Remarks on the high-energy behavior of string scattering amplitudes in warped spacetimes

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The high-energy behavior of string scattering in warped spacetimes is studied to all orders in perturbation theory. If one assumes that the theory is finite, the amplitudes *exactly* fall as powers of momentum.

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I. INTRODUCTION

Recently, the high-energy behavior of type-IIB superstring amplitudes was studied in the case of warped spacetime geometries which are the products of AdS_5 with some five-manifolds [1–5]. One of the most important results is that of Polchinski and Strassler [1]. They proposed a scheme of evaluating high-energy fixed-angle string amplitudes in terms of vertex operators on a spherical world sheet and found that the amplitudes fall as powers of momentum. Thus, a long-standing problem on the way to a string theory description of hadronic processes was solved. In fact, it was already known in the 1970s that the amplitudes of exclusive hadronic processes at large momentum transfer scale as [6]

$$\mathcal{M} \sim p^{-n+4}\left[1+O\left(\frac{1}{p}\right)\right], \tag{1.1}$$

where p is a large momentum scale and n is a total number of hadronic constituents (valence quarks).

In this paper we extend the analysis to higher orders of string perturbation theory. We assume that the perturbation theory in question is a topological expansion as it is in Minkowski space.

II. GENERAL ARGUMENT

As mentioned earlier, if the spacetime geometry is chosen as the product of AdS_5 with a five-manifold K, the high-energy behavior of tree string amplitudes is hard (power law). Let us write the metric as

$$ds^2 = e^{2\varphi}\eta_{\mu\nu}\,dx^\mu\,dx^\nu + R^2\,d\varphi^2 + R^2\,d\Omega_K^2, \tag{2.1}$$

where R is the radius of AdS_5 and $\eta_{\mu\nu}$ is a four-dimensional Minkowski metric. We assume that K does not provide any dimensionfull parameter except R. Moreover, $d\Omega_K^2$ is independent of R.

Unfortunately, full control of type-IIB string theory on curved backgrounds like AdS_5 is beyond our grasp at present. However, we will argue that the scaling behavior can be understood from a nonlinear sigma model perspective which bypasses the known difficulty with Ramond-Ramond (RR) backgrounds. So the part of the world sheet action which is most appropriate for our purposes is simply[1]

$$S_0 = \frac{1}{4\pi\alpha'}\int_{\Sigma_g} d^2z\,d^2\theta\,e^{2\varphi}\eta_{\mu\nu}\bar{D}X^\mu\,DX^\nu, \tag{2.2}$$

where Σ_g is a closed Riemann surface of genus g. The simplest vertex operators predicated on this form of action are $\epsilon_{\mu\nu}(p)\bar{D}X^\mu DX^\nu e^{ip\cdot X}$ dressed by some $e^{-\Delta\varphi}\mathcal{O}_\Delta(\Omega_K)$.[2] Then, integrating over the world sheet Σ_g,

$$V_{\Delta,p} = \int_{\Sigma_g} d^2z\,d^2\theta\,\epsilon_{\mu\nu}(p)\bar{D}X^\mu\,DX^\nu e^{ip\cdot X}\,e^{-\Delta\varphi}\mathcal{O}_\Delta(\Omega_K). \tag{2.3}$$

For reasons that will soon become apparent, we restrict values of Δ's to positive integers. We also discard quantum numbers which are due to the manifold K.

String scattering amplitudes are defined as expectation values of the vertex operators. In the problem of interest, a reasonable first guess for a g-loop amplitude of the $2\to 2$ scattering process is[3]

$$\delta^{(4)}(p_A+\cdots+p_D)\mathcal{M}_g(AB\to CD) = \left\langle \prod_{i=A,\ldots,D} V_{\Delta_i,p_i}\right\rangle, \tag{2.4}$$

where the pair of brackets means integrals over the matter fields, ghost fields as well as moduli space of closed Riemann surfaces of genus g. We do not know the precise details about these integrals. However, what is only important for our purposes is an integral over the bosonic zero modes of X's and φ. Its explicit form is given by[4]

$$\int d^4x\int_{-\infty}^{+\infty} d\varphi\,e^{4\varphi}. \tag{2.5}$$

After performing the integration, the amplitude takes the form

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[1]We use the superspace notations of [7].

[2]To be precise, this form only includes the dominant term in the limit of large φ. Subdominant terms result in $1/p$ corrections in Eq. (1.1), so we suppress them. For more discussion of vertex operators, see, e.g., [8] and references therein.

[3]We omit the explicit dependence on the string coupling constant g_s. For a discussion of this issue, see [1,4].

[4]For a discussion of path integral measure within the sigma model approach, see, e.g., [9].

$$\mathcal{M}_g(AB\to CD)=\frac{1}{2}\Gamma(2-\Delta/2)\left\langle\left[\frac{1}{4\pi\alpha'}\int_{\Sigma_g}d^2z\,d^2\theta\,\mathrm{e}^{2\varphi}\eta_{\mu\nu}\bar{D}X^\mu DX^\nu\right]^{\Delta/2-2}\prod_{i=A,\ldots,D}V_{\Delta_i,p_i}\right\rangle',\tag{2.6}$$

where $\Delta=\Delta_A+\cdots+\Delta_D$. The prime means that the zero modes were integrated out.

One thing about Eq. (2.6) may be disturbing. It seems that the amplitude is divergent for even Δ's because the Γ-function prefactor develops poles.[5] The answer to this puzzle is simple: integration over the matter fields cancels the factor out. We will discuss this later.

We are interested in the hard scattering limit. Its kinematics is very special: there is one independent parameter—the Mandelstam variable s—while the others are functions of it and fixed scattering angle ϕ. This allows us to easily determine the dependence on s of the amplitude for the scalars (dilatons) whose masses are much less than s. In this case we get, from Eq. (2.6) by rescaling $X=\tilde{X}/\sqrt{s}$, $p_i=\sqrt{s}\tilde{p}_i$,

$$\mathcal{M}_g^{(dil)}=c_g^{(dil)}\left(\frac{1}{\sqrt{s}}\right)^{\Delta-4}.\tag{2.7}$$

Here $c_g^{(dil)}$ is given by the right-hand side of Eq. (2.6) with the X's and p_i's replaced by the $\tilde{X}$'s and $\tilde{p}_i$'s. Its independence of s can be understood from general reasoning. $c_g^{(dil)}$ is a function of the Mandelstam variables, defined in terms of rescaled momenta, which are independent of s.

In the case of hadrons, taken as bound states of spinless constituents, this would be the end of the story if the parameters Δ_i were related to the numbers of constituents in the corresponding hadrons as $\Delta_i=n_i$ [1]. Clearly the existence of spin-2 states (gravitons) as follows from the form of the vertex operators (2.3) provides a strong objection to spinless constituents only. Because of spinning constituents, each vertex operator contributes an additional factor $(\sqrt{s})^{\mathrm{S}_i}$, where S_i means its four-dimensional spin. For the graviton vertex operator, it gives $(\sqrt{s})^2$.[6] As a result, we get

$$\mathcal{M}_g^{(grav)}=c_g^{(grav)}\left(\frac{1}{\sqrt{s}}\right)^{\Delta+4}.\tag{2.8}$$

Finally, the desired result is obtained by identifying the (Δ_i-2)'s with the numbers of constituents in the corresponding hadrons as in [1].

Having established the scaling behavior of the g-loop amplitude, we turn to a perturbation series. Assuming that perturbation theory in question is a topological expansion and collecting together the results for each order, the amplitude is simply

$$\mathcal{M}(AB\to CD)=\left(\frac{1}{\sqrt{s}}\right)^{n-4}\sum_{g=0}^{\infty}c_g(g_s,\phi),\tag{2.9}$$

where $n=n_A+\cdots+n_D$. Unfortunately, we cannot, with our present methods, determine the explicit form of the coefficients $c_g(g_s,\phi)$.

At this point, a couple of short remarks is in order.

(i) Our analysis can be easily generalized to include surfaces with boundaries. In this case two new features are especially noteworthy. First, there are vertex operators associated with boundaries. The simplest one to be suggested is a vector state $\epsilon_\mu(p)DX^\mu\mathrm{e}^{ip\cdot X}$ dressed by $\mathrm{e}^{-\Delta\varphi}\mathcal{O}_\Delta(\Omega_{\mathrm{K}})$. Second, perturbation theory is determined by the Euler number χ. Thus, the amplitude takes the form

$$\mathcal{M}(AB\to CD)=\left(\frac{1}{\sqrt{s}}\right)^{n-4}\sum_{\chi=2}^{-\infty}c_\chi(g_s,\phi).\tag{2.10}$$

(ii) Certainly, our derivation of the scaling behavior is valid for any value of the radius of AdS_5 or, equivalently, for any value of the 't Hooft coupling.

III. SIMPLIFIED MODEL

Other approaches to the problem result in the scaling behavior at the tree level [1,4]. A notable difference is the absence in those of nonzero modes of the φ and Ω fields. Nevertheless, as follows from our discussion these nonzero modes do not play a crucial role in the derivation of the scaling behavior and therefore may be discarded. So it is quite natural to pursue this line of thought further.

A reasonable first guess for a g-loop amplitude within the simplified models is

$$\mathcal{M}_g(AB\to CD)=\int d^5\Omega\,d\varphi\,\mathrm{e}^{4\varphi}\mathcal{A}_g(AB\to CD)\times\prod_{i=A,\ldots,D}\mathrm{e}^{-\Delta_i\varphi}\psi_i(\Omega),\tag{3.1}$$

where $\mathcal{A}_g(AB\to CD)$ is the standard string g-loop amplitude with α' replaced by $\hat{\alpha}=\alpha'\mathrm{e}^{-2\varphi}$ and ψ_i are normalized eigenfunctions of the Laplace operator on K.

For the sake of simplicity, let us specialize to the case of scalars—e.g., the dilatons. On the one hand side, in the hard scattering limit the scaling behavior of the amplitude can be derived by rescaling φ as $\varphi\to\varphi+\frac{1}{2}\ln\alpha' s$. At the tree level this method was used in [4] but it also works for Eq. (3.1).

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[5] In fact, the integral over φ diverges at $\varphi=-\infty$. This divergence is due to the net factor $\mathrm{e}^{-\Delta\varphi}$ coming from the vertex operators.

[6] As in QCD [10], one can think of $\epsilon_{\mu\nu}^{(grav)}$ as $(\Pi_i u_i)_{\mu\nu}\epsilon(p)$, where $\epsilon(p)$ plays the same role as the dilaton wave function and u_i are wave functions of free spinning constituents. Note that in QCD the u_i's are the free spinors normalized as $\Sigma_{\mathrm{spin}}u\bar{u}=\gamma\cdot p+m$.

On the other hand, it is known [11] that $\mathcal{A}_g$ is well approximated by its saddle point expression $\mathcal{A}_g \approx (\hat{\alpha} s)^a \mathrm{e}^{-b\hat{\alpha}s}$, where a,b are some positive functions of g and φ. This makes it possible to perform the integration over φ explicitly. As expected, the amplitude falls as a power of $\sqrt{s}$. A by-product of the integration is $\Gamma(a-2+\Delta/2)$. Unlike the Γ-function prefactor in Eq. (2.6), it is now finite. The point is that the integrand becomes nonsingular at $\varphi=-\infty$ after integration over the matter fields and moduli generates a factor $\mathrm{e}^{-bs\alpha' \mathrm{e}^{-2\varphi}}$ which dumps $\mathrm{e}^{-\Delta\varphi}$.

It is worth noting that in Minkowski space the saddle point evaluation is valid for a given order of perturbation theory in the limit of large α's [11]. Thus, a small parameter in question is $1/\alpha$'s. This is not the case for warped geometries. Let us first try to get a heuristic understanding of what happens. $\mathcal{A}_g$ is now expanded in powers of $1/\hat{\alpha}s$. The remainder of the integral is dominated by $\varphi \sim \varphi_*$, where $\alpha'\mathrm{s}\,\mathrm{e}^{-2\varphi_*} \sim \Delta$. Thus, we end up with an expansion in $1/\Delta$. This means that Δ should be large in order for the saddle point evaluation to be valid. Actually one can come to the same conclusion on general grounds. The dependence in the dilaton amplitude of α's is absorbed into the redefinition of φ. Then, as follows from our ansatz (3.1), the amplitude becomes a function of Δ only. If a saddle exists, it may be a good approximation only in the limit of large Δ. The point is that the Δ_i's are restricted to positive integers, so Δ is an integer bounded from below and the limit of small Δ does not exist.[7]

A final remark: it was shown in [12] that states with large quantum numbers are described by special classical solutions of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ nonlinear sigma model in the limit of large 't Hooft coupling. Certainly, no derivation of the high-energy fixed-angle scattering amplitudes of states with large Δ_i's from classical solutions is known. But we believe that the preceding comments are significant hints that it can be done, and this issue is worthy of future study.

IV. CONCLUDING COMMENTS

(i) Here we considered type-IIB string theory; however, similar results will hold for other string theories. In fact, warped geometry in spacetime is sufficient to formally ensure that scattering amplitudes are hard in the high-energy limit at fixed angle. For instance, in the case of bosonic string one can get the scaling behavior of amplitudes by repeating the arguments of Sec. II for the bosonic part of the world sheet action.

Actually, there might be gaps in the above reasoning. First, one must show that the corresponding background is conformal. From this point of view type IIB is of course preferable to the others. Second, the coefficients c_g, c_χ must be finite or, in other words, the theory must be perturbatively renormalizable. Third, the series (2.9), (2.10) must converge. Even if we expect that superstring theory is finite at a given order of perturbation theory, there is no guarantee for convergence. For instance, a rapid growth of the volume of moduli space might be the reason for divergence [13]. As noted earlier, we cannot, with our present methods, determine the explicit form of the coefficients c_g, c_χ and, therefore, address the issue of convergence.

Thus, our general statement is that if the theory is finite, the amplitudes *exactly* fall as powers of momentum.

(ii) A strong belief is that the geometry given by Eq. (2.1) is valid in the limit of large φ or, equivalently, large r, where $r=R\,\mathrm{e}^{\varphi}$. At smaller values of r it is somehow deformed. So it is of some interest to evaluate corrections to the scaling. To do so, let us first consider the simplified model of Sec. III. In order for the saddle point evaluation to be consistent with the deformation of geometry, we should require that φ_* be large. This means that α's$\gg\Delta$. Truncating then the geometry at some small $r=r_0$, we estimate the correction to $\mathcal{M}_g$ as

$$\int_0^{r_0} dr\, r^{3-\Delta} \mathcal{A}_g(AB\to CD) \sim (\alpha_0 s)^a\, \mathrm{e}^{-b\alpha_0 s} \qquad (4.1)$$

and we note that it is just the soft string amplitude with α' replaced by $\alpha_0=\alpha' R^2/r_0^2$. It is indeed subleading to the hard amplitude.

Returning to the settings of Sec. II, one thing that can help with an understanding of what happens in the hard scattering limit is some analogue between φ and the Liouville field of 2D gravity. It was suggested by Polyakov [14] and exploited in [4] for deriving the scaling behavior of the amplitudes. Let us pursue this point of view further. Assuming that φ is slowly varying, we can consider the notion of a low-energy effective action. This action occurs from the right-hand side of Eq. (2.4) after integration over the matter fields, ghosts, and moduli. It is easy to find an effective potential along the lines of Sec. III. It is given by[8]

$$V_{eff}(\varphi) = \mu\, \mathrm{e}^{-2\varphi}, \qquad (4.2)$$

where μ is an effective cosmological constant linearly depending on s. Thus, the effective potential suppresses the path integral for large negative φ. The effect becomes stronger with the growth of s. Finally, only large positive values of φ will be allowed. The known 2D gravity analogue of this is the "Liouville wall" which keeps the theory in the weak coupling regime.

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[7]Strictly speaking, it assumes a single hard process. Landshoff diagrams may result in noninteger Δ's.

[8]We drop possible linear terms. From the viewpoint of 2D gravity these correspond to the dilaton background.

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