-energy calculations in either a purely hadronic or partonic interpretation [42].

Given the fact that the integration contour is in general complex, passing to the Euclidean space and separating the action into the real and imaginary parts becomes a bit inconvenient. Instead, we take the full advantage of the Minkowski space and introduce the auxiliary operator

$$-\mathrm{i}\mathbf{D}_5=\gamma_5(\mathrm{i}\not{d}-\omega U^{5\dagger}-\hat{m}_0+\not{v}-\gamma_5\not{a}-s+i\gamma_5 p)\gamma_5, \tag{14}$$

which corresponds to the Hermitian conjugation in the Euclidean space. Thus, the normal parity action is given by

$$S_{\text{n.p.}}=-\frac{\mathrm{i}}{2}N_c\int d\omega\rho(\omega)\mathrm{Tr}\log(\mathbf{D}\mathbf{D}_5). \tag{15}$$

III. RELATION OF SPECTRAL MOMENTS TO QUARK MASS AND NORMALIZATION

A potential disadvantage of the spectral regularization is that the inverse problem—i.e., the problem of finding the spectral function $\rho(\omega)$ from the known moments—does not always have an easy explicit solution or perhaps has no solution at all. In this section we show how the negative moments and the log moments can be translated into the integrals involving the quark mass function $M(p^2)$ and the quark wave-function renormalization $Z(p^2)$. Let us start with Eq. (1) and assume that the set of spectral conditions is met:

$$\int d\omega\omega^n\rho(\omega)=\delta_{n0}, \quad n=0,1,\dots. \tag{16}$$

Then, the following identity, proved by induction, holds:

$$\int d\omega\frac{\omega^n\rho(\omega)}{\not{p}-\omega}=\not{p}^nS(\not{p})-\not{p}^{n-1} \quad n=1,2,\dots. \tag{17}$$

Rationalizing the denominators yields

$$\int d\omega\omega^n\rho(\omega)\frac{\not{p}+\omega}{p^2-\omega^2}=\not{p}^nZ(p^2)\frac{\not{p}+M(p^2)}{p^2-M(p^2)^2}-\not{p}^{n-1}. \tag{18}$$

We have two cases of odd and even n. For $n=2k$ we find

$$\int d\omega\omega^{2k}\rho(\omega)\frac{\not{p}+\omega}{p^2-\omega^2}=p^{2k}Z(p^2)\frac{\not{p}+M(p^2)}{p^2-M(p^2)^2}-\not{p}p^{2k-2}. \tag{19}$$

Defining

$$L_n(p^2)=\int d\omega\omega^n\rho(\omega)\frac{1}{p^2-\omega^2} \tag{20}$$

and comparing coefficients of powers of $\not{p}$ in Eq. (19) produces the identities

$$L_{2k}(p^2)=p^{2k}Z(p^2)\frac{1}{p^2-M(p^2)^2}-p^{2k-2},$$

$$L_{2k+1}(p^2)=p^{2k}Z(p^2)\frac{M(p^2)}{p^2-M(p^2)^2}. \tag{21}$$

The case $n=2k+1$ produces the same relations.

The following recursion relations follow directly from the spectral conditions (4):

$$\int d\omega\frac{\omega^n\rho(\omega)}{p^2-\omega^2}=p^2\int d\omega\frac{\omega^{n-2}\rho(\omega)}{p^2-\omega^2}, \quad n>2, \tag{22}$$

which are obvious when on the right-hand side we write $p^2=(p^2-\omega^2)+\omega^2$. We now pass to the Euclidean space, $\not{p}^2=p^2\to-p_E^2$, and get

$$\int d\omega\omega^n\log(\omega^2)\rho(\omega)$$

$$=\int_0^\infty dp_E^2L_n(-p_E^2)=-\int_0^\infty dp_E^2p_E^2L_{n-2}(-p_E^2). \tag{23}$$

Thus, we have obtained the log moments in terms of Z and M. The negative moments are simply derivatives of the quark propagator at the origin:

$$\int d\omega\frac{\rho(\omega)}{\omega^n}=-\left(\frac{d}{d\not{p}}\right)^{n-1}S(\not{p})\bigg|_{p=0} \quad n=1,2,\dots. \tag{24}$$

The derivative is computed taking $p^2=\not{p}\not{p}$. Thus, given the quark propagator $S(\not{p})$ we may just use formulas (23), (24) to translate negative moments and log moments without ever having to specify explicitly the spectral function. This is a rather remarkable feature of the spectral approach. The expressions for f_π, $\langle qq\rangle$ (the quark condensate for a single flavor), and B (the vacuum energy density) in the chiral limit are

$$f^2=\frac{4N_c}{(4\pi)^2}\int d\omega\omega^2\rho(\omega)(-\log\omega^2), \tag{25}$$

$$\langle\bar{q}q\rangle=\frac{4N_c}{(4\pi)^2}\int d\omega\omega^3\rho(\omega)(-\log\omega^2), \tag{26}$$

$$-B=\frac{N_FN_c}{(4\pi)^2}\int d\omega\omega^4\rho(\omega)(-\log\omega^2)=\frac{1}{4}\langle\theta_\mu^\mu\rangle, \tag{27}$$

respectively. Here, $\theta^{\mu\nu}$ is the energy-momentum tensor (see also Sec. V C). We get, for instance,

$$f^2=\frac{4N_c}{(4\pi)^2}\int_0^\infty dp_E^2\frac{M(-p_E^2)^2-p_E^2[Z(-p_E^2)-1]}{p_E^2+M(-p_E^2)^2} \tag{28}$$

or

$$\langle\bar{q}q\rangle=\frac{4N_c}{(4\pi)^2}\int_0^\infty dp_E^2p_E^2\frac{Z(-p_E^2)M(-p_E^2)}{p_E^2+M(-p_E^2)^2}. \tag{29}$$

In Eq. (29) we recognize the usual formula for the quark condensate found in nonlocal models. On the other hand, Eq. (28) is different from analogous quark-model expressions [58,59]. The reason is that, strictly speaking, the above formulas should only be used for functions $M(p^2)$ and $Z(p^2)$ complying with the generalized Lehmann representation, Eq. (1), with the spectral density satisfying the spectral conditions.

One can use similar manipulations to get the pion electromagnetic form factor obtained in Ref. [42]. For spacelike momentum $Q^2=-q^2$, we obtain

$$F_V(Q^2)=\frac{4N_c}{(4\pi)^2f_\pi^2}\int_0^1dx\int_0^\infty dp_E^2 \times\frac{M(-P_E^2)^2-P_E^2[Z(-P_E^2)-1]}{P_E^2+M(-P_E^2)^2}, \tag{30}$$

where

$$P_E^2=p_E^2+x(1-x)Q^2. \tag{31}$$

Note that the inversion procedure used in Ref. [42] to determine the spectral density from vector meson dominance (the meson dominance version of the SQM) is linear, whereas written in terms of M and Z becomes highly nonlinear.

IV. CHIRAL ANOMALIES

One of the major advantages of the spectral regularization is that it makes hadronic observables finite and scale independent, a basic requirement of any regularization procedure. However, that does not necessarily mean or imply that the full effective action in the presence of external fields is finite, since even in the case of the vanishing pion fields, $U=1$, we have nonhadronic processes. Actually, it turns out that the photon wave function renormalization [42] is proportional to ρ_0'; thus, it depends on the scale μ and therefore diverges in some regularization schemes (such as the dimensional regularization). This scale dependence arises also in other nonhadronic terms of the effective action.

In Ref. [42] it was checked that the $\pi^0\to2\gamma$ and $\gamma\to3\pi$ decays comply to the correct values expected from the chiral QCD anomaly. With the help of the effective action, Eq. (8), we now want to show that this is also true for all anomalous processes. In order to understand the role of regularization, it is instructive to compute the chiral anomaly first. Next, we will show that in the presence of external fields the anomaly does not depend on the pion field U and thus coincides with the anomaly in QCD due to the spectral conditions $\rho_1=\rho_2=\rho_3=\rho_4=0$.

Under chiral (vector and axial) local transformations the Dirac operator transforms as

$$\mathbf{D}\to e^{+i\epsilon_V(x)-i\epsilon_A(x)\gamma_5}\mathbf{D}e^{-i\epsilon_V(x)-i\epsilon_A(x)\gamma_5}, \tag{32}$$

with

$$\epsilon_V(x)=\sum_a\epsilon_V^a(x)\lambda_a,\quad \epsilon_A(x)=\sum_a\epsilon_A^a(x)\lambda_a. \tag{33}$$

Infinitesimally, we have

$$\delta\mathbf{D}=i[\epsilon_V,\mathbf{D}]-i\{\epsilon_A\gamma_5,\mathbf{D}\}. \tag{34}$$

If we make a chiral transformation of the effective action (8) without any additional regularization, we get

$$\delta S=-iN_c\mathrm{Tr}\int d\omega\rho(\omega)[\delta\mathbf{D}\mathbf{D}^{-1}]. \tag{35}$$

If we assume the cyclic property of the functional trace, we get a contribution from the axial variation only,

$$\begin{aligned}\delta_AS\equiv\mathcal{A}_A&=\int d^4x\,\mathrm{tr}\int d\omega\rho(\omega)\langle 2i\epsilon_A\gamma_5\rangle\\&=\rho_0\int d^4x\,\mathrm{tr}\langle 2i\epsilon_A\gamma_5\rangle,\end{aligned} \tag{36}$$

a result which, due to the infinite dimensional trace [60,61], is ambiguous even in the presence of the spectral regularization. Thus, to get rid of the ambiguity we have to introduce an extra regularization. As is well known, there is no regularization preserving the chiral symmetry; thus, the anomaly is generated.

The calculation can be done by standard methods. A very convenient one is the ζ-function regularization [62], which computes the anomaly directly in terms of the Dirac operator itself (and not its square) and does not require any redefinition of the Dirac γ_5 matrix. This yields the equation

$$\begin{aligned}\delta_AS\equiv\mathcal{A}_A&=\mathrm{Tr}\int d\omega\rho(\omega)(2i\epsilon_A\gamma_5[\mathrm{i}\mathbf{D}]^0)\\&=\int d^4x\mathrm{tr}\int d\omega\rho(\omega)\langle 2i\epsilon_A(x)\gamma_5\langle x|\mathbf{D}^0|x\rangle\rangle,\end{aligned} \tag{37}$$

where the zeroth power of the Dirac operator is understood as an analytical continuation which can be written in terms of the Seeley-DeWitt coefficients for the Dirac operators [62]:

$$\begin{aligned}\langle x|\mathbf{D}^0|x\rangle=\frac{1}{(4\pi)^2}\Big\{&\frac{1}{2}\mathbf{D}^4+\frac{1}{3}(\mathbf{D}^2\Gamma_\mu^2+\Gamma_\mu\mathbf{D}^2\Gamma_\mu+\Gamma_\mu^2\mathbf{D}^2)\\&+\frac{1}{6}(\Gamma_\mu^2\Gamma_\nu^2+(\Gamma_\mu\Gamma_\nu)^2+\Gamma_\mu\Gamma_\nu^2\Gamma_\mu)\Big\},\end{aligned} \tag{38}$$

where $\Gamma_\mu=\frac{1}{2}\{\gamma_\mu,\mathbf{D}\}$ and the operator $\mathbf{D}$ acts to the left. The result for general couplings in four dimensions has been obtained from Ref. [62]. Direct inspection shows that since the ω dependence is given by $i\mathbf{D}=iD-\omega U^5$, the result can be written as a sum of an ω-independent term and a polynomial remainder:

$$\mathcal{A}_A=\int d\omega\rho(\omega)(\mathcal{A}_A[v,a,s,p]+\mathcal{A}_A[v,a,s,p,\omega,U])$$
$$=\rho_0\mathcal{A}_A[v,a,s,p], \quad (39)$$

where the ω-dependent polynomial term vanishes due to the spectral conditions. This shows that the anomaly of the spectral quark model coincides with the anomaly of QCD *after* introducing an additional suitable regularization, regardless of the details of the spectral function. This result is common also to nonlocal models when one evaluates anomalies [63,64]. This is an important point since if the effective action $\Gamma[U,s,p,v,a]$ in Eq. (8) is both chiral symmetric and finite, there is apparently no reason for anomalies. We will see below how and where these divergences arise.

To see now how the standard Wess-Zumino-Witten (WZW) [43,44] term arises in the present context, let us consider for simplicity the chiral limit $\hat{m}_0=0$ and set the external fields to zero and work in flat space, so that $\mathrm{i}\mathbf{D}=\mathrm{i}\partial\!\!\!/$. A convenient representation can be obtained by introducing the field

$$U_t^5=e^{it\sqrt{2}\gamma_5\Phi/f}, \quad (40)$$

interpolating between the vacuum, $U^5_{t=0}=1$, and the full matrix $U^5_{t=1}=U^5$. Then, we have the trivial but useful identity for the vacuum-subtracted action:

$$\Gamma[U,s,\dots]-\Gamma[1,s,\dots]$$
$$=-iN_c\int_0^1 dt\frac{d}{dt}\int_C d\omega\rho(\omega)\mathrm{Tr}\log(\mathrm{i}D-\omega U_t^5)$$
$$=iN_c\int_0^1 dt\int_C d\omega\rho(\omega)\mathrm{Tr}\left[\omega\frac{dU_t^5}{dt}\frac{1}{\mathrm{i}D-\omega U_t^5}\right]. \quad (41)$$

Using the representation in Eq. (42) and the formulas of the Appendix the result can be obtained straightforwardly. Since we are interested in abnormal parity processes, it is enough to identify the terms containing the Levi-Cività tensor $\epsilon_{\mu\nu\alpha\beta}$, which due to the Lorentz invariance requires at least four derivatives. Taking into account the fact that the derivative operator acts to the right we get

$$S_{\rm ab}^{(4)}=-iN_c\int_0^1 dt\int_C d\omega\rho(\omega)\int d^4x\int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-\omega^2]^5}$$
$$\times\mathrm{Tr}\left\{-\omega\gamma_5 U_t^\dagger\frac{dU_t}{dt}\omega[\omega U_t^\dagger \mathrm{i}\partial\!\!\!/ U_t]^4\right\}. \quad (42)$$

After computation of the traces and integrals we finally find

$$\Gamma_{\rm ab}^{(4)}=\rho_0\frac{N_c}{48\pi^2}\int_0^1 dt\int d^4x\,\epsilon_{\mu\nu\alpha\beta}$$
$$\times\left\langle U_t^\dagger\frac{dU_t}{dt}U_t^\dagger\partial^\mu U_tU_t^\dagger\partial^\nu U_tU_t^\dagger\partial^\alpha U_tU_t^\dagger\partial^\beta U_t\right\rangle, \quad (43)$$

which coincides with the WZW term if the spectral normalization condition $\rho_0=1$ is used. External fields can be included again through the use of Eq. (42), yielding the gauged WZW term in the Bardeen-subtracted form. Actually, the difference $\Gamma[U,s,p,v,a]-\Gamma[1,s,p,v,a]$ is finite and preserves gauge invariance but breaks chiral symmetry generating the anomaly of Eq. (39).

Higher-order corrections to the abnormal parity component of the action involve negative spectral moments. For instance, the terms $\mathcal{O}(p^6)$ and higher are regularized, and involve ρ_{-2} for terms with no quark mass terms and ρ_{-1} for terms containing one quark mass. This is in contrast to the approach of Ref. [21] where the infinite-cutoff limit is considered for a constant constituent-quark mass. In this regard let us also note that for the unregularized abnormal parity action one would get the transition form factor

$$F_{\pi\gamma\gamma*}(Q^2)=\frac{8M_Q^2}{(4\pi)^2f_\pi}\int_0^1 dx\frac{1}{(1-x)xQ^2+2M_Q^2},$$

which satisfies the proper anomaly condition $F_{\pi\gamma\gamma*}(0)=1/(4\pi^2f_\pi)$. Again, a log-dependent term is obtained at high virtualities (see also Ref. [32]), in contrast to the correct twist expansion generated by the spectral method [42].

V. LOW-ENERGY CHIRAL EXPANSION OF THE ACTION

The chiral expansion of the action, Eq. (8), corresponds to a counting where the pseudoscalar field U and the curved space-time metric $g^{\mu\nu}$ are zeroth order, the vector and axial fields v_μ and a_μ are first order, and any derivative ∂_μ first order. The external scalar and pseudoscalar fields s and p and the current mass matrix $\hat{m}_0$ are taken to be second order. In chiral quark models at the one-loop level this chiral expansion corresponds to a derivative expansion. With the help of the action of Eq. (8) one can compute the derivative expansion in curved space-time (see the Appendix for details),

$$S=\int d^4x\sqrt{-g}\mathcal{L}(x), \quad (44)$$

where the effective chiral Lagrangian in the Gasser-Leutwyler-Donoghue form [4,5] reads

$$\mathcal{L}=\mathcal{L}^{(0)}+\mathcal{L}^{(2,g)}+\mathcal{L}^{(2,R)}+\mathcal{L}^{(4,g)}+\mathcal{L}^{(4,R)}+\cdots, \quad (45)$$

with the metric (upperscript g) and curvature (upperscript R) terms explicitly separated. The zeroth-order vacuum contribution reads

$$\mathcal{L}^{(0)}=B=\frac{N_F N_c}{(4\pi)^2}\rho_4', \tag{46}$$

where the vacuum constant is given by Eq. (27).

A. Metric contributions

The metric contributions read

$$\mathcal{L}^{(2,g)}=\frac{f^2}{4}\langle D_\mu U^\dagger D^\mu U+(\chi^\dagger U+U^\dagger\chi)\rangle \tag{47}$$

and

$$\mathcal{L}^{(4,g)}=L_1\langle D_\mu U^\dagger D^\mu U\rangle^2+L_2\langle D_\mu U^\dagger D_\nu U\rangle^2+L_3\langle(D_\mu U^\dagger D_\nu U)^2\rangle+L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle\chi^\dagger U+U^\dagger\chi\rangle+L_5\langle D_\mu U^\dagger D^\mu U(\chi^\dagger U+U^\dagger\chi)\rangle+L_6\langle\chi^\dagger U+U^\dagger\chi\rangle^2+L_7\langle\chi^\dagger U-U^\dagger\chi\rangle^2+L_8\langle(\chi^\dagger U)^2+(U^\dagger\chi)^2\rangle-iL_9\langle F^L_{\mu\nu}D_\mu U D_\nu U^\dagger+F^R_{\mu\nu}D_\mu U^\dagger D_\nu U\rangle+L_{10}\langle F^L_{\mu\nu}UF^R_{\mu\nu}U^\dagger\rangle+H_1\langle(F^R_{\mu\nu})^2+(F^L_{\mu\nu})^2\rangle+H_2\langle\chi^\dagger\chi\rangle. \tag{48}$$

We have introduced the standard chiral covariant derivatives and gauge field strength tensors

$$D_\mu U=D^L_\mu U-UD^R_\mu=\partial_\mu U-iA^L_\mu U+iUA^R_\mu,$$
$$F^r_{\mu\nu}=i[D^r_\mu,D^r_\nu]=\partial_\mu A^r_\nu-\partial_\nu A^r_\mu-i[A^r_\mu,A^r_\nu], \tag{49}$$

with $r=L,R$. The pion weak-decay constant and the quark condensate in the chiral limit read

$$f^2=-\frac{4N_c}{(4\pi)^2}\rho_2', \tag{50}$$

$$f^2B_0=-\langle\bar{q}q\rangle=\frac{4N_c}{(4\pi)^2}\rho_3', \tag{51}$$

while the chiral coefficients are[1]

$$L_3=-2L_2=-4L_1=-\frac{N_c}{(4\pi)^2}\frac{\rho_0}{6}, \tag{52}$$

$$L_4=L_6=0, \tag{53}$$

$$L_5=-\frac{N_c}{(4\pi)^2}\frac{\rho_1'}{2B_0}, \tag{54}$$

$$L_7=\frac{N_c}{(4\pi)^2}\frac{1}{2N_F}\left(\frac{\rho_1'}{2B_0}+\frac{\rho_0}{12}\right), \tag{55}$$

$$L_8=\frac{N_c}{(4\pi)^2}\left[\frac{\rho_2'}{4B_0^2}-\frac{\rho_1'}{4B_0}-\frac{\rho_0}{24}\right], \tag{56}$$

$$L_9=-2L_{10}=\frac{N_c}{(4\pi)^2}\frac{\rho_0}{3}, \tag{57}$$

$$H_1=\frac{N_c}{(4\pi)^2}\frac{\rho_0'}{6}, \tag{58}$$

$$H_2=\frac{N_c}{(4\pi)^2}\left(\frac{\rho_2'}{B_0^2}+\frac{\rho_1'}{2B_0}+\frac{\rho_0}{12}\right), \tag{59}$$

where $N_F=2,3$. As we can see, the coefficients $L_1,L_2,L_3,L_4,L_6,L_9,L_{10}$ are pure numbers and coincide for convergent integrals with those expected in the limit where the regularization is removed [21]. The argument anticipating this result in Ref. [42] has to do with the dimensionless character of the low-energy couplings which thus involve the zeroth moment $\rho_0=1$. Note that this remarkable result holds *without* removing the regularization.[2] The fact that H_1 is proportional to ρ_0' corresponds to a scale-dependent or divergent gauge-field wave function and was observed already in Ref. [42]. Hence, the finite piece of H_1 depends on the regularization scheme.

We can use f and L_5 in order to determine L_7, L_8, B_0, and H_2, which immediately yields

[1]The value of L_7 displayed here corresponds to the SU(3) model only. For the U(3) model one gets $L_7=0$ but then the term of Eq. (13) should be added, and the value of L_7 is changed.

[2]Actually, the kinetic energy term obtained in Ref. [21] within the *zeta*-function regularization was scale dependent, so dimensional transmutation sets in. If dimensional regularization is used, it would lead to a $1/\epsilon$ divergence, which after renormalization would also lead to dimensional transmutation. The point of the spectral regularization is that dimensional transmutation is precluded thanks to the spectral conditions, Eqs. (4), and *any* choice of the spectral function yields the same finite result.

TABLE I. The dimensionless low energy constants (multiplied by 10^3) compared with some reference values and other models. The errors for the SQM in the MDM realization reflect the errors in M_S and M_Q of Eq. (86).

$\times 10^3$	SQM (MDM)	ChPT [a]	Large N_c [b]	NJL [c]	Dual large N_c
L_1	0.79	0.53 ± 0.25	0.9	0.96	0.79
L_2	1.58	0.71 ± 0.27	1.8	1.95	1.58
L_3	-3.17	-2.72 ± 1.12	-4.3	-5.21	-3.17
L_4	0	0	0	0	0
L_5	2.0 ± 0.1	0.91 ± 0.15	2.1	1.5	3.17
L_6	0	0	0	0	0
L_7	-0.07 ± 0.01 [d]	-0.32 ± 0.15	-0.3		
L_8	0.05 ± 0.04	0.62 ± 0.20	0.8	0.8	1.18
L_9	6.33	5.93 ± 0.43	7.1	6.7	6.33
L_{10}	-3.17	-4.40 ± 0.70 [e]	-5.4	-5.5	-4.75
L_{11}	1.58	1.85 ± 0.90 [f]	1.6		
L_{12}	-3.17	-2.7 [f]	-2.7		
L_{13}	0.33 ± 0.01	1.7 ± 0.80 [f]	1.1		

[a]The two-loop calculation of Ref. [9].
[b]Reference [14].
[c]Reference [24].
[d]See footnote 1.
[e]References [10,67].
[f]Reference [5].

$$L_7 = -\frac{L_5}{2N_f} + \frac{N_c}{384\pi^2 N_f} \simeq -0.35\times 10^{-3},$$

$$L_8 = \frac{L_5}{2} - \frac{N_c}{384\pi^2} - \frac{f^2}{16B_0^2} \simeq 0.05\times 10^{-3},$$

$$H_2 = -L_5 + \frac{N_c}{192\pi^2} - \frac{f^2}{4B_0^2} \simeq -1.02\times 10^{-3}. \tag{60}$$

The numerical values displayed here have been obtained with the large-N_c value of L_5 from Table I.

B. Curvature contributions

The curvature contributions to the chiral Lagrangian can be written in the form proposed in Ref. [5] and are given by

$$\mathcal{L}^{(2,R)} = H_0 R \tag{61}$$

and

$$\mathcal{L}^{(4,R)} = -L_{11} R\langle D^\mu U^\dagger D_\mu U\rangle - L_{12} R^{\mu\nu}\langle D_\mu U^\dagger D_\nu U\rangle - L_{13} R\langle \chi^\dagger U + U^\dagger \chi\rangle + H_3 R^2 + H_4 R_{\mu\nu} R^{\mu\nu} + H_5 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}. \tag{62}$$

Here $R^\lambda_{\sigma\mu\nu}$, $R_{\mu\nu}$, and R are the Riemann curvature tensor, the Ricci tensor, and the curvature scalar, respectively:[3]

$$-R^\lambda{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\lambda_{\mu\alpha}\Gamma^\alpha_{\nu\sigma} - \Gamma^\lambda_{\nu\alpha}\Gamma^\alpha_{\mu\sigma},$$

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}; \quad R = g^{\mu\nu} R_{\mu\nu}. \tag{63}$$

The Christoffel symbols are specified in Eq. (A22). The curvature terms reflect the composite nature of the pseudoscalar fields, since in the considered model they correspond to the coupling of the gravitational external field at the quark level. After some algebra we get

$$H_0 = -\frac{f^2}{4} N_F/6, \tag{64}$$

$$L_{12} = -2L_{11} = -\frac{N_c}{(4\pi)^2}\frac{\rho_0}{6}, \tag{65}$$

$$L_{13} = -\frac{N_c}{(4\pi)^2}\frac{\rho_1'}{12B_0} = \frac{1}{6}L_5, \tag{66}$$

$$H_3 = +\frac{N_c}{(4\pi)^2} N_F \frac{\rho_0'}{144}, \tag{67}$$

$$H_4 = -\frac{N_c}{(4\pi)^2} N_F \frac{\rho_0'}{90}, \tag{68}$$

$$H_5 = -\frac{N_c}{(4\pi)^2} N_F \frac{7\rho_0'}{720}. \tag{69}$$

Note that there is a finite strong renormalization to Newton's gravitational constant G, since the classical Einstein's Lagrangian is $\mathcal{L} = -R/(16\pi G)$. This correction, proportional to the ratio of the hadronic to the Planck scale $f^2 G\pi/3$, is numerically tiny.

[3]Note the opposite sign of our definition for the Riemann tensor as compared to Ref. [5]. We follow Ref. [65] (see the Appendix).

C. Energy-momentum tensor

Using the action of Eq. (8) one can compute the energy momentum tensor as a functional derivative of the action with respect to an external space-time-dependent metric, $g_{\mu\nu}(x)$, around the flat space-time metric $\eta_{\mu\nu}$ [we take the signature $(+---)$]:

$$\frac{1}{2}\theta^{\mu\nu}(x)=\frac{\delta\Gamma}{\delta g_{\mu\nu}(x)}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}} = -\mathrm{i}\frac{N_c}{2}\int_C d\omega\rho(\omega) \times\langle x|\{O^{\mu\nu},(\mathrm{i}\mathbf{D})^{-1}\}|x\rangle, \tag{70}$$

where

$$O^{\mu\nu}=\frac{\mathrm{i}}{2}(\gamma^\mu\partial^\nu+\gamma^\nu\partial^\mu)-g^{\mu\nu}(\mathrm{i}\partial\!\!\!/-\omega). \tag{71}$$

In the flat space-time limit $g^{\mu\nu}=\eta^{\mu\nu}$, the chiral Lagrangian contains only metric contributions and takes the form given in Refs. [3,4]:

$$\mathcal{L}=\mathcal{L}^{(0)}+\mathcal{L}^{(2)}+\mathcal{L}^{(4)}+\cdots, \tag{72}$$

where

$$\mathcal{L}^{(2)}=\mathcal{L}^{(2,g)}|_{g_{\mu\nu}=\eta_{\mu\nu}},$$

$$\mathcal{L}^{(4)}=\mathcal{L}^{(4,g)}|_{g_{\mu\nu}=\eta_{\mu\nu}}. \tag{73}$$

If we do a derivative expansion (see the Appendix for details), the effective chiral energy-momentum tensor up to and including fourth-order corrections in the chiral counting reads [5]

$$\theta_{\mu\nu}=\theta^{(0)}_{\mu\nu}+\theta^{(2)}_{\mu\nu}+\theta^{(4)}_{\mu\nu}+\cdots, \tag{74}$$

where

$$\theta^{(0)}_{\mu\nu}=-g_{\mu\nu}\mathcal{L}^{(0)}, \tag{75}$$

$$\theta^{(2)}_{\mu\nu}=\frac{f^2}{4}\langle D_\mu U^\dagger D_\nu U\rangle-g_{\mu\nu}\mathcal{L}^{(2)}, \tag{76}$$

$$\theta^{(4)}_{\mu\nu}=-g_{\mu\nu}\mathcal{L}^{(4)}+2L_4\langle D_\mu U^\dagger D_\nu U\rangle\langle\chi^\dagger U+U^\dagger\chi\rangle+L_5\langle D_\mu U^\dagger D_\nu U+D_\nu U^\dagger D_\mu U\rangle\langle\chi^\dagger U+U^\dagger\chi\rangle-2L_{11}(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu) \times\langle D_\alpha U^\dagger D^\alpha U^\dagger\rangle-2L_{13}(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)\langle\chi^\dagger U+U^\dagger\chi\rangle-L_{12}(g_{\mu\beta}g_{\nu\alpha}\partial^2+g_{\mu\nu}\partial_\alpha\partial_\beta-g_{\mu\alpha}\partial_\nu\partial_\beta-g_{\nu\alpha}\partial_\mu\partial_\beta)\langle D^\alpha U^\dagger D^\beta U\rangle. \tag{77}$$

Note that the coefficients L_1–L_{10} appear in $\mathcal{L}^{(4)}$ given by Eq. (48). The terms containing L_{11}–L_{13} cannot be obtained by computing the energy-momentum tensor from the chiral effective Lagrangian in flat-space time (72) and from this viewpoint are genuine quark contributions to $\theta^{\mu\nu}$ in this model. Actually, the difference between computing the energy-momentum tensor from an action at the quark—i.e., starting from Eq. (71)—or at the meson level—i.e., starting from Eq. (72)—is

$$\frac{\delta\Gamma}{\delta g_{\mu\nu}(x)}\bigg|_{g_{\mu\nu}}-\frac{\delta S^g}{\delta g_{\mu\nu}(x)}\bigg|_{g_{\mu\nu}}=\frac{\delta S^R}{\delta g_{\mu\nu}(x)}\bigg|_{g_{\mu\nu}}, \tag{78}$$

with S^g and S^R denoting the metric and curvature contributions to the action, and is precisely related to the curvature terms corresponding to the couplings L_{11}, L_{12}, and L_{13}.

VI. RESULTS FOR THE MESON DOMINANCE MODEL

The meson dominance model (MDM), developed in Ref. [42], offers a particularly simple realization of the SQM and provides an explicit form for the spectral function. The quark propagator becomes

$$S(p)=\int_C d\omega\frac{\rho_V(\omega)p\!\!\!/+\rho_S(\omega)\omega}{p^2-\omega^2}=\frac{Z(p^2)}{p\!\!\!/-M(p^2)}, \tag{79}$$

where

$$\rho_V(\omega)=\frac{1}{2\pi i}\frac{1}{\omega}\frac{1}{(1-4\omega^2/M_V^2)^{5/2}}, \tag{80}$$

$$\rho_S(\omega)=\frac{1}{2\pi i}\frac{12\rho_3'}{M_S^4(1-4\omega^2/M_S^2)^{5/2}}. \tag{81}$$

The vector spectral function $\rho_V(\omega)$ is determined by imposing vector meson dominance of the pion electromagnetic form factor, from which the identity

$$f^2=\frac{N_cM_V^2}{24\pi^2} \tag{82}$$

is deduced. This relation is subject to chiral corrections. It is remarkable that such a simple relation produces a mass of $M_V=826$ MeV for $f_\pi=93$ MeV which agrees with the value recently obtained in Ref. [66]. With this value of f one gets a vacuum energy of $B=-3N_Ff^4/N_c$

$\sim(202\text{–}217\ \mathrm{MeV})^4$ for $N_F=3$. In contrast to $\rho_V(\omega)$, the expression for the scalar spectral function $\rho_S(\omega)$ is an educated guess which satisfies the odd spectral conditions $\rho_1=\rho_3=\cdots=0$ and reproduces the value of the ρ_3' log moment. The preferred value for the vector mass is

$$M_V=770\ \ \mathrm{MeV}, \tag{83}$$

which corresponds to the ρ-meson mass and which is used in the subsequent numerical analysis.

The integration contour C used in the MDM encircles the branch cuts—i.e., starts at $-\infty+i0$, goes around the branch point at $-M_V/2$, and returns to $-\infty-i0$, with the other section obtained by a reflection with respect to the origin [42]. These two sections are connected with semicircles at infinity. The mass function becomes

$$\frac{M(p^2)}{M(0)}=\frac{10p^2}{M_V^2}\frac{\left(\frac{M_S^2}{M_S^2-4p^2}\right)^{5/2}}{\left(\frac{M_V^2}{M_V^2-4p^2}\right)^{5/2}-1}, \tag{84}$$

where the constituent quark mass is[4]

$$M_Q\equiv M(0)=-\frac{48M_V^2\pi^2\langle\bar{q}q\rangle}{5M_S^4N_c}. \tag{85}$$

When $M(p^2)=p^2$, then $Z(p^2)=0$, such that the quark propagator has no poles in the complex p^2 plane. Instead, it has a cut starting at the branch point $p^2=M_V^2/4$. The exponents reproduce accurately the $1/(-p^2)^{3/2}$ behavior in the deep-Euclidean domain. This behavior was seen in the recent QCD lattice simulation in the Landau gauge, linearly extrapolated to the chiral limit [45]. A fit to the data yields [48]

$$M_Q=303\pm24\ \ \mathrm{MeV},$$

$$M_S=970\pm21\ \ \mathrm{MeV}, \tag{86}$$

with the optimum value of χ^2 per degree of freedom equal to 0.72, yielding an impressive agreement of $M(p^2)$ up to $p^2=-16\ \mathrm{GeV}^2$. Although $Z(p^2)$ is not nearly as good (cf. Ref. [48]), leaving room for improvement, we think it worthwhile to pursue the pattern of chiral-symmetry breaking which arises in this particular realization of the SQM. Incidentally, let us note that if the results of Sec. III are used we get

$$f^2=\frac{N_c}{4\pi^2}\int dp_E^2\frac{1}{\left(1+\frac{4p_E^2}{M_V^2}\right)^{5/2}}, \tag{87}$$

which reproduces Eq. (82) and shows the consistency of the approach. For the meson dominance model we get

$$\rho_1'^{\mathrm{MD}}=\frac{8\pi^2\langle\bar{q}q\rangle}{N_cM_S^2}=-\frac{5M_QM_S^2}{6M_V^2},$$

$$\rho_2'^{\mathrm{MD}}=-\frac{4\pi^2f^2}{N_c}=-\frac{M_V^2}{6},$$

$$\rho_3'^{\mathrm{MD}}=-\frac{4\pi^2\langle\bar{q}q\rangle}{N_c}=\frac{5M_QM_S^4}{12M_V^2}. \tag{88}$$

Using these values we get

$$L_5=\frac{N_c}{96\pi^2}\frac{M_V^2}{M_S^2}, \tag{89}$$

$$L_7=\frac{N_c}{32\pi^2N_f}\left(\frac{1}{12}-\frac{M_V^2}{6M_S^2}\right), \tag{90}$$

$$L_8=\frac{N_c}{16\pi^2}\left(-\frac{M_V^{10}}{150M_Q^2M_S^8}+\frac{M_V^2}{12M_S^2}-\frac{1}{24}\right). \tag{91}$$

In the SU(3) case we display our results in Table I. We note that the predictions for $L_{1,2,3,4,6,9,10}$ are common to the scheme of Ref. [21]. The values of $L_{5,7,8}$ are specific both to the SQM and MDM realizations.

In the SU(2) case we have, with the help of the relations given in Ref. [4], to pass form SU(3) to SU(2) [3]. In the absence of meson loop corrections,[5]

$$\bar{l}_1=-\bar{l}_2=-\frac{1}{2}\bar{l}_5=-\frac{1}{4}\bar{l}_6=-N_c, \tag{92}$$

$$\bar{l}_3=\frac{4N_c}{3}+\frac{16N_cM_V^{10}}{75M_Q^2M_S^8}, \tag{93}$$

$$\bar{l}_4=\frac{2M_V^2N_c}{3M_S^2}. \tag{94}$$

The vector and scalar pion radii are given by [3]

$$\langle r^2\rangle_V=\frac{1}{16\pi^2f^2}\bar{l}_6=\frac{6}{M_V^2},$$

$$\langle r^2\rangle_S=\frac{3}{8\pi^2f^2}\bar{l}_4=\frac{6}{M_S^2}, \tag{95}$$

respectively. While the vector pion mean squared radius reproduces the built-in vector meson dominance of the pion

[4]In Ref. [48] there were typographical errors in Eqs. (10.6) and (10.9), which should carry an extra factor of 2 on the right-hand side.

[5]The relations are $\bar{l}_1=192\pi^2(2L_1+L_3)$, $\bar{l}_2=192\pi^2L_2$, $\bar{l}_3=256\pi^2(2L_4+L_5-4L_6-2L_8)$, $\bar{l}_4=64\pi^2(2L_4+L_5)$, $\bar{l}_5=-192\pi^2L_{10}$, $\bar{l}_6=192\pi^2L_9$, $\bar{l}_{11}=192\pi^2L_{11}$, and $\bar{l}_{13}=256\pi^2l_{13}$. The constant l_{12} is not renormalized by the pion loop.

electromagnetic (e.m.) form factor, the scalar radius shows that the scalar mass obtained by a fit to the lattice quark mass function does correspond to the mass of a scalar meson dominating the scalar form factor, $\langle r^2\rangle_S^{1/2}=0.50\pm0.01$ fm.

The scalar (spin-0) and tensor (spin-2) components of the gravitational form factors, θ_0 and θ_2 [5], respectively, produce the same mean-squared radii

$$\langle r^2\rangle_{G,0}=\langle r^2\rangle_{G,2}=\frac{N_c}{48\pi^2 f^2}, \tag{96}$$

regardless of the particular realization of the spectral model. If we saturate the form factors with scalar and tensor mesons f_0 and f_2, we get, for their masses,

$$M_{f_0}=M_{f_2}=4\pi f_\pi\sqrt{3/N_c}=1105\text{–}1168 \text{ MeV}, \tag{97}$$

depending whether we take $f=88$ or 93 MeV, respectively. The experimental value for the lowest tensor meson is $M_{f_2}^{\rm expt}=1270$ MeV. As discussed in Ref. [5], the θ_0 (corresponding to the trace of the energy-momentum tensor) form factor couples to scalars, whereas the θ_2 (corresponding to the traceless combination of $\theta_{\mu\nu}$) form factor couples to tensor (spin-2) mesons.

One message is clear from the present model: the scalar meson of mass M_{f_0} which dominates the energy-momentum tensor does not necessarily coincide with the scalar meson of mass M_S, which dominates the scalar form factor. Actually we have $M_{f_0}=\sqrt{2}M_V$, whereas M_S is a free quantity. This is natural in the spectral approach, where in the chiral limit the scalar form factor F_S involves the odd spectral moments, whereas θ_0 involves the even spectral moments. In particular, the corresponding mean-squared radii are proportional to ρ_1' and ρ_0, respectively. Finally, we note that the numerical value of $\bar{l}_3=4.65$ obtained in MDM amounts to a shift of the pion mass by less than 1% and an increase of f_π yielding 89 MeV as compared to $f=87$ MeV.

VII. LARGE-N_c LIMIT AND DUALITY

Given the fact that our result corresponds to a one-quark-loop approximation, we cannot expect our model to be better than the leading large-N_c contribution to the low-energy parameters, which is made of infinitely many resonance exchanges [15]. On the other hand, the evaluation of these large-N_c contributions requires additional, not necessarily unreasonable, assumptions such as the convergence of an infinite set of states and, moreover, an estimate of the contributions of higher resonances. In practice, one works in the single-resonance approximation (SRA), yielding a reduction of parameters [5,15]:

$$2L_1^{\rm SRA}=L_2^{\rm SRA}=\frac{1}{4}L_9^{\rm SRA}=-\frac{1}{3}L_{10}^{\rm SRA}=\frac{f^2}{8M_V^2}, \tag{98}$$

$$L_5^{\rm SRA}=\frac{8}{3}L_8^{\rm SRA}=\frac{f^2}{4M_S^2}, \tag{99}$$

$$L_3^{\rm SRA}=-3L_2^{\rm SRA}+\frac{1}{2}L_5^{\rm SRA}, \tag{100}$$

$$2L_{13}^{\rm SRA}=3L_{11}^{\rm SRA}+L_{12}^{\rm SRA}=\frac{f^2}{4M_{f_0}^2}, \tag{101}$$

$$L_{12}^{\rm SRA}=-\frac{f^2}{2M_{f_2}^2}, \tag{102}$$

where f, M_V, and M_S should stand for the leading large-N_c contributions to those quantities. To obtain the formulas for L_1–L_{10}, the pseudoscalar and axial meson contributions have been fine-tuned to satisfy the VV-AA and SS-PP two-point correlation-function high-energy-behavior chiral sum rules plus some well-converging high-energy properties of hadronic form factors. (In particular, $M_P/M_S=M_A/M_V=\sqrt{2}$, where M_P is the mass of the excited pion.) Obviously, more short-distance constraints require more resonances. The values of $L_{11,12,13}$ are obtained from the single scalar and tensor resonance exchange [5]. On the one hand, a tensor meson is needed in order to provide a nonvanishing L_{12} as a minimal hadronic ansatz; on the other hand, tensor mesons do contribute also other LECs [68], which is not taken into account in Eq. (102). Thus, to simplify the discussion, in what follows we restrict ourselves to the nongravitational couplings L_1–L_{10}. In practice, phenomenological success is achieved by using the physical values of the parameters. Note that although there is predictive power, it is done in terms of two dimensionless ratios f/M_V and f/M_S. Obviously, in the chiral limit we expect both M_V and M_S to scale with f_π. Therefore, in order to preserve the large-N_c counting rules one should have $M_V=c_V f_\pi/\sqrt{N_c}$ and $M_S=c_S f_\pi/\sqrt{N_c}$ with c_V and c_S denoting some N_c-independent coefficients. Remarkably, in the SQM the low-energy parameters depend on two dimensionless ratios ρ_1'/B_0 and ρ_2'/B_0^2. It is therefore tempting to determine the spectral log moments from large-N_c, arguments, in a model-independent way. Actually in the SRA we note that the ratios L_1:L_2:L_9 of the SQM agree with those of the SRA. The values of L_5 and L_9 can then be used to determine ρ_1', and ρ_2', respectively, yielding

$$\rho_1'^{\rm SRA}=\frac{8\pi^2\langle\bar{q}q\rangle}{N_c M_S^2}, \tag{103}$$

$$\rho_2'^{\rm SRA}=-\frac{4\pi^2 f^2}{N_c}=-\frac{M_V^2}{6}, \tag{104}$$

in agreement with Eqs. (89) and (82). This is not surprising since the physics of the meson dominance version of the SQM and SRA is alike. The only difference is that one cannot deduce from Eq. (104) the value of the constituent-quark mass $M_Q=M(0)$, which is given by the ratio of two negative moments, $M_Q=\rho_{-1}/\rho_{-2}$, Eq. (2). To determine M_Q

would require computing terms of $\mathcal{O}(p^6)$ in the chiral Lagrangian and comparing to the SRA at large N_c.

One can see that it is not possible to match L_8 or L_{10}. The disagreement with the large-N_c values of L_8 and L_{10} has to do with the fact that the SS-PP sum rule and VV-AA second Weinberg sum rule are violated in the present as well as other quark model calculations [69,70] (except for the nonlocal models; see [71,72]). This calls for a modification of our model. The disagreement has to do with the absence of axial-meson exchange in L_{10} (1/4 of the total contribution) and pseudoscalar meson exchange in L_8 (1/4 of the total contribution). On the other hand, for the value of f obtained from Eq. (82) the constants L_1, L_2, L_4, L_5, L_6, L_9 reproduce the large-N_c constraints obtained in Ref. [14]. This agreement is confirmed in Table I if one corrects for the factor $24\pi^2 f_\pi^2/N_c M_V^2 = 1.15$. One could force L_3 to agree with the large-N_c estimate by taking $M_V = M_S$. This agrees with the observation of the chiral unitary approach of Ref. [66]; in the large-N_c limit, the scalar and vector mesons become degenerate.[6] Thus, the marriage of large-N_c in the SRA with our chiral quark-model calculation produces degenerate scalar and vector mesons. Degenerated scalar and vector mesons were suggested very early [73] in the context of superconvergent sum rules and have been interpreted more recently on the basis of mended symmetries [74]. Experimental claims have been raised [75–77] and contested [78]. Direct experimental tests have also been suggested [79].

It is clear that whatever sensible modification of the SQM is considered, it will only affect L_8 and L_{10}, keeping the remaining L's. We leave the explicit construction of such a modified model for a separate study. Regardless of the particular way to achieve this, we may anticipate already the consequences for large N_c in the single-resonance approximation of taking $M_S = M_V = 2\pi f \sqrt{6/N_c}$, yielding the duality relations

$$2L_1 = L_2 = -\frac{1}{2}L_3 = \frac{1}{2}L_5 = \frac{2}{3}L_8 = \frac{1}{4}L_9 = -\frac{1}{3}L_{10} = \frac{N_c}{192\pi^2}. \tag{105}$$

This also implies the set of mass dual relations

$$M_A = M_P = \sqrt{2}M_V = \sqrt{2}M_S = 4\pi\sqrt{3/N_c}f_\pi. \tag{106}$$

The new relation $M_A = M_P$ agrees with the experimental number within the expected 30% of the large-N_c limit. Using Eqs. (95) we obtain

$$\langle r^2\rangle_S^{1/2} = \langle r^2\rangle_V^{1/2} = 1\sqrt{N_c}/2\pi f_\pi. \tag{107}$$

These relations are subject to higher $1/N_c$ and m_π corrections. We may account for the latter by allowing f_π to vary between the physical value and the value in the chiral limit. This yields $\langle r^2\rangle_S^{1/2} = \langle r^2\rangle_V^{1/2} = 0.58$–$0.64$ fm. The value of the scalar radius is compatible with the one obtained in ChPT to two loop [8], 0.78 fm. Going to the SU(2) case, in the dual large-N_c model we get

$$-\bar{l}_1 = \bar{l}_2 = \frac{3}{2}\bar{l}_3 = \frac{3}{2}\bar{l}_4 = \frac{1}{3}\bar{l}_5 = \frac{1}{4}\bar{l}_6 = N_c, \tag{108}$$

whereas the recently extracted values obtained at the two-loop level from analysis of $\pi\pi$ scattering [8] and vector and scalar form factors [7] at the two-loop level are

$$\bar{l}_1 = -0.4 \pm 0.6, \quad \bar{l}_2 = 6.0 \pm 1.3, \quad \bar{l}_3 = 2.9 \pm 2.4,$$
$$\bar{l}_4 = 4.4 \pm 0.2, \quad \bar{l}_5 = 13.0 \pm 1.0, \quad \bar{l}_6 = 16.0 \pm 1.0. \tag{109}$$

The $\bar{l}$ coefficients are in a sense more suitable for comparison with ChPT since the chiral loop generates a constant shift in all of them by the same amount, $c = \log(\mu^2/m^2)$. Thus, it makes sense to compare the differences where chiral logarithms are cancelled. We find

$$\bar{l}_2 - \bar{l}_1 = 2N_c \quad (\text{Exp. } 6.4 \pm 1.4),$$
$$\bar{l}_3 - \bar{l}_1 = \frac{5N_c}{3} \quad (\text{Exp. } 3.3 \pm 2.4),$$
$$\bar{l}_4 - \bar{l}_1 = \frac{5N_c}{3} \quad (\text{Exp. } 4.8 \pm 0.4),$$
$$\bar{l}_5 - \bar{l}_1 = 4N_c \quad (\text{Exp. } 13.4 \pm 1.1),$$
$$\bar{l}_6 - \bar{l}_1 = 5N_c \quad (\text{Exp. } 16.4 \pm 1.1), \tag{110}$$

where the errors have been added in quadrature. As we can see, the agreement is excellent, within the uncertainties, and suggests accuracy of the order of $1/N_c^2$ rather than the standard *a priori* $1/N_c$ error estimate. The constant pion loop shift can be accommodated with a scale $\mu = 513 \pm 200$ MeV, comparable to the ρ meson mass. Taking Eqs. (102), corresponding to the SRA with the physical values $f = 93$ MeV, $M_S = 1000$ MeV, and $M_V = 770$ MeV, as done in Ref. [15], yields $\bar{l}_2 - \bar{l}_1 = 8.3$, $\bar{l}_3 - \bar{l}_1 = 6.2$, $\bar{l}_4 - \bar{l}_1 = 6.2$, $\bar{l}_5 - \bar{l}_1 = 15.2$, $\bar{l}_6 - \bar{l}_1 = 18.7$. More reasonable values are obtained by taking $M_S = 600$ MeV, but then the SRA relation $M_P = \sqrt{2}M_S$ predicts a too low value of the excited pion state. The present discussion favors phenomenologically the dual relations (105) as compared to the SRA relations (102) with physical parameters.

VIII. CONCLUSIONS

In the present work we have studied the chiral expansion of the recently proposed spectral quark model in the presence

[6]For $N_c = 3,10,20,40$, Ref. [66] obtains $M_S/M_V = 0.58,0.84,0.96,0.98$, respectively, with M_S and M_V the real parts of the poles in the second Riemann sheet. We thank J.R. Peláez for providing these numbers.

of electroweak and gravitational external sources. The model is based on a Lehman representation for the quark propagator with an unconventional spectral function, which is genuinely a complex function with cuts in terms of the spectral mass. We have written down the effective action which reproduces the Ward-Takahashi identities presented in the previous work. Thanks to an infinite set of spectral conditions demanded from the powerlike factorization property of form factors at high energies, we have been able to show that the corresponding chiral anomalous contribution to the action is properly normalized without removing the regularization. Moreover, the nonanomalous contribution to the action can be written in the long-wavelength limit in terms of 13 low-energy constants. The numerical values are in reasonable agreement with the phenomenological expectations, although some discrepancies do occur for L_8 and L_{10}. In some cases they can be naturally explained as failures in reproducing some chiral short-distance constraints which suggest that the model needs to be improved. On the other hand, if one tries to match the remaining nongravitational LECs to large-N_c predictions in the single-resonance approximation, a further reduction of parameters takes place. In particular, one finds the best agreement for degenerate scalar and vector mesons.

We have estimated in the framework of chiral quark models the gravitational LECs L_{11}, L_{12}, and L_{13}, describing the coupling to external gravitational sources.[7] These LECs depend on curvature properties of the curved space-time metric. This calculation allows a determination of some matrix elements of the energy-momentum tensor. Our analysis suggests that the scalar meson coupling to the quark condensate $m_0\bar{q}q$ and the scalar meson coupling to the trace of the energy-momentum tensor θ^μ_μ do not necessarily coincide. Clearly, these two operators behave differently under chiral symmetry, since $m_0\bar{q}q$ vanishes in the chiral limit whereas θ^μ_μ does not. This point is in itself rather intriguing and deserves further investigation. We note here that this fact materializes in our model because these two scalar mesons depend on odd and even spectral moments, respectively. On the other hand, we obtain $M_{f_0}=M_{f_2}=\sqrt{2}M_V=\sqrt{2}M_S =4\pi\sqrt{3/N_c}f_\pi$, a very reasonable result if we take into account the large-N_c nature of the one-quark-loop approximation. Further quark-meson duality relations have been discussed, allowing a rather successfully determination of the best known LECs, consistent up to the experimental errors with the best known values up to two-loop accuracy.

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[7]After this paper was submitted we became aware of a previous work [31] where L_{11}–L_{13} were computed using conformal transformation techniques in a different chiral quark model. We thank A. A. Andrianov and V. A. Andrianov for pointing out this paper to us.

APPENDIX: DERIVATIVE EXPANSION AND USEFUL IDENTITIES

Reduction to a vectorlike theory and transformation properties

The Dirac operator can be rewritten as

$$\mathbf{D}=\mathbf{D}_R P_R+\mathbf{D}_L P_L, \tag{A1}$$

with the projection operators on parity,

$$P_R=\frac{1}{2}(1+\gamma_5), \quad P_L=\frac{1}{2}(1-\gamma_5), \tag{A2}$$

such that for a Dirac spinor one has

$$\Psi_R=P_R\Psi, \quad \Psi_L=P_L\Psi. \tag{A3}$$

The right and left Dirac operators are given by

$$i\mathbf{D}_R=i\partial\!\!\!/+A\!\!\!/_R-\mathcal{M},$$

$$i\mathbf{D}_L=i\partial\!\!\!/+A\!\!\!/_L-\mathcal{M}^\dagger, \tag{A4}$$

with

$$\mathcal{M}=s+ip+\omega U, \quad \mathcal{M}^\dagger=s-ip+\omega U^\dagger, \tag{A5}$$

$$A_R^\mu=v^\mu+a^\mu, \quad A_L^\mu=v^\mu-a^\mu. \tag{A6}$$

The quark mass matrix is included in the scalar field s. Under left-right unitary transformations, Ω_L and Ω_R, one has the properties

$$\Psi_R\to\Omega_R\Psi_R, \quad \Psi_L\to\Omega_L\Psi_L, \tag{A7}$$

$$U\to\Omega_L U\Omega_R^\dagger, \quad U^\dagger\to\Omega_R U^\dagger\Omega_L^\dagger, \tag{A8}$$

$$A_R^\mu\to\Omega_R A_R^\mu\Omega_R^\dagger+i\Omega_R\partial^\mu\Omega_R^\dagger, \tag{A9}$$

$$A_L^\mu\to\Omega_L A_L^\mu\Omega_L^\dagger+i\Omega_L\partial^\mu\Omega_L^\dagger. \tag{A10}$$

The chiral covariant derivatives and field strength tensors

$$D_\mu\Psi_R=\partial_\mu\Psi_R-iA_\mu^R\Psi_R,$$

$$D_\mu U=D_\mu^L U-UD_\mu^R=\partial_\mu U-iA_\mu^L U+iUA_\mu^R,$$

$$F_{\mu\nu}^r=i[D_\mu^r,D_\nu^r]=\partial_\mu A_\nu^r-\partial_\nu A_\mu^r-i[A_\mu^r,A_\nu^r],$$

$$r=R,L, \tag{A11}$$

behave as follows under local chiral transformations:

$$D_\mu \Psi_R \to \Omega_R D_\mu \Psi_R, \tag{A12}$$

$$D_\mu \Psi_L \to \Omega_L D_\mu \Psi_L, \tag{A13}$$

$$D_\mu U \to \Omega_L D_\mu U \Omega_R^\dagger, \tag{A14}$$

$$D_\mu U^\dagger \to \Omega_R D_\mu U^\dagger \Omega_L^\dagger. \tag{A15}$$

Coupling the spectral quark model to gravity

The coupling of fermions to gravity is well known (see, e.g., Ref. [80]) but not in the context of chiral quark models. We review it here for completeness and to fix our notation. We use the tetrad formalism of curved space-time (for conventions see, e.g., Ref. [65]). Given the metric tensor we get a local basis of orthogonal vectors (tetrads or vierbein):

$$g^{\mu\nu}(x) = e_A^\mu(x) e_B^\nu(x) \eta^{AB}, \tag{A16}$$

with $\eta^{AB} = \mathrm{diag}(1,-1,-1,-1)$ for a flat Minkowski metric. These vectors satisfy the orthogonality relations

$$\delta_\nu^\mu = \eta^{AB} e_A^\mu e_{\nu B} = e_A^\mu e_\nu^A,$$

$$\delta_B^A = g^{\mu\nu} e_\mu^A e_{\nu B} = e_\mu^A e_B^\mu. \tag{A17}$$

Under the coordinate $x^\mu \to x'^\mu(x)$ and frame $x^A \to \Lambda_B^A x^B$ transformations the transformation properties of the tetrad are

$$e_\mu^A \to \frac{\partial x'^\nu}{\partial x^\mu} e_\nu^A,$$

$$e_\mu^A \to \Lambda_B^A(x) e_\mu^B, \tag{A18}$$

respectively. The tetrads map coordinate tensors into frame tensors (which transform covariantly under local Lorentz transformations)—for instance,

$$T^{AB} = e_\mu^A e_\nu^B T^{\mu\nu}. \tag{A19}$$

Frame tensors are invariant under coordinate transformations $x^\mu \to x'^\mu$. For a general tensor $T_{\nu A}^\alpha$ greek indices transform covariantly under coordinate transformations while latin indices transform covariantly under frame transformations according to Eq. (A18) as follows:

$$T_{\nu A}^\alpha \to \frac{\partial x'_\nu}{\partial x^\mu} \frac{\partial x'^\alpha}{\partial x^\beta} \Lambda_A^B(x) T_{\mu B}^\beta. \tag{A20}$$

The covariant derivative is defined as

$$d_\mu T_{\nu A}^\alpha = \partial_\mu T_{\nu A}^\alpha - \Gamma_{\nu\mu}^\lambda T_{\lambda A}^\alpha + \Gamma_{\mu\lambda}^\alpha T_{\nu A}^\lambda + \omega_{AB\mu} T_\nu^{\alpha B}, \tag{A21}$$

where the Riemann connection is given by the Christoffel symbols

$$\Gamma_{\lambda\mu}^\sigma = \frac{1}{2} g^{\nu\sigma} \{\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda}\}, \tag{A22}$$

which are symmetric in the lower indices, $\Gamma_{\lambda\mu}^\sigma = \Gamma_{\mu\lambda}^\sigma$ (we assume here no torsion). In order to preserve the covariance of the tetrad mapping we must have

$$d_\mu e_{\nu,A} = \partial_\mu e_{A,\nu} - \Gamma_{\nu\mu}^\lambda e_{A,\lambda} + \omega_{AB\mu} e_\nu^B = 0. \tag{A23}$$

In addition, the condition $d_\mu g^{\mu\nu} = 0$, implying

$$d_\mu \eta_{AB} = \omega_{AB\mu} + \omega_{BA\mu} = 0, \tag{A24}$$

requires an antisymmetric spin connection $\omega_{AB\mu} = -\omega_{BA\mu}$, given by

$$\omega_{AB\mu} = e_A^\nu [\partial_\mu e_{B,\nu} - \Gamma_{\nu\mu}^\lambda e_{B,\lambda}]. \tag{A25}$$

The frame and coordinate covariant derivative d_μ is defined according to the spin of the corresponding field. For a spin-0 U, spin-1/2, Ψ, spin-1, A_μ, and spin 3/2, Ψ_μ, fields the transformation properties are

$$U(x) \to U(x),$$

$$\Psi(x) \to S(\Lambda(x)) \Psi(x), \tag{A26}$$

$$A_\mu(x) \to \frac{\partial x'_\nu}{\partial x^\mu} A_\nu(x), \tag{A27}$$

$$\Psi_\mu(x) \to \frac{\partial x'_\nu}{\partial x^\mu} S(\Lambda(x)) \Psi_\nu(x). \tag{A28}$$

For infinitesimal Lorentz transformations $\Lambda_B^A = \delta_B^A + \epsilon_B^A$ with $\epsilon_{AB} = -\epsilon_{BA}$ one has $S(\Lambda) = 1 - (i/4)\sigma_{AB}\epsilon^{AB}$ with σ_{AB} defined below [see Eq. (A34)].

For a scalar (spin-0) field we have the standard definition

$$d_\mu U = \partial_\mu U. \tag{A29}$$

For a (spin-1) vector, one has

$$d_\mu A_\nu = \partial_\mu A_\nu - \Gamma_{\nu\mu}^\lambda A_\lambda, \tag{A30}$$

satisfying the property

$$[d_\mu, d_\nu] A_\alpha = R^\lambda{}_{\alpha\mu\nu} A_\lambda, \tag{A31}$$

with the Riemann curvature tensor given by Eq. (63). The coordinate and Lorentz covariant derivative for Dirac fermions (spin 1/2) is defined as

$$d_\mu \Psi = \partial_\mu \Psi(x) - i\omega_\mu \Psi(x), \tag{A32}$$

where ω_μ is the Cartan spin connection,

$$\omega_\mu = \frac{1}{4} \sigma^{AB} \omega_{AB\mu}, \tag{A33}$$

and

$$\sigma_{AB} = \frac{i}{2} [\gamma_A, \gamma_B], \tag{A34}$$

with the γ_A are fixed x-independent Dirac matrices (we use the conventions of Ref. [81]) satisfying the standard flat-space anticommutation rules

$$\gamma^A\gamma^B+\gamma^B\gamma^A=2\eta^{AB}. \tag{A35}$$

The space-time-dependent Dirac matrices are defined as

$$\gamma_\mu(x)=\gamma_A e_\mu^A(x) \tag{A36}$$

and satisfy

$$\gamma^\mu(x)\gamma^\nu(x)+\gamma^\nu(x)\gamma^\mu(x)=2g^{\mu\nu}(x). \tag{A37}$$

The covariant derivative of a frame (x-independent) Dirac matrix (behaving as the adjoint representation $\Psi\bar{\Psi}$) is

$$d_\mu\gamma_A=\partial_\mu\gamma_A-i[\omega_\mu,\gamma_A]+\omega_{AB\mu}\gamma^B=0. \tag{A38}$$

Thus, we obtain a useful identity for the coordinate (and x-dependent) Dirac matrix:

$$d_\mu\gamma_\nu(x)=0, \tag{A39}$$

which implies that for the free Dirac operator the order is irrelevant, $\not{d}\Psi=\gamma^\mu(x)d_\mu\Psi=d_\mu\gamma^\mu(x)\Psi$. For a mixed (spin-3/2) tensor the frame and coordinate covariant derivative reads

$$d_\mu\Psi_\nu=\Psi_{\nu;\mu}=\partial_\mu\Psi_\nu-\Gamma^\lambda_{\nu\mu}\Psi_\lambda-i\omega_\mu\Psi_\nu. \tag{A40}$$

Applying the previous definition to $d_\mu\Psi$ one gets the useful formulas

$$[d_\mu,d_\nu]\Psi=+\frac{i}{4}\sigma^{\alpha\beta}R_{\alpha\beta\mu\nu}\Psi, \tag{A41}$$

$$d^\mu d_\mu\Psi=\frac{1}{\sqrt{-g}}\{(\partial_\mu-i\omega_\mu)[\sqrt{-g}g^{\mu\nu}(\partial_\nu-i\omega_\nu)]\Psi\}, \tag{A42}$$

where $\sigma^{\alpha\beta}=e_A^\alpha e_B^\beta\sigma^{AB}$ is an antisymmetric x-dependent matrix.

Gauge fields can be included by the standard minimal substitution rule, yielding the covariant derivative for a fermion:

$$\nabla_\mu\Psi=(d_\mu-iA_\mu)\Psi. \tag{A43}$$

With this notation the full Dirac operator in the presence of external vector, axial-vector, scalar, pseudoscalar, and gravitational fields reads as in Eq. (9), where

$$\not{A}=\gamma^\mu(x)A_\mu(x), \tag{A44}$$

and the pseudoscalar Dirac matrix in the curved case is defined as

$$\gamma_5(x)=\frac{1}{4!\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta}\gamma_\mu(x)\gamma_\nu(x)\gamma_\alpha(x)\gamma_\beta(x)$$

$$=\frac{1}{4!}\epsilon^{ABCD}\gamma_A\gamma_B\gamma_C\gamma_D=\gamma_5. \tag{A45}$$

Here $g(x)=\det(g^{\mu\nu})$ since $\det(e_A^\nu)^2=\det(g^{\mu\nu})$ with $\epsilon^{0123}=1$ (both in the frame and in the coordinate sense).

The full coordinate, frame, and chiral gauge covariant derivative for pseudoscalar (spin-0), Dirac spinor (spin-1/2), and a Rarita-Schwinger spinor (spin 3/2) fields are given by the formulas

$$\begin{aligned}\nabla_\mu U&=\mathcal{D}_\mu U=\partial_\mu U-i[v_\mu,U]-i\{a_\mu,U\},\\ \nabla_\mu\Psi&=\mathcal{D}_\mu\Psi=\partial_\mu\Psi-i(\omega_\mu+v_\mu+\gamma_5 a_\mu)\Psi,\\ \nabla_\mu\Psi_\nu&=\partial_\mu\Psi_\nu-i(\omega_\mu+v_\mu+\gamma_5 a_\mu)\Psi_\nu-\Gamma^\lambda_{\nu\mu}\Psi_\lambda,\end{aligned} \tag{A46}$$

and they correspond to replacing the derivative by the frame and coordinate covariant derivative, $\partial_\mu\to d_\mu$, in the chiral covariant derivative D_μ. Note that with this definition neither $\mathcal{D}_\mu\mathcal{D}_\nu\Psi\neq\nabla_\mu\nabla_\nu\Psi$ nor $\mathcal{D}_\mu\mathcal{D}_\nu U$ is coordinate covariant since the second derivative does not include the Riemann connection $\Gamma^\lambda_{\mu\nu}$.

Second-order operator

In the absence of gravitational sources, the normal parity contribution can be obtained from the second-order operator [see Eq. (15)]:

$$\begin{aligned}\mathbf{D}_5\mathbf{D}=&[\not{D}_L^2+\mathrm{i}\mathcal{M}^\dagger\not{D}_L-\mathrm{i}\not{D}_R\mathcal{M}^\dagger+\mathcal{M}^\dagger\mathcal{M}]P_R\\ &+[\not{D}_R^2+\mathrm{i}\mathcal{M}\not{D}_L-\mathrm{i}\not{D}_R\mathcal{M}+\mathcal{M}\mathcal{M}^\dagger]P_L.\end{aligned} \tag{A47}$$

Gravitational fields can be coupled by covariantizing first the Dirac operator—i.e., making $\partial_\mu\to d_\mu$ or $D_\mu\to\mathcal{D}_\mu$—and taking into account that since a spinor field is a coordinate scalar we have

$$\mathcal{D}_\mu\Psi=\nabla_\mu\Psi. \tag{A48}$$

The same reasoning can be applied to the coordinate scalar $\not{\nabla}\Psi$, yielding

$$\mathcal{D}_\mu\not{\nabla}\Psi=\nabla_\mu\not{\nabla}\Psi. \tag{A49}$$

This means that we can assume $\not{\mathcal{D}}_{L,R}=\not{\nabla}_{L,R}$ when acting on spinor field as follows:

$$\begin{aligned}\mathbf{D}_5\mathbf{D}\Psi=&[\not{\nabla}_L^2+\mathrm{i}\mathcal{M}\not{\nabla}_L-\mathrm{i}\not{\nabla}_R\mathcal{M}+\mathcal{M}^\dagger\mathcal{M}]P_R\Psi\\ &+[\not{\nabla}_R^2+\mathrm{i}\mathcal{M}^\dagger\not{\nabla}_L-\mathrm{i}\not{\nabla}_R\mathcal{M}^\dagger+\mathcal{M}\mathcal{M}^\dagger]P_L\Psi.\end{aligned} \tag{A50}$$

If we include the gauge fields, we have two vector like theories with left and right gauge fields V_μ^L and V_μ^R, respectively. Suppressing momentarily the left and right labels we have

$$\not{\mathcal{D}}^2\Psi=\not{\nabla}^2\Psi=\left[\nabla^\mu\nabla_\mu-\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}+\frac{1}{4}R\right]\Psi, \tag{A51}$$

where use of the identity

$$[\nabla_\mu,\nabla_\nu]\Psi=[\mathcal{D}_\mu,\mathcal{D}_\nu]\Psi =[D_\mu,D_\nu]\Psi+\frac{i}{4}\sigma^{\alpha\beta}R_{\alpha\beta\mu\nu}\Psi \tag{A52}$$

has been made. The coordinate and frame invariant Laplacian for a Dirac spinor is given by

$$\nabla^\mu\nabla_\mu\Psi=\frac{1}{\sqrt{-g}}\mathcal{D}_\mu(\sqrt{-g}g^{\mu\nu}\mathcal{D}_\nu\Psi). \tag{A53}$$

Note that for a Dirac spinor field Ψ the operator $\mathcal{D}_\mu$ contains the spin connection. Reinserting the right and left chiral notation the second-order operator takes the suitable form

$$\mathbf{D}_5\mathbf{D}=\frac{1}{\sqrt{-g}}[\mathcal{D}_\mu(\sqrt{-g}g^{\mu\nu}\mathcal{D}_\nu)]+\mathcal{V}, \tag{A54}$$

with

$$\mathcal{V}=\mathcal{V}_R P_R+\mathcal{V}_L P_L \tag{A55}$$

and

$$\mathcal{V}_R=-\frac{1}{2}\sigma^{\mu\nu}F^R_{\mu\nu}+\frac{1}{4}R-\mathrm{i}\gamma^\mu\nabla_\mu\mathcal{M}+\mathcal{M}^\dagger\mathcal{M},$$
$$\mathcal{V}_L=-\frac{1}{2}\sigma^{\mu\nu}F^L_{\mu\nu}+\frac{1}{4}R-\mathrm{i}\gamma^\mu\nabla_\mu\mathcal{M}^\dagger+\mathcal{M}\mathcal{M}^\dagger. \tag{A56}$$

Derivative expansion

We use the proper-time representation

$$\mathrm{Tr}\log(\mathbf{D}_5\mathbf{D})=-\mathrm{Tr}\int_0^\infty\frac{d\tau}{\tau}e^{-\mathrm{i}\tau\mathbf{D}_5\mathbf{D}}+C, \tag{A57}$$

with C and infinite constant. The form of the operator $\mathbf{D}_5\mathbf{D}$ in Eq. (A54) is suitable to make a heat kernel expansion in curved space-time as the one of Ref. [82]. For a review see, e.g., [83] and references therein. In our particular case, before undertaking the heat kernel expansion we separate a ω^2 contribution from the operator $\mathbf{D}_5\mathbf{D}$ which we treat exactly:

$$\langle x|e^{-\mathrm{i}\tau\mathbf{D}_5\mathbf{D}}|x\rangle=e^{-\mathrm{i}\tau\omega^2}\langle x|e^{-\mathrm{i}\tau(\mathbf{D}_5\mathbf{D}-\omega^2)}|x\rangle =\frac{\mathrm{i}}{(4\pi\mathrm{i}\tau)^2}e^{-\mathrm{i}\tau\omega^2}\sum_{n=0}^\infty a_n(x)(\mathrm{i}\tau)^n. \tag{A58}$$

The derivative expansion is done by considering U zeroth order the vector and axial fields v_μ and a_μ first order, and any derivative ∂_μ first order. This implies in particular that $R^{\mu\nu\alpha\beta}$, $R^{\mu\nu}$, and R are taken to be of second order. Finally, the external scalar and pseudoscalar fields s and p are taken to be second order as well. Thus, the multiplicative operator $\mathcal{V}-\omega^2$ is at least first order in the chiral counting. To the computed order $\mathcal{O}(p^4)$ in the heat kernel expansion one has to go up to a_4. The contributions can be separated into the flat-space nonvanishing contributions and the curvature contributions generated by quantum effects. Using the form suggested in [84] we have

$$\begin{aligned}
a_0&=1,\\
a_1&=\omega^2-\mathcal{V}+\frac{1}{6}R,\\
a_2&=\frac{1}{180}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-\frac{1}{180}R_{\mu\nu}R^{\mu\nu}+\frac{1}{12}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\\
&\quad+\frac{1}{30}\nabla^2R-\frac{1}{6}\nabla^2\mathcal{V}+\frac{1}{2}\left[\omega^2-\mathcal{V}+\frac{1}{6}R\right]^2,\\
a_3&=\frac{1}{6}\left[\omega^2-\mathcal{V}+\frac{1}{6}R\right]^3-\frac{1}{12}\nabla^\mu\mathcal{V}\nabla_\mu\mathcal{V}+\mathcal{O}(p^6),\\
a_4&=\frac{1}{24}[\mathcal{V}-\omega^2]^4+\mathcal{O}(p^6),
\end{aligned} \tag{A59}$$

where

$$\mathcal{F}_{\mu\nu}=i[\mathcal{D}_\mu,\mathcal{D}_\nu], \tag{A60}$$

$$\nabla^2\mathcal{V}=\nabla^\mu\nabla_\mu\mathcal{V}. \tag{A61}$$

Clearly, the heat kernel coefficients depend on the spectral mass ω in a polynomial fashion. Using the integrals

$$\int_0^\infty\frac{d\tau}{\tau}(\mathrm{i}\tau)^{z-2}e^{-\mathrm{i}\tau\omega^2}=(\omega^2)^z\Gamma(z-2), \tag{A62}$$

we get for integer $z=n$ and after using the spectral conditions, Eq. (4), the normal parity contribution of the action takes the form

$$\begin{aligned}
-\frac{\mathrm{i}}{2}\mathrm{Tr}\log\mathbf{D}_5\mathbf{D}&=-\frac{1}{2}\frac{N_c}{(4\pi)^2}\int d^4x\sqrt{-g}\int d\omega\rho(\omega)\mathrm{tr}\\
&\quad\times\left\langle-\frac{1}{2}\omega^4\log\omega^2a_0+\omega^2\log\omega^2a_1\right.\\
&\quad\left.-\log(\omega^2/\mu^2)a_2+\frac{1}{\omega^2}a_3+\frac{1}{\omega^4}a_4+\cdots\right\rangle\\
&=\int d^4x\sqrt{-g}(\mathcal{L}^{(2)}+\mathcal{L}^{(4)}+\cdots).
\end{aligned} \tag{A63}$$

After evaluation of the Dirac traces, the second-order Lagrangian is

$$\mathcal{L}^{(2)}=\frac{N_c}{(4\pi)^2}\int \rho(\omega)\left\{-\omega^2\log\omega^2\langle\nabla_\mu U^\dagger\nabla^\mu U\rangle+2\omega^3\log\omega^2\langle m^\dagger U+U^\dagger m\rangle+\omega^2\log\omega^2\frac{1}{12}\langle R\rangle\right\}, \tag{A64}$$

whereas the fourth order becomes

$$\begin{aligned}\mathcal{L}^{(4)}=\frac{N_c}{(4\pi)^2}\int \rho(\omega)\Big\{&+\frac{1}{6}\log\omega^2\langle(F^R_{\mu\nu})^2+(F^L_{\mu\nu})^2\rangle-\log\omega^2\left\langle\frac{7}{720}R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}-\frac{1}{144}R^2+\frac{1}{90}R^{\mu\nu}R_{\mu\nu}\right\rangle\\&-\frac{i}{3}\langle F^R_{\mu\nu}\nabla_\mu U^\dagger\nabla_\nu U+F^L_{\mu\nu}\nabla_\mu U\nabla_\nu U^\dagger\rangle+\frac{1}{12}\langle(\nabla_\mu U\nabla_\nu U^\dagger)^2\rangle-\frac{1}{6}\langle(\nabla_\mu U\nabla^\mu U^\dagger)^2\rangle+\frac{1}{6}\langle\nabla^\mu\nabla^\nu U\nabla_\mu\nabla_\nu U^\dagger\rangle\\&-\frac{1}{6}\langle F^L_{\mu\nu}UF^R_{\mu\nu}U^\dagger\rangle+\log\omega^2\omega^2[2\langle m^\dagger m\rangle+\langle(m^\dagger U+U^\dagger m)^2\rangle]-\frac{1}{2}\omega\langle\nabla_\mu U^\dagger\nabla^\mu U(m^\dagger U+U^\dagger m)\rangle\\&-\log\omega^2\omega\langle\nabla_\mu U^\dagger\nabla^\mu m+\nabla_\mu m^\dagger\nabla^\mu U\rangle-\omega\log\omega^2\frac{1}{6}R\langle U^\dagger m+m^\dagger U\rangle+\frac{1}{6}R\nabla_\mu U^\dagger\nabla^\mu U\Big\rangle.\end{aligned} \tag{A65}$$

Note that up to this order the moments $\rho_0=1$, $\rho_1=0$, and $\rho_2=0$ as well as the log moments ρ_0', ρ_1', and ρ_2' appear.

Equations of motion

We define

$$\chi=2B_0m=2B_0(s+ip). \tag{A66}$$

For on-shell pseudoscalars one may minimize the action at lowest order,

$$S^{(2)}=\frac{f^2}{4}\int d^4x\sqrt{-g}\left\langle\nabla_\mu U^\dagger\nabla^\mu U+(\chi^\dagger U+U^\dagger\chi)-\frac{1}{6}R\right\rangle, \tag{A67}$$

to obtain the equations of motion (EOM). Since U is unitary, $U^\dagger U=1$, we have that the variations on U and $U^\dagger$ are not independent of each other, $\delta U^\dagger U+U^\dagger\delta U=0$. For SU(3) flavor one has, in addition, to impose the condition $\mathrm{Det}\,U=1$. One can treat U and $U^\dagger$ independently by introducing a term in the Lagrangian of the form $\langle\Lambda U^\dagger U-i\lambda\log U\rangle$ where the Lagrange multipliers are Λ, a Hermitian matrix, and λ, a real c number. Thus, the EOM are

$$\nabla^2U=\chi+(\Lambda-i\lambda)U,$$

$$\nabla^2U^\dagger=\chi^\dagger+U^\dagger(\Lambda+i\lambda), \tag{A68}$$

where

$$\nabla^2U=\frac{1}{\sqrt{-g}}D_\mu(\sqrt{-g}g^{\mu\nu}D_\nu U). \tag{A69}$$

Combining these two equations, we get

$$U^\dagger\nabla^2U-\nabla^2U^\dagger U=U^\dagger\chi-\chi^\dagger U-2i\lambda. \tag{A70}$$

Taking the trace and using the condition that for a matrix with $\mathrm{Det}\,U=1$ one has $\langle U^\dagger\nabla_\mu U\rangle=0$ and hence $\langle U^\dagger\nabla^2U-\nabla^2U^\dagger U\rangle=0$, we get

$$\lambda=\frac{1}{6i}\langle U^\dagger\chi-\chi^\dagger U\rangle \tag{A71}$$

and thus

$$U^\dagger\nabla^2U-\nabla^2U^\dagger U=U^\dagger\chi-\chi^\dagger U-\frac{1}{3}\langle U^\dagger\chi-\chi^\dagger U\rangle. \tag{A72}$$

On the other hand, Λ is given by

$$2\Lambda=\nabla^2U^\dagger U+U\nabla^2U^\dagger-(\chi U^\dagger+\chi^\dagger U). \tag{A73}$$

Using the identities deduced form the unitarity condition $U^\dagger U=1$,

$$U^\dagger\nabla_\mu U+\nabla_\mu U^\dagger U=0, \tag{A74}$$

$$U^\dagger\nabla^2U+\nabla^2U^\dagger U=-2\nabla_\mu U^\dagger\nabla^\mu U, \tag{A75}$$

and combining them with the previous Eqs. (A73), (A75), we get the identities

$$\langle\nabla^2U^\dagger\nabla^2U\rangle=\langle(\nabla_\mu U^\dagger\nabla^\mu U)^2\rangle-\frac{1}{4}\langle(\chi^\dagger U-U^\dagger\chi)^2\rangle+\frac{1}{12}\langle\chi^\dagger U-U^\dagger\chi\rangle^2 \quad \text{(A76)}$$

and

$$\langle\chi^\dagger\nabla^2U+\nabla^2U^\dagger\chi\rangle=2\langle\chi^\dagger\chi\rangle-\frac{1}{2}\langle(\chi^\dagger U+U^\dagger\chi)^2\rangle-\langle(\chi^\dagger U+U^\dagger\chi)\nabla^\mu U^\dagger\nabla_\mu U\rangle+\frac{1}{6}\langle\chi^\dagger U+U^\dagger\chi\rangle^2. \quad \text{(A77)}$$

In the case of the U(3) group one has Det $U=e^{i\eta_0/f}\neq 1$ and the last two terms involving $\langle\chi^\dagger U\pm U^\dagger\chi\rangle^2$ in Eqs. (A76) and (A77) should be dropped. [See the discussion before Eq. (59).] The result can be further simplified using the integral identity

$$\int d^4x\sqrt{-g}\langle\nabla^\mu\nabla^\nu U^\dagger\nabla^\mu\nabla^\nu U\rangle=\int d^4x\sqrt{-g}\langle\nabla^2U^\dagger\nabla^2U\rangle+\int d^4x\sqrt{-g}R_{\mu\nu}\langle\nabla^\mu U^\dagger\nabla^\nu U\rangle, \quad \text{(A78)}$$

which can be deduced from Eq. (A31) applied to $\nabla_\mu U$. Finally, we also have the SU(3) identity

$$\langle(\nabla_\mu U^\dagger\nabla_\nu U)^2\rangle=-2\langle\nabla_\mu U^\dagger\nabla^\mu U\rangle+\langle\nabla_\mu U^\dagger\nabla_\nu U\rangle^2+\frac{1}{2}\langle\nabla_\mu U^\dagger\nabla^\mu U\rangle^2. \quad \text{(A79)}$$

Once the identities (A76), (A77), (A78), and (A79) have been used one can make the substitute the coordinate-frame covariant derivative by the covariant derivative—i.e., $\nabla^\mu U=D^\mu U$—since the pseudoscalar matrix U is a coordinate and frame scalar. In that way Eqs. (48) and (62) are deduced.

In four dimensions, one can reduce the form of the curvature contributions to the Lagrangian if the Gauss-Bonnet theorem is used in Eq. (62)—namely, that

$$\kappa=\int d^4x\sqrt{-g}[R^2-4R_{\mu\nu}R^{\mu\nu}+R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}] \quad \text{(A80)}$$

is a topological invariant (the Euler number) and hence

$$\delta\kappa=0 \quad \text{(A81)}$$

under metric deformations $g_{\mu\nu}\to g_{\mu\nu}+\delta g_{\mu\nu}$. This relation was not taken into account in Ref. [5] but it does not affect the calculation of the energy-momentum tensor in flat space, Eq. (76).

Derivative expansion for first-order differential operators

As we see the definition of the action involves the Dirac operator **D** only, which is a first-order differential operator. The derivative expansion of the Dirac operator can be done using the identity

$$\langle x|\frac{1}{i\not{D}-\mathcal{M}-\omega U}|x\rangle=\int\frac{d^4k}{(2\pi)^4}\frac{1}{\not{k}+i\not{D}-\mathcal{M}-\omega U}, \quad \text{(A82)}$$

where the differential operator acts on the right. This formula can be justified by requiring vector gauge invariance of the action [85] or by using the asymmetric version of the Wigner transformation presented in Ref. [62]. Expanding in powers of D and $\mathcal{M}$ and squaring the denominator we get

$$\langle x|\frac{1}{i\not{D}-\mathcal{M}-\omega U}|x\rangle=-\sum_{n=0}^{\infty}\int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{k^2-\omega^2}\right]^{n+1}\times(\not{k}+\omega U^\dagger)[(i\not{D}-\mathcal{M})(\not{k}+\omega U^\dagger)]^n. \quad \text{(A83)}$$

In this way Eq. (44) can be derived.

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