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Chiral soliton model for arbitrary colors and flavors

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The semiclassical quantization of the chiral soliton model is studied for an arbitrary number of colors and flavors. The quantum numbers of the baryons in the soliton model are derived and are shown to agree with those in the constituent quark model for normal as well as exotic baryons. The general analysis elucidates the correct definition of exoticness for the three-flavor case, and allows one to interpret the soliton state in terms of quark model variables. The quantum numbers of all the allowed soliton states for the physically relevant case of three flavors are derived.

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I. INTRODUCTION

The discovery of the Θ^+ baryon [1] has led to renewed interest in the chiral soliton model for baryons [2]. The Θ^+ is an $S=+1$ baryon, and is a member of a flavor $SU(3)$ $\overline{\mathbf{10}}_{1/2}$ multiplet. The baryon states in the chiral soliton model are obtained by quantizing the rotational motion of the soliton. The lowest energy states are the $\mathbf{8}_{1/2}$ and $\mathbf{10}_{3/2}$, which contain the nucleon and Δ baryons [3,4]. The higher rotational excitations include the $\overline{\mathbf{10}}_{1/2}$, $\mathbf{27}_{1/2}$ and $\mathbf{27}_{3/2}$ exotic baryons [5,6].

The chiral soliton model constructs normal and exotic baryons from a soliton made up of meson fields. In the nonrelativistic quark model, baryons are $qqq(q\bar{q})^E$ states, where exoticness E is the minimum number of quark-antiquark pairs necessary to construct a baryon with given flavor quantum numbers [7]. At first sight, the two models do not appear to have much in common. However, it is remarkable that the completely symmetric spin-flavor states in the quark model match the rotational states in the chiral soliton model. This fact allows one to relate the two models to each other, and to determine quark substructure (and exoticness) in the soliton model, which does not contain explicit quark degrees of freedom.

The connection between chiral soliton and quark models is clearer if they are studied as a function of the number of colors and flavors. In the case of three flavors, the antisymmetric product $[\mathbf{3}\times\mathbf{3}]_A=\overline{\mathbf{3}}$, and one cannot distinguish the flavor quantum numbers of an antiquark from those of two antisymmetrized quarks. However, if one treats the number of flavors F as arbitrary, the antiquark is mimicked by $F-1$ antisymmetrized quarks, and it is possible to separate the soliton flavor quantum numbers into quark and antiquark contributions in an unambiguous way. Once this is done, one can apply the results to the physically relevant case of three flavors.

In this paper, we derive the states obtained by collective coordinate quantization of the chiral soliton model [2], for an arbitrary number of colors N_c and flavors F, and discuss the importance of the exoticness quantum number E.

II. SOLITON QUANTIZATION

QCD has a $SU(F)_L\times SU(F)_R$ chiral symmetry, which is spontaneously broken to the diagonal $SU(F)$ flavor group. The Goldstone bosons are elements of an $SU(F)$ matrix $U(\mathbf{x},t)$, and the dynamics is given by a chiral Lagrangian L_χ,

$$L_\chi=\frac{f_\pi^2}{4}\operatorname{Tr}\partial^\mu U\partial_\mu U^{-1}+\cdots \tag{1}$$

where f_π is the pion decay constant and the ellipsis denotes terms of higher order in the derivative expansion. The topology of the $SU(F)$ manifold allows for the possibility of solitons. The standard hedgehog configuration for the (static) soliton is

$$U_0(\mathbf{x}) = \left(\begin{array}{c|c} e^{i\tau\cdot\hat{\mathbf{x}}F(r)} & 0 \\ \hline 0 & \mathbb{1} \end{array}\right), \tag{2}$$

with non-trivial fields only in the upper 2×2 block of the $F\times F$ matrix U. The shape function $F(r)$ is determined by solving the non-linear classical field equations of L_χ. The soliton has winding number one, and has been argued to have baryon number one [3], even though it is made up purely of meson fields.

The chiral Lagrangian has an expansion in powers of ∂/Λ_χ, where $\Lambda_\chi\sim4\pi f_\pi$ is the scale of chiral symmetry breaking, so $F(r)$ varies over a typical scale $r\sim\Lambda_\chi^{-1}$. All space derivative terms in L_χ are equally important, and one cannot determine the shape (or even whether the soliton is stable) from the first few terms in L_χ. Nevertheless, assuming the existence of the soliton with some arbitrary shape function $F(r)$ allows one to compute the quantum numbers of the low-lying states in the baryon spectrum in the $N_c\to\infty$ limit. For large N_c, the pion decay constant is of order $\sqrt{N_c}$, so that L_χ is of order N_c. As a result, the mass and moment of inertia of the soliton are both of order N_c. The low-lying states are given by quantizing the rotational motion of the soliton, and have mass-splittings relative to the lowest baryon state which are order $1/N_c$. The semiclassical expansion of the effective theory is an expansion in powers of $1/N_c$, or equivalently, in powers of time-derivatives. All results in the soliton model which do not depend on details of the soliton shape function $F(r)$ can be derived directly from QCD using the $1/N_c$ expansion [8,9].

The collective coordinates for the standard soliton configuration are translations, space rotations and flavor rotations. Translations produce a $\mathbf{P}^2/2M$ shift in the energy, but do not affect the quantum numbers of the soliton, and will be neglected here. Space rotations are generated by J^i, and flavor rotations by T^a. The flavor generators are normalized to $\mathrm{Tr}\, T^a T^b = \delta^{ab}/2$ in the fundamental representation. We will need the decomposition of the flavor group $SU(F) \to SU(2)\times SU(F-2)\times U(1)$, where $SU(2)$ isospin acts on the first two flavors and is generated by I^a; $SU(F-2)$ acts on the remaining flavors and is generated by S^a; and the $U(1)$ generator is

$$T_Y = \sqrt{\frac{F-2}{4F}}\mathcal{Y}, \tag{3}$$

where

$$\mathcal{Y} = \left(\begin{array}{cc|cccc} 1 & & & & \\ & 1 & & & \\ \hline & & y & & \\ & & & \ddots & \\ & & & & y \end{array}\right), \quad y = -\frac{2}{F-2}. \tag{4}$$

For three flavors, $SU(F-2)$ is absent, and $\mathcal{Y}=3Y$, where Y is the usual $SU(3)$ hypercharge. For two flavors, $SU(F-2)$ and $U(1)$ are both absent.

The baryon quantum numbers are determined by quantizing the rotational motion of the soliton. The soliton in a rotated configuration is described by the matrix $A \in SU(F)$, where $U = AU_0(\mathbf{x})A^{-1}$. Transforming the soliton by the flavor transformation $V \in SU(F)$ gives $A \to VA$, and transforming by the space rotation W gives $A \to AW^{-1}$. Space rotations are equivalent to right-multiplication of A by W^{-1}, because spin and isospin are locked together by the $\boldsymbol{\tau}\cdot\mathbf{x}$ dependence of the soliton configuration Eq. (2) which satisfies $(\mathbf{I}+\mathbf{J})U_0(\mathbf{x}) = 0$.

Quantizing the collective coordinate A leads to a tower of states. Different collective coordinates A and Ah lead to the same soliton configuration U if $U_0 = hU_0h^{-1}$. The elements h which leave the soliton invariant form the little-group of the solution, and are given by $\mathbf{I}+\mathbf{J}$, S^a, and $\mathcal{Y}$. The quantization of the soliton is a simple generalization of that for a symmetric top, and will not be reproduced here. One finds that the allowed states of the Skyrme model have wave functions $\sqrt{\dim R}D^{(R)}_{ab}(A)$, where R is an irreducible $SU(F)$ representation, and a and b label states in R. The soliton transforms like $|Ra\rangle$ under flavor, where a is the particular element of R. Each possible choice of $b \in R$ gives a multiplet $R \in SU(F)$. Not all possible choices of R,b are allowed because there is a little-group constraint. In the case of three flavors, the constraint is that the state $|Rb\rangle$ in $SU(3)$ must have hypercharge $3Y = \mathcal{Y} = N_c$ [3], and that the soliton spin is given by the isospin of the state $|Rb\rangle$.

The little-group constraint for arbitrary F derived in Ref. [5] generalizes the hypercharge constraint for three flavors [3], and is: Decompose the representation R of $SU(F)$ into representations of $SU(2)\times SU(F-2)\times U(1)_{\mathcal{Y}}$. The allowed wavefunctions are those for which the state $|Rb\rangle$ is an $SU(F-2)$ singlet and has $\mathcal{Y}=N_c$. The soliton spin is given by the isospin of $|Rb\rangle$. In this paper, we work out the consequences of this constraint for arbitrary flavors, starting with $F \geq 5$ and then restricting to the special cases $F=2,3,4$. The cases $F=2,3$ are already known, but the general F analysis allows one to better understand properties of the allowed baryon states.

III. ARBITRARY *F*

An irreducible representation of $SU(F)$ is described by the Dynkin weight $(n_1, n_2, \ldots, n_{F-1})$, i.e. a Young tableau with n_1 columns of one box, n_2 columns with two boxes, etc. Each box in the Young tableau corresponds to an (upper) index on the $SU(F)$ tensor. Indices in a given column are totally antisymmetrized. We will refer to a column with n boxes as an $[n]$ column, to emphasize the antisymmetry in the n indices. A particular state is described by choosing values for each index (or box), i.e. deciding whether to set it to u, d, s, etc. For example,

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & \multicolumn{3}{c}{} \\ \cline{1-2} 3 & \multicolumn{4}{c}{} \\ \cline{1-1} 4 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \tag{5}$$

is the $T^{111[12][1234]}$ element of the (3,1,0,1) representation of $SU(5)$, and is a state with hypercharge $7+2y = 17/3$ using Eq. (4), since each index 1, 2 has $\mathcal{Y}=1$, and each index 3,4,5 has hypercharge $y = -2/3$.

We now proceed to solve the little-group constraint. We need to find an $SU(F)$ state $|Rb\rangle$, denoted by a Young tableau with a choice of indices for each box, such as Eq. (5). The hypercharge constraint says that this state has $\mathcal{Y}=N_c$, and must be an $SU(F-2)$ singlet. Each such state we find gives an allowed soliton state. The $SU(F)$ representation of the soliton is given by the tableau R, and the spin is given by the isospin of the specific state $|Rb\rangle$.

Consider a generic Young tableau, and pick a choice of indices for each box. Each index chosen to be 1, 2 (i.e. u, d) contributes 1 to $\mathcal{Y}$, and each index chosen to be $3,\ldots,F$ contributes $y<0$. Thus, the minimum number of boxes in the Young tableau is equal to N_c. The N_c-box states $|Rb\rangle$ must have all indices set equal to 1, 2. Since one can antisymmetrize in at most two indices if they are restricted to have at most two values, the allowed tableaux can only contain $[1]$ and $[2]$ columns, if they are to contain a state with $\mathcal{Y}=N_c$. The allowed Dynkin weights are $w = (n_1, n_2, 0, \ldots, 0)$, where n_1 is the number of $[1]$ columns and n_2 is the number of $[2]$ columns, and $n_1 + 2n_2 = N_c$ is the total number of boxes. The allowed $|Rb\rangle$ states are:

$$\begin{array}{l} \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&i\\\hline 2&2&2&2&2&2\\\cline{1-6}\end{array} \\[4pt] \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&i&i&i\\\hline 2&2&2&2&2\\\cline{1-5}\end{array} \\ \vdots \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline i&i&i&i&i&i&i&i&i&i&i&i&i\\\hline\end{array} \end{array} \tag{6}$$

where i can take on the values 1, 2. Each of these states has $\mathcal{Y}=N_c$, and isospin $n_1/2$. The $SU(F-2)$ constraint is satisfied automatically, since no index transforms under $SU(F-2)$, so each of these states is an $SU(F-2)$ singlet. A [2] column with indices 1 and 2 is the antisymmetric combination $ud-du$, and has isospin zero, so the above states have isospin $1/2,3/2,\ldots,N_c/2$. The spin of the soliton is given by the isospin of the $|Rb\rangle$ state, so we have a tower of states $w=(n_1,n_2,0,\ldots,0)$, with $n_1+2n_2=N_c$ and spin $n_1/2$, which is the usual non-exotic tower of soliton states.

There are additional states in the rotational spectrum of the soliton. The hypercharge constraint can be satisfied by representations with more than N_c boxes, by choosing some of the boxes of the state $|Rb\rangle$ to be 1, 2, and the rest to be $3,\ldots,F$, so that the net $\mathcal{Y}$ is N_c. Since 1, 2 have $\mathcal{Y}=1>0$ and $3,\ldots,F$ have $\mathcal{Y}=y<0$, one can obtain a net $\mathcal{Y}=N_c$ by choosing N_c boxes with index 1, 2, plus some additional boxes whose $\mathcal{Y}$ adds up to zero. All additional boxes with values $3,\ldots,F$ must form an $SU(F-2)$ singlet. The only way to form an $SU(F-2)$ singlet is to completely antisymmetrize $F-2$ boxes using the ϵ-symbol of $SU(F-2)$, i.e. they must have the form

$$\begin{array}{|c|}\hline 3\\\hline \cdot\\\hline \cdot\\\hline F\\\hline\end{array} \tag{7}$$

Thus, anytime we add an index $\geqslant 2$, we also must add one each of all the remaining indices $3,\ldots,F$ in an $[F-2]$ column, as well as two more boxes with values 1, 2 to satisfy the hypercharge constraint. Let r' denote the number of extra sets of $3,\ldots,F$ indices added. Then, the hypercharge and $SU(F-2)$ constraints require that all tableaux have $N_c+r'F$ boxes, where $r'\geqslant 0$ is an integer. Each such tableau must have at least r' columns with length greater than or equal to $F-2$, to accomodate r' structures of the form Eq. (7).

The possible Young tableaux depend on how the N_c+2r' boxes with labels 1, 2 are added to the r' $[F-2]$ columns of Eq. (7). The allowed tableaux are given by first constructing the tableau with r' $[F-2]$ columns side-by-side, and then adding the remaining N_c+2r' boxes with labels 1, 2. There are two possibilities:

(i) The 1, 2 boxes go into completely separate columns. These additional columns can only contain one or two boxes each (i.e., they are $[1]$ or $[2]$ columns), since one cannot antisymmetrize in three or more indices if each index only takes on the values 1, 2.

(ii) Some of the 1, 2 boxes are added to $[F-2]$ columns, to form $[F-1]$ or $[F]$ columns, depending on whether one or two boxes are added. As in case (i), one cannot add more than two boxes with values 1, 2 to a single column.

The above conditions imply that the only allowed columns are [1], [2], $[F-2]$, $[F-1]$ and $[F]$ columns, and the total number of boxes is $N_c+r'F$, with r' being the total number of $[F-2]$, $[F-1]$ and $[F]$ columns. There can be no columns with lengths between 3 and $F-3$. This restriction is crucial for an understanding of exoticness and is why we first consider the case of arbitrary F. The value of r' is closely related to exoticness E.

In $SU(F)$, $[F]$ columns are flavor singlet, and can be thrown away. Thus, the allowed tableaux (after throwing away $[F]$ columns) contain only [1], [2], $[F-2]$, and $[F-1]$ columns, and have N_c+rF boxes, where the integer r, $0\leqslant r\leqslant r'$, is equal to the number of $[F-2]$ and $[F-1]$ columns. Translating from tableaux to Dynkin weights shows that the allowed $SU(F)$ weights are $w=(n_1,n_2,0\ldots,0,n_{-2},n_{-1})$, where we have relabelled $n_{F-2}\rightarrow n_{-2}$ and $n_{F-1}\rightarrow n_{-1}$. The four non-negative integers n_i satisfy

$$n_{-1}+n_{-2}=r,$$

$$n_1+2n_2+(F-2)n_{-2}+(F-1)n_{-1}=N_c+rF,$$

$$n_1+2n_2-2n_{-2}-n_{-1}=N_c, \tag{8}$$

where the first relation defines r as the number of $[F-1]$ and $[F-2]$ columns, and the second relation sets the total number of boxes equal to N_c+rF. The third relation is a linear combination of the first two.

The soliton spin is given by the isospin of the $SU(F-2)$ singlet, $\mathcal{Y}=N_c$ state that we have constructed. The isospin transformation properties of the state are given by dropping all boxes with indices $3,\ldots,F$, since these are all $SU(2)$ singlets. An $[F-2]$ column has the form Eq. (7), is an $SU(2)$ singlet, and can be dropped. The $[2]$ columns also are $SU(2)$ singlets, and can be dropped. Only the $[1]$ and $[F-1]$ columns of the Young tableau are left. Each $[1]$ column has isospin 1/2. Each $[F-1]$ column has the form Eq. (7) plus one box set equal to 1, 2, and so transforms as isospin 1/2. The n_1 $[1]$ columns are completely symmetrized, and so have isospin $n_1/2$. The n_{-1} $[F-1]$ columns are completely symmetrized, and so have isospin $n_{-1}/2$. There is no symmetry relation between the $[1]$ and $[F-1]$ columns. The allowed isospins for $|Rb\rangle$ are given by the tensor product of isospins $n_1/2$ and $n_{-1}/2$, so the allowed soliton spins are $j=(n_1/2)\otimes(n_{-1}/2)$. Each flavor and spin representation occurs at most once in the collective coordinate quantization. To get multiple copies of the same state, such as two $(\mathbf{8}_{1/2})$ states, requires vibrational excitations of the soliton.

The above analysis gives the classification of states for $F\geqslant 5$ presented in Ref. [7]: The soliton states have Dynkin weights $w=(n_1,n_2,0\ldots,0,n_{-2},n_{-1})$ which satisfy Eq. (8), and spins j in the tensor product $n_1/2\otimes n_{-1}/2$. Taking the last relation in Eq. (8) modulo two shows that the soliton is a fermion or boson depending on whether N_c is odd or even. The cases $F=4,3,2$ are all special and must be considered individually.

A. $F=4$

For $F=4$, the above analysis remains valid. However, one cannot distinguish columns with 2 and $F-2$ boxes in the final Young tableau. The total number of columns with two boxes is the sum of the [2] and $[F-2]$ columns, and will be denoted by $n_0=n_2+n_{-2}$. With this substitution, the allowed weights for $F=4$ are $w=(n_1,n_0,n_{-1})$ with spins $(n_1/2)\otimes(n_{-1}/2)$. Equation (8) also is modified to account for the equality $2=F-2$. The second relation in Eq. (8) becomes

$$n_1+2n_2+2n_{-2}+3n_{-1}=N_c+4r, \tag{9}$$

which gives

$$n_1+2n_0+3n_{-1}=N_c+4r, \tag{10}$$

a relation written solely in terms of "observable" properties of the Dynkin weight $w=(n_1,n_0,n_{-1})$. The first relation in Eq. (8) involves the "unobservable" integer n_{-2}. It leads to an inequality,

$$n_{-1}\leqslant r\leqslant n_0+n_{-1}, \tag{11}$$

since $n_{-2}\geqslant 0$.

Equation (10) modulo two again shows that the soliton is a fermion or boson depending on whether N_c is odd or even. Equation (11) is necessary and sufficient for there to be two positive integers $n_{\pm 2}\geqslant 0$ which satisfy $r=n_{-1}+n_{-2}$ and $n_0=n_2+n_{-2}$. As an example of why the inequality in Eq. (11) is needed, consider the flavor representation $w=(1,0,2)=\overline{\mathbf{36}}$ for $N_c=3$. This representation does not contain any $SU(F-2)$ singlets with $\mathcal{Y}=3$, and so it is not an allowed soliton flavor state for $N_c=3$, $F=4$. Although it satisfies the condition Eq. (10), with $r=1$, it violates the inequality of Eq. (11).

B. $F=3$

Soliton quantization for $F=3$ has been discussed before [3–6,10], but the general result derived below is new. For $F=3$, the general $SU(3)$ representation is conventionally denoted by (p,q), which is a traceless tensor $T^{a_1\ldots a_p}_{b_1\ldots b_q}$ with p upper and q lower indices. It corresponds to a Young tableau with p columns of one box, and q columns of two boxes.

We will derive the states for this case directly, rather than from the $F\geqslant 5$ results. The $SU(F-2)$ constraint is absent, and the hypercharge constraint implies that there must be a state with $3Y\equiv\mathcal{Y}=N_c$. The weights of the (p,q) representation in $SU(3)$ have the form shown in Fig. 1, with the well-known properties described in the figure caption. The maximum hypercharge is given by choosing all the upper indices of the tensor equal to 1, 2, and the lower indices equal to 3, and is given by $3Y_{\max}=p+2q$. One moves from a given hypercharge level to the next lower level by replacing one of the upper indices by 3, or one of the lower indices by 1, 2, so that $3Y\to 3Y-3$. The minimum hypercharge level is reached when the upper indices are all 3 and the lower indices are all 1, 2, so $3Y_{\min}=-2p-q$. The total number of hypercharge levels is $p+q+1$.

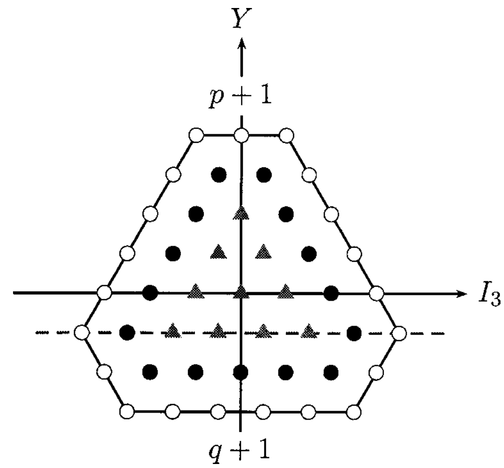


FIG. 1. (Color online) $SU(3)$ weight diagram for the $(p=2,q=5)$ representation. The upper edge has $p+1$ weights, and the lower edge has $q+1$ weights. The outermost layer (open circle) has multiplicity one, the next layer (solid circle) has multiplicity two, and the multiplicities increase by one until one gets to a triangular layer (red triangle), after which they stay constant. The horizontal blue dashed line is drawn through the corners of the weight diagram.

The little group constraint is that there must be a $3Y=N_c$ level in the weight diagram, which requires

$$p+2q=N_c+3r, \quad 0\leqslant r\leqslant p+q, \tag{12}$$

and the states we need are r steps below the maximum Y states. Since $Y_{\min}<0$ and $N_c>0$, there are always states with $3Y=N_c$ if Eq. (12) is satisfied. The three integers p, q and r are related to the n_i of the general $F\geqslant 5$ analysis by

$$\begin{aligned} p&=n_1+n_{-2},\\ q&=n_{-1}+n_2,\\ r&=n_{-1}+n_{-2}, \end{aligned} \tag{13}$$

where the first two relations follow since [1] and $[F-2]$ columns, as well as [2] and $[F-1]$ columns, are indistinguishable for $F=3$. It is also convenient to define $s=n_1+n_2\equiv p+q-r$.

The allowed spins are in the product $n_1/2\otimes n_{-1}/2$, so the minimum spin j is equal to $j_{\min}\equiv|n_1-n_{-1}|/2=|p-r|/2=|s-q|/2$. To determine the maximum spin, we need to determine the maximum isospin of the states r steps below the maximum Y states. The horizontal dashed line in Fig. 1 through the corners of the weight diagram is q steps below the upper edge. If $r\leqslant q$, then the states we need are above (or on) the horizontal dashed line, and if $r\geqslant q$, the states are below (or on) the horizontal dashed line. We consider the two cases separately.

Case $(r \leqslant q)$. The upper edge states have isospin $I = p/2$. As one drops down in *Y*, one gets states with several different values of *I*, $I = I_{\min} \oplus \cdots \oplus I_{\max}$. $I_{\max}$ increases by 1/2 for each step down in *Y*, since the weight diagram gets one weight wider. Thus, the states with $3Y = N_c$, which are *r* steps down, have $I_{\max} = (p+r)/2$. The value of $I_{\min}$ decreases by 1/2 at each step, until one reaches the apex of the "triangular layer" shown by the triangles in Fig. 1, which is min(*p*,*q*) levels down from the upper edge. After that point, $I_{\min}$ increases by 1/2 at each step, since the triangular layer gets wider as one moves down in *Y*. Thus $I_{\min}$ keeps decreasing by 1/2 at each step until either *r* or min(*p*,*q*) is reached, whichever comes first, after which it increases by 1/2 at each step. Since we have assumed $r \leqslant q$, $I_{\min}$ decreases by 1/2 for the first min(*p*,*r*) steps after which it increases by 1/2. If $r \leqslant p$, then $I_{\min} = p/2 - r/2$. If $r \geqslant p$, $I_{\min} = p/2 - p/2 + (r-p)/2 = (r-p)/2$, so in either case, $I_{\min} = |p-r|/2$. The values for $I_{\max}$ and $I_{\min}$ are the same as those in $(p/2) \otimes (r/2)$, so the allowed isospin (and hence spin) states are $(p/2) \otimes (r/2)$.

Case $(r \geqslant q)$. The states *r* levels down from the upper edge are $s = p+q-r$ levels above the lower edge. Thus, one can apply the previous argument, now moving up from the minimum *Y* states instead of down from the maximum *Y* states. The solution is given by the previous case with $p \to q$, $r \to s \equiv p+q-r$.

In summary: The allowed $SU(3)$ Skyrme states are (p,q) with

$$p + 2q = N_c + 3r, \quad j = \begin{cases} \dfrac{p}{2} \otimes \dfrac{r}{2} & \text{if } r \leqslant q, \\ \dfrac{q}{2} \otimes \dfrac{p+q-r}{2} & \text{if } r \geqslant q. \end{cases} \tag{14}$$

Using the above equations modulo two shows that the soliton is a fermion or boson depending on whether N_c is odd or even.

For $N_c = 3$, the $r = 0$ states with 3 boxes are:

(a) $(1,1) \to \mathbf{8}_{1/2}$.

(b) $(3,0) \to \mathbf{10}_{3/2}$.

The $r = 1$ states with 6 boxes are:

(c) $(0,3) \to \overline{\mathbf{10}}_{1/2}$.

(d) $(2,2) \to \mathbf{27}_{1/2}, \mathbf{27}_{3/2}$. Note that since both *p* and *r* are not zero, there are several spin states with the same flavor representation. Equation (14) gives the allowed spins as $1 \otimes 1/2 = 1/2 \oplus 3/2$.

(e) $(4,1) \to \mathbf{35}_{3/2}, \mathbf{35}_{5/2}$.

(f) $(6,0) \to \mathbf{28}_{5/2}$. This is an example where one needs to use the $r > q$ case, since $p = 6$, $q = 0$, $r = 1$. Using the $r \leqslant q$ formula would give both $\mathbf{28}_{5/2}$ and $\mathbf{28}_{7/2}$ states.

One can similarly work out the states for higher values of *r*.

C. $F = 2$

For $F = 2$, the only constraint is that $I = J$, so there is an infinite tower of states with all possible values $I = J = j$. Witten has argued that one must restrict to $2I = 2J$ even states for N_c even, and $2I = 2J$ odd states for N_c odd [3].

IV. ENERGY

The rotational energy of the soliton is given by the Hamiltonian,

$$H = M_0 + \frac{1}{2I_1}\mathbf{J}^2 + \frac{1}{2I_2}\left(\mathbf{T}^2 - \mathbf{J}^2 - \frac{F-2}{4F}N_c^2\right), \tag{15}$$

with corrections of order $1/N_c^2$. The mass M_0 and moments of inertia $I_{1,2}$ are order N_c. The flavor Casimir $\mathbf{T}^2$ is required to determine the rotational energy of the soliton. The Casimir of the $SU(F)$ representation $w = (n_1, n_2, 0 \ldots, 0, n_{-2}, n_{-1})$ for $F \geqslant 5$ is

$$\begin{aligned} \mathbf{T}^2 = {} & \frac{1}{2} n_1 \left(1 + \frac{n_1}{2}\right) + \frac{N_c(N_c+2F)(F-2)}{4F} \\ & + 2r^2 + (2F + N_c - 4) r \\ & + \left(\frac{5}{2} - 2r - \frac{1}{2}N_c - F\right) n_{-1} + \frac{3}{4}(n_{-1})^2 \end{aligned} \tag{16}$$

using the Freudenthal formula, where Eq. (8) has been used to eliminate $n_{\pm 2}$ in favor of N_c and *r*. The expression does not have a nice form, because *r* is not symmetrically defined with respect to charge conjugation. Using the variables

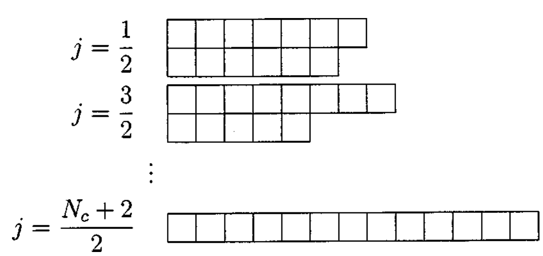
$$j_q = n_1/2, \quad j_{\bar{q}} = n_{-1}/2, \quad E = n_{-1} + 2n_{-2} \tag{17}$$

defined in Ref. [7], instead of $n_{\pm 1}$ and *r*, leads to the formula

$$\begin{aligned} \mathbf{T}^2 = {} & \frac{N_c(N_c+2F)(F-2)}{2F} + E(E + 2F + N_c - 4) + j_q(j_q+1) \\ & + j_{\bar{q}}(j_{\bar{q}}+1) \end{aligned} \tag{18}$$

which has a nice physical interpretation in terms of quarks.

The variables j_q, $j_{\bar{q}}$ and E arise naturally in a quark model construction. The baryon is made of $N_c + E$ quarks and E antiquarks, where exoticness E is the *minimum* number of $q\bar{q}$ pairs required to construct a baryon with the desired flavor quantum numbers. The completely symmetric spin-flavor states for $N_c + E$ quarks are:



(19)

where each Young tableau has $N_c + E$ boxes. There can be at most two boxes in any column, since there are only two quark spin states. The allowed quark states are flavor representations $(n_1,n_2,0,\ldots,0)$ with $n_1+2n_2=N_c+E$, and spin $j_q=n_1/2$. Similarly, the antiquarks form the flavor representation $(0,\ldots,0,n_{-2},n_{-1})$ with $n_{-1}+2n_{-2}=E$ and spin $j_{\bar{q}}=n_{-1}/2$. The exotic baryon representation obtained by combining the quarks and antiquarks has flavor weight $w=(n_1,n_2,0,\ldots,0,n_{-2},n_{-1})$, since flavor singlet $q\bar{q}$ pairs which can annihilate are excluded [7]. The other states in the tensor product of $(n_1,n_2,0,\ldots,0)$ and $(0,\ldots,0,n_{-2},n_{-1})$ have index contractions, and so contain flavor singlet $q\bar{q}$ pairs. The allowed exotic baryon spins are given by $j_q\otimes j_{\bar{q}}$. Note that the allowed states in the quark model precisely match those in the soliton model.

For $F\geqslant 5$, the "observable" integers $n_{\pm 1}$, $n_{\pm 2}$ in the weight w define $j_q,j_{\bar{q}},E$ uniquely using Eq. (17). Thus, one can write the quantum numbers of the solition states in terms of quark model variables j_q, $j_{\bar{q}}$ and E. The Casimir Eq. (18) then has a simple interpretation in terms of the hyperfine interactions of quarks and antiquarks, and the constituent mass of the quarks [7].

For $F<5$, it is still useful to convert to quark model variables, even though the conversion is not as simple as Eq. (17), because the quark and antiquark contributions are not separated in the weight w.

$F=4$

For $F=4$, the Casimir formula Eq. (16) continues to hold for $w=(n_1,n_0,n_{-1})$, where Eq. (10) has been used to eliminate n_0 in favor of r. One still can define

$$j_q=n_1/2,\quad j_{\bar{q}}=n_{-1}/2, \tag{20}$$

and exoticness $E=n_{-1}+2n_{-2}=2r-n_{-1}$ still can be determined from the flavor weight and N_c,

$$E=\frac{n_1+n_{-1}-N_c}{2}+n_0. \tag{21}$$

Equation (18) remains valid.

B. $F=3$

For three flavors, the (p,q) Casimir is

$$\mathbf{T}^2=\frac{1}{3}(p^2+pq+q^2)+(p+q). \tag{22}$$

The weight (p,q) and N_c are the "observable" properties of the soliton state, and determine p,q,r. Equations (13), (17) can be solved to give the n_i in terms of p,q,r,E:

$$\begin{aligned} n_1&=-E+p+r,\\ n_{-1}&=-E+2r,\\ n_2&=E+q-2r,\\ n_{-2}&=E-r, \end{aligned} \tag{23}$$

but now E is not determined uniquely by p,q and N_c. Since $n_i\geqslant 0$, one has

$$r\leqslant E\leqslant r+p,\quad 2r-q\leqslant E\leqslant 2r. \tag{24}$$

Definitions Eqs. (13), (17) imply that $E=r+n_{-2}$, so that $E\neq r$ [7,11]. This distinction is important. It was assumed previously in the literature that exoticness was related to the number of boxes in the Young tableau. A tableau with N_c+3r boxes was said to have exoticness r. However, this interpretation is incorrect [7].

Since exoticness is the minimum value of E for which one can construct a given baryon state in a quark model, Eq. (24) gives [7]

$$E=E_{\text{min}}=\max(r,2r-q)=\begin{cases} r & \text{if } r\leqslant q,\\ 2r-q & \text{if } r\geqslant q,\end{cases} \tag{25}$$

as the minimum allowed value for E.

If $r\leqslant q$, then $E=r$, and Eq. (23) gives

$$\begin{aligned} n_1&=p,\\ n_{-1}&=r,\\ n_2&=q-r,\\ n_{-2}&=0, \end{aligned} \tag{26}$$

with $j_q=p/2$ and $j_{\bar{q}}=r/2$.

If $r\geqslant q$, then $E=2r-q$, and Eq. (23) gives

$$\begin{aligned} n_1&=p+q-r,\\ n_{-1}&=q,\\ n_2&=0,\\ n_{-2}&=r-q, \end{aligned} \tag{27}$$

with $j_q=(p+q-r)/2$ and $j_{\bar{q}}=q/2$.

Equations (26), (27) allow one to interpret the soliton state in terms of quark variables. Combining j_q and $j_{\bar{q}}$ gives the same spin states as Eq. (14), which is a non-trivial check on the analysis. Equation (18) remains valid, and agrees with Eq. (22) for both $r\leqslant q$ and $r\geqslant q$.

C. $F=2$

For $F=2$, the quarks are in the representation $(2j)$ with spin j. In this case, for states with spin of order one, $E=r=0$, $j_q=j$, $j_{\bar{q}}=0$. The Casimir is $\mathbf{T}^2=j(j+1)$, and Eq. (18) remains valid.

V. DISCUSSION AND CONCLUSIONS

Collective coordinate quantization is valid for baryons with $E=0$, where the Casimir $\mathbf{T}^2$ is of order N_c^0, so that the rotational energy is order $1/N_c$. However, for $E\neq 0$ baryon exotics, the Casimir is order N_c, and the rotational energy is order N_c^0, which is the same order as the vibrational energies. In this case, one has to include vibrational-rotational mixing to correctly compute the energies, and Eq. (15) is no longer valid [12]. Nevertheless, the structure of the energy still has the form Eq. (18), though the coefficients of the E, E^2, $j_q(j_q+1)$ and $j_{\bar{q}}(j_{\bar{q}}+1)$ terms no longer have the values given in Eq. (18) [7]. The interpretation of the soliton states presented here remains valid. Including vibrational modes will produce additional states with the quantum numbers of a soliton plus some number of mesons, which can be thought of as soliton-meson bound states.

In summary, we have computed the allowed baryon states in the chiral soliton model for an arbitrary number of colors and flavors. Explicit results are given for the case of three colors and flavors. The analysis presented in this paper provides insight into the connection between the soliton and quark models, for normal as well as for exotic baryons, since there is a non-trivial map between the states and their energies in the two models.

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