# Tree-level *S* matrix of Yang-Mills theory

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We further investigate the procedure for computing tree-level amplitudes in Yang-Mills theory from connected instantons in the B model on $\mathbb{P}^{3|4}$, emphasizing that the problem of calculating Feynman diagrams is recast into the problem of finding solutions to a certain set of algebraic equations. We show that the B model correctly reproduces all 6-particle amplitudes, including non-MHV amplitudes with three negative and three positive helicity gluons. As a further check, we also show that *n*-particle amplitudes obtained from the B model obey a number of properties required of gauge theory, such as parity symmetry (which relates an integral over degree *d* curves to one over degree $n-d-2$ curves) and the soft and collinear gluon poles.

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## I. INTRODUCTION

In [1] Witten proposed a remarkable connection between scattering amplitudes in Yang-Mills (YM) theory and a certain topological string theory, the B model on $\mathbb{P}^{3|4}$ (recent related work includes [2–9]). This conjecture leads to the following formula (equivalent to one first written down in [2] and studied further in [3,4]) for the *n*-particle amplitude, written in a manifestly $\mathcal{N}=4$ supersymmetric notation:

$$A_n = i(2\pi)^4 g_{\rm YM}^{n-2} \sum_{d=1}^{n-3} \int d\mathcal{M}_{n,d} \prod_{i=1}^{n} \delta^2(\lambda_i^a - \xi_i P_i^a) \times \prod_{k=0}^{d} \delta^2\left(\sum_{i=1}^{n} \xi_i \sigma_i^k \tilde{\lambda}_i^{\dot{a}}\right) \delta^4\left(\sum_{i=1}^{n} \xi_i \sigma_i^k \eta_{iA}\right). \quad (1.1)$$

The details of this formula will be clarified in Sec. II, but we have written it down here in order to stress its simplicity and importance. We believe that Eq. (1.1) encapsulates the *complete n*-particle tree-level *S* matrix of YM theory (for any gauge group), thereby providing an exact solution of classical YM theory in four dimensions. This formula sums up a huge number of Feynman diagrams (see for example Fig. 1) into an expression which fits on a single line. In this paper we provide strong evidence supporting our confidence in this formula and explore some of its structure.

The formula (1.1) was derived by considering the contribution to the scattering amplitude from a single connected instanton (=holomorphic curve in $\mathbb{P}^{3|4}$) of degree *d* in the topological B model. [As explained in [1], counting the fermionic modes reveals that the degree *d* is related to the total helicity $\Sigma h_i$ of all *n* particles according to $d=1/2(n-\Sigma h_i-2)$.] In [1] Witten speculated that one might have to consider, in addition to Eq. (1.1), contributions from collections of disconnected instantons of degrees $d_i$ with $\Sigma d_i = d$. (See Fig. 2 for a schematic depiction for the $n=6$, $d=3$ amplitude.)

However, it was found in [2,4] that the formula (1.1) correctly reproduces the known YM result for the mostly minus MHV (maximally helicity violating) amplitudes (sometimes called ''googly MHV'' or $\overline{\rm MHV}$) that are related to the mostly plus MHV amplitudes by complex conjugation. Even though the googly MHV amplitudes are calculated from an integral over the moduli space of instantons of arbitrarily high degree, precise agreement was found with gauge theory without the need for additional contributions.

More recently, a novel method for calculating YM tree amplitudes, also motivated by the B model on $\mathbb{P}^{3|4}$, was proposed in a very interesting paper [6]. The starting point for their proposal involved considering only completely disconnected instantons (i.e., *d* instantons of degree 1). Remarkably, it was found that their rule also gives correct gauge theory amplitudes. The B model seems to give two separately correct methods for calculating YM tree amplitudes, rather than a set of contributions which need to be summed (see Fig. 2).

The proposal of [6] involves a diagrammatic expansion which bears no apparently obvious connection to the formula (1.1), except that they both seem to be correct. It would be very interesting to understand directly the relation between these two methods. Moreover, if the B model on $\mathbb{P}^{3|4}$ gives us two not obviously equivalent formulas for YM amplitudes, then it will likely give us an infinite family of formulas (which roughly speaking weight the different types of diagrams in Fig. 2 differently). Undoubtedly we have only encountered the tip of the iceberg connecting the topological B model to Yang-Mills amplitudes.

So far the formula (1.1) had only been checked for MHV and $\overline{\rm MHV}$ amplitudes [2,4]. In Sec. III of this paper we confirm that the formula also gives the correct 6-particle non-MHV amplitudes. In Sec. IV we check that for any *n* and *d*, Eq. (1.1) satisfies a number of properties required of general Yang-Mills amplitudes, such as the soft and collinear gluon limits. Of particular importance is parity symmetry, which requires that Eq. (1.1) should be invariant under $\lambda \leftrightarrow \tilde{\lambda}$. This non-manifest symmetry of Eq. (1.1) is proved explicitly in

 

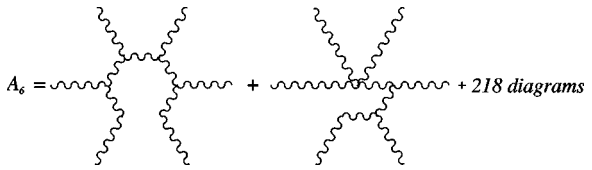


FIG. 1. The standard computation of a six-gluon tree amplitude requires summing 220 Feynman diagrams (in conventional gauges) [10].

Sec. IV B below. We conclude with a list of open questions and puzzles. First, however, we turn our attention to the details of Eq. (1.1) and highlight a crucial fact about the formula: namely, that it is not really an integral at all.

## II. MAIN FORMULA

In this section we first clarify the ingredients appearing in the formula (1.1) and then investigate some of its mathematical properties. The quantity $A_n$ in Eq. (1.1) denotes the color-stripped $n$-particle partial amplitude (see for example [11]), and we employ the spinor helicity notation in writing $A_n$ as a function of $(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}}, \eta_{iA})$, $i=1,...,n$, where $\lambda$ and $\tilde{\lambda}$ are commuting real two-component spinors of positive and negative chirality, respectively,[1] and $\eta_A$ is the four-component Grassmann coordinate of $\mathcal{N}=4$ superspace.

The $P^a$ are two degree $d$ polynomials in $\sigma$ which we parametrize as

$$P_i^a = \sum_{k=0}^{d} a_k^a \sigma_i^k \tag{2.1}$$

in terms of $2d+2$ coefficients (moduli) $a_k^a$. When needed, we will follow the conventions of [2,4] in denoting $P_i^1 = A_i$ and $P_i^2 = B_i$. The measure for integration in Eq. (1.1) is

$$d\mathcal{M}_{n,d} = \frac{d^{2d+2}a\, d^n\sigma\, d^n\xi}{\mathrm{vol}[\mathrm{GL}(2)]} \prod_{i=1}^{n} \frac{1}{\xi_i(\sigma_i - \sigma_{i+1})}. \tag{2.2}$$

The factor of 1/vol[GL(2)] is included because the integrand is invariant under a certain GL(2) symmetry and so the integral would otherwise be infinite. Practically, the consequence of this factor is simply that we can choose to fix four of the variables (which we will take to be any one of the $a$'s and any three of the $\sigma$'s) at the expense of introducing a Jacobian factor of

$$J = a(\sigma_i - \sigma_j)(\sigma_j - \sigma_k)(\sigma_k - \sigma_i). \tag{2.3}$$

The choice of which $a$ and which three $\sigma$'s to leave unintegrated is arbitrary and does not affect the final result.

[1]For simplicity we work in signature $+\ +\ -\ -$, where such spinors are possible. It is straightforward to analytically continue the tree-level YM amplitudes to signature $-\ +\ +\ +$ if desired.

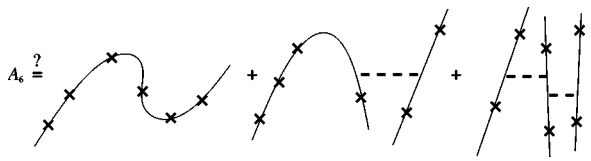


FIG. 2. Schematic depiction of how one might have thought to organize the calculation of the 6-particle mostly minus MHV ($d=3$) amplitude in the B model on $\mathbb{P}^{3|4}$. The dark $\times$'s mark the insertions of the 6 external particles, the dotted line is a twistor space propagator (constructed in [6]), and the solid lines represent instantons (i.e., holomorphic curves in $\mathbb{P}^{3|4}$) of degree $d=1,2,3$ (schematically encoded in the waviness of the curve). Although one might have expected that it would be necessary to sum together contributions of all three types, we find that the *single* diagram of the first type (studied here and in [2,4]) and the sum of the 21 diagrams of the third type (studied in [6]) *separately* give the correct gauge theory answer.

### A. Key point

The single most important fact about the integral (1.1) is that it is not really an integral. To see this, let us start by showing that Eq. (1.1) respects momentum conservation. Taking a particular linear combination of the quantities set to zero by the delta functions gives

$$0 = \sum_{k=0}^{d} a_k^a \left( \sum_{i=1}^{n} \xi_i \sigma_i^k \tilde{\lambda}_i^{\dot{a}} \right) = \sum_{i=1}^{n} \xi_i P_i^a \tilde{\lambda}_i^{\dot{a}} = \sum_{i=1}^{n} \lambda_i^a \tilde{\lambda}_i^{\dot{a}} = \sum_{i=1}^{n} p_i^{a\dot{a}}, \tag{2.4}$$

where we used the definition (2.1) and some more delta functions from Eq. (1.1). Therefore, the delta functions in Eq. (1.1) indeed force overall momentum conservation.

At the practical level, this means we can "pull out" the overall factor of $\delta^4(\Sigma p_i)$ at the expense of introducing a Jacobian, by using an identity such as

$$\prod_{i=1}^{n} \delta\left( \frac{\lambda_i^2}{\lambda_i^1} - \frac{B_i}{A_i} \right) \prod_{k=0}^{d} \delta^2\left( \sum_{i=1}^{n} \frac{\tilde{\lambda}_i^{\dot{a}} \lambda_i^1 \sigma_i^k}{A_i} \right)$$
$$= A_1 A_2 \delta^4\left( \sum_{i=1}^{n} p_i \right) \prod_{i=3}^{n} \delta\left( \frac{\lambda_i^2}{\lambda_i^1} - \frac{B_i}{A_i} \right) \prod_{k=1}^{d} \delta^2\left( \sum_{i=1}^{n} \frac{\tilde{\lambda}_i^{\dot{a}} \lambda_i^1 \sigma_i^k}{A_i} \right) \tag{2.5}$$

(where we used $\xi_i = \lambda_i^1 / A_i$). In writing this identity we have made a particular choice of which four delta functions to pull out. There is however no canonical choice, and different choices are useful for different calculations (and lead to different Jacobians), so it is convenient to leave momentum conservation slightly scrambled into the delta functions in Eq. (1.1). Note that supermomentum conservation $\delta^8(\Sigma \lambda_i^a \eta_{iA})$ pulls out similarly.

Let us now return to the claim that Eq. (1.1) is not really an integral. The measure $d\mathcal{M}_{n,d}$ in Eq. (2.2) has $(2d+2) + (n) + (n) - (4) = 2n + 2d - 2$ integration variables, while the integrand in Eq. (1.1) has $2n+2d+2$ delta functions. If we "pull out" the overall momentum conservation delta functions, then for any $n$ and $d$ there are precisely as many

integration variables as delta functions. Therefore the entire integral is supported on a discrete set of points, and the formula (1.1) is just a recipe to solve the $2n+2d+2$ polynomial equations

$$\lambda_i^a = \xi_i \sum_{k=0}^{d} \sigma_i^k a_k^a, \quad a=1,2, \;\; i=1,...,n,$$

$$0 = \sum_{i=1}^{n} \xi_i \sigma_i^k \tilde{\lambda}_i^{\dot{a}}, \quad \dot{a}=1,2, \;\; k=0,...d, \tag{2.6}$$

for the variables $(a_k^a, \sigma_i, \xi_i)$, and then to sum a certain Jacobian [obtained in the usual way from Eq. (1.1)] over the collection of roots.

One of the most interesting questions about the system (2.6) is, what is the number of roots, $N_{n,d}$, for general $n$ and $d$? At this point all we know for sure is that

$$N_{n,1} = N_{n,n-3} = 1, \quad N_{6,2} = 4. \tag{2.7}$$

The first two cases are the MHV and $\overline{\text{MHV}}$ amplitudes previously studied in the literature, and $N_{6,2}$ is the non-MHV 6-particle amplitude discussed in the following section. In Sec. IV we prove that $N_{n,n-d-2} = N_{n,d}$. Certainly it would be very interesting to have a better understanding of the mathematics underlying Eqs. (2.6). In particular, it would be especially interesting to learn how $N_{n,d}$ grows with $n$ and $d$.

### B. Complex puzzle

*A priori*, the moduli $a_k^a$ of the curve and the coordinates $\sigma_i$ on $\mathbb{P}^1$ should all be complex variables. In order to evaluate the integral (1.1) it is necessary to specify an integration contour in this $2n+2d-2$ complex dimensional space. In spacetime signature $++--$ it makes sense to take $\lambda$ and $\tilde{\lambda}$ to be independent real variables, and it is natural to choose the integration contour for which all of the $a_k^a$ and $\sigma_i$ are real.

For both the MHV ($d=1$) and $\overline{\text{MHV}}$ ($d=n-3$) cases, the unique root of Eqs. (2.6) indeed has the property that $\sigma_i$ and $a_k^a$ are real. However, for the 6-particle amplitude with $d=2$, which we discuss in Sec. III, there is a puzzle. Depending on the choice of $\lambda$ and $\tilde{\lambda}$, there can be four real roots, two real roots and one complex conjugate pair, or two complex conjugate pairs. The YM tree amplitude, which is always real [forgetting the $i$ in front of Eq. (1.1)], is reproduced only if all four roots are summed over, regardless of whether they are real or complex.

The lesson from this analysis is that restricting Eq. (1.1) to the contour where all $a$'s and $\sigma$'s are real does not give the correct gauge theory scattering amplitudes. In fact, we do not know how to write any contour which makes the integral formula (1.1) valid for arbitrary choices of $\lambda$ and $\tilde{\lambda}$. This amplifies the comment we made at the beginning of the previous subsection: the formula (1.1) is not really an integral. To overcome this problem we avoid thinking about Eq. (1.1) as an honest integral, but instead view it as a recipe for finding the solution (which in general can be complex) of Eq. (2.6) and then summing a Jacobian over the set of roots.

### C. Diagrammatic expansion?

From this new standpoint, let us ask ourselves whether the formula might have another, more natural interpretation. The fact that the computation of a scattering amplitude from the formula (1.1) reduces to summing a certain quantity over a finite set of points is reminiscent of some sort of diagrammatic expansion, where, for example, Eq. (2.7) suggests that there is a single diagram for mostly plus and mostly minus MHV amplitudes, while four diagrams contribute to the 6-point non-MHV amplitudes.

It is tempting to wonder whether there is any connection between such ''diagrams'' and the new diagrammatic expansion for YM scattering amplitudes which was recently proposed in [6]. According to their proposal, $A_{n,d}$ is associated with the collection of trees with $n$ cyclically labeled external legs and $d$ vertices, such that each vertex has at least 3 legs. For general $n$ and $d$ there are

$$\frac{1}{d}\binom{n-3}{d-1}\binom{n+d-2}{d-1}$$

such graphs,[2] which in all cases except the trivial case $d=1$ is larger than Eq. (2.7). (We have written the number of diagrams in $\mathcal{N}=4$ superspace. For particular choices of helicities of the external particles there are frequently fewer diagrams.)

However, the diagrams of [6] have an additional symmetry in the form of an arbitrary spinor $\eta^{\dot{a}}$ which drops out only after summing together all of the graphs. The number of diagrams is not gauge invariant, and special choices of $\eta$ can set whole classes of diagrams to zero. In contrast, our ''diagrams'' have no residual manifest symmetry—the GL(2) cancels out diagram by diagram (root by root) and does not change their number. Maybe there is some choice of $\eta$ for which the diagrams of [6] reduce, in number and in value, to the contributions obtained from the roots of our formula (1.1).

We believe it is more likely that the topological B model has some huge symmetry group which relates the formula (1.1), with its associated ''diagrams,'' to the diagrammatic expansion of [6]. Their parameter $\eta$ is a small remnant of that huge symmetry.

## III. SIX-PARTICLE NON-MHV AMPLITUDES

In the previous section we introduced the formula (1.1) and discussed its basic properties. But what is the connection between Eq. (1.1) and the $n$-particle scattering amplitude in gauge theory? In [1] it was shown that a prescription equivalent to the $d=1$ case of Eq. (1.1) reproduces the mostly plus

[2]The counting of these graphs is equivalent to a combinatorial problem which appeared in Plutarch's biographical notes on Hipparchus [12]. We are grateful to C. Herzog for many fun and enlightening discussions regarding the combinatorics.

MHV amplitudes in YM theory. In [2,4] it was shown that the formula also works for mostly minus MHV amplitudes. These have $d=n-3$ and are related (in Minkowski signature) to MHV amplitudes by complex conjugation.

Although the latter check involved an apparently nontrivial integral over the moduli space of curves of arbitrary degree in $\mathbb{P}^{3|4}$, the question of whether Eq. (1.1) is correct for genuinely non-MHV amplitudes was left open. The simplest amplitudes which are neither MHV nor googly MHV are those with $n=6$ particles and $d=2$. Since we work in a manifestly $\mathcal{N}=4$ formalism, our results apply simultaneously to all possible helicity orderings (when all six particles are gluons, there are three cyclically independent helicity orderings: $+++---$, $++-+--$ and $+-+-+-$).

In this paper we report that the formula (1.1), in the case $n=6$ and $d=2$, precisely matches the 6-gluon scattering amplitudes first computed by Mangano, Parke, and Xu [13]. We originally obtained this result numerically, by (1) choosing at random a collection of $(\lambda_i,\tilde{\lambda}_i)$ [subject to overall momentum conservation (2.4)], (2) numerically solving the polynomial equations (2.6), which were always observed to have four roots, and then (3) summing the Jacobian obtained from Eq. (1.1) over the four roots. The whole calculation takes only a few seconds on a fast computer and can be repeated as often as desired for different $(\lambda_i,\tilde{\lambda}_i)$. The result was always found to agree spectacularly with the formula given in [13]. Note that all three independent helicity configurations can be checked at the same time since the choice of helicities only affects the fermion determinant and does not change the value of the roots.

The only puzzle we encountered is that occasionally, for some $(\lambda_i,\tilde{\lambda}_i)$, the roots are complex, as we discussed in Sec. II B. Precise agreement with gauge theory was nevertheless always found by doing the most naive thing possible and summing over all four roots, whether real or complex.

Unfortunately, it seems rather difficult to construct an analytic proof that the formula (1.1) is correct for the case $n=6$, $d=2$. Let us now outline the best line of attack that we know of at the moment. We will not give precise formulas for each intermediate step because they are extremely lengthy and moreover because we are hopeful that a more clever way of analyzing the equations will become available. We believe that only after the mathematical structure of Eqs. (2.6) is better understood (for arbitrary $n$ and $d$) will it be clear how best to organize this calculation analytically.

### A. Constructing a Groebner basis: A sketch

The most interesting result of the numerical analysis is that the number of roots is $N_{6,2}=4$, which does not appear obvious from Eqs. (2.6). Recall that we can fix one of the $a$'s and three of the $\sigma$'s (say $\sigma_1$, $\sigma_2$ and $\sigma_3$) using the GL(2) symmetry. The remaining $2n+2d-2=14$ "integration variables" are fixed by solving Eqs. (2.6). In fact, it turns out to be possible to express all of the $a$'s, $\xi$'s and two of the remaining three $\sigma$'s as rational functions of the final $\sigma$ (say $\sigma_6$). Moreover, one can extract from Eqs. (2.6) a single equation which is quartic in $\sigma_6$ and does not depend on any of the other "integration variables." The coefficients of this quartic polynomial are themselves polynomials in $\sigma_1$, $\sigma_2$ and $\sigma_3$ and the covariant kinematic quantities $[ij]$ and $\langle ij\rangle$. The four roots of this master quartic equation determine the solutions for all 14 variables.

Here is a schematic description of how to derive this quartic equation. Choose some subset $S$ of the equations and use them to solve for the $a$'s and $\xi$'s in terms of the $\sigma$'s. Plugging the solution into the remaining equations gives polynomial equations just on the $\sigma$'s. This process can be repeated many times by starting with different sets $S$ of equations, leading to a large number of polynomial equations on $\sigma_4$, $\sigma_5$ and $\sigma_6$. The game then is to find the common roots of these polynomial equations. In mathematical language, we need to construct a Groebner basis for the ideal generated by these polynomials. Let us now be a little more specific.

Start with the equations on the top line of Eqs. (2.6). By eliminating $\xi_i$ between the $a=1$ and $a=2$ versions of this equation, one arrives at the six equations

$$\lambda_i^2\sum_{k=0}^{2} a_k^1\sigma_i^k=\lambda_i^1\sum_{k=0}^{2} a_k^2\sigma_i^k,\quad i=1,...,6, \tag{3.1}$$

which are conveniently expressed in matrix notation as

$$\begin{pmatrix} \lambda_1^1 & \lambda_1^1\sigma_1 & \lambda_1^1\sigma_1^2 & \lambda_1^2 & \lambda_1^2\sigma_1 & \lambda_1^2\sigma_1^2 \\ \lambda_2^1 & \lambda_2^1\sigma_2 & \lambda_2^1\sigma_2^2 & \lambda_2^2 & \lambda_2^2\sigma_2 & \lambda_2^2\sigma_2^2 \\ \lambda_3^1 & \lambda_3^1\sigma_3 & \lambda_3^1\sigma_3^2 & \lambda_2^2 & \lambda_2^2\sigma_2 & \lambda_2^2\sigma_3^2 \\ \lambda_4^1 & \lambda_4^1\sigma_4 & \lambda_4^1\sigma_4^2 & \lambda_4^2 & \lambda_4^2\sigma_4 & \lambda_4^2\sigma_4^2 \\ \lambda_5^1 & \lambda_5^1\sigma_5 & \lambda_5^1\sigma_5^2 & \lambda_5^2 & \lambda_5^2\sigma_5 & \lambda_5^2\sigma_5^2 \\ \lambda_6^1 & \lambda_6^1\sigma_6 & \lambda_6^1\sigma_6^2 & \lambda_6^2 & \lambda_6^2\sigma_6 & \lambda_6^2\sigma_6^2 \end{pmatrix} \begin{pmatrix} a_0^2 \\ a_1^2 \\ a_2^2 \\ a_0^1 \\ a_1^1 \\ a_2^1 \end{pmatrix} = 0. \tag{3.2}$$

A nontrivial solution exists if and only if the determinant of this matrix is zero:

$$0=X=\sum_{i,j,k,l,m,n}\epsilon_{ijklmn}\mathcal{V}(i,j,k,l,m,n)\langle il\rangle\langle jm\rangle\langle kn\rangle. \tag{3.3}$$

Here $\mathcal{V}$ is the cyclic product of $\sigma$'s (not the Vandermonde matrix):

$$\mathcal{V}(i,j,k,l,m,n)=(\sigma_i-\sigma_j)(\sigma_j-\sigma_k)(\sigma_k-\sigma_l)(\sigma_l-\sigma_m)\times(\sigma_m-\sigma_n)(\sigma_n-\sigma_i). \tag{3.4}$$

Another way to think about this equation is as follows. Since one of the $a$'s is fixed by the GL(2) symmetry, we really only are allowed to solve for five of the $a$'s. If we choose any five equations of Eqs. (3.1) to solve for the five $a$'s and then plug the solution into the sixth equation, we find the condition that Eq. (3.4) should vanish.

Next we turn our attention to the equations on the second line of Eqs. (2.6). These are six ($\dot{a}=1,2,k=0,1,2$) homogeneous linear equations on the six variables $\xi_i$. When cast in matrix form, the relevant matrix is precisely the transpose of

Eq. (3.2), but with $\lambda \leftrightarrow \tilde{\lambda}$. A nontrivial solution exists if and only if the corresponding determinant vanishes:

$$0 = \tilde{X} = \sum_{i,j,k,l,m,n} \epsilon_{ijklmn} \mathcal{V}(i,j,k,l,m,n)[ij][kl][mn]. \tag{3.5}$$

So far we have obtained (subject to $X=0$) a unique solution for all of the moduli $a_k^{\dot{a}}$ and (subject to $\tilde{X}=0$) a unique solution for all of the $\xi_i$. The final step is to require that these solutions be compatible, in that they obey the top line of Eqs. (2.6). There are a many such compatibility conditions that one can form, depending on which five of the six equations of Eqs. (3.1) one uses to solve for the moduli and which five of the six equations from the second line of Eqs. (2.6) that one uses to solve for the $\xi_i$. These equations are polynomials in $\sigma_4$, $\sigma_5$ and $\sigma_6$ whose coefficients depend on $\lambda$, $\tilde{\lambda}$ and the fixed values of $\sigma_1$, $\sigma_2$ and $\sigma_3$.

However, these equations (as well as the $X=0=\tilde{X}$ equations) all have spurious roots at $\sigma_4=\sigma_5=\sigma_6=\sigma_i$ for $i=1,2,3$. To eliminate these roots one constructs a linear combination of these equations (with coefficients involving powers of $\sigma_4$ and $\sigma_5$), with the coefficients chosen so that the result factors into a single quartic polynomial $q(\sigma_6)$ without the spurious roots times a high-degree polynomial with only spurious roots.

In the previous few paragraphs we have explained in words the process of constructing a Groebner basis for the ideal generated by the polynomials (2.6). Once the roots are found, it remains to evaluate the Jacobian. At the end of the day, the amplitude can be written schematically as a rational function in $\sigma_6$, summed over $\sigma_6$ satisfying some quartic polynomial:

$$A_{6,2} = \sum_{\{\sigma_6 : q(\sigma_6)=0\}} \frac{p(\sigma_6)}{r(\sigma_6)}. \tag{3.6}$$

Abel's theorem guarantees that the result of this sum is a rational function of the coefficients of the polynomials $p$, $q$ and $r$, and it is easy to check numerically that the result precisely matches the gauge theory amplitude of Mangano, Parke, and Xu [13]. More generally, Abel's theorem guarantees that for any $n$ and $d$, Eq. (1.1) turns into a rational function of the covariant quantities $\langle ij \rangle$ and $[ij]$ once all of the roots of Eqs. (2.6) are summed over.

### B. Analysis for special λ

Although the $n=6$, $d=2$ amplitude is complicated in general, instructive analytic expressions can be obtained by considering special cases. For example, let us here consider the case $\lambda_1^2=\lambda_4^2=0$ and $[15]=0$. For this degenerate case, numerical investigation reveals that there are only three roots (one is a double root—the statement that $N_{4,2}=4$ is always true when one counts multiplicities). Let us demonstrate analytically how to find these three roots.

We fix the GL(2) symmetry by setting $a_0^1=1$ and $\sigma_i=\{0,1,-1\}$ for $i=1,2,3$. Also, without loss of generality we can rescale the $\lambda$'s to set $\lambda_i^1=1$. From the $A_i\lambda_i=B_i$ equations for $i=2,3,4,5,6$ we can solve for the moduli $a_1^1$, $a_2^1$, $a_1^2$, $a_2^2$ and $\sigma_4$ in terms of $\sigma_5$, $\sigma_6$. The first solution is $\sigma_4=0$, and the other one is (with $\sigma_{ij}=\sigma_i-\sigma_j$)

$$\sigma_4 = \frac{\lambda_3\lambda_5\lambda_6\sigma_{53}\sigma_{63}\sigma_{56} + \lambda_2[2\lambda_3\lambda_6\sigma_5^2\sigma_{62}\sigma_{63} - \lambda_5\sigma_{52}(\lambda_6\sigma_{56}\sigma_{63} + 2\lambda_3\sigma_{53}\sigma_6^2)]}{\lambda_3\lambda_5\lambda_6\sigma_{53}\sigma_{63}\sigma_{56} + \lambda_2[\lambda_5\lambda_6\sigma_{52}\sigma_{62}\sigma_{56} + 2\lambda_3(-\sigma_6\lambda_5\lambda_{52}\sigma_{53} + \lambda_6\sigma_5\sigma_{62}\sigma_{63})]}. \tag{3.7}$$

The $\sigma_4=0$ root gives a unique solution for $\sigma_5$, $\sigma_6$ when we plug the expressions for $a_1^1$, $a_2^1$, $a_1^2$ and $a_2^2$ into the equations following from the second line of Eqs. (2.6). The nonzero $\sigma_4$ root gives a simple solution for $\sigma_6$,

$$\sigma_6 = \frac{[65](\lambda_3-\lambda_2)\lambda_6}{2[45]\lambda_2\lambda_3 + [65](2\lambda_2\lambda_3 - \lambda_2\lambda_6 - \lambda_3\lambda_6)}, \tag{3.8}$$

and a quadratic equation on $\sigma_5$. In other words, the analogue of the fourth order polynomial described in the previous subsection factorizes into a quadratic one and the square of a linear one. Solving the equations and plugging them into the Jacobian gives a result which agrees numerically with the known gauge theory result.

## IV. CHECKS ON $n$-PARTICLE AMPLITUDES

To summarize, we now know that the formula (1.1) correctly reproduces all MHV and $\overline{\text{MHV}}$ amplitudes, as well as all 6-particle amplitudes. The nontriviality of these checks makes it implausible that some complication arises for further amplitudes which might render Eq. (1.1) invalid. Nevertheless, it would certainly be satisfying to prove that the formula (1.1) is correct, perhaps by showing that it satisfies the recursion relation of [14]. Since we do not have a complete proof yet, we will content ourselves with tabulating several consistency checks that Eq. (1.1) is indeed the tree-level $S$ matrix of YM theory for arbitrary $n$ and $d$.

### A. Some properties of gauge theory scattering amplitudes

Color-ordered partial amplitudes in YM theory satisfy a number of important properties, including the following:

(i) Cyclicity:

$$A(2,3,...,n,1) = A(1,2,...,n), \tag{4.1}$$

(ii) Reflection:

$$A(n,n-1,...,1) = (-1)^n A(1,2,...,n), \tag{4.2}$$

(iii) Conjugation: Parity symmetry implies that the amplitude is invariant under interchanging each helicity $+\leftrightarrow-$ and simultaneously interchanging $\lambda\leftrightarrow\tilde{\lambda}$.[3] The $\mathcal{N}=4$ supersymmetric version of this statement is

$$A(\lambda,\tilde{\lambda},\eta)=\int d^{4n}\psi\exp\left[i\sum_{i=1}^{n}\eta_{iA}\psi_i^A\right]A(\tilde{\lambda},\lambda,\psi). \tag{4.3}$$

(iv) Dual Ward (or sub-cyclic) identity:

$$\sum_{C(1,\ldots,n-1)}A(1,2,3,\ldots,n)=0, \tag{4.4}$$

where $n$ is held fixed in the last position and $C(1,\ldots,n-1)$ denotes the set of cyclic permutations of $\{1,\ldots,n-1\}$. This identity expresses decoupling of the U(1) degree of freedom [15].

(v) In [16] it was proved that YM amplitudes satisfy the following generalization of (iv):

$$\sum_{\mathrm{Perm}(i,j)}A(i_1,\ldots,i_m,j_1,\ldots,j_k,n+1)=0,$$
$$1\leqslant m\leqslant n-1,\quad m+k=n, \tag{4.5}$$

where the sum is taken over permutations of the set $(i_1,\ldots,i_m,j_1,\ldots,j_k)$ which preserve the order of the $(i_1,\ldots,i_m)$ and $(j_1,\ldots,j_k)$ separately.

(vi) Soft-gluon limit: In the limit $p_1\to 0$, the amplitude behaves as

$$A(1^+,2,\ldots,n)\to\frac{\langle n2\rangle}{\langle n1\rangle\langle 12\rangle}A(2,\ldots,n). \tag{4.6}$$

Of course a conjugated version of this equation should also hold in the case when particle 1 has negative helicity. We do not consider that case directly in this paper, since it follows as a result of (iii) above.

(vii) Collinear limit: In the limit $p_1\to zp$ and $p_2\to(1-z)p$ for $z\in(0,1)$ and some $p$ with $p^2=0$, the amplitude behaves as

$$A(1^+,2^+,3,\ldots,n)\to\frac{1}{\sqrt{z(1-z)}}\frac{1}{\langle 12\rangle}A(p^+,3,\ldots,n). \tag{4.7}$$

Again it follows from (iii) that there is an obvious conjugate to this relation for the case when particles 1 and 2 both have negative helicity. The final case, when particles 1 and 2 have opposite helicity, is

$$A(1^+,2^-3,\ldots,n)\to\frac{z^2}{\sqrt{z(1-z)}}\frac{1}{[12]}A(p^+,3,\ldots,n)+\frac{(1-z)^2}{\sqrt{z(1-z)}}\frac{1}{\langle 12\rangle}A(p^-,3,\ldots,n). \tag{4.8}$$

(viii) Multi-particle poles: Color-ordered amplitudes can only have poles in channels corresponding to a sum of cyclically adjacent momenta going on-shell [11]. If we denote $p_{1,m}=p_1+p_2+\cdots+p_m$, then the amplitude factors in the $p_{1,m}^2\to 0$ limit according to

$$A_n(1,\ldots,n)\to\sum_{\chi=\pm}A_{m+1}(1,\ldots,m,p^\chi)\times\frac{i}{p_{1,m}^2}A_{n-m+1}(m+1,\ldots,n,p^{-\chi}). \tag{4.9}$$

Properties (i), (ii), (iv) and (v) are manifest in Eq. (1.1) due to the way the $\sigma_i$ enter in Eq. (2.2). Indeed they follow so trivially from Eq. (2.2) that the reader may well wonder why we have bothered to mention them. We have done so only because not all of these properties are immediately obvious from the Feynman diagram expansion of gauge theory amplitudes. (These properties are also not all manifest in the diagrammatic prescription of [6].)

Of the remaining properties, (iii), (vi) and (vii) will be proved in the following subsections. The final property (viii) regarding multi-particle poles will not be addressed here. Indeed, note that a proof that Eq. (1.1) satisfies (viii) would essentially be a proof that Eq. (1.1) is correct—since a tree-level YM amplitude is uniquely fixed by its poles (and their residues).

### B. Parity symmetry

The parity symmetry (4.3) is obvious in gauge theory but not manifest in the formula (1.1).[4] On the individual component amplitudes $A_{n,d}$, Eq. (4.3) says that

$$A_{n,d}(\lambda,\tilde{\lambda},\eta)=\int d^{4n}\psi\exp\left[i\sum_{i=1}^{n}\eta_{iA}\psi_i^A\right]\tilde{A}_{n,n-d-2}(\tilde{\lambda},\lambda,\psi), \tag{4.10}$$

thereby relating an integral over the moduli space of degree $d$ curves to an integral over the moduli space of degree $n-d-2$ curves.

The proof of Eq. (4.10) is fairly straightforward. We start by looking for a way to relate the bosonic part of the amplitudes:

[3]This transformation makes sense with our choice of signature (see footnote 1). In Minkowski signature we would also have to take the complex conjugate of the amplitude.

[4]The parity symmetry was also very recently discussed in [9] in the framework of [3].

$$A_{n,d}(\lambda,\tilde{\lambda}) = \int d\mathcal{M}_{n,d} \prod_{i=1}^{n} \delta^2(\lambda_i^a - \xi_i P_i^a) \times \prod_{k=0}^{d} \delta^2\left(\sum_{i=1}^{n} \xi_i \sigma_i^k \tilde{\lambda}_i^{\dot{a}}\right),$$

$$\tilde{A}_{n,n-d-2}(\tilde{\lambda},\lambda) = \int d\tilde{\mathcal{M}}_{n,n-d-2} \prod_{i=1}^{n} \delta^2(\tilde{\lambda}_i^{\dot{a}} - \tilde{\xi}_i \tilde{P}_i^{\dot{a}}) \times \prod_{l=0}^{n-d-2} \delta^2\left(\sum_{i=1}^{n} \tilde{\xi}_i \tilde{\sigma}_i^l \lambda_i^a\right). \quad (4.11)$$

Here $d\tilde{\mathcal{M}}_{n,n-d-2}$ and $\tilde{P}$ are obvious generalizations of Eqs. (2.2) and (2.1):

$$d\tilde{\mathcal{M}}_{n,n-d-2} = \frac{d^{2(n-d-2)+2}\tilde{a}\, d^n\tilde{\sigma}\, d^n\tilde{\xi}}{\mathrm{vol}[\mathrm{GL}(2)]} \prod_{i=1}^{n} \frac{1}{\tilde{\xi}_i(\tilde{\sigma}_i - \tilde{\sigma}_{i+1})},$$

$$\tilde{P}_i^{\dot{a}} = \sum_{l=0}^{n-d-2} \tilde{a}_l^{\dot{a}} \tilde{\sigma}_i^l. \quad (4.12)$$

We will show that after integrating out the moduli $a$, the first set of delta functions in $A$ exactly transforms into the second set of delta functions in $\tilde{A}$ (and vice versa) when one makes the change of variables

$$\tilde{\sigma}_i = \sigma_i, \quad \tilde{\xi}_i = \frac{1}{\xi_i \Pi_{j\neq i}(\sigma_i - \sigma_j)}. \quad (4.13)$$

The Jacobian for this coordinate transformation is unity, but we will pick up a simple Jacobian from manipulating the bosonic delta functions. This Jacobian will exactly cancel a similar fermionic determinant.

Let us begin by studying the quantity

$$p_m = \sum_{i=1}^{n} \frac{\sigma_i^m}{\Pi_{j\neq i}(\sigma_i - \sigma_j)}. \quad (4.14)$$

We claim that $p_m$ is a polynomial in the $\sigma_i$'s of degree $m-n+1$. To see this, consider $p_m$ as an analytic function of $z=\sigma_n$ (this can of course be repeated for all of the $\sigma$'s). It looks like $p_m(z)$ might have poles at the other $\sigma_i$, but in fact it is easy to see that the residue is always zero. So $p_m(z)$ has no poles, and grows at infinity like $z^{m-n+1}$, so it must be a polynomial of degree $m-n+1$. In particular, $p_m$ vanishes for $m<n-1$, and $p_{n-1}=1$.

Now consider the first type of delta function in $A$ (we focus on one value of $a$ and restore covariance later),

$$I = \int d^{d+1}a \prod_{i=1}^{n} \delta(\lambda_i - \xi_i P_i), \quad (4.15)$$

and take linear combinations of the delta functions according to the $n\times n$ matrix with entries

$$M_{mi} = \frac{\sigma_i^m}{\xi_i \Pi_{j\neq i}(\sigma_i - \sigma_j)}, \quad i=1,\ldots,n, \quad m=0,\ldots,n-1. \quad (4.16)$$

That is, we write

$$I = \int d^{d+1}a (\det M) \prod_{m=0}^{n-1} \delta\left(\sum_{i=1}^{n} M_{mi}(\lambda_i - \xi_i P_i)\right). \quad (4.17)$$

The second term in the delta function is now

$$\sum_{i=1}^{n} \frac{\sigma_i^m}{\xi_i \Pi_{j\neq i}(\sigma_i - \sigma_j)} \xi_i \sum_{k=0}^{d} a_k \sigma_i^k = \sum_{k=0}^{d} a_k p_{k+m}, \quad (4.18)$$

using the definitions (2.1), (4.14) and (4.16). Then recalling that $p_{k+m}$ is zero for $m<n-d-1$, we can split the delta functions into two kinds:

$$I = (\det M) \prod_{m=0}^{n-d-2} \delta\left(\sum_{i=1}^{n} \tilde{\xi}_i \sigma_i^m \lambda_i\right) \int d^{d+1}a \times \prod_{m=n-d-1}^{n-1} \delta\left(\sum_{i=1}^{n} \tilde{\xi}_i \sigma_i^m \lambda_m - \sum_{k=0}^{d} a_k p_{k+m}\right). \quad (4.19)$$

The $d+1$ moduli now appear linearly in the last $d+1$ delta functions and can be integrated out trivially. The Jacobian for this is just 1, because $p_{k+m}$ is a triangular matrix with diagonal entries $p_{n-1}=1$. Finally, we conclude that

$$I = \int d^{d+1}a \prod_{i=1}^{n} \delta(\lambda_i - \xi_i P_i) = \left[V \prod_{i=1}^{n} \tilde{\xi}_i\right] \prod_{m=0}^{n-d-2} \delta\left(\sum_{i=1}^{n} \tilde{\xi}_i \sigma_i^m \lambda_i\right), \quad (4.20)$$

where $V$ is the Vandermonde determinant of all of the $\sigma$'s and the term in brackets comes from evaluating $\det(M)$.

The next step is to simply apply Eq. (4.20) in reverse to get

$$\prod_{k=0}^{d} \delta\left(\sum_{i=1}^{n} \xi_i \sigma_i^k \tilde{\lambda}_i\right) = \left[V \prod_{i=1}^{n} \xi_i\right]^{-1} \int d^{(n-d-2)+1}\tilde{a} \times \prod_{i=1}^{n} \delta(\tilde{\lambda}_i - \tilde{\xi}_i \tilde{P}_i). \quad (4.21)$$

Finally we can combine Eqs. (4.20) and (4.21) and restore the $a$ and $\dot{a}$ indices to arrive at

$$\int d\mathcal{M}_{n,d}\prod_{i=1}^{n}\delta^2(\lambda_i^a-\xi_i P_i^a)\prod_{k=0}^{d}\delta^2\left(\sum_{i=1}^{n}\xi_i\sigma_i^k\tilde{\lambda}_i^{\dot{a}}\right)$$
$$=\int d\tilde{\mathcal{M}}_{n,n-d-2}\left[\prod_{i=1}^{n}\frac{\tilde{\xi}_i}{\xi_i}\right]^2\prod_{i=1}^{n}\delta^2(\tilde{\lambda}_i^{\dot{a}}-\tilde{\xi}_i\tilde{P}_i^a)$$
$$\times\prod_{l=0}^{n-d-2}\delta^2\left(\sum_{i=1}^{n}\xi_i\sigma_i^l\lambda_i^a\right). \tag{4.22}$$

We have now related the bosonic integral over degree $d$ curves to the bosonic integral over degree $n-d-2$ curves, up to a factor which with the help of Eq. (4.13) can be written as

$$\left[\prod_{i=1}^{n}\frac{\tilde{\xi}_i}{\xi_i}\right]^2=\left[V\prod_{i=1}^{n}\tilde{\xi}_i\right]^4. \tag{4.23}$$

In fact, this is precisely the factor which should arise from the fermionic Fourier transform in the formula (4.10):

$$\int d^{4n}\psi\exp\left[i\sum_{i=1}^{n}\eta_{iA}\psi_i^A\right]\prod_{l=0}^{n-d-2}\delta^4\left(\sum_{i=1}^{n}\tilde{\xi}_i\sigma_i^l\psi_i^A\right)$$
$$=\left[V\prod_{i=1}^{n}\tilde{\xi}_i\right]^4\prod_{k=0}^{d}\delta^4\left(\sum_{i=1}^{n}\xi_i\sigma_i^k\eta_{iA}\right). \tag{4.24}$$

This completes the proof that Eq. (1.1) satisfies the conjugation property (4.10).

Incidentally, the above arguments show that given any solution of Eqs. (2.6) one can construct a solution of the conjugate equations

$$\tilde{\lambda}_i^{\dot{a}}=\tilde{\xi}_i\sum_{l=0}^{n-d-2}\tilde{\sigma}_i^l\tilde{a}_l^{\dot{a}},\quad \dot{a}=1,2,\quad i=1,\ldots,n,$$

$$0=\sum_{i=1}^{n}\tilde{\xi}_i\tilde{\sigma}_i^l\lambda_i^a,\quad a=1,2,\quad l=0,\ldots,n-d-2, \tag{4.25}$$

by taking $\tilde{\sigma}$ and $\tilde{\xi}$ to be given by Eqs. (4.13). It is not necessary to independently specify the $\tilde{a}_l^{\dot{a}}$ since the top equations in Eqs. (4.25) determine them uniquely in terms of $(\tilde{\sigma}_i,\tilde{\xi}_i)$. Thus, we have shown that

$$N_{n,n-d-2}=N_{n,d}, \tag{4.26}$$

and, moreover, that the contribution to $A_{n,d}$ from any given root is exactly the conjugate of the contribution of that root to $A_{n,n-d-2}$. The relation (4.26) is reminiscent of the relation between Betti numbers for a manifold of dimension $n-2$ as well as of the relation between Hodge numbers under mirror symmetry. It would be interesting to find a relation between $N_{n,d}$ and some invariants of $\mathbb{P}^{3|4}$ (perhaps Gromov-Witten invariants) or of its moduli space of holomorphic curves.

### C. Soft gluon limit

For both the soft gluon and collinear limits, comparing the left- and right-hand sides of Eqs. (4.6) and (4.7) reveals that we will have to perform two integrals and eliminate two delta functions. Clearly we want to eliminate the appearance of gluon 1 on the right-hand side, so we should eliminate the two delta functions $\delta^2(\lambda_1^a-\xi_1 P_1^a)$ (for $a=1,2$) by performing the integrals over $\xi_1$ and $\sigma_1$. In general there can be several roots which contribute to this integral. However, we are only interested in roots which in the desired limit give rise to a pole in the amplitude. We will argue that only one root contributes to the coefficient of this pole.

A prototype for both the soft gluon and collinear limits involves an integral of the form

$$I_i=\lim_{\langle 1i\rangle\to 0}\int\frac{d\sigma_1}{\sigma_1-\sigma_i}f(\sigma_1)\delta\left(\frac{\langle i1\rangle}{\lambda_1^1\lambda_i^1}-\left[\frac{B_1}{A_1}-\frac{B_i}{A_i}\right]\right). \tag{4.27}$$

Specifically, we are interested in the poles of this integral. We do not yet need the explicit form of $A$, $B$ or $f$, and need only to make assumptions which are completely reasonable for the application at hand: $B/A$ is a rational function of $\sigma$ with isolated roots, and the function $f$ has no poles in $\sigma_1$.

The quantity in brackets in Eq. (4.27) vanishes when $\sigma_1=\sigma_i$ and hence can be written as

$$\left[\frac{B_1}{A_1}-\frac{B_i}{A_i}\right]=(\sigma_1-\sigma_i)F(\sigma_1-\sigma_i,\sigma_i) \tag{4.28}$$

for some $F$. Changing integration variables from $\sigma_1$ to $w=\sigma_1-\sigma_i$ gives

$$I_i=\lim_{\langle 1i\rangle\to 0}\int\frac{dw}{w}\delta(g(w)),\quad g(w)=\frac{\langle i1\rangle}{\lambda_1^1\lambda_i^1}-wF(w,\sigma_i). \tag{4.29}$$

In the limit $\langle 1i\rangle\to 0$ the roots of $g(w)$ are easy to analyze. There is one root (which we will call $w=w_0$) for which $w$ is small (of the same order as $\langle 1i\rangle$), and there may be other roots for which $F(w,\sigma_i)$ is small. We assume there is no degeneracy among the possible roots. Integrating the delta function gives a factor of $1/g'(w)$, which is a number of order 1 at any of the roots. Therefore, the only pole in the integral $I_i$ comes from the factor of $1/w$ evaluated on the root $w=w_0\to 0$.

The value of $w_0$ is given by the implicit equation

$$w_0=\frac{\langle i1\rangle}{\lambda_1^1\lambda_i^1}\frac{1}{F(w_0,\sigma_i)}, \tag{4.30}$$

with $F(w_0,\sigma_i)$ being of order unity. The contribution of this root to the integral is

$$I_i=\frac{1}{w}\left(\frac{\partial g}{\partial w}\right)^{-1}\Bigg|_{w=w_0}=\frac{1}{w_0}[F(w_0,\sigma_i)+w_0\partial_w F(w_0,\sigma_i)]^{-1}. \tag{4.31}$$

Since $F$ is a rational function and $F(w_0,\sigma_i)$ is of order unity, the derivative $\partial_w F(w_0,\sigma_i)$ cannot blow up. Therefore the second term in brackets can be ignored as $w_0\to 0$, so using Eq. (4.30) we arrive at the formula

$$I_i = \lim_{\langle 1i\rangle\to 0}\int \frac{d\sigma_1}{\sigma_1-\sigma_i} f(\sigma_1)\,\delta\left(\frac{\langle i1\rangle}{\lambda_1^1\lambda_i^1}-\left[\frac{B_1}{A_1}-\frac{B_i}{A_i}\right]\right) = \frac{\lambda_1^1\lambda_i^1}{\langle i1\rangle} f(\sigma_i), \tag{4.32}$$

which is valid under the assumptions on $f$, $A$ and $B$ given above.

Now let us turn our attention to the soft gluon limit (4.6). First we set the helicity of gluon 1 to $+1$ by setting $\eta_1=0$ in $A_{n,d}$. This destroys the $i=1$ term in the third delta function in Eq. (1.1). In the second delta function, the $i=1$ term also vanishes trivially in the soft limit since $\tilde\lambda_1^{\dot a}\to 0$. Particle number 1 therefore only appears in the integrals

$$\int d\sigma_1 d\xi_1 \frac{1}{\xi_1(\sigma_1-\sigma_2)(\sigma_n-\sigma_1)}\,\delta(\lambda_1^1-\xi_1 A_1)\times\delta(\lambda_1^2-\xi_1 B_1). \tag{4.33}$$

The $\xi_1$ integral is trivial and leads to

$$\frac{1}{\sigma_n-\sigma_2}\frac{1}{(\lambda_1^1)^2}\int d\sigma_1\left[\frac{1}{\sigma_1-\sigma_2}-\frac{1}{\sigma_1-\sigma_n}\right]\delta\left(\frac{\lambda_1^2}{\lambda_1^1}-\frac{B_1}{A_1}\right). \tag{4.34}$$

Now we are completely free to subtract from the argument of the delta function an amount which is equal to zero in the form of $\lambda_i^2/\lambda_i^1-B_i/A_i$ for any $i\neq 1$. (This is guaranteed to be zero by the other $\lambda_i^a-\xi_i P_i^a$ delta functions.) Then we simply apply the formula (4.32), once with $i=2$ and once with $i=n$, to obtain the factor

$$\frac{1}{\sigma_n-\sigma_2}\frac{1}{(\lambda_1^1)^2}(I_2-I_n)=\frac{1}{\sigma_n-\sigma_2}\frac{1}{(\lambda_1^1)^2}\left[\frac{\lambda_1^1\lambda_2^1}{\langle 21\rangle}-\frac{\lambda_1^1\lambda_n^1}{\langle n1\rangle}\right] = \frac{1}{\sigma_n-\sigma_2}\left[\frac{\langle n2\rangle}{\langle 21\rangle\langle n1\rangle}\right]. \tag{4.35}$$

The factor of $1/(\sigma_n-\sigma_2)$ is needed to write the correct measure factor (2.2) for the $(n-1)$-particle amplitude $A(2,\ldots,n)$. Gluon number one has now completely disappeared from the integral, leaving only the overall factor in brackets, in agreement with Eq. (4.6).

### D. Collinear limit

First we consider the factor

$$\frac{1}{\sigma_n-\sigma_2}\frac{1}{(\lambda_1^1)^2}\int d\sigma_1\left[\frac{1}{\sigma_1-\sigma_2}-\frac{1}{\sigma_1-\sigma_n}\right]\delta\left(\frac{\lambda_1^2}{\lambda_1^1}-\frac{B_1}{A_1}\right), \tag{4.36}$$

which arises exactly as in the previous subsection. However, whereas we could there use Eq. (4.32) for both $i=2$ and $i=n$ (since $\langle 12\rangle$ and $\langle 1n\rangle$ were both going to zero), here we can only use Eq. (4.32) for $i=2$ since only $\langle 12\rangle\to 0$ in the collinear limit. Therefore the second term in brackets in Eq. (4.36) gives no contribution to the pole, and we only pick up the factor

$$\frac{1}{\sigma_n-\sigma_2}\frac{\lambda_2^1}{\lambda_1^1}\frac{1}{\langle 21\rangle}=\frac{1}{\sigma_n-\sigma_2}\left[\sqrt{\frac{1-z}{z}}\frac{1}{\langle 21\rangle}\right]. \tag{4.37}$$

At this stage the integrals over the variables $\sigma_1$ and $\xi_1$ associated with gluon number 1 have been performed, but those associated with gluon 2 remain and we must rewrite the $\lambda_2$ dependence in terms of $\lambda=\lambda_2/\sqrt{1-z}$. In the $\xi_2$ integral this is accomplished by rescaling $\xi_2$ in order to obtain

$$\int\frac{d\xi_2}{\xi_2}\,\delta(\lambda_2^1-\xi_2 A_2)\,\delta(\lambda_2^2-\xi_2 B_2) = \left[\frac{1}{1-z}\right]\int\frac{d\xi_2'}{\xi_2'}\,\delta(\lambda^1-\xi_2' A_2)\,\delta(\lambda^2-\xi_2' B_2). \tag{4.38}$$

The last delta functions to check are the ones of the form

$$\begin{aligned}\delta^2\left(\sum_{i=1}^n \xi_i\sigma_i^k\tilde\lambda_i^{\dot a}\right) &= \delta^2\left(\xi_1\sigma_1^k\tilde\lambda_1^{\dot a}+\xi_2\sigma_2^k\tilde\lambda_2^{\dot a}+\sum_{i=3}^n\xi_i\sigma_i^k\tilde\lambda_i^{\dot a}\right)\\ &= \delta^2\left(\frac{\lambda_1^1}{A_1}\sigma_1^k\tilde\lambda_1^{\dot a}+\xi_2\sigma_2^k\tilde\lambda_2^{\dot a}+\sum_{i=3}^n\xi_i\sigma_i^k\tilde\lambda_i^{\dot a}\right)\\ &= \delta^2\left(\frac{z}{1-z}\frac{\lambda_2^1}{A_2}\sigma_2^k\tilde\lambda_2^{\dot a}+\xi_2\sigma_2^k\tilde\lambda_2^{\dot a}+\sum_{i=3}^n\xi_i\sigma_i^k\tilde\lambda_i^{\dot a}\right),\end{aligned} \tag{4.39}$$

where in the first line we used the fact that we already integrated out $\xi_1$ setting it to $\xi_1=\lambda_1^1/A_1$, in the second line we used the fact that we integrated out $\sigma_1$ setting $\sigma_1=\sigma_2$, and in the third line we used the fact that $\lambda_1\tilde\lambda_1=[z/(1-z)]\lambda_2\tilde\lambda_2$. Of course, we know that $\xi_2$ will eventually be set by a delta function to the value $\lambda_2^1/A_2$, so we may as well write the final line as

$$\delta^2\left(\frac{1}{1-z}\xi_2\sigma_2^k\tilde\lambda_2^{\dot a}+\sum_{i=3}^n\xi_i\sigma_i^k\tilde\lambda_i^{\dot a}\right) = \delta^2\left(\xi_2'\sigma_2^k\lambda^{\dot a}+\sum_{i=3}^n\xi_i\sigma_i^k\tilde\lambda_i^{\dot a}\right), \tag{4.40}$$

keeping in mind that $\tilde\lambda_2=\sqrt{1-z}\tilde\lambda$ and $\xi_2=\sqrt{1-z}\xi_2'$.

What remains has precisely the structure of the amplitude $A(p,3,\ldots,n)$, together with the extra factors in brackets from Eqs. (4.37) and (4.38), in complete agreement with the collinear limit (4.7). The conjugate of this equation follows from the parity transformation discussed in Sec. IV B. The most notable fact following from that analysis is that the pole arises from the root satisfying $\sigma_1-\sigma_2\simeq[12]$. One might

attempt to prove the last collinear limit (4.8) by combining the above discussion with this observation. Then, the different $z$ dependence might arise from the $\lambda$ dependence of the fermionic integrals.

## V. CONCLUSIONS AND SPECULATIONS

In this paper we have presented strong evidence that the formula (1.1) encodes the complete tree-level $S$ matrix of Yang-Mills theory in four dimensions. Explicit calculation has now shown that Eq. (1.1) agrees with YM theory for all MHV and $\overline{\text{MHV}}$ amplitudes, as well as all 6-particle non-MHV amplitudes. Moreover the analysis of Sec. IV shows that for any $n$, Eq. (1.1) satisfies a number of important properties required of gauge theory amplitudes, including parity symmetry. Many interesting directions remain open.

Of primary importance is to understand the connection between the formula (1.1), which was obtained in [2] following the suggestion in [1] that one should consider a single instanton of degree $d$ in the topological B model on $\mathbb{P}^{3|4}$, and the diagrammatic procedure of [6], in which arbitrary amplitudes are built out of $d$ disconnected amplitudes, each of degree 1. We suspect that formulating a proof that Eq. (1.1) factorizes correctly onto multiparticle poles would essentially amount to proving the equivalence of Eq. (1.1) and the rules of [6], simply because the factorization properties are completely manifest in the latter.

The numerical coefficient in front of Eq. (1.1) was fixed by comparing with gauge theory. We have not computed this coefficient independently in the B model. It is conceivable that the degree $d$ contribution and the separated degree 1 contributions (as well as other contributions) have to be added together to fully reproduce the normalization of the gauge theory scattering amplitudes. It is also possible that the B model has some huge symmetry group which relates the connected instanton contribution (1.1) to the fully disconnected instantons of [6].

Of course, even forgetting for the moment about the B model, it would also be very interesting to prove rigorously that the formula (1.1) is the tree-level $S$ matrix of Yang-Mills theory. To this end it would be useful to understand better the mathematical structure of Eqs. (2.6), and in particular to learn how many roots they have for general $n$ and $d$ (i.e., what is the degree of the corresponding Groebner basis). These numbers might be related to some interesting invariants of $\mathbb{P}^{3|4}$ or of its moduli space of holomorphic curves, and perhaps the equality of $N_{n,d}$ and $N_{n,n-d-2}$ could be understood in this language.

Finally, all of our considerations have applied to the tree-level $S$ matrix in gauge theory. An obvious next step of great interest would be to see what light the topological B model can shed on one-loop calculations [17].

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