Physical renormalization condition for the quark-mixing matrix

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We investigate the renormalization of the quark-mixing matrix in the electroweak standard model. The corresponding counterterms are gauge independent as can be shown using an extended BRS symmetry. Using rigid $SU(2)_L$ symmetry, we prove that the ultraviolet-divergent parts of the invariant counterterms are related to the field renormalization constants of the quark fields. We point out that for a general class of renormalization schemes rigid $SU(2)_L$ symmetry cannot be preserved in its classical form, but is renormalized by finite counterterms. Finally, we discuss a genuine physical renormalization condition for the quark-mixing matrix that is gauge independent and does not destroy the symmetry between quark generations.

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I. INTRODUCTION

Presently, the parameters of the quark-mixing matrix (QMM) are being precisely measured at the B factories. When calculating precision observables involving the QMM, in general the renormalization of the QMM is required. This was first realized in Ref. [1] for the Cabibbo angle in the standard model (SM) with two fermion generations. An example where the counterterms for the QMM for three generations have been taken into account can be found in Ref. [2]. In the SM the effects of the renormalization of the QMM are numerically small, since the masses of all down-type quarks are small compared to the W-boson mass [3]. However, a consistent renormalization of the QMM should be formulated for conceptual reasons. Moreover, the renormalization of mixing matrices may become phenomenologically relevant in extensions of the SM.

The most straightforward way to renormalize the QMM is to directly fix the four independent parameters of the QMM, three angles, and a *CP*-violating phase, by choosing four suitable observables—e.g., four specific W-boson decays [4]. However, the counterterms determined in this way depend on the chosen observables, and the symmetry between the amplitudes involving different generations is destroyed. A symmetric renormalization condition can be obtained naturally using the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme (see, e.g., Refs. [5], [6]). This, however, is not a physical condition and depends on an arbitrary renormalization scale. Moreover, in this scheme, the renormalized *S*-matrix elements exhibit singularities of the form $1/(m_{q,i}^2-m_{q,k}^2)$ in the limit of degenerate up-type or down-type quark masses $m_{q,i}\approx m_{q,k}$; i.e., the limit of degenerate quark masses, where the QMM is equal to the unit matrix and need not be renormalized, is not approached smoothly.

A renormalization condition for the QMM in the on-shell scheme was first proposed in Refs. [3], [7]. In this proposal, the counterterms of the QMM are determined from the field renormalization constants of the quark fields in the on-shell renormalization scheme. This prescription is simple, does not introduce a renormalization scale, and is smoothly connected to the limit of degenerate quark masses. Later it was discovered [8], however, that the renormalization condition of Refs. [3], [7] leads to gauge-parameter-dependent counterterms for the QMM and thus to gauge-parameter-dependent *S*-matrix elements. In Ref. [8] a modified renormalization condition was proposed based on field renormalization constants defined at zero momentum. This scheme gives gauge-parameter-independent results at the one-loop level, but leads to singularities in the *S*-matrix elements for degenerate quark masses. Moreover, it is not clear whether it can be generalized beyond one-loop order.

It was also suggested to split off the gauge-parameter-dependent part of the on-shell quark-field renormalization constants as far as the definition of the QMM counterterm is concerned—i.e., to define the QMM counterterm from the quark-field renormalization constants calculated in the 't Hooft-Feynman gauge [9]. This scheme corresponds exactly to the original one of Refs. [3], [7]. It is gauge-parameter independent by definition, but of course depends implicitly on the choice of the 't Hooft-Feynman gauge. Generalizing this philosophy, it was argued in Ref. [6] that any renormalization scheme for the QMM may be viewed as a gauge-invariant scheme by definition. This is possible since any scheme is related to the (gauge-invariant) $\overline{\mathrm{MS}}$ scheme by ultraviolet- (UV-) finite matrices which can be chosen to match any renormalization condition.

In Ref. [10] desirable properties for the renormalization condition of the QMM have been formulated. These are UV finiteness, gauge-parameter independence, and unitarity of the renormalized QMM. In addition, the renormalization condition should be physically motivated and treat all generations on an equal footing. The requirement that the renormalized amplitudes approach the limit of degenerate up-type or down-type masses smoothly is also implicitly contained in this paper. A renormalization condition was formulated that obeys all these properties and is also applicable to the lepton mixing in Majorana-neutrino theories. In this scheme, the renormalized QMM is fixed by matching the matrix elements for W-boson decay in the SM with those in reference theories with zero mixing and different assignments of down-type quarks to the generations. The unitarity of the renormalized QMM is obtained by subtracting the unitarity-violating part from the counterterm obtained from the reference theory.

All the mentioned prescriptions have only been used at the one-loop level, and it is not clear how they can be consistently generalized to higher orders. Recently, a renormalization prescription for the QMM has been proposed [11] that could overcome all these weaknesses. This renormalization condition has been introduced via a two-step procedure, and Ref. [11] leaves a lot of questions open. In the present paper we rederive the renormalization condition of Ref. [11] in a different way and put it on a more sound basis.

Before we consider explicit renormalization conditions for the QMM we first investigate the consequences of the symmetries of the theory on the QMM and its renormalization. In gauge theories, the gauge-parameter dependence of Green functions can be controlled by extending the gauge parameter ξ to a Becchi-Rouet-Stora (BRS) doublet (ξ,χ), where χ is a Grassmann-valued parameter. Gauge-parameter dependence of Green functions and counterterms is determined by an extended Slavnov-Taylor (ST) identity [12,13]. By solving the extended ST identity it is seen that, in general, physical parameters and their counterterms have to be gauge-parameter independent. Finally, it is possible to prove gauge-parameter independence of physical *S*-matrix elements [13,14].

This formalism has been first applied to the renormalization of the QMM in Ref. [8], yielding the result that counterterms to the QMM are gauge-parameter independent. As an additional constraint the authors of Ref. [8] require that the Ward-Takahashi identity of gauge invariance in the background-field gauge be preserved in its classical form to all orders. As we show, the gauge-parameter dependence of *S*-matrix elements and counterterms is governed by the BRS invariance only. Thus, the use of the Ward-Takahashi identity is not adequate in this context. In particular, invariance of the Ward-Takahashi identity implies that the renormalization of the QMM is related to the renormalization of quark fields. If the renormalization of the QMM is required to be gauge independent in this context, the renormalization of the quark fields must be gauge independent, too. Thus, a complete on-shell renormalization scheme is not admissible in this approach. Therefore, the authors of Ref. [8] modify the on-shell conditions for quark fields to renormalizations at zero momentum, at least as far as the renormalization of the QMM is concerned.

In the present paper we investigate the consequences of the symmetries of the theory on the QMM and its renormalization in the general linear R_ξ gauge. As already mentioned, the relevant symmetry to control the gauge-parameter dependence of counterterms as well as of physical *S*-matrix elements is BRS symmetry in form of the ST identity. On the other hand, rigid invariance restricts the UV divergences of the invariant counterterms and, in particular, relates the divergent parts of the field renormalization constants and of the counterterms of the QMM. However, finite field redefinitions can be introduced that renormalize the Ward identity of rigid invariance. These are, in particular, needed in the complete on-shell scheme in order to have enough freedom to introduce complete on-shell conditions in agreement with gauge-parameter independence of the QMM.

This paper is organized as follows: in Sec. II we show that the counterterms for the QMM are gauge-parameter independent as a consequence of the BRS invariance of the SM. Using rigid symmetry, we prove in Sec. III that the UV-divergent parts of the invariant counterterms of the QMM are related to the field renormalization constants of the quark fields. The free parameters of the QMM and their counterterms are elaborated in Sec. IV. Finally, we discuss a physical renormalization condition for the QMM in Sec. V. The Appendix contains some discussion of absorptive parts.

II. IMPLICATIONS OF BRS INVARIANCE ON COUNTERTERMS

The interplay between the renormalization of the QMM and the field renormalization of the quark fields makes it particularly difficult to disentangle the gauge-parameter dependence of the different kinds of counterterms. The relevant symmetry that governs the gauge-parameter dependence is the BRS symmetry. This is considered in this section.

Following closely the conventions of Ref. [7] the part of the classical action of the SM relevant for quark mixing reads

$$\begin{aligned}\Gamma_{\mathrm{cl}}^{\mathrm{quark}}=\int \mathrm{d}^4x\Bigg\{ & \mathrm{i}\overline{Q}_i^{\mathrm{L}}\gamma_\mu D^\mu_{ij}Q^{\mathrm{L}}_j+\mathrm{i}\overline{u}_i^{\mathrm{R}}\gamma_\mu D^\mu_{ij}u^{\mathrm{R}}_j+\mathrm{i}\overline{d}_i^{\mathrm{R}}\gamma_\mu D^\mu_{ij}d^{\mathrm{R}}_j\\ & -m_{\mathrm{d},i}(\overline{d}_i^{\mathrm{L}}d_i^{\mathrm{R}}+\overline{d}_i^{\mathrm{R}}d_i^{\mathrm{L}})-m_{\mathrm{u},i}(\overline{u}_i^{\mathrm{L}}u_i^{\mathrm{R}}+\overline{u}_i^{\mathrm{R}}u_i^{\mathrm{L}})\\ & -\frac{e}{\sqrt{2}M_{\mathrm{W}}s_{\mathrm{w}}}[\overline{Q}_i^{\mathrm{L}}\mathbf{V}_{ij}\Phi m_{\mathrm{d},j}d_j^{\mathrm{R}}\\ & +\overline{Q}_i^{\mathrm{L}}\mathbf{V}^\dagger_{ij}(\mathrm{i}\sigma^2)\Phi^* m_{\mathrm{u},j}u_j^{\mathrm{R}}+\mathrm{H.c.}]\Bigg\},\end{aligned}\tag{2.1}$$

where the left-handed quarks (isospin doublets) are denoted by $Q_i^{\mathrm{L}}=(u_i^{\mathrm{L}},d_i^{\mathrm{L}})^{\mathrm{T}}$, the right-handed quarks (isospin singlets) by $u_i^{\mathrm{R}},d_i^{\mathrm{R}}$, and the Highs doublet, with vacuum expectation value $(0,v/\sqrt{2})^{\mathrm{T}}$ subtracted, by Φ. The indices i,j run over N quark families. The sine and cosine of the weak mixing angle are defined as usual by $c_{\mathrm{w}}=\sqrt{1-s_{\mathrm{w}}^2}=M_{\mathrm{W}}/M_{\mathrm{Z}}$, and σ^2 is a Pauli matrix.

The matrix $\mathbf{V}$ is a unitary $2N\times 2N$ matrix in the quark-family and isospin space and is composed as

$$\mathbf{V}_{ij}=\begin{pmatrix} V_{ij} & 0\\ 0 & \delta_{ij}\end{pmatrix},\qquad \mathbf{V}^\dagger\mathbf{V}=1.\tag{2.2}$$

This matrix includes the QMM V_{ij} which is a unitary $N\times N$ matrix depending on $N(N-1)/2$ angles and $(N-1)(N$

$-2)/2$ phases. These are the Cabibbo angle for $N=2$ and the three angles and one phase of the Cabibbo-Kobayashi-Maskawa matrix for $N=3$. In case of *CP* conservation all phases vanish.

The covariant derivatives of the quark fields read

$$
\begin{aligned}
D_{ij}^{\mu} Q_j^{\mathrm{L}} = \Bigg\{ & \delta_{ij}\partial^{\mu} + \mathrm{i}\frac{e}{c_{\mathrm{w}}}\delta_{ij}\frac{Y_{\mathrm{w}}^{Q}}{2}(s_{\mathrm{w}}Z^{\mu}+c_{\mathrm{w}}A^{\mu}) \\
& -\mathrm{i}\frac{e}{s_{\mathrm{w}}}[V_{ij}I_{\mathrm{w}}^{+}W^{+\mu}+V_{ij}^{\dagger}I_{\mathrm{w}}^{-}W^{-\mu} \\
& +\delta_{ij}I_{\mathrm{w}}^{3}(c_{\mathrm{w}}Z^{\mu}-s_{\mathrm{w}}A^{\mu})]\Bigg\} Q_j^{\mathrm{L}}, \\
D_{ij}^{\mu} q_j^{\mathrm{R}} = \Bigg[& \partial^{\mu} + \mathrm{i}\frac{e}{c_{\mathrm{w}}}\frac{Y_{\mathrm{w}}^{q}}{2}(s_{\mathrm{w}}Z^{\mu}+c_{\mathrm{w}}A^{\mu})\Bigg]\delta_{ij}q_j^{\mathrm{R}}, \\
q_j^{\mathrm{R}} = {} & u_j^{\mathrm{R}}, d_j^{\mathrm{R}},
\end{aligned}
\tag{2.3}
$$

where $V_{ij}^{\dagger}=V_{ji}^{*}$, and the generators of the $\mathrm{SU}(2)_{\mathrm{L}}$ gauge group are defined by $I_{\mathrm{w}}^{a}=\sigma^{a}/2$ with the Pauli matrices σ^{a}. For convenience, we replaced in Eqs. (2.3) the generators $I_{\mathrm{w}}^{1,2}$ by

$$
I_{\mathrm{w}}^{\pm}=\frac{1}{\sqrt{2}}(I_{\mathrm{w}}^{1}\pm \mathrm{i}I_{\mathrm{w}}^{2}). \tag{2.4}
$$

The hypercharges of the fields Q_j^{L} and q_j^{R} are denoted by Y_{w}^{Q} and Y_{w}^{q}, respectively. Note that the QMM V_{ij} appears in Eq. (2.3) only in terms involving W bosons.

The classical action, partially given by Eq. (2.1), is complete in the sense that it is the most general field polynomial that is consistent with power counting and respects all symmetry requirements of the underlying theory. In particular, invariance under BRS symmetry implies unitarity of the QMM. Besides BRS symmetry, rigid $\mathrm{SU}(2)_{\mathrm{L}}$ symmetry and local $\mathrm{U}(1)_{Y}$ gauge symmetry of hypercharge are the relevant symmetries of the SM (see Ref. [15]). Since the BRS symmetry controls the gauge-parameter dependence of the QMM, we focus on BRS transformations in the following and come back to gauge transformations later.

The BRS transformations of the quark fields take the form

$$
\begin{aligned}
\mathbf{s}Q_i^{\mathrm{L}} = \Bigg\{ & -\mathrm{i}\frac{e}{c_{\mathrm{w}}}\delta_{ij}\frac{Y_{\mathrm{w}}^{Q}}{2}(s_{\mathrm{w}}c_{\mathrm{Z}}+c_{\mathrm{w}}c_{\mathrm{A}}) \\
& +\mathrm{i}\frac{e}{s_{\mathrm{w}}}[V_{ij}I_{\mathrm{w}}^{+}c_{+}+V_{ij}^{\dagger}I_{\mathrm{w}}^{-}c_{-} \\
& +\delta_{ij}I_{\mathrm{w}}^{3}(c_{\mathrm{w}}c_{\mathrm{Z}}-s_{\mathrm{w}}c_{\mathrm{A}})]\Bigg\} Q_j^{\mathrm{L}}, \\
\mathbf{s}q_i^{\mathrm{R}} = {} & -\mathrm{i}\frac{e}{c_{\mathrm{w}}}\frac{Y_{\mathrm{w}}^{q}}{2}(s_{\mathrm{w}}c_{\mathrm{Z}}+c_{\mathrm{w}}c_{\mathrm{A}})\delta_{ij}q_j^{\mathrm{R}},
\end{aligned}
\tag{2.5}
$$

where c_{A}, c_{Z}, and $c_{\pm}$ are Faddeev-Popov ghost fields. The BRS transformations for vector-boson and scalar fields have the usual form and can be found, for instance, in Refs. [15], [16].

Since the BRS transformations of the quark fields are composite operators and receive quantum corrections, we couple them to external fields $\psi_{q,i}^{\mathrm{L}}$ and Ψ_i^{R},

$$
\Gamma_{\mathrm{cl}}^{\mathrm{ext}}=\int \mathrm{d}^4x\left[\cdots+\sum_{i=1}^{N}(\overline{\psi}_{\mathrm{u},i}^{\mathrm{L}}\mathbf{s}u_i^{\mathrm{R}}+\overline{\psi}_{\mathrm{d},i}^{\mathrm{L}}\mathbf{s}d_i^{\mathrm{R}}+\overline{\Psi}_i^{\mathrm{R}}\mathbf{s}Q_i^{\mathrm{L}}+\mathrm{H.c.})\right], \tag{2.6}
$$

and add $\Gamma_{\mathrm{cl}}^{\mathrm{ext}}$ to the classical action. The auxiliary nonpropagating fields $\psi_{q,i}^{\mathrm{L}}$ and $\Psi_i^{\mathrm{R}}=(\psi_{\mathrm{u},i}^{\mathrm{R}},\psi_{\mathrm{d},i}^{\mathrm{R}})^{\mathrm{T}}$ have ghost charge -1 and are BRS invariants—i.e., $\mathbf{s}\psi_{q,i}^{\mathrm{L/R}}=0$. For quantization, BRS transformations are encoded in the ST identity

$$
\begin{aligned}
\mathcal{S}(\Gamma)=\int \mathrm{d}^4x\Bigg[\cdots+\sum_{i=1}^{N}\Bigg(&\frac{\delta\Gamma}{\delta\overline{\psi}_{\mathrm{u},i}^{\mathrm{L}}}\frac{\delta\Gamma}{\delta u_i^{\mathrm{R}}}+\frac{\delta\Gamma}{\delta\overline{\psi}_{\mathrm{d},i}^{\mathrm{L}}}\frac{\delta\Gamma}{\delta d_i^{\mathrm{R}}} \\
&+\frac{\delta\Gamma}{\delta\overline{\Psi}_i^{\mathrm{R}}}\frac{\delta\Gamma}{\delta Q_i^{\mathrm{L}}}+\mathrm{H.c.}\Bigg)\Bigg]=0,
\end{aligned}
\tag{2.7}
$$

where Γ is the generating functional of one-particle irreducible Green functions. The ellipses in Eqs. (2.6) and (2.7) denote the contributions from vector-boson, scalar, lepton, and ghost fields (see Refs. [15], [16] for details). As usual, the linearized ST operator $\mathbf{s}_{\Gamma}$ is defined by the expansion

$$
\mathcal{S}(\Gamma+\Delta)=\mathcal{S}(\Gamma)+\mathbf{s}_{\Gamma}\Delta+\mathcal{O}(\Delta^2). \tag{2.8}
$$

All counterterms compatible with the ST identity are called *BRS-invariant* counterterms in the following. These comprise counterterms that are invariant under rigid symmetry, but also those that are not (see Sec. III). According to the definition of the BRS-invariant counterterms, they are invariant under the linearized ST operator

$$
\mathbf{s}_{\Gamma_{\mathrm{cl}}}\Gamma_{\text{BRS-inv},i}=0. \tag{2.9}
$$

These counterterms can be expressed in form of differential operators $\mathcal{D}_i$ that commute with the linearized ST operator:

$$
\Gamma_{\text{BRS-inv},i}=\delta Z_i(\xi)\mathcal{D}_i\Gamma_{\mathrm{cl}},\quad \mathcal{D}_i\mathcal{S}(\mathcal{F})-\mathbf{s}_{\mathcal{F}}\mathcal{D}_i\mathcal{F}=0, \tag{2.10}
$$

where $\mathcal{F}$ is an arbitrary field polynomial. In order to investigate the gauge-parameter dependence of the renormalization constants $\delta Z_i(\xi)$, we introduce BRS-varying gauge parameters following Refs. [12], [13]:

$$
\mathbf{s}\xi=\chi,\quad \mathbf{s}\chi=0, \tag{2.11}
$$

where ξ is a gauge parameter, and the new auxiliary constant field χ is Grassmann valued and has ghost charge -1. We include the BRS transformation of ξ into an extended ST identity:

$$
\mathcal{S}^{\chi}(\Gamma)=\mathcal{S}(\Gamma)+\chi\partial_{\xi}\Gamma=0. \tag{2.12}
$$

This construction allows us to classify the counterterms into genuine gauge-parameter-dependent and -independent ones and to prove the gauge-parameter independence of the S-matrix elements (see Ref. [14] for details). It is important to realize that the introduction of the additional constant field χ is only an auxiliary construction and does not affect the χ-independent part of Γ, since the part of Γ involving the BRS doublet (ξ, χ) is free of anomalies by construction, and χ does not appear in ST identities for physical Green functions.

In the following, we are searching for counterterms that are invariant with respect to the extended ST identity (2.12),

$$\mathbf{s}^{\chi}_{\Gamma_{\rm cl}}\Gamma^{\chi}_{\text{BRS-inv},i}=0, \tag{2.13}$$

and investigate the consequences for the gauge-parameter dependence of the renormalization constants $\delta Z_i(\xi)$. In general, the counterterms (2.10) violate the extended ST identity:

$$\mathbf{s}^{\chi}_{\Gamma_{\rm cl}}\Gamma_{\text{BRS-inv},i}=\chi(\partial_\xi \ln \delta Z_i(\xi))\Gamma_{\text{BRS-inv},i}\,. \tag{2.14}$$

There are two possibilities to construct counterterms compatible with the extended ST identity.

(i) If a local field polynomial $\hat{\Delta}_i$ exists such that

$$\Gamma_{\text{BRS-inv},i}=\delta Z_i(\xi)\mathbf{s}_{\Gamma_{\rm cl}}\hat{\Delta}_i\,, \tag{2.15}$$

we are able to build an invariant counterterm for the extended ST identity (2.12) by defining

$$\Gamma^{\chi}_{\text{BRS-inv},i}:=\Gamma_{\text{BRS-inv},i}-\chi(\partial_\xi\delta Z_i(\xi))\hat{\Delta}_i\,. \tag{2.16}$$

In this case there is no restriction on the gauge-parameter dependence of the renormalization constant $\delta Z_i(\xi)$. Typical examples of such counterterms are field renormalizations of the matter fields, gauge-fixing, and ghost terms, and in the case of the minimal supersymmetric standard model soft-supersymmetry-breaking terms if they are introduced according to Ref. [17].

(ii) If $\Gamma_{\text{BRS-inv},i}$ cannot be written in form of Eq. (2.15), the renormalization constant δZ_i must be gauge-parameter independent, and the counterterm of the extended ST identity reads

$$\Gamma^{\chi}_{\text{BRS-inv},i}:=\Gamma_{\text{BRS-inv},i}\quad\text{with}\quad \partial_\xi\delta Z_i=0. \tag{2.17}$$

Examples of such counterterms are those for the physical parameters of the SM, like the gauge couplings and masses (cf. Refs. [18–21]).

As a result, the counterterms split into two classes: those of the first class can be written as a $\mathbf{s}_{\Gamma_{\rm cl}}$ variation and are in general gauge-parameter dependent; those of the second class cannot be written in the form of Eq. (2.15) and thus must be genuinely gauge-parameter independent, provided that appropriate renormalization conditions are chosen that do not lead to an artificial gauge-parameter dependence. A careful separation between both classes is necessary in order to draw conclusions on gauge-parameter dependence.

For the renormalization of the QMM the following counterterms are relevant.

(i) The field renormalizations of the quark fields result from $\mathbf{s}_{\Gamma_{\rm cl}}$ variations as

$$\begin{aligned}\Gamma^{q,\mathrm{L/R}}_{\text{BRS-inv},ij}&=-\delta Z^{q,\mathrm{L/R}}_{ij}\mathbf{s}_{\Gamma_{\rm cl}}(\overline{\psi}^{\mathrm{R/L}}_{q,i}q^{\mathrm{L/R}}_j)+\text{H.c.}\\&=\delta Z^{q,\mathrm{L/R}}_{ij}\mathcal{N}^{q,\mathrm{L/R}}_{ij}\Gamma_{\rm cl}+\text{H.c.},\end{aligned} \tag{2.18}$$

with

$$\mathcal{N}^{q,\mathrm{L/R}}_{ij}=\int \mathrm{d}^4x\left[(q^{\mathrm{L/R}}_j)^{\mathrm{T}}\frac{\delta}{\delta q^{\mathrm{L/R}}_i}-\overline{\psi}^{\mathrm{R/L}}_{q,i}\frac{\delta}{\delta(\overline{\psi}^{\mathrm{R/L}}_{q,j})^{\mathrm{T}}}\right]. \tag{2.19}$$

Thus, the field renormalization constants of the quark fields are in general gauge-parameter dependent, which is a well-known fact. The field renormalization constants $Z^{q,\mathrm{L/R}}$ are in general complex $N\times N$ matrices. The field-number operator $\mathcal{N}^{q,\mathrm{L/R}}_{ij}$ commutes with the ST operator [cf. (Eq. 2.10)].

Equivalently, field renormalization can be introduced using the field redefinitions

$$\begin{aligned}q^{\mathrm{L/R}}_i&\to Z^{q,\mathrm{L/R}}_{ij}q^{\mathrm{L/R}}_j=(\delta_{ij}+\delta Z^{q,\mathrm{L/R}}_{ij})q^{\mathrm{L/R}}_j,\\ \overline{\psi}^{\mathrm{R/L}}_{q,i}&\to\overline{\psi}^{\mathrm{R/L}}_{q,j}(Z^{q,\mathrm{L/R}})^{-1}_{ji}=\overline{\psi}^{\mathrm{R/L}}_{q,j}(\delta_{ji}-\delta Z^{q,\mathrm{L/R}}_{ji}),\end{aligned} \tag{2.20}$$

and the one obtained by Hermitian adjungation. The ST operator is invariant under these transformations. However, the rigid transformations are not invariant under Eqs. (2.20) as will be stressed in the next section.

(ii) On the other hand, the counterterms corresponding to the parameters θ_n of the QMM and the quark masses $m_{q,k}$ cannot be written in form of a $\mathbf{s}_{\Gamma_{\rm cl}}$ variation. Therefore, they are genuinely gauge-parameter-independent quantities. These counterterms read

$$\Gamma^{\theta_n}_{\text{BRS-inv}}=\delta\theta_n\frac{\partial}{\partial\theta_n}\Gamma_{\rm cl}\,,\quad \Gamma^{m_{q,k}}_{\text{BRS-inv}}=\delta m_{q,k}\frac{\partial}{\partial m_{q,k}}\Gamma_{\rm cl}\,, \tag{2.21}$$

and can be introduced by parameter redefinitions

$$\theta_n\to\theta_n+\delta\theta_n\,,\quad m_{q,k}\to m_{q,k}+\delta m_{q,k}\,, \tag{2.22}$$

with

$$\partial_\xi\delta\theta_n=\partial_\xi\delta m_{q,k}=0. \tag{2.23}$$

III. RESTRICTIONS FROM RIGID $\mathrm{SU}(2)_\mathrm{L}$ INVARIANCE

In this section, we want to construct all invariant counterterms of the SM relevant for our case. Invariant counterterms correspond to local field operators that respect the defining symmetries of the underlying model. The relevant symmetries for the renormalization of the QMM in the SM are the extended BRS symmetry as discussed in the previous section but also the (spontaneously broken) rigid $\mathrm{SU}(2)_\mathrm{L}$ gauge symmetry. The latter is usually expressed in form of a Ward identity. In the following, we construct the invariant counterterms from the BRS-invariant counterterms by requiring rigid $SU(2)_L$ gauge symmetry in addition. Like in the case of the BRS-invariant counterterms [cf. Eqs. (2.10)], we define the invariant counterterms by invariant operators that commute with both the ST and Ward operators. Finally, we introduce finite field redefinitions for the quark fields, which leads to a renormalization of $SU(2)_L$ gauge symmetry. We show that these new parameters are required to impose on-shell renormalization conditions for the quark fields.

Before discussing the invariant counterterms, we would like to comment on symmetry breaking resulting from the chosen regularization. If we assume an invariant regularization scheme for the renormalization in the SM, we need only invariant counterterms for the renormalization procedure. Unfortunately for the SM there is no regularization method known which respects all symmetries owing to the so-called γ_5 problem. Therefore, we require that the symmetry breaking owing to the regularization be restored in a first step by introducing symmetry-restoring counterterms. These counterterms need not respect the symmetries of the underlying model and can also include UV divergences. In the following, we assume that this has been done and the symmetries are restored. Hence, the remaining UV divergences respect the symmetry identities and can be absorbed by invariant counterterms only.

The rigid $SU(2)_L$ Ward identities take the form

$$\mathcal{W}_a\Gamma=\int d^4x\,\delta_a^{\mathrm{rig}}\phi_k\frac{\delta\Gamma}{\delta\phi_k}=0+\mathcal{O}(\chi),\quad a=1,2,3,4, \tag{3.1}$$

where ϕ_k runs over all fields. The Ward operators $\mathcal{W}_a$, $a=1,2,3$, in Eq. (3.1) respect the algebra of the $SU(2)_L$ group,

$$[\mathcal{W}_a,\mathcal{W}_b]=\mathrm{i}\epsilon_{abc}\mathcal{W}_c,\quad a,b,c=1,2,3, \tag{3.2}$$

and commute with the Ward operator $\mathcal{W}_4$ of $U(1)_Y$ symmetry of hypercharge. The Ward identity has to be fulfilled by the χ-independent part of Γ, while the unphysical part involving the field χ need not be rigidly invariant. Using the definition

$$\mathcal{W}_\pm=\frac{1}{\sqrt{2}}(\mathcal{W}_1\pm\mathrm{i}\mathcal{W}_2), \tag{3.3}$$

in the classical approximation the rigid transformations for $\mathcal{W}_\pm$ take the form

$$\delta_+^{\mathrm{rig}}u_i^{\mathrm{L}}=\frac{\mathrm{i}}{\sqrt{2}}V_{ij}d_j^{\mathrm{L}},\quad \delta_+^{\mathrm{rig}}\overline{u}_i^{\mathrm{L}}=0,$$

$$\delta_-^{\mathrm{rig}}u_i^{\mathrm{L}}=0,\quad \delta_-^{\mathrm{rig}}\overline{u}_i^{\mathrm{L}}=-\frac{\mathrm{i}}{\sqrt{2}}\overline{d}_j^{\mathrm{L}}V_{ji}^\dagger,$$

$$\delta_+^{\mathrm{rig}}d_i^{L}=0,\quad \delta_+^{\mathrm{rig}}\overline{d}_i^{\mathrm{L}}=-\frac{\mathrm{i}}{\sqrt{2}}\overline{u}_j^{\mathrm{L}}V_{ji},$$

$$\delta_-^{\mathrm{rig}}d_i^{\mathrm{L}}=\frac{\mathrm{i}}{\sqrt{2}}V_{ij}^\dagger u_j^{\mathrm{L}},\quad \delta_-^{\mathrm{rig}}\overline{d}_i^{\mathrm{L}}=0,$$

$$\delta_+^{\mathrm{rig}}\psi_{\mathrm{u},i}^{\mathrm{R}}=\frac{\mathrm{i}}{\sqrt{2}}V_{ij}\psi_{\mathrm{d},j}^{\mathrm{R}},\quad \delta_+^{\mathrm{rig}}\overline{\psi}_{\mathrm{u},i}^{\mathrm{R}}=0,$$

$$\delta_-^{\mathrm{rig}}\psi_{\mathrm{u},i}^{\mathrm{R}}=0,\quad \delta_-^{\mathrm{rig}}\overline{\psi}_{\mathrm{u},i}^{\mathrm{R}}=-\frac{\mathrm{i}}{\sqrt{2}}\overline{\psi}_{\mathrm{d},j}^{\mathrm{R}}V_{ji}^\dagger,$$

$$\delta_+^{\mathrm{rig}}\psi_{\mathrm{d},i}^{\mathrm{R}}=0,\quad \delta_+^{\mathrm{rig}}\overline{\psi}_{\mathrm{d},i}^{\mathrm{R}}=-\frac{\mathrm{i}}{\sqrt{2}}\overline{\psi}_{\mathrm{u},j}^{\mathrm{R}}V_{ji},$$

$$\delta_-^{\mathrm{rig}}\psi_{\mathrm{d},i}^{\mathrm{R}}=\frac{\mathrm{i}}{\sqrt{2}}V_{ij}^\dagger\psi_{\mathrm{u},j}^{\mathrm{R}},\quad \delta_-^{\mathrm{rig}}\overline{\psi}_{\mathrm{d},i}^{\mathrm{R}}=0. \tag{3.4}$$

The rigid transformations for the auxiliary fields ψ_q^{R} are defined such that the ST operator is invariant under Eqs. (3.4). The rigid transformations of the other fields take their usual form (see, e.g., Ref. [16]).

Counterterms that respect both the ST identity *and* the Ward identities (3.1) of rigid $SU(2)_L$ symmetry in the classical form (3.4) are called *invariant* counterterms. Like in Sec. II, we discuss the gauge-parameter-dependent and -independent counterterms separately.

(i) In order to generate quark-field counterterms invariant under BRS *and* rigid symmetries, we search for all field redefinitions (2.20) that leave the rigid Ward identities invariant. While the Ward operators $\mathcal{W}_{3,4}$ are not affected by the replacements (2.20), $\mathcal{W}_\pm$ in Eqs. (3.4) are in general modified by Eqs. (2.20). Requiring that the rigid transformations (3.4) be invariant under the replacement (2.20), the field renormalizations (2.20) of the left-handed fields are restricted by $(Z_{\mathrm{inv}}^{\mathrm{u,L}})^{-1}VZ_{\mathrm{inv}}^{\mathrm{d,L}}=V$, resulting in

$$Z_{\mathrm{inv}}^{\mathrm{d,L}}=V^\dagger Z_{\mathrm{inv}}^{\mathrm{u,L}}V. \tag{3.5}$$

Since the field renormalizations of the right-handed fields are not restricted by rigid invariance, the corresponding BRS-invariant counterterms are also invariant counterterms—i.e., $Z_{\mathrm{inv}}^{q,\mathrm{R}}=Z^{q,\mathrm{R}}$. Furthermore, the operators $\mathcal{N}_{ij}^{q,\mathrm{L/R}}$ corresponding to the invariant counterterms $Z_{\mathrm{inv}}^{q,\mathrm{L/R}}$ commute with the Ward operators.

(ii) The renormalization operators $\delta m_{q,k}\partial/\partial m_{q,k}$ of Eqs. (2.21) commute with the ST and Ward operators and, hence, generate invariant counterterms:

$$\Gamma_{\mathrm{inv}}^{m_{q,k}}=\delta m_{q,k}\frac{\partial}{\partial m_{q,k}}\Gamma_{\mathrm{cl}}. \tag{3.6}$$

On the other hand, the operators that correspond to the renormalization of the angles and phases of the QMM do not commute with the Ward operators $\mathcal{W}_\pm$. In order to disentangle the counterterms to the QMM from the counterterms of the field renormalizations, we construct the invariant operators that commute with both the ST operator and the Ward operators as

$$\mathcal{D}_{\theta_n}=\frac{\partial}{\partial\theta_n}+\left[\frac{1}{2}\mathcal{N}^{\mathrm{u,L}}_{ij}\frac{\partial V_{ik}}{\partial\theta_n}V^{\dagger}_{kj}-\frac{1}{2}\mathcal{N}^{\mathrm{d,L}}_{ij}V^{\dagger}_{ik}\frac{\partial V_{kj}}{\partial\theta_n}+\mathrm{H.c.}\right]. \tag{3.7}$$

These operators define the gauge-parameter-independent, invariant counterterms to the QMM:

$$\Gamma^{\theta_n}_{\mathrm{inv}}=\delta\theta_n\mathcal{D}_{\theta_n}\Gamma_{\mathrm{cl}}, \tag{3.8}$$

where the parameters θ_n run over the angles and phases of the QMM. This renormalization includes both renormalization transformations of the mixing angles and field renormalizations,

$$\begin{aligned}
&\theta_n\to\theta_n+\delta\theta_n,\\
&u^{\mathrm{L}}_i\to\left[\delta_{ij}+\frac{1}{2}\delta\theta_n(\partial_{\theta_n}V_{ik})V^{\dagger}_{kj}\right]u^{\mathrm{L}}_j,\\
&d^{\mathrm{L}}_i\to\left[\delta_{ij}-\frac{1}{2}\delta\theta_n V^{\dagger}_{ik}(\partial_{\theta_n}V_{kj})\right]d^{\mathrm{L}}_j,\\
&\overline{\psi}^{\mathrm{R}}_{\mathrm{u},i}\to\overline{\psi}^{\mathrm{R}}_{\mathrm{u},j}\left[\delta_{ji}-\frac{1}{2}\delta\theta_n(\partial_{\theta_n}V_{jk})V^{\dagger}_{ki}\right],\\
&\overline{\psi}^{\mathrm{R}}_{\mathrm{d},i}\to\overline{\psi}^{\mathrm{R}}_{\mathrm{d},j}\left[\delta_{ji}+\frac{1}{2}\delta\theta_n V^{\dagger}_{jk}(\partial_{\theta_n}V_{ki})\right],
\end{aligned} \tag{3.9}$$

and the corresponding Hermitian adjoint transformations.

Both $\delta m_{q,k}$ and $\delta\theta_n$ are genuine gauge-parameter-independent counterterms.

The relation (3.5) and the renormalization of the QMM (3.9) restrict the UV-divergence structure of the invariant counterterms.

The remaining field renormalization parameters of Eqs. (2.20)—i.e., those that do not respect Eq. (3.5)—belong to a new type of renormalization constants—namely, finite field redefinitions.

(iii) We introduce finite field redefinitions for the left-handed down-type quarks since $\delta Z^{\mathrm{d,L}}_{\mathrm{inv}}$ is constrained by Eq. (3.5). The finite field redefinitions of the down-type quarks compatible with the ST identity read

$$\begin{aligned}
&d^{\mathrm{L}}_i\to R^{\mathrm{fin}}_{ij}d^{L}_j=(\delta_{ij}+\delta R^{\mathrm{fin}}_{ij})d^{\mathrm{L}}_j,\\
&\overline{\psi}^{\mathrm{R}}_{\mathrm{d},i}\to\overline{\psi}^{\mathrm{R}}_{\mathrm{d},j}(R^{\mathrm{fin}})^{-1}_{ji}=\overline{\psi}^{\mathrm{R}}_{\mathrm{d},j}(\delta_{ji}-\delta R^{\mathrm{fin}}_{ji}),
\end{aligned} \tag{3.10}$$

with an arbitrary complex $N\times N$ matrix δR^{fin}. Since the replacement (3.10) is done everywhere—i.e., in the action, in the ST operator, and in the Ward operators—it does not disturb the validity of the symmetry requirements of the theory. While the ST operator and $\mathcal{W}_{3,4}$ stay unchanged, the renormalized rigid transformations corresponding to $\mathcal{W}_{\pm}$ are modified. The renormalized Ward operators $\mathcal{W}_{\pm}$ are obtained from Eq. (3.4) by the substitution (3.10). As shown in Sec. II, these field redefinitions are in general gauge-parameter dependent. Furthermore, these renormalization constants do not include UV divergences and are only needed to satisfy on-shell renormalization conditions.

With the so-defined invariant counterterms and finite field redefinitions, we are able to define a more convenient set of renormalization constants:

$$\begin{aligned}
&V_{ij}\to V_{ij}+\delta V_{ij},\\
&q^{\mathrm{L/R}}_i\to Z^{q,\mathrm{L/R}}_{ij}q^{\mathrm{L/R}}_j=(\delta_{ij}+\delta Z^{q,\mathrm{L/R}}_{ij})q^{\mathrm{L/R}}_j,\\
&\overline{\psi}^{\mathrm{R/L}}_i\to\overline{\psi}^{\mathrm{R/L}}_j(Z^{q,\mathrm{L/R}})^{-1}_{ji}=\overline{\psi}^{\mathrm{R/L}}_j(\delta_{ji}-\delta Z^{q,\mathrm{L/R}}_{ji}),
\end{aligned} \tag{3.11}$$

with

$$\begin{aligned}
&\delta V=\delta\theta_n(\partial_{\theta_n}V),\\
&\delta Z^{\mathrm{u,L}}(\xi)=\delta Z^{\mathrm{u,L}}_{\mathrm{inv}}(\xi)+\frac{1}{2}\delta\theta_n(\partial_{\theta_n}V)V^{\dagger},\\
&\delta Z^{\mathrm{d,L}}(\xi)=V^{\dagger}\delta Z^{\mathrm{u,L}}_{\mathrm{inv}}(\xi)V-\frac{1}{2}\delta\theta_n V^{\dagger}(\partial_{\theta_n}V)+\delta R^{\mathrm{fin}}(\xi),\\
&\delta Z^{q,\mathrm{R}}(\xi)=\delta Z^{q,\mathrm{R}}_{\mathrm{inv}}(\xi).
\end{aligned} \tag{3.12}$$

In Eqs. (3.12) we have indicated the gauge-parameter dependence of the renormalization constants explicitly. The definition of the renormalization constant δV in Eqs. (3.12) implies that the renormalized QMM stays unitary in all orders by construction as is required by BRS invariance of the theory.

Using Eqs. (3.12) we can express the UV divergences of the QMM in terms of UV divergences of left-handed field redefinitions:

$$\begin{aligned}
&\delta V=\delta Z^{\mathrm{u,L}}V-V\delta Z^{\mathrm{d,L}}+\text{finite terms},\\
&\delta V=-(\delta Z^{\mathrm{u,L}})^{\dagger}V+V(\delta Z^{\mathrm{d,L}})^{\dagger}+\text{finite terms},
\end{aligned} \tag{3.13}$$

where we used $(\partial_{\theta_n}V)V^{\dagger}=-V(\partial_{\theta_n}V^{\dagger})$. As the UV divergences satisfy also the extended ST identity, the gauge-parameter-dependent part of field redefinitions cancels in Eqs. (3.13) in such a way that the UV divergences of the QMM are gauge independent. Since in the $\overline{\mathrm{MS}}$ scheme only the UV-divergent parts and gauge-independent constants are subtracted, δV can be consistently determined by $\overline{\mathrm{MS}}$ field renormalizations:

$$\begin{aligned}
\delta V^{\overline{\mathrm{MS}}}\overset{!}{=}-\frac{1}{2}\{&[(\delta Z^{\mathrm{u,L},\overline{\mathrm{MS}}})^{\dagger}-\delta Z^{\mathrm{u,L},\overline{\mathrm{MS}}}]V\\
&+V[\delta Z^{\mathrm{d,L},\overline{\mathrm{MS}}}-(\delta Z^{\mathrm{d,L},\overline{\mathrm{MS}}})^{\dagger}]\}.
\end{aligned} \tag{3.14}$$

From Eqs. (3.12) it can be seen that the renormalization scheme which uses an $\overline{\mathrm{MS}}$ subtraction for δV but on-shell conditions for $\delta Z^{q,\mathrm{L/R}}$ is fully consistent [5,6], yielding, however, contrary to the pure $\overline{\mathrm{MS}}$ scheme renormalized Ward identities of rigid symmetry.

In Refs. [3], [7], the relation (3.14) is used as a renormalization condition also for the finite parts,[1]

$$\delta V \overset{!}{=} -\frac{1}{2}\{[(\delta Z^{\mathrm{u,L}})^\dagger - \delta Z^{\mathrm{u,L}}]V + V[\delta Z^{\mathrm{d,L}} - (\delta Z^{\mathrm{d,L}})^\dagger]\}, \tag{3.15}$$

which is equivalent to

$$\delta R^{\mathrm{fin}} - (\delta R^{\mathrm{fin}})^\dagger \overset{!}{=} 0. \tag{3.16}$$

This renormalization condition sets the anti-Hermitian part of the renormalization constants $\delta R^{\mathrm{fin}}(\xi)$ to zero. In order to satisfy the on-shell conditions, $\delta Z^{\mathrm{u,L}}$ and $\delta Z^{\mathrm{d,L}}$ are needed as independent counterterms at our disposal (see Sec. IV). As Eq. (3.12) shows, the gauge-dependent parts of $\delta Z^{\mathrm{u,L}}(\xi)$ and $\delta Z^{\mathrm{d,L}}(\xi)$ can in general be absorbed in $\delta Z^{\mathrm{u,L}}_{\mathrm{inv}}(\xi)$ and $\delta R^{\mathrm{fin}}(\xi)$. However, if $\delta R^{\mathrm{fin}} - (\delta R^{\mathrm{fin}})^\dagger = 0$, the renormalization condition (3.15) requires one to adjust $\delta\theta_n$. This leads in general to a gauge-parameter-dependent $\delta\theta_n$, which is inconsistent with the extended ST identity. For this reason the renormalization condition (3.15) yields gauge-parameter-dependent results for physical matrix elements, as has been confirmed by an explicit one-loop calculation in Ref. [8].

To circumvent these problems, the authors of Ref. [8] give up the complete on-shell conditions, but propose renormalization conditions respecting the Ward identity (3.1) in its classical form—i.e., $\delta R^{\mathrm{fin}} = 0$—and resulting in gauge-parameter-independent counterterms to QMM by using fermion-field renormalization constants fixed at zero momentum. In this scheme, gauge independence of the QMM has been confirmed by an explicit one-loop calculation, but could not be confirmed to all orders.

In Ref. [8] the motivation for constructing a scheme in accordance with the classical Ward identity is based on the theorem that any renormalization prescription that preserves the rigid Ward identity in its classical form leads to a gauge-parameter-independent definition of the QMM. Note that this theorem requires the assumptions that the extended ST identity is satisfied and that the Ward operator commutes with the extended ST operator, as can be seen in the proof of this theorem in Ref. [8]. However, in this form the theorem of Ref. [8] is only of little use, especially in the on-shell scheme where the Ward identity has to be renormalized as discussed before. Preserving the rigid Ward identity means to take $\delta R^{\mathrm{fin}}(\xi) = 0$. Then, the rigid Ward identity is preserved in its classical form for any $\delta\theta_n$ irrespective of its gauge-parameter dependence. As we have shown in Sec. II, it is the extended BRS invariance alone which controls gauge-parameter dependence or independence of Green functions and S-matrix elements and which yields in the present case the gauge independence of counterterms to the QMM.

For this reason we do not establish conditions which are motivated by rigid invariance like Eq. (3.15) or the scheme of Ref. [8] but fix the counterterms of the QMM directly on physical matrix elements (see Sec. V).

[1]Note that our notation differs from the one of Ref. [7] by a factor of 2 in the definition of the field renormalization constants.

IV. PARAMETERS OF THE QUARK-MIXING MATRIX

Before we turn to the definition of a physical renormalization condition for the QMM in Sec. V, we investigate the different types of parameters included in the QMM and the field redefinitions $Z^{q,\mathrm{L/R}}$, and study how these parameters contribute to the bilinear action and to the $\mathrm{W}^+\overline{\mathrm{u}}_i\mathrm{d}_j$ vertex.

The SM allows the generalized field redefinitions (3.11) of the quark fields where $Z^{q,\mathrm{L/R}}$ are general complex $N\times N$ matrices. A general complex matrix can be decomposed into a Hermitian and a unitary matrix,

$$Z^{q,\mathrm{L/R}} = U^{q,\mathrm{L/R}} H^{q,\mathrm{L/R}}, \tag{4.1}$$

with

$$(H^{q,\mathrm{L,R}})^\dagger = H^{q,\mathrm{L/R}}, \quad (U^{q,\mathrm{L/R}})^\dagger U^{q,\mathrm{L/R}} = 1. \tag{4.2}$$

Applying the field redefinitions (3.11) to the bilinear part of the classical action (2.1) yields

$$\Gamma^{\text{field-red}}_{\text{bil}} = \int d^4x \sum_{q=u,d} [\mathrm{i}\overline{q}^{\mathrm{L}}_i X^{q,\mathrm{L}}_{ij} \partial\!\!\!/ q^{\mathrm{L}}_i + \mathrm{i}\overline{q}^{\mathrm{R}}_i X^{q,\mathrm{R}}_{ij} \partial\!\!\!/ q^{\mathrm{R}}_i + (\overline{q}^{\mathrm{L}}_i M^q_{ij} q^{\mathrm{R}}_j + \mathrm{H.c.})], \tag{4.3}$$

with

$$X^{q,\mathrm{L/R}} = (H^{q,\mathrm{L/R}})^\dagger H^{q,\mathrm{L/R}},$$
$$M^q = (H^{q,\mathrm{L}})^\dagger (U^{q,\mathrm{L}})^\dagger M^q_{\mathrm{diag}} U^{q,\mathrm{R}} H^{q,\mathrm{R}}, \tag{4.4}$$

and $M^q_{\mathrm{diag}} = \mathrm{diag}(m_{q,1},\ldots,m_{q,N})$.

Inspecting this result, we see that the Hermitian parts of the matrices $Z^{q,\mathrm{L}}$ and $Z^{q,\mathrm{R}}$ can be determined on kinetic terms, while the unitary parts can be fixed on mass terms up to a common complex diagonal matrix $\mathrm{diag}[\exp(\mathrm{i}\tilde{\varphi}^q_1),\ldots,\exp(\mathrm{i}\tilde{\varphi}^q_N)]$, which can be extracted as

$$U^{q,\mathrm{L}}_{ij} = \sum_{k=1}^N e^{\mathrm{i}\tilde{\varphi}^q_i}\delta_{ik}\tilde{U}^{q,\mathrm{L}}_{kj}, \quad U^{q,\mathrm{R}}_{ij} = \sum_{k=1}^N e^{\mathrm{i}\tilde{\varphi}^q_i}\delta_{ij}\tilde{U}^{q,\mathrm{R}}_{kj}. \tag{4.5}$$

The diagonal matrix can be parametrized as

$$e^{\mathrm{i}\tilde{\varphi}^q_i}\delta_{ij} = \left[\exp\left(\mathrm{i}\sum_{n=1}^{N-1}\varphi^q_n T^{\mathrm{diag}}_n\right)\exp(\mathrm{i}\varphi^q_0 T^{\mathrm{diag}}_0)\right]_{ij}, \tag{4.6}$$

where the matrices T^{diag}_n denote the $N-1$ traceless diagonal generators of $\mathrm{SU}(N)$ and $T^{\mathrm{diag}}_0 = 1/\sqrt{2N}$.

The number of physical parameters of the QMM for N quark families is determined as follows: BRS invariance implies that the QMM is a general unitary matrix. A general unitary matrix Y has N^2 parameters. We can use the diagonal matrices that are not fixed on bilinear terms to remove $2N-1$ phases from Y and obtain the usual form of the QMM including only physical parameters:

$$
\begin{aligned}
V&=\exp\left(-\mathrm{i}\sum_{n=1}^{N-1}\varphi_n^{\mathrm{u}}T_n^{\mathrm{diag}}\right)\exp(-\mathrm{i}\varphi_0^{\mathrm{u}}T_0^{\mathrm{diag}})Y\\
&\quad\times\exp\left(\mathrm{i}\sum_{n=1}^{N-1}\varphi_n^{\mathrm{d}}T_n^{\mathrm{diag}}\right)\exp(\mathrm{i}\varphi_0^{\mathrm{d}}T_0^{\mathrm{diag}})\\
&=\exp(-\mathrm{i}(\varphi_0^{\mathrm{u}}-\varphi_0^{\mathrm{d}})T_0^{\mathrm{diag}})\exp\left(-\mathrm{i}\sum_{n=1}^{N-1}\varphi_n^{\mathrm{u}}T_n^{\mathrm{diag}}\right)Y\\
&\quad\times\exp\left(\mathrm{i}\sum_{n=1}^{N-1}\varphi_n^{\mathrm{d}}T_n^{\mathrm{diag}}\right),
\end{aligned}
\tag{4.7}
$$

where we used the fact that T_0^{diag} is the only generator that commutes with all generators of the SU(N) group. Thus, the QMM has $N^2-(2N-1)=(N-1)^2$ physical parameters. These consist of $N(N-1)/2$ angles ϑ_m of an $N\times N$ orthogonal matrix and of $(N-1)(N-2)/2$ complex phase factors $\exp(\mathrm{i}\varphi_l)$.

Similar to the decomposition (4.1) of the field redefinition matrices, the corresponding counterterms $\delta Z^{q,\mathrm{L/R}}$ can be written as a linear combination of Hermitian and anti-Hermitian matrices:

$$
\delta Z^{q,\mathrm{L/R}}=\sum_{n=0}^{N^2-1}\delta z_n^{q,\mathrm{L/R}}T_n, \tag{4.8}
$$

where $\delta z_n^{q,\mathrm{L/R}}$ are complex numbers and $T_n,\ n=1,\ldots,N^2-1$ are the generators of SU(N) and $T_0=\mathbf{1}/\sqrt{2N}$. In the same way as we have discussed for the matrices $Z^{q,\mathrm{L/R}}$ in the beginning of this section, not all of the $\delta z_n^{q,\mathrm{L}}$ and $\delta z_n^{q,\mathrm{R}}$ can be determined on the bilinear terms:

$$
\begin{aligned}
\delta\Gamma_{\mathrm{bil}}^{\text{field-red}}&=\int \mathrm{d}^4x\sum_{q=u,d}[\mathrm{i}\bar{q}_i^{\mathrm{L}}\delta X_{ij}^{q,\mathrm{L}}\partial\!\!\!/ q_i^{\mathrm{L}}+\mathrm{i}\bar{q}_i^{\mathrm{R}}\delta X_{ij}^{q,\mathrm{R}}\partial\!\!\!/ q_i^{\mathrm{R}}\\
&\quad+(\bar{q}_i^{\mathrm{L}}\delta M_{ij}^{q}q_j^{\mathrm{R}}+\mathrm{H.c.})],
\end{aligned}
\tag{4.9}
$$

with

$$
\begin{aligned}
\delta X^{q,\mathrm{L/R}}&=(\delta Z^{q,\mathrm{L/R}})^\dagger+\delta Z^{q,\mathrm{L/R}},\\
\delta M^{q}&=(\delta Z^{q,\mathrm{L}})^\dagger M_{\mathrm{diag}}^{q}+M_{\mathrm{diag}}^{q}\delta Z^{q,\mathrm{R}}.
\end{aligned}
\tag{4.10}
$$

Common imaginary parts of the coefficients $\delta z_n^{q,\mathrm{L}}$ and $\delta z_n^{q,\mathrm{R}}$ corresponding to the diagonal generators T_n^{diag} remain as free parameters. Splitting off these free parameters, we obtain

$$
\begin{aligned}
\delta Z^{q,\mathrm{L}}&=\delta\tilde{Z}^{q,\mathrm{L}}+\mathrm{i}\sum_{n=0}^{N-1}c_n^{q}T_n^{\mathrm{diag}},\\
\delta Z^{q,\mathrm{R}}&=\delta\tilde{Z}^{q,\mathrm{R}}+\mathrm{i}\sum_{n=0}^{N-1}c_n^{q}T_n^{\mathrm{diag}},
\end{aligned}
\tag{4.11}
$$

with free real parameters c_n^q, and $\delta\tilde{Z}^{q,\mathrm{L}}$ and $\delta\tilde{Z}^{q,\mathrm{R}}$ having fixed values for the imaginary parts of the diagonal entries. One can choose for example real diagonal entries for the left-handed field renormalization constants $\delta\tilde{Z}_{ii}^{q,\mathrm{L}}=(\delta\tilde{Z}_{ii}^{q,\mathrm{L}})^*$.

Now we turn to the counterterms of the QMM. Owing to BRS invariance $(V+\delta V)$ is a unitary matrix. The unitarity constraint reads

$$
(V+\delta V)(V+\delta V)^\dagger=(V+\delta V)^\dagger(V+\delta V)=1, \tag{4.12}
$$

which implies

$$
\delta VV^\dagger+V\delta V^\dagger=-\delta V\delta V^\dagger,\quad \delta V^\dagger V+V^\dagger\delta V=-\delta V^\dagger\delta V. \tag{4.13}
$$

These equations are implicitly solved by decomposing $\delta VV^\dagger$ into a Hermitian and an anti-Hermitian part:

$$
\delta V=-\frac{1}{2}\delta V\delta V^\dagger V+\delta\tilde{V}\quad\text{with}\quad \delta\tilde{V}V^\dagger=-(\delta\tilde{V}V^\dagger)^\dagger. \tag{4.14}
$$

Equation (4.14) can be solved perturbatively for δV once $\delta\tilde{V}$ is given. Therefore, $\delta\tilde{V}$ has the same number of independent parameters as δV or V—namely, $(N-1)^2$.

In order to formulate a renormalization condition for the QMM, we investigate the $\mathrm{W}^+\bar{\mathrm{u}}_i\mathrm{d}_j$ vertex. Including counterterms, this vertex reads

$$
\Gamma_{\mathrm{W\bar{u}d}}=\frac{e}{\sqrt{2}s_{\mathrm{w}}}\int d^4x\,\bar{u}_i^{\mathrm{L}}\gamma^\mu W_\mu^+(V+\delta F_{\mathrm{ct}})_{ij}d_j^{\mathrm{L}}, \tag{4.15}
$$

with the matrix

$$
\delta F_{\mathrm{ct}}=V\left(\delta Z_{\mathrm{W}}+\frac{\delta e}{e}-\frac{\delta s_{\mathrm{w}}}{s_{\mathrm{w}}}\right)+(\delta Z^{\mathrm{u,L}})^\dagger V+V\delta Z^{\mathrm{d,L}}+\delta V. \tag{4.16}
$$

Inserting the decomposition (4.11) as well as Eq. (4.14), this can be written as

$$
\begin{aligned}
\delta F_{\mathrm{ct}}&=V\left(\delta Z_{\mathrm{W}}+\frac{\delta e}{e}-\frac{\delta s_{\mathrm{w}}}{s_{\mathrm{w}}}\right)+(\delta\tilde{Z}^{\mathrm{u,L}})^\dagger V+V\delta\tilde{Z}^{\mathrm{d,L}}\\
&\quad-\frac{1}{2}\delta V\delta V^\dagger V+\delta\tilde{Y},
\end{aligned}
\tag{4.17}
$$

where we defined

$$
\delta\tilde{Y}=-\mathrm{i}\sum_{n=1}^{N^2-1}(c_n^{\mathrm{u}}T_n^{\mathrm{diag}}V-Vc_n^{\mathrm{d}}T_n^{\mathrm{diag}})-\mathrm{i}(c_0^{\mathrm{u}}-c_0^{\mathrm{d}})V+\delta\tilde{V}. \tag{4.18}
$$

While δZ_{W}, δe, δs_{w}, and $\delta\tilde{Z}^{q,\mathrm{L}}$ are fixed from other vertex functions, we have the N^2 parameters of the matrix $\delta\tilde{Y}$ at our disposal for renormalization conditions of the QMM: $(N-1)^2$ free parameters from $\delta\tilde{V}$, $2(N-1)$ real constants c_n^q from the traceless diagonal generators T_n^{diag}, and one real constant c_0^q from $T_0^{\rm diag}$. These parameters are just sufficient to fix a general unitary matrix. For later use we note that Eqs. (4.18) and (4.14) imply the anti-Hermiticity of $\delta\widetilde{Y}V^\dagger$:

$$\delta\widetilde{Y}V^\dagger = -(\delta\widetilde{Y}V^\dagger)^\dagger. \tag{4.19}$$

V. PHYSICAL RENORMALIZATION OF THE QUARK-MIXING MATRIX

In the previous section we found that the field renormalization constant $\delta Z^{q,\mathrm{L/R}}$ can be determined on the quark self-energies only up to some unphysical phases. These phases can be used to extend the QMM to a general unitary matrix. In this section we formulate a physical renormalization condition for this unitary matrix and thus for the QMM.

As we have already seen in Sec. II, the counterterms to the QMM are genuinely gauge-parameter independent. In order not to introduce an artificial gauge-parameter dependence, the renormalization conditions have to be chosen properly. If gauge-parameter-independent matrix elements that involve the QMM are available, these matrix elements can be used to determine δV. If we ignore for the moment the instability of the W bosons and the quarks, such matrix elements are those of the decays $\mathrm{W}^+\to \mathrm{u}_i\overline{\mathrm{d}}_j$ or $\overline{\mathrm{u}}_i\to \mathrm{W}^-\overline{\mathrm{d}}_j$ if a top quark is involved. Both types of decays are related by crossing symmetry.

Before we come to the actual renormalization condition, we want to add some comments on the difficulties related to unstable particles. Matrix elements to the decays $\mathrm{W}^+\to \mathrm{u}_i\overline{\mathrm{d}}_j$ or $\overline{\mathrm{u}}_i\to \mathrm{W}^-\overline{\mathrm{d}}_j$ suffer from the fact that the external particles are unstable. Contrary to stable particles it is not known how to construct gauge-parameter-independent matrix elements with unstable particles at external legs. Nevertheless, we use these matrix elements for a renormalization condition for the QMM. The problem related to the instability of the external particles manifests itself in contributions of the order of the decay width of these particles, which we cannot control. Several attempts to obtain gauge-parameter-independent matrix elements involving internal or external unstable particles have been undertaken in the literature (see, e.g., Refs. [22–25]). For the case where no gauge-parameter-independent matrix element is available, a fully consistent prescription has been given in Ref. [13], which defines renormalization conditions for gauge-parameter-independent counterterms on arbitrary gauge-parameter-dependent Green functions. In this section we treat the external particles as stable and ignore the problems related to their finite decay widths. This approach implies that absorptive parts should be disregarded in the following results. Some remarks on the absorptive parts are made at the end of this section and in the Appendix.

Following closely the notation of Ref. [7], the lowest-order matrix element for the decay $\mathrm{W}^+\to \mathrm{u}_i\overline{\mathrm{d}}_j$ reads

$$\mathcal{M}_{0,ij} = V_{ij}\mathcal{M}^-_{1,ij}, \tag{5.1}$$

with the standard matrix element

$$\mathcal{M}^-_{1,ij} = -\frac{e}{\sqrt{2}s_\mathrm{w}}\overline{u}(p_{\mathrm{u},i})\not{\varepsilon}(p_\mathrm{W})\omega_- v(p_{\mathrm{d},j}), \tag{5.2}$$

and the chiral projector $\omega_- = (1-\gamma_5)/2$. The matrix element including radiative corrections can be written as

$$\mathcal{M}_{ij} = \sum_{a=1}^{2}\sum_{\sigma=\pm} F^\sigma_{a,ij}\mathcal{M}^\sigma_{a,ij}, \tag{5.3}$$

with four standard matrix elements $\mathcal{M}^\sigma_a$. The corresponding four gauge-invariant form factors F^σ_a are functions of M^2_W, $m^2_{\mathrm{u},i}$, and $m^2_{\mathrm{d},j}$. The form factor F^-_1, the only one appearing at lowest order and thus involving overall UV divergences and counterterms, can be decomposed as

$$F^-_1 = V + \delta F^-_{\mathrm{loop},1} + \delta F_{\mathrm{ct}} = V + \sum_{l\geqslant 1}(\delta F^{-(l)}_{\mathrm{loop},1} + \delta F^{(l)}_{\mathrm{ct}}), \tag{5.4}$$

where $\delta F^{-(l)}_{\mathrm{loop},1}$ summarizes the lth-order loop contributions including also counterterm insertions of lower orders and $\delta F^{(l)}_{\mathrm{ct}}$ includes the lth-order overall counterterms as defined in Eq. (4.16).

Since the form factor F^-_1 depends on the counterterms of the QMM, it can be used to define a proper renormalization condition for the QMM. Owing to loop corrections this form factor is a general complex $N\times N$ matrix. We decompose[2] this form factor into a unitary matrix Y and a Hermitian matrix H:

$$F^-_1 = HY \quad \text{with} \quad H^\dagger = H, \quad Y^\dagger Y = 1. \tag{5.5}$$

As discussed at the end of Sec. IV, we have enough parameters at our disposal for the renormalization of the QMM to fix a general unitary matrix. This allows us to require that the unitary part of the form factor F^-_1 not receive quantum corrections in higher orders—i.e.,

$$Y \overset{!}{=} V. \tag{5.6}$$

The renormalization condition (5.6) can be written as

$$F^-_1 \overset{!}{=} HV. \tag{5.7}$$

Using the Hermiticity of H, we obtain, as the final result for the renormalization condition,

$$F^-_1 V^\dagger - V(F^-_1)^\dagger \overset{!}{=} 0 \tag{5.8}$$

or, using Eq. (5.4),

$$(V + \delta F^-_{\mathrm{loop},1} + \delta F_{\mathrm{ct}})V^\dagger \overset{!}{=} V(V + \delta F^-_{\mathrm{loop},1} + \delta F_{\mathrm{ct}})^\dagger. \tag{5.9}$$

[2]The decomposition $F^-_1 = YH'$ leads to the same renormalization condition (5.11).

For the l-loop contribution this reads

$$(\delta F_{\text{loop},1}^{-(l)} + \delta F_{\text{ct}}^{(l)}) V^{\dagger} \overset{!}{=} V(\delta F_{\text{loop},1}^{-(l)} + \delta F_{\text{ct}}^{(l)})^{\dagger}. \tag{5.10}$$

Inserting Eq. (4.17) and using the anti-Hermiticity of $\delta\tilde{Y}V^{\dagger}$ and the fact that $\delta e/e$ and $\delta s_{\text{w}}/s_{\text{w}}$ are real, this equation can be solved for $\delta\tilde{Y}^{(l)}$:

$$\begin{aligned}\delta\tilde{Y}^{(l)} = & -\frac{1}{2}[(\delta\tilde{Z}^{\text{u,L}(l)})^{\dagger} - \delta\tilde{Z}^{\text{u,L}(l)}]V \\ & -\frac{1}{2}V[\delta\tilde{Z}^{\text{d,L}(l)} - (\delta\tilde{Z}^{\text{d,L}(l)})^{\dagger}] - \frac{1}{2}(\delta Z_{\text{W}}^{(l)} - \delta Z_{W}^{*(l)}) \\ & -\frac{1}{2}[\delta F_{\text{loop},1}^{-(l)} - V(\delta F_{\text{loop},1}^{-(l)})^{\dagger}V]. \end{aligned} \tag{5.11}$$

We can rewrite Eq. (5.11) into standard form by absorbing the undetermined phases into the field renormalization and find, dropping the loop index l:

$$\begin{aligned}\delta\tilde{V} = & -\frac{1}{2}[(\delta Z^{\text{u,L}})^{\dagger} - \delta Z^{\text{u,L}}]V - \frac{1}{2}V[\delta Z^{\text{d,L}} - (\delta Z^{\text{d,L}})^{\dagger}] \\ & -\frac{1}{2}(\delta Z_{\text{W}} - \delta Z_{\text{W}}^{*}) - \frac{1}{2}[\delta F_{\text{loop},1}^{-} - V(\delta F_{\text{loop},1}^{-})^{\dagger}V]. \end{aligned} \tag{5.12}$$

This together with Eq. (4.14) determines the counterterms to the QMM. As already mentioned, our conventions differ by a factor of 2 in the definition of the field renormalization constants from those of Ref. [7]. We note that Eq. (5.12) implicitly requires us to fix the c_n^q such that $\delta V = \delta\theta_n \partial V/\partial\theta_n$; i.e., the free parameters in $\delta Z^{q,\text{L}}$ must be fixed such that the renormalized QMM can be expressed in terms of the corresponding renormalized physical parameters (see also Ref. [26]).

Let us now discuss the renormalization condition (5.12), leaving aside absorptive parts: it is a physical renormalization condition that satisfies all requirements mentioned in the Introduction. The corresponding renormalized QMM is gauge independent, unitary by construction, and symmetric with respect to the fermion generations. Moreover, the renormalized matrix elements approach the limit of degenerate fermion masses smoothly. An apparent drawback of the renormalization condition (5.12) is that it requires the calculation of the vertex form factor $\delta F_{\text{loop},1}^{-}$. This, however, is anyhow needed for all processes involving the QMM. The renormalization condition (5.12) is equivalent to the one given by Zhou in Ref. [11]. However, while we impose a renormalization condition on a physical matrix element (5.8) that preserves the unitarity of the QMM, Zhou requires a condition that violates unitarity. In a second step he corrects this by extracting the unitarity-preserving part of a counterterm, a procedure that was also used in the renormalization scheme proposed in Ref. [10]. As a further remark, we note that exactly the same renormalization condition (5.12) is obtained from the decays $\text{W}^{+}\to\text{u}_i\overline{\text{d}_j}$ and $\overline{\text{u}_i}\to\text{W}^{-}\overline{\text{d}_j}$ or $\text{W}^{-}\to\overline{\text{u}_i}\text{d}_j$ and $\text{u}_i\to\text{W}^{+}\text{d}_j$.

If we take into account absorptive parts, we encounter a number of problems and drawbacks: the counterterm (5.12) obtained from the decays $\text{W}^{+}\to\text{u}_i\overline{\text{d}_j}$ and $\overline{\text{u}_i}\to\text{W}^{-}\overline{\text{d}_j}$ differs from the one obtained analogously from the decays $\text{W}^{-}\to\overline{\text{u}_i}\text{d}_j$ and $\text{u}_i\to\text{W}^{+}\text{d}_j$ owing to Eq. (A6), and the counterterms to the QMM become complex also in the case of *CP* conservation. Moreover, when absorptive parts of the loop corrections and of the Lehmann-Symanzik-Zimmermann (LSZ) factors are included in the calculation of *S*-matrix elements, the limit of degenerate fermion masses is no longer approached smoothly. This problem is related to the lack of a consistent definition of *S*-matrix elements for unstable external particles and applies to all other existing renormalization prescriptions for the QMM once absorptive parts are taken into account. One proposal for the modification of the LSZ factors in the presence of external unstable particles [25] and the renormalization of the QMM in this approach is discussed in some detail in the appendix. However, it also does not solve the mentioned problems.

Therefore, we advocate to only include dispersive parts in Eq. (5.12).

VI. CONCLUSIONS

We have studied the renormalization of the quark-mixing matrix and the corresponding restrictions from BRS invariance and rigid $\text{SU}(2)_{\text{L}}$ symmetry.

We started from the fact that the gauge-parameter dependence of counterterms and physical *S*-matrix elements can be controlled using a modified Slavnov-Taylor identity, where the gauge parameter ξ is extended to a BRS doublet by introducing an auxiliary field χ. Using this formalism, it can be seen that counterterms that cannot be written as a BRS variation must be genuinely gauge-parameter independent; the others may be gauge-parameter dependent. While the quark field renormalization constants are generally gauge-parameter dependent, the counterterms to the QMM do not depend on the gauge parameters if appropriate physical renormalization conditions are imposed.

In order to satisfy complete on-shell renormalization conditions in the quark sector, finite gauge-parameter-dependent field redefinitions must be introduced. Imposing complete on-shell conditions without these finite field redefinitions induces an artificial gauge-parameter dependence in the counterterms to the QMM, as found by an explicit one-loop calculation [8] for the renormalization prescription of Refs. [3], [7]. These finite field redefinitions appear explicitly in the rigid $\text{SU}(2)_{\text{L}}$ transformations, resulting in a renormalization of the rigid symmetry. From rigid $\text{SU}(2)_{\text{L}}$ invariance we found relations between the ultraviolet-divergent parts of the invariant counterterms.

Finally, we proposed a physical renormalization condition for the QMM based on the decays $\text{W}^{+}\to\text{u}_i\overline{\text{d}_j}$ and $\overline{\text{t}}\to\text{W}^{-}\overline{\text{d}_j}$. This condition fixes all counterterms properly and yields gauge-parameter-independent results for physical *S*-matrix elements up absorptive parts related to the presence of external unstable particles. Moreover, it is symmetric with respect to the fermion generations and, at least for the nonabsorptive parts, avoids unphysical singularities in the limit of degenerate quark masses.

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APPENDIX: RENORMALIZATION OF THE QUARK-MIXING MATRIX IN THE PRESENCE OF LSZ FACTORS INCLUDING ABSORPTIVE PARTS

In Ref. [25] it has been argued that in the case of unstable external fermions different sets of LSZ factors have to be introduced for incoming and outgoing fermions in order to obtain gauge-parameter-independent amplitudes. Although this approach has a certain appeal, it is still far from a description of amplitudes for unstable particles. The purpose of this appendix is to investigate the renormalization of the QMM in this approach.

The matrix-valued LSZ factors for all left- and right-handed incoming fermions and out-going antifermions are denoted by $\delta Z^{q,\mathrm{L}}$ and $\delta Z^{q,\mathrm{R}}$ and those for the outgoing fermions and incoming antifermions by $\delta \bar{Z}^{q,\mathrm{L}}$ and $\delta \bar{Z}^{q,\mathrm{R}}$. Similarly, the LSZ factor for the incoming W^+ boson and outgoing W^- boson is given by δZ_{W} and the one for the incoming W^- boson and outgoing W^+ boson by $\delta \bar{Z}_{\mathrm{W}}$. These LSZ factors are suitable for *S*-matrix elements and involve enough freedom to include all absorptive parts. However, they should not be used as field renormalization constants in the renormalized Lagrangian, since they would violate its hermiticity. In the following, the constants δZ and $\delta \bar{Z}$ can be understood to include both the LSZ factors and field-renormalization constants or only the LSZ factors assuming that field renormalization has already been performed.

The LSZ factors are fixed by imposing on-shell conditions for incoming and outgoing fermions [25]. From the resulting explicit expressions for $\delta Z^{q,\mathrm{L/R}}$ and $\delta \bar{Z}^{q,\mathrm{L/R}}$ in terms of self-energies given in Ref. [25] and those of the self-energies we find the relations

$$\delta \bar{Z}^{q,\mathrm{L/R}} = (\delta Z^{q,\mathrm{L/R}})^{\mathrm{T}}\big|_{V\to V^*} \tag{A1}$$

or, equivalently,

$$\widetilde{\mathrm{Re}}\, \delta \bar{Z}^{q,\mathrm{L/R}} = \widetilde{\mathrm{Re}}(\delta Z^{q,\mathrm{L/R}})^{\dagger}, \quad \widetilde{\mathrm{Im}}\, \delta \bar{Z}^{q,\mathrm{L/R}} = -\widetilde{\mathrm{Im}}(\delta Z^{q,\mathrm{L/R}})^{\dagger}, \tag{A2}$$

using $\widetilde{\mathrm{Re}}$ and $\widetilde{\mathrm{Im}}$ as defined in Ref. [7], which acts only on the loop integrals but not on the QMM; i.e., $\widetilde{\mathrm{Re}}$ projects on dispersive parts and $\widetilde{\mathrm{Im}}$ on absorptive parts. As a consequence of Eqs. (A2), the barred matrices $\delta \bar{Z}^{q,\mathrm{L}}$ and $\delta \bar{Z}^{q,\mathrm{R}}$ agree with $(\delta Z^{q,\mathrm{L}})^{\dagger}$ and $(\delta Z^{q,\mathrm{R}})^{\dagger}$ in the dispersive parts but have a different sign for the absorptive parts. Analogous relations hold between δZ^*_{W} and $\delta \bar{Z}_{\mathrm{W}}$.

In the approach of Ref. [25] the quantum corrections to the decays $\mathrm{W}^+\to \mathrm{u}_i \bar{\mathrm{d}}_j$ and $\bar{\mathrm{u}}_i \to \mathrm{W}^- \bar{\mathrm{d}}_j$ are given by Eqs. (5.3), (5.4), and (4.16) with $(\delta Z^{\mathrm{u,L}})^{\dagger}$ replaced by $\delta \bar{Z}^{\mathrm{u,L}}$. Using the physical renormalization condition (5.8) leads to

$$\begin{aligned}\delta \tilde{V} = &-\frac{1}{2}[\delta \bar{Z}^{\mathrm{u,L}} - (\delta \bar{Z}^{\mathrm{u,L}})^{\dagger}]V - \frac{1}{2}V[\delta Z^{\mathrm{d,L}} - (\delta Z^{\mathrm{d,L}})^{\dagger}]\\ &-\frac{1}{2}(\delta Z_{\mathrm{W}} - \delta Z^*_{\mathrm{W}})V - \frac{1}{2}[\delta F^-_{\mathrm{loop},1} - V(\delta F^-_{\mathrm{loop},1})^{\dagger}V]\end{aligned} \tag{A3}$$

instead of Eq. (5.12). This renormalization condition is equivalent to the one given by Zhou in Ref. [11].

Using the decays $\mathrm{W}^-\to \mathrm{d}_j \bar{\mathrm{u}}_i$ and $\mathrm{u}_i \to \mathrm{W}^+ \mathrm{d}_j$ with an analogous renormalization condition, yields instead

$$\begin{aligned}\delta \tilde{V} = &-\frac{1}{2}[(\delta Z^{\mathrm{u,L}})^{\dagger} - \delta Z^{\mathrm{u,L}}]V - \frac{1}{2}V[(\delta \bar{Z}^{\mathrm{d,L}})^{\dagger} - \delta \bar{Z}^{\mathrm{d,L}}]\\ &-\frac{1}{2}(\delta \bar{Z}^*_{\mathrm{W}} - \delta \bar{Z}_{\mathrm{W}})V - \frac{1}{2}[(\delta G^-_{\mathrm{loop},1})^{\dagger} - V\delta G^-_{\mathrm{loop},1}V],\end{aligned} \tag{A4}$$

where $\delta G^-_{\mathrm{loop},1}$ is defined in analogy to $\delta F^-_{\mathrm{loop},1}$. These matrices are related by

$$\delta G^-_{\mathrm{loop},1} = (\delta F^-_{\mathrm{loop},1})^{\mathrm{T}}\big|_{V\to V^*} \tag{A5}$$

or, equivalently,

$$\widetilde{\mathrm{Re}}\, \delta G^-_{\mathrm{loop},1} = \widetilde{\mathrm{Re}}(\delta F^-_{\mathrm{loop},1})^{\dagger}, \quad \widetilde{\mathrm{Im}}\, \delta G^-_{\mathrm{loop},1} = -\widetilde{\mathrm{Im}}(\delta F^-_{\mathrm{loop},1})^{\dagger}, \tag{A6}$$

as can be seen from the structure of the explicit expressions. The counterterms (A3) and (A4) involve the same dispersive parts but opposite absorptive parts. In the case of *CP* conservation the renormalization conditions (A3) and (A4) violate the orthogonality and reality of the QMM. These drawbacks can be cured by omitting all absorptive parts in the counterterms for $\delta \tilde{V}$. In this case the expressions (A3) and (A4) become equivalent, and the counterterm simplifies to

$$\begin{aligned}\delta \tilde{V} = -\widetilde{\mathrm{Re}}\Big\{&\frac{1}{2}[(\delta Z^{\mathrm{u,L}})^{\dagger} - \delta Z^{\mathrm{u,L}}]V + \frac{1}{2}V[\delta Z^{\mathrm{d,L}} - (\delta Z^{\mathrm{d,L}})^{\dagger}]\\ &+\frac{1}{2}[\delta F^-_{\mathrm{loop},1} - V(\delta F^-_{\mathrm{loop},1})^{\dagger}V]\Big\}.\end{aligned} \tag{A7}$$

Alternatively, this result can be directly obtained from the renormalization condition

$$[F_1^- + (G_1^-)^\dagger]V^\dagger = V[(F_1^-)^\dagger + G_1^-], \tag{A8}$$

where $G^-_{\mathrm{loop},1}$ is the matrix replacing $F^-_{\mathrm{loop},1}$ in these decays, and the relations (A2) and (A6) without the need to use the $\widetilde{\mathrm{Re}}$ prescription.

As a consequence, also in this approach the most natural strategy is to discard all absorptive parts in the renormalization constants of the QMM. Then, the counterterms for $\delta\tilde{V}$ are the same as those introduced in Eq. (5.12) with the same merits and drawbacks.

Finally, we note that one could construct a counterterm that yields S-matrix elements for the decays $\mathrm{W}^+ \to \mathrm{u}_i\overline{\mathrm{d}_j}$ and $\overline{\mathrm{t}} \to \mathrm{W}^-\overline{\mathrm{d}}_j$ where the limit of degenerate fermion masses is approached smoothly by including appropriate absorptive parts of the LSZ factors in Eq. (A7) as

$$\delta\tilde{V} = -\frac{1}{2}[\delta\overline{Z}^{\mathrm{u,L}} - \delta Z^{\mathrm{u,L}}]V - \frac{1}{2}V[\delta Z^{\mathrm{d,L}} - \delta\overline{Z}^{\mathrm{d,L}}] - \frac{1}{2}[\delta F^-_{\mathrm{loop},1} - V(\delta F^-_{\mathrm{loop},1})^\dagger V]. \tag{A9}$$

With this counterterm only the combinations $\delta\overline{Z}^{\mathrm{u,L}} + \delta Z^{\mathrm{u,L}}$ and $\delta\overline{Z}^{\mathrm{d,L}} + \delta Z^{\mathrm{d,L}}$ that are nonsingular for degenerate fermion masses appear in the renormalized S-matrix elements for $\mathrm{W}^+ \to \mathrm{u}_i\overline{\mathrm{d}_j}$ and $\overline{\mathrm{u}_i} \to \mathrm{W}^-\overline{\mathrm{d}_j}$. However, the absorptive parts in Eq. (A9) violate the unitarity of the renormalized QMM and are thus not admissible. Moreover, even for these counterterms the S-matrix elements for the decays $\mathrm{W}^- \to \overline{\mathrm{u}_i}\mathrm{d}_j$ and $\mathrm{u}_i \to \mathrm{W}^+\mathrm{d}_j$ are not smooth in the degenerate-quark-mass limit.

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