Localization of non-Abelian gauge fields on domain walls at weak coupling: D-brane prototypes

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Building on our previous results (Ref. [4]), we study D-brane and string prototypes in weakly coupled (3+1)-dimensional supersymmetric field theory engineered to support (2+1)-dimensional domain walls, "non-Abelian" strings and various junctions. Our main but not exclusive task is the study of localization of non-Abelian gauge fields on the walls. The model we work with is $\mathcal{N}=2$ QCD, with the gauge group SU(2) $\times$U(1) and $N_f=4$ flavors of fundamental hypermultiplets (referred to as quarks), perturbed by the Fayet-Iliopoulos term of the U(1) factor. In the limit of large but almost equal quark mass terms a set of vacua exists in which this theory is at weak coupling. We focus on these vacua (called the quark vacua). We study elementary BPS domain walls interpolating between selected quark vacua, as well as their bound state, a composite wall. The composite wall is demonstrated to localize a non-Abelian gauge field on its world sheet. Next, we turn to the analysis of recently proposed "non-Abelian" strings (flux tubes) which carry orientational moduli corresponding to rotations of the "color-magnetic" flux direction inside a global O(3). We find a 1/4-BPS solution for the string ending on the composite domain wall. The end point of this string is shown to play the role of a non-Abelian (dual) charge in the effective world volume theory of non-Abelian (2+1)-dimensional vector fields confined to the wall.

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I. INTRODUCTION

String theory which emerged from dual hadronic models in the late 1960s and 1970s, elevated to the "theory of everything" in the 1980s and 1990s when it experienced an unprecedented expansion, has seemingly entered a "return-to-roots" stage. Results and techniques of string-D-brane theory, being applied to non-Abelian field theories (both, supersymmetric and nonsupersymmetric), have recently generated numerous predictions for gauge theories at strong coupling. If the latter are, in a sense, dual to string-D-brane theory — as is generally believed to be the case — they must support domain walls (of the D-brane type) [1], and we know, they do [2,3]. In addition, string-D-brane theory teaches us that a fundamental string that starts on a confined quark, can end on the domain wall [1].

In our previous paper [4] we embarked on the studies of field-theoretic prototypes of D branes and strings. To this end we considered (2+1)-dimensional domain walls in (3+1)-dimensional $\mathcal{N}=2$ SQCD with the SU(2) gauge group (and $N_f=2$ flavors of fundamental hypermultiplets—quarks), perturbed by a small mass term of the adjoint matter. In fact, our analysis reduced to that of the effective low-energy $\mathcal{N}=2$ SQED with a (generalized) Fayet-Iliopoulos term. We found 1/2 BPS-saturated domain wall solution interpolating between two quark vacua at weak coupling. The main finding was the *localization* of a U(1) gauge field on this domain wall. We also demonstrated that the Abrikosov-Nielsen-Olesen magnetic flux tube can end on the wall.

The goal of the present work is the extension of the above results. Now we want to consider composite walls, analogs of a stack of D branes, to see that they localize non-Abelian gauge fields, say U(2). Our second task—as important to us as the first one—is the study of non-Abelian flux tubes, and, especially, how they end on the walls. In this way we continue the line of research initiated in Refs. [5–7].

The setup that will provide us with the appropriate tools is the same as in the previous paper [4], namely $\mathcal{N}=2$ SQCD analyzed by Seiberg and Witten [8,9]. Compared to Ref. [4] we will deal with a somewhat different version, however. We will start from the SU(3) theory with four "quark" hypermultiplets ($N_f=4$) in the fundamental representation. The SU(3) gauge group will be spontaneously broken down to SU(2)$\times$U(1) at a large scale m where $m\sim m_A=m$, $A=1,2,3,4$, and m_1, m_2, m_3 and m_4 are the mass terms ascribed to the four quark flavors that are present in the model. Generically, all m_A's are different, but we will choose a nongeneric configuration.

Although SU(3) $\mathcal{N}=2$ SQCD provides a conceptual skeleton for our setup, in essence its role is to stay behind the scene, as a motivating factor. Since the gauge SU(3) group will be broken at the largest scale relevant to the model, and the bulk of our present work refers to lower scales, in practice our setup is based on SU(2)$\times$U(1) gauge model with four quark hypermultiplets and unbroken $\mathcal{N}=2$. The underlying SU(3) $\mathcal{N}=2$ SQCD which one may or may not keep in mind in reading this paper, will be referred to as the "prototheory."

With four flavors, the SU(2) subsector is conformal; therefore the problem we address can be fully analyzed in the weak coupling regime. In fact, two of four quarks will be just spectators while the other two will play a nontrivial role in the solution. The role of the spectators is to ensure the conformal regime (see below).

As was mentioned, we will deal with the gauge symmetry breaking pattern of a hierarchical type. First, at a large scale $\sim m \gg \Lambda_{\rm SU(3)}$ the gauge group SU(3) is broken down to a subgroup SU(2)$\times$U(1) by the vacuum expectation values

(VEV's) of the adjoint scalars.[1] [Here $\Lambda_{SU(3)}$ is the dynamical scale parameter of SU(3).] Second, at a lower scale $\sim\sqrt{\mu m_A}$, in the presence of the adjoint mass term $\mu\,\mathrm{Tr}\,\Phi^2$, the light squark fields acquire VEV's of the color-flavor diagonal form (''color-flavor locking''),

$$q_k^A=\delta_A^k\sqrt{\mu m},\quad k,A=1,2. \tag{1.1}$$

In each vacuum to be considered below, the squark fields of two (out of four) flavors will be condensed, so that we can label each vacuum by a set of two numbers, (AB), showing that the flavors A and B are condensed. For instance, we will speak of 12-vacuum, 13-vacuum and so on.

The basic idea of the gauge field localization on the domain walls is that the quark fields (almost) vanish inside the wall. Therefore, the gauge group $SU(2)\times U(1)$, being Higgsed in the vacua to the right and to the left of the wall, is restored inside the wall. Correspondingly, dual gauge bosons, being confined outside the wall are unconfined (or less confined), inside, thus leading to localization [2].

In fact, there is another scale in the problem which plays an important role in the aforementioned hierarchy. In dealing with domain walls we cannot consider the limit in which *all* quark masses are exactly equal. In this limit the pairs of appropriate vacua coalesce, and we have no domain walls interpolating between them. Therefore, we consider the limit of almost coinciding quark mass terms,

$$m_1=m_2,\quad m_3=m_4;\quad \Delta m\equiv m_1-m_3;\quad |\Delta m|\ll m. \tag{1.2}$$

The resulting hierarchy

$$|\Delta m|,\ m\gg\Lambda_{SU(3)},$$
$$\Lambda_{SU(2)}\ll\sqrt{\mu m}\ll|\Delta m|\ll m \tag{1.3}$$

is exhibited in Fig. 1, together with the behavior of the corresponding gauge couplings. Here

$$m=\frac{m_1+m_3}{2}. \tag{1.4}$$

Note that $\Lambda_{SU(2)}$ is a would-be SU(2) dynamical scale. It is of the order of

$$\Lambda_{SU(2)}\sim|\Delta m|\exp(-4\pi^2/g^2_{SU(2)}). \tag{1.5}$$

[1]The generic pattern of the SU(3) gauge group breaking by the adjoint VEV's is $SU(3)\to U(1)\times U(1)$. This case essentially reduces to the problem which had been considered previously [2,4,10]—localization of the Abelian gauge fields on the wall. Here we are interested in localization of the non-Abelian gauge fields. Therefore, we will deal with a special regime in which SU(3) $\to SU(2)\times U(1)$. In Refs. [11,12] (see also [13]) it was shown that some of $\mathcal{N}=1$ vacua of SU(N) $\mathcal{N}=2$ SQCD can preserve a non-Abelian subgroup.

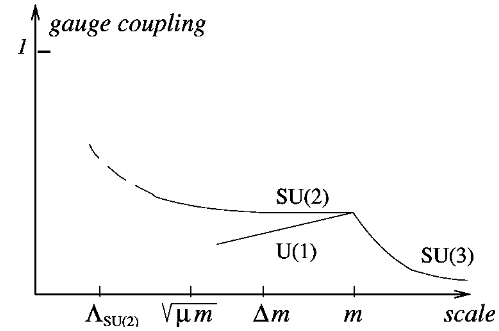

FIG. 1. The scale hierarchy: illustrating the fact that the gauge couplings never become large in the problem at hand. According to Eq. (2.6) $\mu m=\xi/6$.

That is where the SU(2) gauge coupling could have exploded. However, the problem under consideration is insensitive to this scale, as we will explain in detail in due course.

The theory at hand has domain walls of distinct types. Assume that in the vacuum to the left of the wall the squarks with the flavor indices 1 and 2 condense. If in the vacuum to the right of the wall the condensed squarks are 1 and 3, we will call such a wall *elementary*. If, on the other hand, the condensed squarks to the right of the wall are 3 and 4, this wall is obviously composite—it ''consists'' of two elementary walls, $12\to13$ and $13\to34$; see Fig. 2.

The domain wall which localizes SU(2) gauge fields is *not* elementary. It is a bound state of two elementary domain walls placed at one and the same position. This is in accordance with the string-brane picture in which SU(2) gauge theory is localized on the world volume of a stack of two coinciding D-branes. If, however, the two D-branes are separated, then in string theory the SU(2) gauge group is broken to $U(1)^2$, while the masses of the ''charged'' W bosons are linear in the brane separations. We will recover this picture in our field-theoretical setup.

The first stage of the spontaneous symmetry breaking, $SU(3)\to SU(2)\times U(1)$, is well studied in the literature, and presents no interest for our purposes. Therefore, our dynamical analysis will start in essence from the $SU(2)\times U(1)$ model. If one wishes, one can keep in mind that this latter model is originally embedded in SU(3) $\mathcal{N}=2$ SQCD, the ''prototheory,'' but this is not crucial.

Next, we turn to the analysis of recently proposed non-Abelian strings (flux tubes) which carry orientational moduli corresponding to rotations of the ''color-magnetic'' flux di-

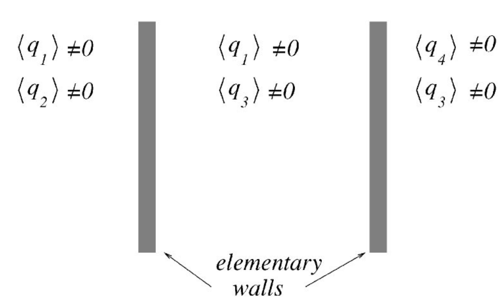

FIG. 2. Two elementary walls which comprise a composite $12\to34$ wall.

rection inside a global SU(2) [6] (similar results in three dimensions were obtained in Ref. [5]).[2] We find a 1/4-BPS solutions for the non-Abelian string ending on the composite domain wall. The end point of the string is shown to play the role of a non-Abelian charge in the effective world volume theory of non-Abelian (2+1)-dimensional gauge fields confined to the wall.

II. THEORETICAL SETUP: SU(2)×U(1) $\mathcal{N}=2$ SQCD

In this section we will describe the model we will work with [a descendant of SU(3) Seiberg-Witten model with four matter hypermultiplets] and appropriate pairs of vacua which are connected by elementary and composite walls.

A. The model

As was mentioned in Sec. I, the model we will deal with derives from $\mathcal{N}=2$ SQCD with the gauge group SU(3) and four flavors of the quark hypermultiplets. At a generic point on the Coulomb branch of this theory, the gauge group is broken down to $U(1)\times U(1)$. We will be interested, however, in a particular subspace of the Coulomb branch, on which the gauge group is broken down to $SU(2)\times U(1)$. We will enforce[3] this regime by a special choice of the quark mass terms; see Eq. (1.2).

The breaking $SU(3)\to SU(2)\times U(1)$ occurs at the scale m which is supposed to lie very high, $m\gg\Lambda_{SU(3)}$. Correspondingly, the masses of the gauge bosons from $SU(3)/SU(2)\times U(1)$ and their superpartners, proportional to m, are very large, and so are the masses of the third color component of the matter fields in the fundamental representation. We will be interested in phenomena at the scales $\ll m$. Therefore, our starting point is in fact the $SU(2)\times U(1)$ model with four matter fields in the doublet representation of SU(2), as it emerges after the $SU(3)\to SU(2)\times U(1)$ breaking. These matter fields are also coupled to the U(1) gauge field.

The field content of $SU(2)\times U(1)$ $\mathcal{N}=2$ SQCD with four flavors is as follows. The $\mathcal{N}=2$ vector multiplet consists of the U(1) gauge fields A_μ and SU(2) gauge field A^a_μ, (here $a=1,2,3$), their Weyl fermion superpartners $(\lambda^1_\alpha,\ \lambda^2_\alpha)$ and $(\lambda^{1a}_\alpha,\ \lambda^{2a}_\alpha)$, and complex scalar fields a, and a^a, the latter in the adjoint of SU(2). The spinorial index of λ's runs over $\alpha=1,2$. In this sector the global $SU(2)_R$ symmetry inherent to the model at hand manifests itself through rotations $\lambda^1\leftrightarrow\lambda^2$.

The quark multiplets of $SU(2)\times U(1)$ theory consist of the complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) and the Weyl fermions ψ^{kA} and $\tilde{\psi}_{Ak}$, all in the fundamental representation of SU(2) gauge group. Here $k=1,2$ is the color index while A is the flavor index, $A=1,2,3,4$. Note that the scalars q^{kA} and $\bar{\tilde{q}}^{kA}\equiv\overline{\tilde{q}_{Ak}}$ form a doublet under the action of the global $SU(2)_R$ group.

The original SU(3) theory was perturbed by adding a small mass term for the adjoint matter, via the superpotential $\mathcal{W}=\mu\,\mathrm{Tr}\,\Phi^2$. Generally speaking, this superpotential breaks $\mathcal{N}=2$ down to $\mathcal{N}=1$. The Coulomb branch shrinks to a number of *isolated* $\mathcal{N}=1$ vacua [11,12]. In the limit of $\mu\to 0$ these vacua correspond to special singular points on the Coulomb branch in which pair of monopoles or dyons or quarks become massless. The first three of these points (often referred to as the Seiberg-Witten vacua) are always at strong coupling. They correspond to $\mathcal{N}=1$ vacua of pure SU(3) gauge theory.

The massless quark points—they present vacua of a distinct type, to be referred to as the quark vacua—may or may not be at weak coupling depending on the values of the quark mass parameters m_A. If $m_A\gg\Lambda_{SU(3)}$, the quark vacua do lie at weak coupling. Below we will be interested only in the quark vacua assuming that the condition $m_A\gg\Lambda_{SU(3)}$ is met.

In the low-energy $SU(2)\times U(1)$ theory, which is our starting point, the perturbation $\mathcal{W}=\mu\,\mathrm{Tr}\,\Phi^2$ can be truncated, leading to a crucial simplification. Indeed, since the $\mathcal{A}$ chiral superfield, the $\mathcal{N}=2$ superpartner of the U(1) gauge field,[4]

$$\mathcal{A}\equiv a+\sqrt{2}\lambda^2\theta+F_a\theta^2, \tag{2.1}$$

it not charged under the gauge group $SU(2)\times U(1)$, one can introduce the superpotential linear in $\mathcal{A}$,

$$\mathcal{W}_{\mathcal{A}}=-\frac{1}{\sqrt{2}}\xi\mathcal{A}. \tag{2.2}$$

It is rather obvious that $\mathcal{W}_{\mathcal{A}}$ is indeed a linear truncation of $\mathcal{W}=\mu\,\mathrm{Tr}\,\Phi^2$. A remarkable feature of the superpotential (2.2) is that it does *not* break $\mathcal{N}=2$ supersymmetry [15,16]. Keeping higher order terms in $\mu\,\mathrm{Tr}\,\Phi^2$ would inevitably explicitly break $\mathcal{N}=2$. For our purposes it is crucial that the model we will deal with is *exactly* $\mathcal{N}=2$ supersymmetric.

The bosonic part of our $SU(2)\times U(1)$ theory has the form[5] [6]

[2] A very fresh publication [14], which appeared after the completion of the present paper, also examines strings and their relation to monopoles.

[3] In certain vacua to be considered in this paper the gauge group is further broken to $U(1)\times U(1)$ at a much lower scale $|\Delta m|$; see Sec. II B.

[4] The superscript 2 in Eq. (2.1) is the global $SU(2)_R$ index of λ rather than λ squared.

[5] Here and below we use a formally Euclidean notation, e.g., $F^2_{\mu\nu}=2F^2_{0i}+F^2_{ij}$, $(\partial_\mu a)^2=(\partial_0 a)^2+(\partial_i a)^2$, etc. This is appropriate since we are going to study static (time-independent) field configurations, and $A_0=0$. Then the Euclidean action is nothing but the energy functional. Furthermore, we define $\sigma^{\alpha\dot{\alpha}}=(1,-i\vec{\tau})$, $\bar{\sigma}_{\dot{\alpha}\alpha}=(1,i\vec{\tau})$. Lowing and raising of spinor indices is performed by virtue of the antisymmetric tensor defined as $\varepsilon_{12}=\varepsilon_{\dot{1}\dot{2}}=1$, $\varepsilon^{12}=\varepsilon^{\dot{1}\dot{2}}=-1$. The same raising and lowering convention applies to the flavor SU(2) indices f, g, etc.

$$S=\int d^4x\left[\frac{1}{4g_2^2}(F^a_{\mu\nu})^2+\frac{1}{4g_1^2}(F_{\mu\nu})^2+\frac{1}{g_2^2}|D_\mu a^a|^2+\frac{1}{g_1^2}|\partial_\mu a|^2+|\nabla_\mu q^A|^2+|\nabla_\mu\bar{\tilde{q}}^A|^2+V(q^A,\tilde{q}_A,a^a,a)\right]. \tag{2.3}$$

Here D_μ is the covariant derivative in the adjoint representation of SU(2), while

$$\nabla_\mu=\partial_\mu-\frac{i}{2}A_\mu-iA^a_\mu\frac{\tau^a}{2}, \tag{2.4}$$

where we suppress the color SU(2) indices, and τ^a are the SU(2) Pauli matrices. The coupling constants g_1 and g_2 correspond to the U(1) and SU(2) sectors, respectively. With our conventions the U(1) charges of the fundamental matter fields are $\pm 1/2$.

The potential $V(q^A,\tilde{q}_A,a^a,a)$ in the Lagrangian (2.3) is a sum of D and F terms,

$$\begin{aligned}V(q^A,\tilde{q}_A,a^a,a)=&\frac{g_2^2}{2}\left(\frac{1}{g_2^2}\varepsilon^{abc}\bar{a}^b a^c+\bar{q}_A\frac{\tau^a}{2}q^A-\tilde{q}_A\frac{\tau^a}{2}\bar{\tilde{q}}^A\right)^2\\&+\frac{g_1^2}{8}(\bar{q}_A q^A-\tilde{q}_A\bar{\tilde{q}}^A)^2+\frac{g_2^2}{2}|\tilde{q}_A\tau^a q^A|^2\\&+\frac{g_1^2}{2}|\tilde{q}_A q^A-\xi|^2+\frac{1}{2}\sum_{A=1}^{4}\{|(a+\sqrt{2}m_A\\&+\tau^a a^a)q^A|^2+|(a+\sqrt{2}m_A+\tau^a a^a)\bar{\tilde{q}}_A|^2\},\end{aligned} \tag{2.5}$$

where the sum over repeated flavor indices A is implied, and we introduced a *constant* ξ related to μ as follows:

$$\xi=6\mu m. \tag{2.6}$$

The first and second lines represent D terms, the third line the $F_{\mathcal{A}}$ terms, while the fourth and the fifth lines represent the squark F terms. As we know [6,13,15–17], this theory supports BPS vortices.

Bearing in mind that we have four flavors we conclude that the SU(2) coupling does not run: SU(2) theory with $N_f=4$ is conformal. Hence, g_2^2 is given by its value at the scale m,

TABLE I. Field content of the model under consideration.

$\mathrm{SU}(2)_C$ repr.→ / Spin↓	singlet		fundamental	/anti	adjoint	
0		a	q^A	$\tilde{q}_A$		a^a
1/2	λ^1	λ^2	ψ^A	$\tilde{\psi}_A$	λ^{1a}	λ^{2a}
1	A_μ				A^a_μ	

$$\frac{8\pi^2}{g_2^2}=2\ln\frac{m}{\Lambda_{\mathrm{SU}(3)}}+\cdots. \tag{2.7}$$

At large m the SU(2) sector is indeed weakly coupled.

The U(1) coupling undergoes an additional renormalization from scale m down to the scale determined by the masses of light states in the low-energy theory (the latter are of the order of $\sqrt{\mu m}\sim\sqrt{\xi}$; see Sec. II B). At the scale m the both couplings, g_2^2 and g_1^2 unify since at this scale they belong to SU(3). Note that in passing from the SU(3) theory to $\mathrm{SU}(2)\times\mathrm{U}(1)$ we changed the normalization of the eighth generator of SU(3) which became the generator of U(1); see Eq. (2.4). This change of normalization implies that the unification condition takes the form

$$\frac{8\pi^2}{g_1^2(m)}=3\frac{8\pi^2}{g_2^2(m)}. \tag{2.8}$$

The one-loop coefficient of the β function for the U(1) theory is $\beta_0=2\times 2n_e^2N_f$ where the first factor of 2 reflects the difference in normalizations of the SU(N) versus U(1) generators, the extra factor of two comes from the fact that for each flavor we deal with matter doublets, and, finally, the electric charge $n_e=1/2$, see Eq. (2.4). Thus, evolving g_1^2 from m down to $\sqrt{\mu m}$ we get

$$\frac{8\pi^2}{g_1^2(\sqrt{\mu m})}=6\ln\frac{m}{\Lambda_{\mathrm{SU}(3)}}+2\ln\frac{m}{\mu}+\cdots. \tag{2.9}$$

Clearly, this coupling is even smaller than that of the SU(2) sector.

To make readers' journey through this work easier we display in Table I the field content of the model.

It is also instructive to summarize the symmetries of the model and patterns of their breaking; see Table II. Besides the gauge symmetries, of importance are the global symme-

TABLE II. Pattern of the symmetry breaking.

$\mathcal{N}=2$ SUSY	unbroken
$\mathrm{SU}(2)_R$	unbroken
$\{\mathrm{U}(1)\times\mathrm{SU}(2)\}_G\times\mathrm{SU}(2)_{f_{12}}\times\mathrm{SU}(2)_{f_{34}}\times\mathrm{U}(1)$	→ $\mathrm{U}(1)_{\mathrm{diag}}\times\mathrm{SU}(2)_{\mathrm{diag}}\times\mathrm{SU}(2)_{f_{34}}$, 12 vacuum; $\mathrm{U}(1)_{\mathrm{diag}}$, 13 vacuum

tries of the model. Our "proto-SU(3)-model" (mentioned in passing) had $SU(3)_c$ broken down to $SU(2)\times U(1)$. This breaking occurs at the scale m. The resulting superpotential

$$\mathcal{W}=\frac{1}{\sqrt{2}}\sum_{A=1}^{4}(\tilde{q}_A\mathcal{A}q^A+\tilde{q}_A\mathcal{A}^a\tau^a q^A)+m_1\sum_{A=1,2}\tilde{q}_A q^A + m_3\sum_{A=3,4}\tilde{q}_A q^A-\frac{1}{\sqrt{2}}\xi\mathcal{A}, \tag{2.10}$$

has, in addition, a large global $SU(2)_{f_{12}}\times SU(2)_{f_{34}}\times U(1)$ flavor symmetry. At the scale Δm the color SU(2) may or may not be broken. In the 13, 14, 23, and 24 vacua it is broken by the vacuum expectation value of a^3 down to U(1), paving the way to monopoles with typical sizes $\sim(\Delta m)^{-1}$. In the 12 and 34 vacua $\langle a^3\rangle$ does not develop, and we can descend further, down the scale ξ. At this scale all gauge symmetries, in all six vacua under consideration, are fully Higgsed. The Abrikosov-Nielsen-Olesen (ANO, Ref. [18]) strings are supported. The transverse size of these strings, $\sim\xi^{-1/2}$, is much larger than $(\Delta m)^{-1}$. In fact, we will deal with two distinct types of strings which generalize their more primitive ANO counterparts. They correspond (in the quasi-classical limit) to distinct types of winding, see Sec. V for further details.

B. The vacuum structure and excitation spectrum

This section briefly outlines the vacuum structure and the excitation mass spectrum of our basic $SU(2)\times U(1)$ model (for further details, including those referring to the full SU(3) theory, see Refs. [6,11–13]). First, we will examine relevant vacua.

The vacua of the theory (2.3) are determined by the zeros of the potential (2.5). We will assume the conditions (1.3) to be met. Then, besides three strong-coupling vacua which exist in pure SU(3) $\mathcal{N}=2$ Yang-Mills theory, we have eight vacua in which one quark flavor is condensed, and six vacua in which two quark flavors develop nonvanishing VEV's. For our problem—domain walls and flux tubes at weak coupling—we will choose these latter six vacua. They are 12-, 34-, 13-, 14-, 23- and 24-vacua. In the first two SU(2) gauge symmetry is unbroken by adjoint scalars while in the last four vacua it is broken by them at the scale Δm.

Each of the above six vacua (AB) is labeled by a triplet of gauge-invariant order parameters I_1, I and I_3 defined as

$$\langle\tilde{q}_A q^B\rangle\equiv I_1\delta_A^B,$$

$$\langle\tilde{q}_A a q^B\rangle\equiv-\sqrt{2}I\times I_1\delta_A^B,$$

$$\langle\tilde{q}_A a^a\tau_a q^B\rangle\equiv\frac{1}{\sqrt{2}}I_3\times I_1(\tau_3)_A^B, \tag{2.11}$$

where summation over the SU(2) color indices is implied. The corresponding vacuum structure is exhibited in Fig. 3.

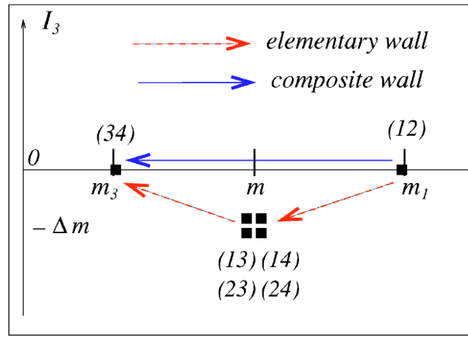

FIG. 3. The vacuum structure on the (I,I_3) plane.

1. SU(2) symmetric vacua

Let us first consider the 12 (or 34) vacuum. The adjoint fields develop the following VEV's:

$$\langle a^3\rangle=0,\quad \langle a\rangle=-\sqrt{2}m_1=-\sqrt{2}m+O(\Delta m), \tag{2.12}$$

where m is defined in Eq. (1.4). If the values of the mass parameters $m_{1,3}$ and μ are real, we can exploit the freedom of rotations in SU(2) and U(1) to make the quark VEV's real too. Then in the case at hand they take the color-flavor locked form

$$\langle q^{kA}\rangle=\langle\bar{\tilde{q}}^{kA}\rangle=\sqrt{\frac{\xi}{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad k=1,2,\quad A=1,2. \tag{2.13}$$

This particular form of the squark condensates is dictated by the third line in Eq. (2.5). Note that the squark fields stabilize at non-vanishing values entirely due to the U(1) factor—the second term in the third line.

The gauge invariants corresponding to the vacuum (2.13) are

$$I_1=\frac{\xi}{2},\quad I=m_1,\quad I_3=0. \tag{2.14}$$

At first site it might seem that, say, the field configuration

$$\langle q^{kA}\rangle=\langle\bar{\tilde{q}}^{kA}\rangle=\sqrt{\frac{\xi}{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad k=1,2,\quad A=1,2, \tag{2.15}$$

which also provides a (12)-vacuum solution, presents another vacuum. This is obviously not the case, since it is nothing but a gauge copy of Eq. (2.13). The gauge invariants obtained from Eq. (2.15) are the same as in Eq. (2.14).

Let us move on to the issue of the excitation spectrum in this vacuum. The mass matrix for the gauge fields (A_μ^a, A_μ) can be read off from the quark kinetic terms in Eq. (2.3) and has the form

$$\mathcal{M}_V^2=\xi\begin{pmatrix}g_2^2&0\\0&g_1^2\end{pmatrix}. \tag{2.16}$$

Thus all three SU(2) gauge bosons become massive, with one and the same mass

$$M_{1,2,3}=g_2\sqrt{\xi}. \tag{2.17}$$

The equality of the masses is no accident. The point is that our model actually has the symmetry $\mathrm{SU}(2)_c\times\mathrm{SU}(2)_{f_{1,2}}\times\mathrm{SU}(2)_{f_{3,4}}$ where the first flavor SU(2) corresponds to rotations of the $A=1,2$ flavors, while the second flavor SU(2) to rotations of the $A=3,4$ flavors. The pattern of the spontaneous breaking is such that the diagonal SU(2) from the product $\mathrm{SU}(2)_c\times\mathrm{SU}(2)_{f_{1,2}}$ remains an unbroken global SU(2) symmetry of the theory. Sure enough, $\mathrm{SU}(2)_{f_{3,4}}$ is also unbroken, since the $A=3,4$ flavors play a passive role of spectators in the 12-vacuum.

The mass of the U(1) gauge boson is

$$M_{\mathrm{U}(1)}=g_1\sqrt{\xi}. \tag{2.18}$$

From the mass scale m down to the scale $|\Delta m|$ the gauge coupling g_2^2 does not run because of conformality. Below $|\Delta m|$, the conformality is broken: two quark flavors out of four have mass of the order of Δm and, hence, decouple. Therefore, g_2^2 runs, generating a dynamical mass scale (1.5). At the mass scale $M_{1,2,3}=g_2\sqrt{\xi}$ this last running gets frozen. Since $M_{1,2,3}\gg\Lambda_{\mathrm{SU}(2)}$, by assumption, the running of g_2^2 in the interval from $|\Delta m|$ down to $\sqrt{\xi}$ can be neglected. Therefore, we can treat g_2^2 in the above relations as a scale independent constant (coinciding with the gauge constant normalized at m). The mass spectrum of the adjoint scalar excitations is the same as for the gauge bosons. This is enforced by $\mathcal{N}=2$.

What is the mass spectrum of the quark (squark) excitations? These fields are color doublets. To ease the notation it will be convenient (sometimes) to use subscripts r and b (red and blue) for the color indices of q and $\tilde{q}$. It is rather obvious that $q_r^{(A=1)}$, $q_b^{(A=2)}$, $\tilde{q}^r_{(A=1)}$ and $\tilde{q}^b_{(A=2)}$ are "eaten up" in the Higgs mechanism. The remaining four superfields, $q_r^{(A=2)}$, $q_b^{(A=1)}$, $\tilde{q}^r_{(A=2)}$ and $\tilde{q}^b_{(A=1)}$ split into two groups—a singlet under the residual global SU(2) with the mass (2.18), and a triplet under the residual global SU(2) with the mass (2.17). Altogether we have $1+3=4$ long massive $\mathcal{N}=2$ supermultiplets with mass squared proportional to ξ.

As for the spectator quark flavors (those that do not condense in the given vacuum), the quarks and squarks of the third and fourth flavors are much heavier. They have masses $\sim|\Delta m|$, as is clear from Eq. (2.5). Assuming the limit (1.2), we include the spectator quarks and squarks in the low-energy theory (2.3). In particular, each spectator quark flavor with the mass term m_3—remember, we have two of those—produces two $\mathcal{N}=2$ multiplets. The first one, with the mass $|m_3-m_1|=|\Delta m|$, is formed from the r-components of the spectator quark, while the second one, with the same mass, is formed from its b-components. Thus, each supermultiplet contains four bosonic and four fermionic (real) degrees of freedom (short $\mathcal{N}=2$ supermultiplet). Altogether we have four such supermultiplets—two doublets of the global SU(2). There are no massless excitations in this vacuum.

The 34-vacuum is similar. The only difference is that the value of the gauge invariant I in the 34-vacuum is $I=m_3$.

2. SU(2) nonsymmetric vacua

As an example of such vacuum we will consider the 13-vacuum. Now, in contradistinction with the previous case, both adjoint fields, $\langle a^3\rangle$ and $\langle a\rangle$, develop vacuum expectation values, so that Eq. (2.12) must be replaced by

$$\langle a^3\rangle=-\frac{m_1-m_3}{\sqrt{2}}\equiv-\frac{\Delta m}{\sqrt{2}},$$
$$\langle a\rangle=-\frac{m_1+m_3}{\sqrt{2}}\equiv-\sqrt{2}m. \tag{2.19}$$

The above vacuum values of the adjoint scalars follow from examination of the last two lines in Eq. (2.5). The squark fields in the vacuum are similar to those in Eq. (2.13), with the replacement of the second flavor by the third one, namely,

$$\langle q^{kA}\rangle=\langle\bar{\tilde{q}}^{kA}\rangle=\sqrt{\frac{\xi}{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix},$$
$$k=1,2,\quad A=1,3, \tag{2.20}$$

up to gauge copies. The gauge invariant order parameters are

$$I_1=\frac{\xi}{2},\quad I=m,\quad I_3=-\Delta m, \tag{2.21}$$

see Fig. 3.

Next, let us examine the excitation spectrum in this vacuum. In this vacuum the gauge group of our model is fully Higgsed, too—all four gauge bosons acquire masses. No "custodial" global SU(2) survives, however. Correspondingly, the masses of the gauge bosons $A_\mu^{1\pm i2}$ on the one hand, and A_μ^3 on the other, split.

More concretely,

$$M(A_\mu^{1\pm i2})=M(a^{1\pm i2})=\Delta m\gg\xi, \tag{2.22}$$

while the masses of A_μ and A_μ^3 (and the same for a and a^3) are given by the same values as in the SU(2)-symmetric vacua,

$$M_{\mathrm{U}(1)}=g_1\sqrt{\xi},$$
$$M_3=g_2\sqrt{\xi}. \tag{2.23}$$

The mass matrix for the lightest quarks has the size 8×8, including four (real) components of the q_r^1 quark and four components of the q_b^3 quark. It has two vanishing eigenvalues associated with two states "eaten" by the Higgs mechanism for two U(1) gauge factors, and two nonzero eigenvalues coinciding with masses (2.23). Each of these nonzero eigenvalues corresponds to three quark eigenvectors. Altogether we have two long $\mathcal{N}=2$ multiplets with masses (2.23), each one containing eight bosonic and eight fermionic states.

Let us remember that, in the limit $m_1=m_2$ and $m_3=m_4$, the 13-vacuum coalesce with three others, namely, the 14-, 23- and 24-vacua; see Fig. 3. This means that we have massless multiplets in these vacua. In fact, the common position of these vacua on the Coulomb branch is the root of a Higgs branch. This Higgs branch has dimension eight, cf. [11,13]. To see this observe, that in the (A,B) vacuum with $A=1,2$ and $B=3,4$ we have 16 real quark scalar variables (q_r^1, q_r^2, q_b^3 and q_b^4) subject to two D-term conditions and four F-term conditions. Also we have to subtract two U(1) phases. Overall we have $16-6-2=8$ which gives us the dimension of the Higgs branch. This dimension should be a multiple of four since the Higgs branches are hyper-Kähler manifolds [9].

III. THE $\mathcal{N}=2$ CENTRAL CHARGES RELEVANT TO THE PROBLEM

The model under consideration supports, in various limits, all three classes of topological defects that are under scrutiny in the current literature: domain walls, strings and monopoles. Below we will explore BPS-saturated defects, with a special emphasis on various junctions. The domain walls and strings are 1/2 BPS, the wall-string junctions and the string-string junctions are 1/4 BPS.

It is instructive to begin from the discussion of corresponding central charges in $\mathcal{N}=2$ superalgebra. While a part of the material below is a mini-review, in the analysis of the monopole central charge we will add a bifermion term which was routinely omitted previously. It was omitted for a good reason, though: for a free monopole the contribution of this bifermion term vanishes. It is crucial, however, for the confined monopoles to which we will turn below.

A. (1,0) and (0,1) central charges

These central charges are saturated by domain walls. They appear in the anticommutators $\{Q_\alpha^f Q_\beta^g\}$ (remember, $f,g=1,2$ are $SU(2)_R$ indices). Since $\{Q_\alpha^f Q_\beta^g\}\sim\int d\Sigma_{\alpha\beta}$ where $d\Sigma_{\alpha\beta}$ is the element of the area of the domain wall in question ($d\Sigma_{\alpha\beta}=d\Sigma_{\beta\alpha}$), the (1/2,1/2) central charges must be symmetric with respect to the interchange $f\leftrightarrow g$. More precisely,

$$\{Q_\alpha^f Q_\beta^g\}=-4\Sigma_{\alpha\beta}\bar{\mathcal{Z}}^{fg}, \tag{3.1}$$

where

$$\Sigma_{\alpha\beta}=-\frac{1}{2}\int_{\text{wall}} dx_{[\mu}dx_{\nu]}(\sigma^\mu)_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha}_{\beta}\sim\text{wall area}, \tag{3.2}$$

while the central charge $\bar{\mathcal{Z}}^{fg}$ ought to be an $SU(2)_R$ vector.

In the string-theory context the (1,0) and (0,1) central charges were first discussed in Ref. [19]. The discovery of the corresponding field-theoretic anomaly in supersymmetric Yang-Mills (SYM) theories [2] paved the way to multiple explorations and uses of the (1,0) and (0,1) central charges in field theory (e.g. [20–22] to name just a few).

Two observations severely constrain the form of the central charge $\bar{\mathcal{Z}}^{fg}$: first, it must be (anti)holomorphic in fields; second, it must be a $SU(2)_R$ vector. As a result, the most general form compatible with the above observations is[6]

$$\mathcal{Z}^{fg}=\Delta\left(-\frac{4}{\sqrt{2}}\xi^{fg}\mathcal{A}+\frac{c_1}{16\pi^2}\lambda^f\lambda^g+\frac{c_2}{16\pi^2}\lambda^{af}\lambda^{ag}\right), \tag{3.3}$$

where Δ means the difference of the expectation values of the operator in parentheses in two vacua between which the wall in question interpolates. Furthermore, the parameter $\bar\xi^{fg}$ is an $SU(2)_R$ matrix (related to a real vector $\vec\xi$) introduced in Ref. [16]. In the model under consideration $\xi^{fg}=(\xi/2)\,\mathrm{diag}\{1,-1\}$, see the remark after Eq. (3.4).

The last two terms in Eq. (3.3), containing numerical coefficients $c_{1,2}$, present a quantum anomaly, a generalization of that of Ref. [2]. The coefficients $c_{1,2}$ are readily calculable in terms of the Casimir operators of the gauge group of the model under consideration; they also depend on the matter content. We will not dwell on them here because, given our hierarchy of parameters (1.3), the anomalous terms in $\mathcal{Z}^{fg}$ will play no role. A rather straightforward algebra (in conjunction with known results) yields us the coefficient in front of $\xi^{fg}\mathcal{A}$ quoted in Eq. (3.3). To this end we combine Eq. (2.10) above with Eqs. (3.19) and (3.20) from Ref. [23]. In our normalization the BPS wall tension reduces,[7] e.g., to $T_{\mathrm{w}}=|\mathcal{Z}^{11}|$.

B. (1/2, 1/2) central charge

This central charge is saturated by strings (flux tubes).[8] It appears in the anticommutator $\{Q_\alpha^f\bar{Q}_{\dot\beta g}\}$. This central charge is not holomorphic, and has no particular symmetry with respect to permutations of the $SU(2)_R$ indices f and g.

It is well known that the (1/2, 1/2) central charge exists also in $\mathcal{N}=1$ supersymmetric QED (SQED) with the Fayet-Iliopoulos term, see Ref. [27] and especially Ref. [28], specifically devoted to this issue. In Ref. [28] it is shown, in particular, that if the spontaneous breaking of U(1) is due to the Fayet-Iliopoulos term [29], then the corresponding ANO string is saturated in $\mathcal{N}=1$, and the string tension is given by the value of the central charge. In Ref. [24] it was proven that at weak coupling this is the *only* mechanism leading to

[6]Derivation of Eq. (3.3) also exploits the specific feature of $\mathcal{N}=2$ theories that each given superfield enters in the superpotential linearly; see Eq. (2.10). This implies, in particular, that in any given vacuum (at the classical level) $\mathcal{W}_2+\mathcal{W}_3\equiv 0$ where $\mathcal{W}_{2,3}$ are the quadratic and cubic parts of the superpotential.

[7]For generic matrices $\mathcal{Z}^{fg}$ it is the eigenvalue of $\mathcal{Z}^{fg}$ that counts.

[8]It is also instrumental in the issue of BPS-saturated vortices [21] and wall junctions [24,25]. One can trace this line of reasoning to Ref. [26].

BPS strings in $\mathcal{N}=1$ theories, which are, thus, by necessity, the ANO strings.[9]

Our emphasis will be on "non-Abelian," rather than ANO strings (the meaning of "non-Abelian-ness" is explained in Sec. V). It is only the extended, $\mathcal{N}=2$, supersymmetry that can make them BPS-saturated. Following Ref. [16], the (1/2,1/2) anticommutator in the $\mathcal{N}=2$ model at hand can be written as follows:

$$\{Q_\alpha^f \bar{Q}_{\dot\beta g}\} = 2\,\delta_g^f (\sigma^\mu)_{\alpha\dot\beta} P_\mu + 4i(\sigma^\mu)_{\alpha\dot\beta}\xi_g^f \int d^3x \frac{1}{2}\varepsilon_{0\mu\delta\gamma}F_{\delta\gamma}, \tag{3.4}$$

where P_μ is energy-momentum operator while[10]

$$\xi_g^f = (\tau^m/2)_g^f \xi^m.$$

Moreover, the vector ξ^m is a $SU(2)_R$ triplet of generalized Fayet-Iliopoulos parameters (in our model only $\xi^1 \equiv \xi$ is nonzero).

The second term in Eq. (3.4) is the (1/2, 1/2) central charge. It is worth emphasizing that it is only the U(1) field $F_{\delta\gamma}$ that enters; the SU(2) gauge field does not contribute to this central charge for rather evident reasons. The central charge is obviously proportional to L, the length of the string, times the magnetic flux of the string $\int d^2x\vec{B}$ directed along the string axis. With the normalizations accepted throughout this paper one can write for the BPS string tension

$$T_s = \left|\int d^2x \vec{B}\right| |\vec{\xi}| = 2\pi\xi, \tag{3.5}$$

where the last equality refers to the elementary strings. For the ANO string the flux is twice larger, so that $T_{ANO} = 4\pi\xi$, see Sec. V.

C. The Lorentz-scalar central charge

As is well known [31], this central charge is possible only because of the extended nature of supersymmetry, $\mathcal{N}=2$. It appears in the anticommutator $\{Q_\alpha^f Q^g{}_\beta\}$ and has the structure

$$\{Q_\alpha^f Q^g{}_\beta\} = \varepsilon_{\alpha\beta}\varepsilon^{fg} 2Z, \tag{3.6}$$

where Z is an $SU(2)_R$ singlet while the factor of 2 on the right-hand side is a traditional normalization. It is most convenient to write Z in terms of the topological charge, an integral over the topological density,

$$Z = \int d^3x\, \zeta^0(x). \tag{3.7}$$

In the model at hand

$$\zeta^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu\left(\frac{i}{g_2^2}a^a F_{\rho\sigma}^a + \frac{i}{g_1^2}aF_{\rho\sigma} + \frac{c}{4\pi^2}\lambda_\alpha^f s^{\alpha\beta}\lambda_\beta^g \varepsilon_{fg} + \frac{2cg_2^2}{4\pi^2}\psi_\alpha^A s^{\alpha\beta}\tilde{\psi}_{A\beta}\right), \tag{3.8}$$

where $s_{\alpha\beta}$ is a symmetric matrix corresponding to the Lorentz representation (1,0) in the spinorial notation,

$$s_{\alpha\beta} = (\sigma^{[\rho})_{\alpha\dot\alpha}(\bar{\sigma}^{\sigma]})^{\dot\alpha}_\beta, \tag{3.9}$$

while the square brackets in the superscripts denote antisymmetrization with respect to ρ and σ. Moreover, c is a numerical coefficient.

Two comments are in order here. First, the first two terms in Eq. (3.8) present the conventional "monopole" central charge which is routinely discussed in numerous reviews. It emerges from the canonic (anti)commutators at the tree level. In generic models it is in fact the (1,0) gauge field strength tensor which appears in these classical terms in the first line in Eq. (3.8), i.e., the (anti)self-dual combination. In the model at hand only the magnetic field survives in the expression for the central charge; therefore, we dropped the electric component.

The last two (bifermion) terms in Eq. (3.8) are due to an anomaly, which is, in a sense, an $\mathcal{N}=2$ counter-partner to that of Ref. [2]. They will be discussed in more detail in the accompanying paper [32] where the value of the coefficient c will be determined. They were unknown previously playing no role in the routine monopole analysis. They do play a crucial role, however, for the Higgs phase monopoles (confined monopoles), to be discussed in brief in Sec. VI. In fact, this anomaly must match the recently obtained anomalous central charge [33] in two-dimensional O(3) sigma model. More on that will be said in Sec. VI and Ref. [32].

IV. DOMAIN WALLS

In this section we study BPS domain walls between various vacua described in Secs. II B 1 and II B 2. First, we derive the first order equations for the BPS walls and, second, find and analyze their solutions. Our final goal is to work out the solution for the composite wall 12→34 on which we will eventually get localized non-Abelian gauge fields.

A. First-order equations for elementary and composite walls

Let us note that the structure of the vacuum condensates in all six vacua considered in Secs. II B 1 and II B 2 suggests that we can search for the domain wall solutions using the *ansatz*

$$q^{kA} = \bar{\tilde{q}}^{kA} \equiv \frac{1}{\sqrt{2}}\varphi^{kA}, \tag{4.1}$$

where we introduce a new complex field φ^{kA}, $k=r,b$. Note that the above *ansatz* violates holomorphy in the space of fields inherent to F terms: superpotentials and certain other

[9] Note, however, that the (1/2,1/2) central charge is missing in the general analysis of Ref. [30].

[10] Note that the definition of ξ_g^f in Ref. [16] differs by a factor of 2.

expressions derivable from them. This is why some expressions presented below which should be holomorphic on general grounds, do not look holomorphic on the *ansatz* (4.1).

Within this *ansatz* the effective action (2.3) becomes

$$S=\int d^4x\Bigg\{\frac{1}{4g_2^2}(F^a_{\mu\nu})^2+\frac{1}{4g_1^2}(F_{\mu\nu})^2+\frac{1}{g_2^2}|D_\mu a^a|^2+\frac{1}{g_1^2}|\partial_\mu a|^2+|\nabla_\mu\varphi^A|^2+\frac{g_2^2}{8}(\bar{\varphi}_A\tau^a\varphi^A)^2+\frac{g_1^2}{8}(|\varphi^A|^2-2\xi)^2+\frac{1}{2}|(a^a\tau^a+a+\sqrt{2}m_A)\varphi^A|^2\Bigg\}, \tag{4.2}$$

where we use the same notation as in Eq. (2.4).

For the time being let us drop the gauge field in Eq. (4.2). It is irrelevant for the "standard" domain wall. If we assume that all fields depend only on the coordinate $z\equiv x_3$, the Bogomolny completion[11] of the wall energy functional can be written as

$$T_{\rm w}=\int dz\Bigg\{\bigg|\partial_z\varphi^A\pm\frac{1}{\sqrt{2}}(a^a\tau^a+a+\sqrt{2}m_A)\varphi^A\bigg|^2+\bigg|\frac{1}{g_2}\partial_z a^a\pm\frac{g_2}{2\sqrt{2}}(\bar{\varphi}_A\tau^a\varphi^A)\bigg|^2+\bigg|\frac{1}{g_1}\partial_z a\pm\frac{g_1}{2\sqrt{2}}(|\varphi^A|^2-2\xi)\bigg|^2\pm\sqrt{2}\xi\partial_z a\Bigg\}. \tag{4.3}$$

In the above expression we have omitted another full-derivative boundary term proportional to $(\partial/\partial z)\Sigma Q(\partial W/\partial Q)$ where the sum runs over all superfields. Since in all vacua $(\partial W/\partial Q)=0$, this term produces no impact whatsoever.

Putting mod-squared terms to zero gives us the first-order Bogomolny equations, while the surface term [the last one in Eq. (4.3)] gives the wall tension. Assuming for definiteness that $\Delta m>0$ and choosing the upper sign in Eq. (4.3) we get the BPS equations,

$$\partial_z\varphi^A=-\frac{1}{\sqrt{2}}(a_a\tau^a+a+\sqrt{2}m_A)\varphi^A,$$

$$\partial_z a^a=-\frac{g_2^2}{2\sqrt{2}}(\bar{\varphi}_A\tau^a\varphi^A),$$

$$\partial_z a=-\frac{g_1^2}{2\sqrt{2}}(|\varphi^A|^2-2\xi). \tag{4.4}$$

Tensions of the walls satisfying the above equations are given by the surface term in Eq. (4.3).[12] Say, for the elementary walls $12\to 1B$ or $12\to B2$ $(B=3,4)$, this gives

$$T_{\rm w}^{(12\to 1B)}=T_{\rm w}^{(12\to B2)}=(\Delta m)\xi, \tag{4.5}$$

where we use the fact that

$$\Delta a\equiv(a)_{13}-(a)_{12}=\frac{m_1-m_3}{\sqrt{2}}=\frac{\Delta m}{\sqrt{2}}.$$

Remember, the elementary walls are those for which both vacua, initial and final, have a common flavor. The wall $12\to 34$ can be considered as a bound state of two elementary walls $12\to 1B$ and $1B\to CB$ $(C=3,4,\ B\neq C)$. For the composite walls Eq. (4.3) implies

$$T_{\rm w}^{(12\to 34)}=2(\Delta m)\xi, \tag{4.6}$$

since Δa is twice larger. We see that this wall has twice the tension of the elementary walls. This means that the bound state is marginally stable; the elementary BPS components forming the composite wall do not interact. Equations (4.5) and (4.6) and the subsequent statement are valid up to nonperturbative effects residing in the anomalous terms in Eq. (3.3). For further discussion see Sec. VII.

B. Elementary domain walls

It is time to explicitly work out the solution to the first-order equations (4.4) for the domain wall interpolating between the vacua (12) and $(1B)$ where $B=3$ or 4. We assume that

$$m\gg\Delta m\gg\sqrt{\xi}=\sqrt{6\mu m}. \tag{4.7}$$

This condition allows us to find analytic domain wall solutions. In addition, it makes transparent the physical reason for the gauge field localization on domain walls [4]. Accepting Eq. (4.7) we guarantee, as will be shown shortly, that the quark fields (almost) vanish inside the composite $(12\to 34)$-wall, to be treated in Sec. IV C. The only gauge symmetry breaking surviving inside this wall is that induced by the VEV of the SU(2) singlet adjoint field a.

Let us choose the wall $12\to 14$ for definiteness. The boundary conditions for the fields a^3 and a are obviously as follows (cf. Secs. II B 1 and II B 2):

$$a^3(-\infty)=0,\quad a(-\infty)=-\sqrt{2}m_1,$$

$$a^3(\infty)=-\frac{1}{\sqrt{2}}\Delta m,\quad a(\infty)=-\sqrt{2}m. \tag{4.8}$$

We see that the range of variation of the fields a^3 and a inside the wall is of the order of Δm. Minimization of their kinetic energies implies then that these fields are slowly

[11]The Bogomolny completion is routinely used in such problems after its introduction in Ref. [34] and the subsequent identification of the central charges of various superalgebras with topological charges [35].

[12]It is easy to check that the very same result follows from the central charge (3.3).

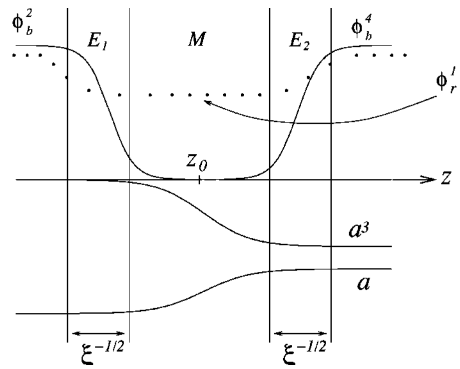

FIG. 4. Internal structure of the 12→14 domain wall: two edges (domains $E_{1,2}$) of the width $\sim\xi^{-1/2}$ are separated by a broad middle band (domain M) of the width R; see Eq. (4.11).

varying. Therefore, we may safely assume that the wall is thick on the scale of $\xi^{-1/2}$; the wall size $R \gg 1/\sqrt{\xi}$. This fact will be confirmed shortly, see also the previous investigation [4].

On the contrary, the quark fields vary inside small regions of the order of $1/\sqrt{\xi}$—this scale is determined by the masses of the light quarks (2.23). In particular, φ_b^2 varies from its VEV in the 12-vacuum [see Eq. (2.13)],

$$\langle \varphi_b^2 \rangle = \sqrt{\xi},$$

at $z = -\infty$ to zero near the left edge of the wall (Fig. 4), whereas φ_b^4 varies from zero to its VEV in the 14-vacuum,

$$|\langle \varphi_b^4 \rangle| = \sqrt{\xi},$$

near the right edge of the wall. The φ_r^1 quark field does not vanish inside the wall because it has a nonzero VEV

$$\langle \varphi_r^1 \rangle = \sqrt{\xi}$$

in both vacua, initial and final. It acquires a constant value φ_{r0}^1 inside the wall which will be determined shortly.

With these values of the quark fields inside the wall, the last two equations in Eq. (4.4) tell us that the fields a^3 and a are linear function of z (cf. Ref. [4]). The solutions for a^3 and a take the form

$$a = -\sqrt{2}\left(m - \frac{\Delta m}{2}\,\frac{z - z_0 - R/2}{R} \right),$$

$$a^3 = -\frac{1}{\sqrt{2}}\Delta m \frac{z - z_0 + R/2}{R}, \qquad (4.9)$$

where the collective coordinate z_0 is the position of the wall center, while R is the wall thickness (Fig. 4). It is worth remembering that Δm is assumed positive. The solution (4.9) is valid in a wide domain of z,

$$|z - z_0| < \frac{R}{2}, \qquad (4.10)$$

except narrow areas of size $\sim 1/\sqrt{\xi}$ near the edges of the wall at $z - z_0 = \pm R/2$. Substituting the solution (4.9) into the last two equations in Eq. (4.4) we get

$$R = \frac{\Delta m}{\xi}\left(\frac{1}{g_1^2} + \frac{1}{g_2^2} \right). \qquad (4.11)$$

At the same time, the solution for r-quark inside the wall is

$$\varphi_r^1 = \varphi_{r0}^1 = \sqrt{\frac{2\xi}{g_2^2\left(\frac{1}{g_1^2} + \frac{1}{g_2^2} \right)}} \approx \sqrt{\frac{\xi}{2}}. \qquad (4.12)$$

We see that the r-quark field inside the wall differs from its value in the bulk, generally speaking. Only if we take $g_1 = g_2$ [which is *not* what comes out from the SU(3) "proto-model," see Eq. (2.8)] φ_{r0}^1 becomes equal to $\sqrt{\xi}$, its value in the bulk. Since $\Delta m/\sqrt{\xi} \gg 1$, the result (4.11) shows that $R \gg 1/\sqrt{\xi}$, justifying our approximation.

As a test of the validity of the solution above, let us verify that the solution $\varphi_r^1 = \mathrm{const}$ satisfies the first of equations (4.4) inside the wall. Substituting solutions (4.9) for the a fields in this equation we get $\partial_z \varphi_r = 0$, in full accord with our solution (4.12). Furthermore, we can now use the first relation in Eq. (4.4) to determine the tails of the b-components of the 2,4-squark fields inside the wall.

To this end, consider first the left edge (the domain E_1 in Fig. 4) at $z - z_0 = -R/2$. Substituting the above solution for a's in the equation for φ_b^2 we arrive at

$$\varphi_b^2 = \sqrt{\xi} e^{-(\Delta m/2R)(z - z_0 + R/2)^2}. \qquad (4.13)$$

This behavior is valid in the domain M, at $(z - z_0 + R/2) \gg 1/\sqrt{\xi}$, and shows that the field of the second quark flavor tends to zero exponentially inside the wall, as was expected.

By the same token, we can consider the behavior of b components of the fourth flavor squark field near the right edge of the wall at $z - z_0 = R/2$. The first equation in Eq. (4.4) for $A = 4$ implies

$$\varphi_b^4 = e^{i\sigma}\sqrt{\xi} e^{-(\Delta m/2R)(z - z_0 - R/2)^2}, \qquad (4.14)$$

which is valid in the domain M provided that

$$(R/2 - z + z_0) \gg 1/\sqrt{\xi}.$$

Here σ is an Abelian wall modulus (of the phase type) similar to that discovered in our previous work [4] where the reader can find extensive explanations as to its origin. Inside the wall the fourth quark fields tend to zero exponentially too.

In the domains near the wall edges,

$$z - z_0 = \pm R/2,$$

the fields $\varphi_{r,b}^A$ as well as a^3 and a smoothly interpolate between their VEV's in the given vacua and the inside-the-wall behavior determined by Eqs. (4.9), (4.13), and (4.14). It is

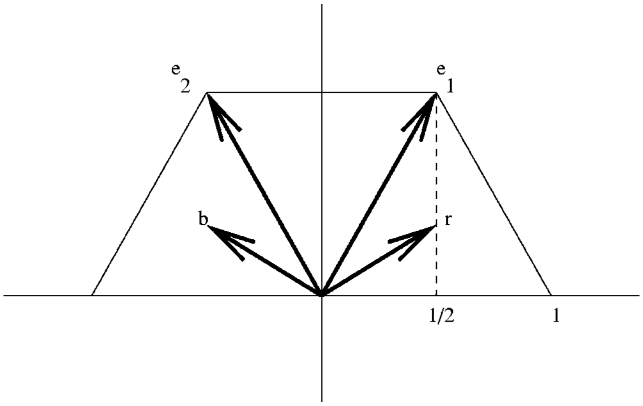

FIG. 5. Root and weight vectors of the SU(3) algebra.

not difficult to check that these domains produce contributions to the wall tension of the order of $\xi^{3/2}$, which makes them negligible.

A comment is in order here regarding the collective coordinates characterizing the elementary domain wall. We have two collective coordinates in our wall solution: the position of the center z_0 and the phase σ. In the effective low-energy theory on the wall world volume they become (pseudo)scalar fields of the world volume (2+1)-dimensional theory, $\zeta(t,x,y)$ and $\sigma(t,x,y)$, respectively. The target space of the second field is S_1, as is obvious from Eqs. (4.14).

In (2+1)-dimensional theory on the wall the compact (pseudo)scalar is equivalent to a U(1) gauge field via the relation [36]

$$F_{nm}^{(2+1)} = \text{const} \times \varepsilon_{nmk} \partial^k \sigma, \tag{4.15}$$

where $n,m,k=1,2,3$.

We see that our elementary domain wall localizes the U(1) gauge field on its world volume, as was expected, and in full accord with the string and D-brane notions. The physical reason for this localization was first suggested in [2] and then elaborated in detail in [4] for the case of $\mathcal{N}=2$ QCD with the SU(2) gauge group (a model effectively reducible to SQED). In this particular aspect—the gauge field localization on the elementary wall—the present $SU(2)\times U(1)$ model has slight distinctions compared to that of Ref. [4] that are worth mentioning.

In the bulk the gauge symmetry is broken down to $U(1)^2$ by the VEV of the a^3 adjoint field, and then, at a much lower scale, it is completely broken by the squark condensation. At the same time, inside the wall the only nonvanishing squark field is the r-component of the first quark flavor. Therefore, inside the wall the U(1) factor orthogonal to the r-th weight vector of the gauge group SU(3) is restored. This U(1) factor is associated with e_2-root of the gauge group; see Fig. 5 where we imagine an embedding of the $SU(2)\times U(1)$ gauge group in the SU(3) gauge group of our underlying "protomodel." Thus we have a localization of the e_2 gauge field on $12\to 14$ wall.

Note that this field is dual to the one present in the bulk [2,4,10]. This means that if we put the e_2-monopole at a certain point in the bulk, the e_2-string will be attached to this monopole because monopoles are in the confining phase in the quark vacua (see [13] and a brief review in Sec. V below on the flux tubes in $SU(2)\times U(1)$ $\mathcal{N}=2$ QCD). As we will see in Sec. V, it is this string that will end on our elementary walls, the string end point playing the role of a (dual) electric charge for the (2+1)-dimensional U(1) gauge field (4.15) living on the wall world volume.

In conclusion of this section it is worth noting that the scalar $\zeta(x_n)$ and the gauge field $A_m(x_n)$ form the bosonic part of $\mathcal{N}=2$ vector supermultiplet in $2+1$ dimensions.

C. Composite walls $(12\to 34)$ (bound states of the type $(12\to 14)+(14\to 34)$)

In this section we will consider the composite domain wall interpolating between the vacua 12 and 34.

The boundary conditions for all fields at $z=-\infty$ are given by their VEV's in the 12-vacuum

$$\begin{aligned} &a^3(-\infty)=0,\\ &a(-\infty)=-\sqrt{2}m_1,\\ &\varphi_r^1(-\infty)=\varphi_b^2(-\infty)=\sqrt{\xi},\\ &\varphi_r^3(-\infty)=\varphi_b^4(-\infty)=0, \end{aligned} \tag{4.16}$$

while at $z=\infty$ they are given by VEV's in the 34-vacuum,

$$\begin{aligned} &a^3(\infty)=0,\\ &a(\infty)=-\sqrt{2}m_3,\\ &\varphi_r^1(\infty)=\varphi_b^2(\infty)=0,\\ &\varphi_r^3(\infty)=\varphi_b^4(\infty)=\sqrt{\xi}. \end{aligned} \tag{4.17}$$

Now all quark fields (nearly) vanish inside the wall. The solution for the a fields in the middle domain M (Fig. 4) is given by

$$\begin{aligned} &a=-\sqrt{2}\left(m_1-\Delta m\frac{z-z_0+\tilde{R}/2}{\tilde{R}}\right),\\ &a^3=0, \end{aligned} \tag{4.18}$$

where we introduce the thickness $\tilde{R}$ of the composite wall, to be considered large, $\tilde{R}\gg 1/\sqrt{\xi}$, see below. The equation for a^3 in Eq. (4.4) is trivially satisfied, while the equation for a yields

$$\tilde{R}=\frac{2\Delta m}{g_1^2\xi}, \tag{4.19}$$

demonstrating that indeed that $\tilde{R}\gg 1/\sqrt{\xi}$. Note, that for a particular (unrealistic) case $g_1=g_2$ [which we do *not* consider, since, according to Eq. (2.8), $g_1\neq g_2$] the size of the composite wall is equal to that of the elementary ones; see Eq. (4.11).

Substituting the above solutions in the first two equations in Eq. (4.4) we determine the falloff of the quark fields inside the wall. Namely, near the left edge

$$\varphi_r^1 = \sqrt{\xi} e^{-(\Delta m/2\tilde{R})(z-z_0+\tilde{R}/2)^2},$$
$$\varphi_b^2 = \sqrt{\xi} e^{-(\Delta m/2\tilde{R})(z-z_0+\tilde{R}/2)^2}, \qquad (4.20)$$

while near the right one

$$\varphi^{kB} = \sqrt{\xi}(\tilde{U})^{kB} e^{-(\Delta m/2\tilde{R})(z-z_0+\tilde{R}/2)^2}, \quad B=3,4, \qquad (4.21)$$

where the matrix $\tilde{U}$ is a matrix from the U(2) global flavor group, which takes into account possible flavor rotations inside the flavor pair $B=3,4$. It can be represented as a product of a U(1) phase factor and a matrix U from SU(2)

$$\tilde{U} = e^{i\sigma_0} U. \qquad (4.22)$$

This matrix is parametrized by four phases, σ_0 plus three phases residing in the matrix U.

The occurrence of these four wall moduli—one related to U(1) and three to SU(2)—can be illustrated by the argument which runs parallel to that outlined in Ref. [4]. Indeed, in both vacua, 12 or 34, taken separately, one can always use the symmetries of the theory to render the vacuum matrix $\{\phi^{kA}\}$ diagonal,

$$\{\varphi^{kA}\}_{\rm vac} = \sqrt{\xi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A=1,2 \ \text{ or } \ A=3,4, \qquad (4.23)$$

with the real parameter $\sqrt{\xi}$ in front. When both 12- and 34-vacua get involved—as is the case in the problem of the composite wall—a necessity arises of taking into account their relative alignment. The most concise way to see how these moduli emerge is through examination of a (nonlocal) *gauge-invariant* order parameter[13]

$$\mathcal{O}_A^B(t,x,y) \equiv \frac{1}{\xi}\Big\langle \bar{\varphi}_A(t,x,y|z=-L) \times \exp\Big\{ i\int_{-L}^{L} dz \Big(\frac{\tau^a}{2} A_3^a(t,x,y|z) + \frac{1}{2}A_3(t,x,y|z)\Big)\Big\} \varphi^B(t,x,y|z=L)\Big\rangle, \qquad (4.24)$$

where

$$A=1,2; \quad B=3,4$$

and L is a large parameter which we are supposed to take to infinity at the very end (in practice, $L \gg \tilde{R}$).

[13]The definition below is restricted to the *ansatz* (4.1). In defining the nonlocal gauge invariant order parameter relevant to the domain walls this is by no means necessary. The general definition is similar to that in Eq. (4.24) with the replacement $\bar{\varphi} \to \tilde{q}$.

The order parameter $\mathcal{O}_A^B$ is non-singlet with respect to the global U(2) inherent to our model upon its complete Higgsing. In both vacua, 12 and 34, the order parameter $\mathcal{O}_A^B$ equals to unit matrix—this is quite evident. However, is *not* trivial on the 12→34 wall. On the wall $\mathcal{O}_A^B(t,x,y)$ reduces to a constant U(2) matrix (independent of t,x,y) of the form

$$\mathcal{O}_A^B = \tilde{U}_A^B. \qquad (4.25)$$

Applying all available symmetries of the model at hand, the best we can do is to reduce the number of parameters residing in $\mathcal{O}_A^B$ to four: one U(1) phase and three parameters of global SU(2). There are no massless moduli in both vacua, initial and final; thus all of these four parameters are collective coordinates of the wall.

Below we will identify these four moduli with (2+1)-dimensional gauge fields living on the wall world volume via duality relations of the type presented in Eq. (4.15).

Thus, we get four gauge fields localized on the wall. The physical interpretation of this result is as follows. The quark fields are condensed outside the 12→34 wall while inside they vanish. This means that dual gauge fields are severely confined outside the wall while inside the confinement becomes inoperative. This is precisely the mechanism of the gauge field localization suggested in Ref. [2].

D. Effective field theory on the wall

In this section we work out the (2+1)-dimensional low-energy theory of the moduli on the wall. First we will discuss the elementary walls and then focus on the composite wall 12→34.

1. Elementary walls

In this section we will deal with the elementary domain walls 1,2→1,B with $B=3$ or 4. Our task is to work out the effective (2+1)-dimensional theory for the wall collective coordinates (which become the world-volume fields). For the elementary walls the overall situation is quite similar to that discussed in our previous work [4]. Therefore, we will be rather fragmentary.

As was elucidated in Sec. IV B, the elementary wall has two bosonic collective coordinates, z_0 and σ, plus their fermionic counterparts $\eta^{\alpha f}$. We make slowly varying fields dependent on $t,x,y \equiv x_n (n=0,1,2)$,

$$z_0 \to \zeta(x_n), \quad \sigma \to \sigma(x_n), \quad \eta^{\alpha f} \to \eta^{\alpha f}(x_n). \qquad (4.26)$$

We can limit ourselves to the bosonic fields $\zeta(x_n)$ and $\sigma(x_n)$—the residual supersymmetry will allow us to readily reconstruct the fermion part of the effective action.

The fields $\zeta(x_n)$ and $\sigma(x_n)$ are in one-to-one correspondence with the zero modes in the wall background; therefore, they have no potential terms in the world sheet theory, only kinetic. Our immediate task is to derive these kinetic terms essentially repeating the procedure of Ref. [4]. For $\zeta(x_n)$ this is trivial. Substituting the wall solution (4.9), (4.13), and

(4.14) in the action (4.2) and accounting for the x_n dependence of $\zeta(x_n)$, with no further delay we arrive at

$$\frac{T_{\rm w}}{2}\int d^3x(\partial_n\zeta)^2. \tag{4.27}$$

This answer is quite general and would be valid for the translational modulus in any model.

As far as the kinetic term for $\sigma(x_n)$ is concerned an additional (albeit modest) effort is needed. We start from Eq. (4.14) for the quark fields on the right edge of the wall which depend on the phase σ parametrizing a relative phase orientation of the fourth flavor with regards to the second one. To calculate the corresponding kinetic term we have to modify our *ansatz* for the gauge fields, namely,

$$A_n^3=-\chi_3(z)\partial_n\sigma(x_n),$$
$$A_n=\chi_0(z)\partial_n\sigma(x_n). \tag{4.28}$$

We introduce extra profile functions $\chi_0(z)$ and $\chi_3(z)$, much in the same way it was done in [4]. They have no role in the static wall solution *per se*. However, in constructing the kinetic part of the world-volume theory for the moduli fields their occurrence cannot be avoided.

These new profile functions give rise to their own action, which must be subject to minimization. The gauge potentials (4.28) are introduced in order to cancel the x dependence of the quark fields far from the wall (in the final quark vacuum at $z\to\infty$) emerging through the x dependence of $\sigma(x_n)$; see Eq. (4.14).

Now let us turn to the kinetic terms in the (2+1)-dimensional effective action coming from the quark kinetic terms in Eq. (4.2). For the first flavor we have

$$\nabla_n q^{r1}=-\frac{i}{2}(\partial_n\sigma)(\chi_0-\chi_3)\varphi^{r1}. \tag{4.29}$$

This expression is valid far away from the edges of the domain wall, that is to say, in the middle domain M, where q^{r1} is a nonvanishing constant (4.12), and at $z\to\pm\infty$ where q^{r1} tends to its vacuum expectation value $\sqrt{\xi}$. To ensure the finiteness of the kinetic energy of the first quark flavor we impose the following boundary conditions on the functions χ_0 and χ_3:

$$\chi_0\to\chi_3,\ z\to\pm\infty. \tag{4.30}$$

For the second flavor we have

$$\nabla_n q^{b2}=-\frac{i}{2}(\partial_n\sigma)(\chi_0+\chi_3)\varphi(z_-), \tag{4.31}$$

where we introduced the quark profile function given by

$$\varphi(z)=\begin{cases}\sqrt{\xi}, \\ \sqrt{\xi}e^{-(\Delta m/2R)z^2},\end{cases}\quad z\to\mp\infty, \tag{4.32}$$

and the shorthand

$$z_\pm=z-z_0\mp\frac{R}{2} \tag{4.33}$$

is implied. $z_\pm$ are the coordinates which vanish at the wall edges.

To make the kinetic energy of the second quark flavor finite we impose the boundary conditions

$$\chi_0\to\chi_3\to 0,\quad z\to-\infty. \tag{4.34}$$

A parallel procedure for those quarks that have nonvanishing VEV's in the final vacuum leads us to

$$\nabla_n q^{bB}=i(\partial_n\sigma)\left(1-\frac{\chi_0+\chi_3}{2}\right)\varphi(-z_+). \tag{4.35}$$

This gives us the desired boundary conditions for the functions χ at $z\to+\infty$,

$$\chi_0\to\chi_3\to 1,\ \ z\to+\infty. \tag{4.36}$$

Now we are ready to assemble all necessary elements. Substituting Eqs. (4.29), (4.31) and (4.35) in the action and taking into account the kinetic term for the gauge fields we arrive at

$$S_{2+1}^{\sigma}=\int dz\Bigg\{\frac{1}{g_2^2}(\partial_z\chi_3)^2+\frac{1}{g_1^2}(\partial_z\chi_0)^2+\frac{1}{2}(\chi_0-\chi_3)^2(\varphi^{r1})^2$$
$$+2\left(1-\frac{\chi_0+\chi_3}{2}\right)^2\varphi(-z_+)^2$$
$$+\frac{1}{2}(\chi_0+\chi_3)^2\varphi(z_-)^2\Bigg\}\int d^3x\frac{1}{2}(\partial_n\sigma)^2. \tag{4.37}$$

The expression in the integral over z must be viewed as an action for the χ profile functions. To get the classical solution for the BPS wall *and* the wall world-volume theory of the moduli fields we must minimize this action. The minimization leads to two second-order equations for the functions χ_0 and χ_3. The solutions to these equations are linear in the middle domain M, for both functions,

$$\chi_{0,3}=\frac{z-z_0+R/2}{R}. \tag{4.38}$$

Furthermore, outside the domain wall the both functions exponentially approach their boundary values (4.36), (4.34). This exponential approach is controlled by the photon mass (2.18) for the U(1) field and the W-boson mass (2.17) for the SU(2) field (cf. [4]). Substituting the solution (4.38) in the χ action (4.37) and taking into account Eq. (4.11) we finally obtain

$$S_{2+1}^{\sigma}=\frac{\xi}{\Delta m}\int d^3x\frac{1}{2}(\partial_n\sigma)^2. \tag{4.39}$$

As has been already mentioned previously, the compact scalar field $\sigma(t,x,y)$ can be reinterpreted as a dual to the (2+1)-dimensional Abelian gauge field living on the wall, see Eq. (4.15). Note, that Eqs. (4.28) and (4.38) demonstrate that the particular combination of two U(1) gauge fields which is localized inside the wall is the combination with the following alignment:

$$A_n^3 = -A_n\,.$$

This combination corresponds to the e_2-root of the SU(3) gauge group of the "prototheory;" see Fig. 5. In particular, the φ^1 quark which has a nonvanishing r-component inside the wall is not charged under this combination.

The result presented in Eq. (4.39) implies that the coupling constant of the effective U(1) theory on the wall

$$e_{2+1}^2 = 4\pi^2 \frac{\xi}{\Delta m}. \tag{4.40}$$

This statement will help us make the definition of the (2+1)-dimensional gauge field outlined in Eq. (4.15) more precise,

$$F_{nm}^{(2+1)} = \frac{e_{2+1}^2}{2\pi}\varepsilon_{nmk}\partial^k \sigma. \tag{4.41}$$

As a result, the effective low-energy theory of the moduli fields on the wall takes the form

$$S_{2+1} = \int d^3x \left\{ \frac{T_w}{2}(\partial_n \zeta)^2 + \frac{1}{4e_{2+1}^2}[F_{nm}^{(2+1)}]^2 + \text{fermion terms} \right\}. \tag{4.42}$$

The elementary wall at hand is 1/2 BPS-saturated—it breaks four supercharges out of eight present in $\mathcal{N}=2$ theory. Thus we have four fermion fields residing on the wall, $\eta^{\alpha f}$ ($\alpha, f = 1,2$). Because of the (2+1)-dimensional Lorentz invariance of the on-the-wall theory we are certain that these four fermion moduli fields form two (two-component) Majorana spinors. Thus, the field content of the world-volume theory we obtained is in full accord with the representation of the (2+1)-dimensional extended supersymmetry: one complex scalar field plus one Dirac two-component fermion field. Minimal supersymmetry in 2+1 dimensions (with two supercharges) would require one real scalar field and one Majorana two-component fermion field. It is natural that we recover extended supersymmetry: there are eight supercharges in our microscopic theory; the domain wall at hand is 1/2 BPS; hence, we end up with four supercharges in the world-volume theory.

2. The composite wall

Let us pass to the discussion of the effective world-volume theory on the composite domain wall 12→34. The emphasis will be put on novel elements appearing in the theory of the moduli fields on the composite wall which were absent in the case of the elementary walls.

The first technical modification compared to Sec. IV D 1 is that now we have four independent compact moduli, rather than one—three residing in the matrix U, Eq. (4.22) plus σ_0. Therefore, in order to cancel the x_n-dependence of the quark fields far away from the wall, in the final vacuum, we have to introduce in the *ansatz* four gauge fields,

$$A_n = \chi_0(z)\partial_n \sigma^0(x_n),$$
$$\frac{\tau^a}{2}A_n^a = -i\chi(z)[\partial_n U(x_n)]U^{-1}(x_n). \tag{4.43}$$

Here χ and χ_0 are the profile functions for SU(2) and U(1) gauge fields, respectively. Calculations of the quark and gauge kinetic terms run parallel to those in Sec. IV D 1 leading us to a key formula

$$\begin{aligned} S_{2+1}^{\rm cm} = {} & \int dz \left[\frac{1}{g_1^2}(\partial_z \chi_0)^2 + (1-\chi_0)^2\varphi^2(-z_+) + \chi_0^2 \varphi^2(z_-) \right] \int d^3x \frac{1}{2}(\partial_n \sigma_0)^2 \\ & + \int dz \left[\frac{1}{g_2^2}(\partial_z \chi)^2 + (1-\chi)^2 \varphi^2(-z_+) + \chi^2\varphi^2(z_-) \right] \int d^3x\, \mathrm{Tr}[(U^{-1}\partial_n U)(U^{-1}\partial_n U)], \end{aligned} \tag{4.44}$$

where the superscript cm stands for compact moduli. The boundary conditions for the functions χ and χ_0 must be chosen to ensure finiteness of energy in the domains far away from the wall. This gives

$$\chi_0 \to \chi \to 0, \quad z \to -\infty,$$
$$\chi_0 \to \chi \to 1, \quad z \to +\infty. \tag{4.45}$$

Equation (4.44) can be considered as an action functional for χ and χ_0.

The functions χ and χ_0 are determined by minimization of the above action functional which gives a second-order equation for each function. We will not present them here, since the reader can trivially get them himself or herself by minimization. The solutions in the middle domain have the already familiar linear form,

$$\chi_0 = \chi = \frac{z - z_0 + \tilde{R}/2}{\tilde{R}}, \tag{4.46}$$

where the size of the composite wall $\tilde{R}$ is given in Eq. (4.19). Outside the wall the functions χ and χ_0 exponentially approach their boundary values (4.45). The rate of approach is determined by the photon mass (2.18) for the function χ_0, while it is determined by the W-boson mass (2.17) for the function χ. Substituting the solution (4.46) back in the action (4.44) we obtain the following kinetic term:

$$S_{2+1}^{\rm cm}=\frac{\xi}{2\Delta m}\int d^3x\left\{\frac{1}{2}(\partial_n\sigma_0)^2+\frac{g_1^2}{g_2^2}{\rm Tr}[(U^{-1}\partial_nU)(U^{-1}\partial_nU)]\right\}. \qquad (4.47)$$

Next, as in the elementary wall case, we can try to dualize the moduli residing in U, as well as σ_0, to convert them in $(2+1)$-dimensional gauge fields

$$F_{nm}^{(2+1)}=\frac{e_{2+1}^2}{2\pi}\varepsilon_{nmk}\partial^k\sigma_0,$$

$$\frac{\tau^a}{2}F_{nm}^{(2+1)a}=-i\frac{g_{2+1}^2}{2\pi}\varepsilon_{nmk}U^{-1}\partial_kU. \qquad (4.48)$$

Assembling all the above elements we obtain the action of the world-volume effective theory,

$$S_{2+1}=\int d^3x\left\{\frac{1}{2e_{2+1}^2}(\partial_na_{2+1})^2+\frac{1}{2g_{2+1}^2}(D_na_{2+1}^a)^2+\frac{1}{4e_{2+1}^2}[F_{nm}^{(2+1)}]^2+\frac{1}{4g_{2+1}^2}[F_{nm}^{(2+1)a}]^2+\text{fermion terms}\right\}, \qquad (4.49)$$

of which a few comments are in order immediately.

The first comment refers to four noncompact a,a^a moduli which emerged in Eq. (4.49) seemingly out of blue. We can use gauge transformation in the world volume theory to put two of them to zero, say $a_{2+1}^{1,2}=0$. The other two a_{2+1}^3 and a_{2+1} should be identified with (linear combinations of) two centers of the elementary walls comprising our composite wall.[14] More exactly, as a_{2+1} has no interactions whatsoever it is to be be identified with the center of mass of the composite wall,

$$a_{2+1}=\sqrt{\xi\Delta m}\,e_{2+1}\frac{1}{\sqrt{2}}(\zeta_1+\zeta_2)=\pi\xi(\zeta_1+\zeta_2), \qquad (4.50)$$

while a_{2+1}^3 can be identified with the relative separation between the elementary walls,

$$a_{2+1}^3=\sqrt{\xi\Delta m}\,g_{2+1}\frac{1}{\sqrt{2}}(\zeta_1-\zeta_2)=\pi\xi\frac{g_1}{g_2}(\zeta_1-\zeta_2), \qquad (4.51)$$

where we use the fact that the tension of elementary walls is $T_w=\xi\Delta m$.

The second comment is devoted to a technical (but very important) element of the derivation of Eq. (4.49). In fact, this world-volume action was obtained, through the calculational procedure described above, only at the quadratic level (i.e. omitting non-Abelian nonlinearities). This is rather obvious as our derivation, and in particular the identification (4.48) and the effective action (4.47), limits itself to the quadratic approximation. Higher-than-quadratic terms in the $(3+1)$-dimensional action, such as the commutator term in the gauge field strength tensor, would produce four-derivative terms in the $(2+1)$-dimensional theory (4.47). Such terms were explicitly omitted in the derivation above. To recover non-quadratic (truly non-Abelian) terms in Eq. (4.49) we use gauge invariance on the world volume.

The final remark is about the values of the coupling constants in the $(2+1)$-dimensional ("macroscopic") theory in relation to the $(3+1)$-dimensional ("microscopic") parameters. The U(1) and SU(2) gauge coupling constants in Eq. (4.49) are given by

$$e_{2+1}^2=2\pi^2\frac{\xi}{\Delta m},$$

$$g_{2+1}^2=2\pi^2\frac{g_1^2}{g_2^2}\frac{\xi}{\Delta m}. \qquad (4.52)$$

Our domain wall is a 1/2-BPS object so it preserves four supercharges on its world volume. Thus, we must have the extended $\mathcal{N}=2$ supersymmetry, with four supercharges, in the $(2+1)$-dimensional world-volume theory. This is in accord with Eq. (4.49) in which the U(1) and SU(2) gauge fields are combined with the scalars a_{2+1} and a_{2+1}^a to form the bosonic parts of $\mathcal{N}=2$ vector multiplets.

Now let us discuss the possibility of spontaneous gauge symmetry breaking in the world volume theory. Clearly if the adjoint scalar a_{2+1}^a develops a VEV, the SU(2) gauge symmetry is spontaneously broken in our world volume theory (4.49) on the composite wall. We can always use gauge rotations to direct a_{2+1}^a VEV along third axis in the color space, $\langle a_{2+1}^3\rangle\neq 0$. Then identification (4.51) shows that the separation $l=z_1-z_2$ between two elementary walls which form our composite wall is nonvanishing. In particular, the mass of the $(2+1)$-dimensional W-boson is given by the separation between elementary walls,

$$m_W^{2+1}=\pi\xi\frac{g_1}{g_2}l. \qquad (4.53)$$

The mass grows linearly with l. This is completely consistent with similar result for D-branes obtained in string theory.

Now let us discuss how can one see this gauge symmetry breaking in the $(3+1)$-dimensional bulk theory. Let us split our composite wall in two elementary ones, say, $12\to14$ and $14\to34$. Now we pull these two elementary walls apart making the separation much larger than the wall thickness, $l\gg R$. The separation l must be much larger than R because

[14]Note that in Sec. IV C we worked out the solution for the composite wall as a bound state of two elementary walls at zero separation. However, in fact, this bound state is marginally unstable and has a zero mode associated with the possibility of arbitrary separation between components.

the scale $1/R$ plays a role of an ultraviolet cutoff in the world-volume theory (4.49). Clearly two well separated elementary walls have only two phase collective coordinates, one per each wall; see Sec. IV B. If we dualize these phases we get two Abelian gauge fields in the effective world volume theory. This corresponds to breaking of $SU(2)\times U(1)$ gauge symmetry down to $U(1)\times U(1)$ in Eq. (4.49). At separations $l\gg R$ the masses of two $(2+1)$-dimensional W-bosons become much larger than Δm [see Eq. (4.53)] and they cannot be seen in the effective low-energy world-volume theory.

One may ask where do two extra ''non-Abelian'' phases of the composite wall disappear at separations larger than the elementary wall width. Of course, they do not disappear. They just pass into Goldstone modes in the intermediate 14-vacuum. Remember that the intermediate 14-vacuum has a Higgs branch; see Sec. II B 2. The two extra phases are now associated with the massless moduli on this Higgs branch. At zero separation these phases are collective coordinates of the composite wall. They belong to the $(2+1)$-dimensional world-volume theory. At large separations they become bulk excitations living in the intermediate vacuum. We will return to this issue in Sec. VI D where we will show that only Abelian strings can end on the composite wall when the separation between its components gets larger than the thickness of the individual components.

To conclude this section, we reiterate that the consideration presented above is certainly not a ''rigorous derivation'' of the non-Abelian gauge invariance in the effective world-volume action (4.49). Rather, it can be viewed as a motivated argument. Our derivation is carried out only at the quadratic level and does not take into account non-Abelian nonlinearities. We identify four compact collective coordinates, to be dualized into four gauge fields living on the wall. We also calculate their kinetic terms which fix the values of the 3D gauge coupling constants. Direct calculation of cubic and quartic interaction terms, i.e., a *bona fide* complete derivation, goes beyond the scope of this paper. This is a task for future work.

The gauge invariance in Eq. (4.47) is not apparent since Eq. (4.47) is written in terms of *gauge invariant* phases. The gauge invariance of the world-volume theory appears only in Eq. (4.49), after dualization. There are quite compelling albeit indirect arguments showing that our proposal [i.e. the $SU(2)\times U(1)$ gauge theory (4.49)] is the correct generalization of Eq. (4.47). First, the number of fields matches. We have four compact phases and two noncompact centers. Upon dualization, they fit into a vector multiplet of 3D $\mathcal{N}=2$ theory with the $SU(2)\times U(1)$ gauge group. Say, if the gauge group were $U(1)^4$, we would need four phases and four noncompact coordinates, which we do not have. Thus, the non-Abelian gauge symmetry of the world-volume theory, in effect, is supported by supersymmetry. Second, there are ony two distinct coupling constants in Eq. (4.47), rather than four. This also indicates that three phases, upon dualization, should be unified in the SU(2) gauge theory.

V. NON-ABELIAN FLUX TUBES IN $\mathcal{N}=2$ QCD

In string theory gauge fields are localized on D-branes because fundamental open strings can end on D branes [1]. In Ref. [4] we demonstrated that this picture is also valid in field theory, in the Abelian gauge field case. Namely, the Abelian flux tube was shown to end on the domain wall. The reason for such behavior is easy to understand. In the Higgs vacuum (in which electric charges condense), the magnetic field is trapped into flux tubes. However, inside the wall quark fields (almost) vanish. Therefore, the magnetic flux which is carried by the string in the bulk can spread over inside the wall. The magnetic fields become electric upon dualization. The string end point plays the role of the electric charge for the gauge field localized on the wall [4].

Our task is to generalize this picture to cover the case of the non-Abelian gauge fields. The main goal is finding a solution for a 1/4-BPS string-wall junction, in which a string carrying a non-Abelian flux can end on the composite wall $12\to34$. We start implementation of the string-wall junction program in earnest in Secs. V B and VI. Meanwhile, a brief introduction in non-Abelian flux tubes will be in order, to acquaint the reader with the necessary machinery. An advanced investigation of non-Abelian flux tubes in various regimes will be described elsewhere [32].

Vortices in non-Abelian theories were studied in many papers in recent years [13,38–43]. However, in all these examples of vortex solutions, the string flux is always directed in the Cartan subalgebra of the gauge group. This implies a (hidden) Abelian nature of these strings. Clearly these strings cannot be used for our purposes because their end points on domain walls cannot act as sources for non-Abelian fields.

Only recently a special regime was found [6] in which flux tubes acquire orientational zero modes which allow one to freely rotate the string flux inside a non-Abelian group. This special regime is associated with the presence of a certain combination of global gauge and flavor symmetry not broken by VEV's of scalar fields. Below we will show that precisely this regime is realized in 12- and 34-vacua of the theory under consideration.

The theory analyzed in Ref. [6] is $\mathcal{N}=2$ QCD with the SU(3) gauge group broken down to $SU(2)\times U(1)$, with four quark flavors, all with the same mass. We review the string solution found in this paper and adapt the analysis to our $SU(2)\times U(1)$ model. To begin with, however, we present some general arguments in a simple toy model.

A. How ''non-abelian'' are non-abelian strings we deal with?

In Ref. [24] it was proven that the only BPS-saturated strings at weak coupling in $\mathcal{N}=1$ theories are those of the ANO type, occurring in U(1) theories. Here we speak of ''non-Abelian'' BPS strings in $\mathcal{N}=2$. A natural question to ask is in which sense the BPS flux tubes under consideration are non-Abelian strings. A conceptual answer can be given in a simplified model which does not even need to be supersymmetric.

Indeed, let us consider a (non-supersymmetric) model which generalizes that of Abrikosov-Nielsen-Olesen, and has two gauge groups, SU(2) and U(1), and scalar fields of two flavors.[15] Denote two SU(2) doublet fields by $\varphi_i^{(1)}$ and $\phi_j^{(2)}$, $i,j=1,2$. Then, introduce a 2×2 matrix field

$$\Phi=\begin{pmatrix} \phi_1^{(1)} & \phi_1^{(2)} \\ \phi_2^{(1)} & \phi_2^{(2)} \end{pmatrix}. \tag{5.1}$$

The covariant derivatives are defined in such a way that they act from the *left*,

$$\nabla_\mu \Phi \equiv \left(\partial_\mu - \frac{i}{2}A_\mu - iA_\mu^a \frac{\tau^a}{2}\right)\Phi. \tag{5.2}$$

We assume the action to have the form

$$S=\int d^4x\left[\frac{1}{4g_2^2}(F_{\mu\nu}^a)^2+\frac{1}{4g_1^2}(F_{\mu\nu})^2 + \mathrm{Tr}(\nabla_\mu\Phi)^\dagger(\nabla_\mu\Phi)+V(\Phi)\right], \tag{5.3}$$

where, for the time being, the potential function V is assumed to be gauge invariant as well as invariant under the *global* U(2)

$$\Phi \to \Phi U_R. \tag{5.4}$$

Here U_R is a constant matrix from U(2), and the multiplication is performed from the *right*. The action (5.3) is invariant under the local U(2),

$$\Phi \to U_L(x)\Phi, \tag{5.5}$$

with A_μ and A_μ^a transformed appropriately, *and* under the global U(2), Eq. (5.4).

Models of the type (5.3) were engineered long ago [44] with the purpose of providing a set up for the spontaneous breaking of the local (gauge) group G down to a diagonal global G. Indeed, with an appropriate choice of the potential function V, one can ensure the vacuum expectation value of Φ to be diagonal,

$$\Phi_{\rm vac}=v\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \quad v\neq 0. \tag{5.6}$$

Then, since this vacuum is obviously invariant under the combined multiplication

$$\Phi \to U_L \Phi U_R, \tag{5.7}$$

with $U_R=U_L^\dagger$, the diagonal global U(2) will be preserved.

Now we can discuss topological defects of the string type. Defects of the ANO type are always possible. Indeed, put the SU(2) gauge field to zero (and temporarily forget about it whatsoever). A nontrivial topology will be realized through the U(1) winding of Φ,

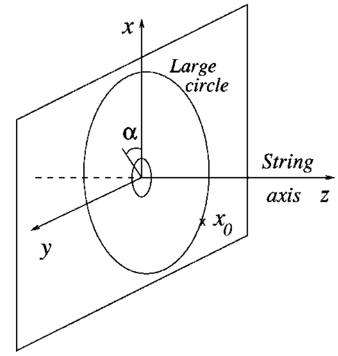

FIG. 6. Geometry of the string.

$$\Phi(x)=e^{i\alpha(x)}v, \quad |x|\to\infty, \tag{5.8}$$

and

$$A_\ell = -2\varepsilon_{\ell k}\frac{x_k}{r}, \tag{5.9}$$

where α is the phase in the perpendicular plane, and r is the distance from the string axis in the perpendicular plane. Since $\pi_1(\mathrm{U}(1))=Z$, we will get in this way a set of the ANO flux tubes with the arbitrary windings.

These are not the strings we are after, however. At first sight, the presence of the SU(2) gauge symmetry, in addition to U(1), does not create any new possibilities. Indeed, $\pi_1(\mathrm{SU}(2))$ is trivial; one can readily unwind windings in SU(2) relevant to strings.

Nevertheless, the fact that SU(2) has a center, Z_2, does create a new possibility.[16] To see that this is the case, let us examine the following topology. A large circle in the plane perpendicular to the string axis is depicted in Fig. 6. Assume that one starts from a certain point on this circle and makes a full rotation around the string. Introduce the winding in SU(2), and assume the full rotation above to bring us to the same element up to the center, namely,

$$\Phi(x)=e^{i\vec\omega(x)\vec\tau/2}\Phi(x_0)\to -\Phi(x_0) \quad \text{at} \quad x\to x_0 \quad \text{after rot.}, \tag{5.10}$$

see Fig. 7. The condition (5.10) *per se* is forbidden, since it results in a discontinuity of the Φ field. One can eliminate this discontinuity by supplementing the SU(2) winding above by a U(1) winding with the condition

$$\Phi(x)=e^{i\alpha(x)/2}\Phi(x_0)\to -\Phi(x_0) \quad \text{at} \quad x\to x_0 \quad \text{after rot.} \tag{5.11}$$

The formula

$$\Phi(x)=\exp\left(i\alpha(x)\frac{1\pm\tau^3}{2}\right)\Phi(x_0), \quad \alpha(x)\to 2\pi \quad \text{after rot.}, \tag{5.12}$$

[15]This model is also a version of the Higgs sector of the standard model.

[16]Of course, any element of U(1) can be considered as a center, since this group is Abelian.

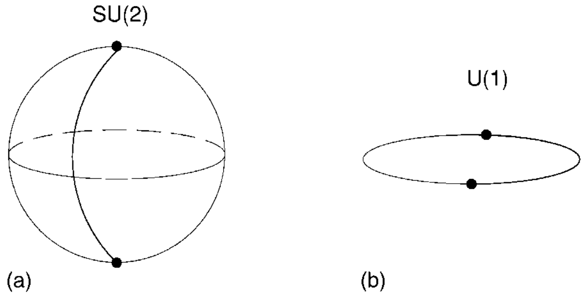

FIG. 7. Topology of the (1,0) and (0,1) strings. The trajectories in the group spaces corresponding to circumnavigating along the large circle in Fig. 6 are denoted by bold lines.

summarizes this pattern. Depending on the choice of the sign in the exponent, only the r or only the b components of the fields $\phi^{(1,2)}$ have a nontrivial winding. It is clear from Fig. 7 that one cannot unwind it. It is also clear that the fluxes corresponding to the fields A_μ and A_μ^3 are half the flux of the U(1) field of the ANO string. We will refer to such strings as (1,0) and (0,1)—the first winding number corresponding to the index r, the second to b. This notation seems rather awkward given the way we introduced the setup. It emerges naturally within the historical line of development, however; see below. The standard ANO string emerges as a sum of the (1,0) and (0,1) strings. The question as to which strings are more favorable energetically depends on dynamical details. We will return to it later.

A remarkable feature of the (1,0) and (0,1) strings is the appearance of non-Abelian moduli which are absent in the ANO strings. Indeed, while the vacuum field (5.6) is invariant under the global SU(2) by virtue of Eq. (5.7), the string configuration (5.12) is not. Therefore, if there is a solution of the form (5.12) there is a family of solutions obtained from Eq. (5.12) by the replacement

$$\Phi(x)\to\Omega\,\phi(x)\Omega^\dagger \tag{5.13}$$

where Ω is an x-independent matrix from U(2). Generally speaking, it is parametrized by four parameters. The U(1) factor is nothing but a shift of the origin of the angle α, however; one should not count it. Thus, what remains is SU(2). Moreover, in fact, it is SU(2)/U(1), as is clearly seen from Eq. (5.12). [Rotations around the third axis in the SU(2) space leave the solution intact.] SU(2)/U(1) is the target space of the CP^1 [or $O(3)$] sigma model which, thus, provides the adequate description of the moduli dynamics [5,6].

It is just this aspect that allows us to refer to the strings above as "non-Abelian." They are as non-Abelian as it gets at weak coupling.

Note that the stability of the (1,0) and (0,1) strings under consideration would be impossible without the presence of the U(1) factor.

In conclusion, it is instructive to ask what happens if we explicitly break the SU(2) flavor symmetry of the model (5.3) by introducing unequal masses to the fields $\phi^{(1)}$ and $\phi^{(2)}$, namely,

$$S_{\mathcal{M}}=\int d^4x\,\mathrm{Tr}(\Phi\mathcal{M}^2\Phi^\dagger), \tag{5.14}$$

where the mass matrix has the form

$$\mathcal{M}^2=\begin{pmatrix} m_1^2 & 0\\ 0 & m_2^2\end{pmatrix},\quad m_1^2\neq m_2^2. \tag{5.15}$$

Intuitively it is clear that if Δm^2 is small, the only change is as follows: the SU(2) symmetry of the CP^1 model must be broken, producing quasi-moduli from the would-be moduli of CP^1. What survives is U(1), rather than SU(2). Later, after re-introduction of supersymmetry, we will see that this process is nothing but the transition from the $\mathcal{N}=2$ CP^1 model to the one with the twisted mass.[17] Anticipating our further needs we present here the bosonic part of the CP^1 model with the twisted mass,

$$\mathcal{L}_{\mathrm{CP(1),t.m.}}=G\{\partial_\mu\bar{\phi}\partial^\mu\phi-|\tilde{m}|^2\bar{\phi}\phi\}, \tag{5.16}$$

where G is the metric on the target space,

$$G\equiv\frac{2}{g^2}\frac{1}{(1+\phi\bar{\phi})^2}, \tag{5.17}$$

and

$$\chi\equiv 1+\phi\bar{\phi}.$$

(It is useful to note that the Ricci tensor $R=2\chi^{-2}$.) The twisted mass parameter $\tilde{m}$ introduced in Eq. (5.16) is related to the mass splitting Δm of the microscopic theory, $\tilde{m}=\Delta m$.

Anticipating further applications, we hasten to add that the $\mathcal{N}=2$ superalgebra of the CP^1 model (which is our macroscopic theory) is centrally extended, namely,

$$\{Q_L\bar{Q}_R\}=-i\tilde{m}q_{\mathrm{U(1)}}-\tilde{m}\int dz\,\partial_z h+\frac{1}{\pi}\int dz\,\partial_z(\chi^{-2}\bar{\Psi}_R\Psi_L), \tag{5.18}$$

where the first term is proportional to the U(1) charge $q_{\mathrm{U(1)}}$, while in the second term $h=-(2/g^2)(1/\chi)$ presents the main impact of the twisted mass. The last term is the central charge anomaly established in Ref. [33]. It is proportional to the difference between the bifermion condensates in the final and initial vacua. The central charge anomaly becomes important in the limit $\tilde{m}\to 0$ corresponding to $m_1\to m_2$ in the microscopic theory. Then the classical terms $\sim q_{\mathrm{U(1)}}$ and $\sim h$ vanish, and the central charge is entirely determined by the anomaly.

[17] $\mathcal{N}=2$ sigma models with twisted mass were first constructed in Ref. [45]. The superspace/superfield description was developed in Refs. [46,47]. In particular, the notion of a twisted chiral superfield was introduced in the second of these works. The word "twisted" appears for the first time in the given context in Ref. [47].

The above central charge is in one-to-one correspondence with the BPS kinks in the CP^1 model. Sure enough, it must (and does) have a counterpart in the microscopic theory, see Sec. III C. Projecting them onto one another allows one to establish relations between the parameters of the microscopic and macroscopic theories [32], for instance,

$$\frac{1}{g_{CP^1}^2}=\frac{2\pi}{g_2^2}. \tag{5.19}$$

Let us note that g_2^2 in runs according to the formula of asymptotic freedom down to ξ (at $\Delta m=0$), where it is frozen in the bulk. The asymptotic freedom running of g_2^2 is taken over and matched by that of $g_{CP^1}^2$ in the macroscopic (world-volume) theory.

B. Back to strings in $\mathcal{N}=2$

For definiteness let us consider strings in the 12-vacuum. To find the BPS string solutions we use the same *ansatz* as in Eq. (4.1) and also put adjoint fields, which are irrelevant for the string solutions, equal to their VEV's (2.12). With these simplifications our theory (4.2) becomes

$$S=\int d^4x\left[\frac{1}{4g_2^2}(F_{\mu\nu}^a)^2+\frac{1}{4g_1^2}(F_{\mu\nu})^2+|\nabla_\mu\varphi^A|^2+\frac{g_2^2}{8}(\bar{\varphi}_A\tau^a\varphi^A)^2+\frac{g_1^2}{8}(\bar{\varphi}_A\varphi^A-2\xi)^2\right]. \tag{5.20}$$

Clearly, only those two flavors $A=1,2$ which develop VEV's in the 12-vacuum will play a role in the classical vortex solution. Other flavors remain vanishing on the solution. Hence, we consider the quark fields φ^{kA} to be 2×2 matrices in this section. Note, however, that the additional two flavors are crucial in quantum theory, in keeping the SU(2) interactions weak.

The string tension can be written in the manner of Bogomolny [34] as follows:

$$T=\int d^2x\left\{\left[\frac{1}{\sqrt{2}g_2}F_3^{*a}\pm\frac{g_2}{2\sqrt{2}}(\bar{\varphi}_A\tau^a\varphi^A)\right]^2+\left[\frac{1}{\sqrt{2}g_1}F_3^*\pm\frac{g_1}{2\sqrt{2}}(|\varphi^A|^2-2\xi)\right]^2+|\nabla_1\varphi^A\pm i\nabla_2\varphi^A|^2\pm\xi F_3^*\right\}, \tag{5.21}$$

where

$$F_3^*\equiv\frac{1}{2}\epsilon_{ij}F_{ij}\quad(i,j=1,2), \tag{5.22}$$

plus the same for F_3^{*a}, are the coordinates in the plane orthogonal to the string axis directed along the third (i.e., z) axis. The Bogomolny representation implies the first-order equations for the BPS strings,

$$F_3^{*a}+\frac{g_2^2}{2}\varepsilon(\bar{\varphi}_A\tau^a\varphi^A)=0,\quad a=1,2,3;$$

$$F_3^*+\frac{g_1^2}{2}\varepsilon(|\varphi^A|^2-2\xi)=0;$$

$$(\nabla_1+i\varepsilon\nabla_2)\varphi^A=0, \tag{5.23}$$

where

$$\varepsilon=\pm1$$

is the sign of the total flux specified below.

We first review the U(1)×U(1) string solutions found [13] in the unequal quark mass case, and then show that in the limit of equal quark masses additional orientational zero modes arise making the string non-Abelian [6]. For unequal quark masses some of the orientational moduli become quasi-moduli, corresponding to passing from the CP^1 sigma model with no twisted mass to that with a twisted mass; see Sec. V A.

The U(1)×U(1) strings can be recognized, with no effort, as particular solutions of Eqs. (5.23). To construct them we further restrict the gauge fields A_μ^a to a single (third) color component A_μ^3 (by setting $A_\mu^1=A_\mu^2=0$), and consider only the quark fields of the 2×2 color-flavor diagonal form,

$$\varphi^{kA}(x)\neq0,\quad\text{for}\quad k=A=1,2, \tag{5.24}$$

with vanishing other components. For the unequal masses the relevant topological classification is

$$\pi_1\left(\frac{\mathrm{U}(1)\times\mathrm{U}(1)}{Z_2}\right)=Z^2, \tag{5.25}$$

and the allowed strings form a lattice labeled by two integer winding numbers. To be more specific, assume that the first flavor winds n times while the second flavor winds k times. The solutions of Eq. (5.23) are sought for using a ''natural'' *ansatz*,

$$\varphi^{kA}(x)=\begin{pmatrix}e^{in\alpha}\phi_1(r) & 0\\ 0 & e^{ik\alpha}\phi_2(r)\end{pmatrix},$$

$$A_i^3(x)=-\varepsilon\epsilon_{ij}\frac{x_j}{r^2}[(n-k)-f_3(r)],$$

$$A_i(x)=-\varepsilon\epsilon_{ij}\frac{x_j}{r^2}[(n+k)-f(r)], \tag{5.26}$$

where (r,α) are the polar coordinates in the (12)-plane while the profile functions ϕ_1, ϕ_2 for the scalar fields and f_3, f for the gauge fields depend only on r $(i,j=1,2)$.

With this *ansatz*, the first-order equations (5.23) take the form [13]

$$r\frac{d}{dr}\phi_1(r) - \frac{1}{2}[f(r)+f_3(r)]\phi_1(r)=0,$$

$$r\frac{d}{dr}\phi_2(r) - \frac{1}{2}[f(r)-f_3(r)]\phi_2(r)=0,$$

$$-\frac{1}{r}\frac{d}{dr}f(r)+\frac{g_1^2}{2}[\phi_1(r)^2+\phi_2(r)^2-2\xi]=0,$$

$$-\frac{1}{r}\frac{d}{dr}f_3(r)+\frac{g_2^2}{2}[\phi_1(r)^2-\phi_2(r)^2]=0. \tag{5.27}$$

The profile functions in these equations are determined by the following boundary conditions:

$$f_3(0)=\varepsilon_{n,k}(n-k), \quad f(0)=\varepsilon_{n,k}(n+k),$$

$$f_3(\infty)=0, \quad f(\infty)=0 \tag{5.28}$$

for the gauge fields. The boundary conditions for the quark fields are

$$\phi_1(\infty)=\sqrt{\xi}, \quad \phi_2(\infty)=\sqrt{\xi},$$

$$\phi_1(0)=0, \quad \phi_2(0)=0 \tag{5.29}$$

for both n and k nonvanishing, while, say, for $k=0$ the only condition at $r=0$ is $\phi_1(0)=0$. Here the sign of the string flux is defined as

$$\varepsilon=\varepsilon_{n,k}=\frac{n+k}{|n+k|}=\mathrm{sgn}(n+k)=\pm 1. \tag{5.30}$$

The tension of a (n,k)-string is determined by the flux of the U(1) gauge field alone and is given by the formula [6,13]

$$T_{n,k}=2\pi\xi|n+k|. \tag{5.31}$$

Note that F_3^{*3} does not enter the central charge of the $\mathcal{N}=2$ algebra and, therefore, does not affect the string tension. The stability of the string in this case is due to the U(1) factor of the $SU(2)\times U(1)$ group only. Note also that (1,0) and (0,1)-strings are exactly degenerate.

For a generic (n,k)-string equations (5.27) do not reduce to the standard Bogomolny equations. For instance, for the (1,1)-string these equations reduce to two Bogomolny equations while for the (1,0) and (0,1) strings they do not. Numerical solutions for the two "elementary" (1,0) and (0,1) strings were obtained in Ref. [6].

The charges of (n,k)-strings can be plotted on the Cartan plane of the SU(3) algebra of the "prototheory." We shall use the convention of labeling the flux of a given string by the magnetic charge of the monopole which produces this flux and can be attached to its end. This is possible, since both, string fluxes and the monopole charges, are elements of the group $\pi_1(U(1)^2)=Z^2$. This convention is quite convenient because specifying the flux of a given string automatically fixes the charge of the monopole that it confines.

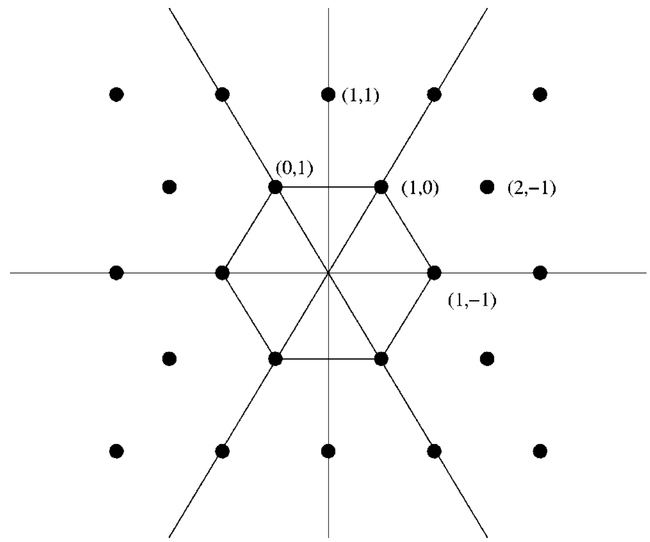

FIG. 8. Lattice of (n,k) vortices.

Our strings are formed as a result of the quark condensation; the quarks have electric charges equal to the weights of the SU(3) algebra. The Dirac quantization condition tells us [13] that the lattice of the (n,k)-strings is formed by roots of the SU(3) algebra. The lattice of (n,k)-strings is shown in Fig. 8. Two strings, (1,0) and (0,1), are the "elementary" or "minimal." They are BPS-saturated. All other strings can be considered as bound states of these "elementary" strings. If we plot two lines along the charges of these "elementary" strings (Fig. 8) they divide the lattice into four sectors. It turns out [13] that the strings in the upper and lower sectors are BPS but they are marginally unstable. At the same time, the strings in the right and left sectors are (meta)stable bound states of the "elementary" ones but they are not BPS.

Now, let us generalize the string solutions (5.26) to the case of the equal quark masses, when the $SU(2)\times U(1)$ gauge group is not broken by the difference of the quark masses, as is the case in the 12-vacuum. The relevant homotopy group in this case is the fundamental group

$$\pi_1\left(\frac{SU(2)\times U(1)}{Z_2}\right)=Z, \tag{5.32}$$

replacing Eq. (5.25). This means that the lattice of (n,k)-strings reduces to a tower labeled by one integer $(n+k)$. For instance, the $(1,-1)$-string becomes completely unstable. On the restored SU(2) -group manifold it corresponds to a winding along the meridian on the sphere S_3. Clearly this winding can be shrunk to nothing by contracting the loop towards the north or south poles [37].

On the other hand, the (1,0) and (0,1) strings cannot be shrunk because their winding is half a circle (Fig. 7). They have the same tension

$$T_1=2\pi\xi \tag{5.33}$$

for equal quark masses and, thus, apparently belong to a doublet of SU(2).

Below we will show that there is a continuous deformation of the (1,0)-string solution transforming it into a (0,1)-string. This deformation leaves the string tension unchanged and, therefore, corresponds to an orientational zero mode [6].

First let us fix the unitary gauge (at least globally, which is enough for our purposes) by imposing the condition that the squark VEV's are given precisely by Eq. (2.13), and so all gauge phases vanish. Now transform the (1,0)-string solution (5.26) into the unitary gauge, which corresponds to the singular gauge, in which the string flux comes from the singularity of the gauge potential at zero. In this gauge the solution (5.26) for the (1,0)-string takes the form

$$\varphi^{kA}=\begin{pmatrix}\phi_1(r) & 0\\ 0 & \phi_2(r)\end{pmatrix},$$

$$A_i^3(x)=\epsilon_{ij}\frac{x_j}{r^2}f_3(r),\quad A_i(x)=\epsilon_{ij}\frac{x_j}{r^2}f(r). \tag{5.34}$$

Now, please, observe that a global diagonal subgroup in the product of gauge and flavor symmetries $SU(2)_C\times SU(2)_F$ is not broken by the quark VEV's. Namely,

$$U_{C+F}\langle q\rangle U_{C+F}^{-1}=\langle q\rangle, \tag{5.35}$$

where U_{C+F} is a global rotation in SU(2) while the quark VEV matrix is given by Eq. (2.13). We refer to this unbroken group as $SU(2)_{C+F}$.

Let us apply this global rotation to the (1,0) string solution (5.34). We find

$$\varphi^{kA}=U_{C+F}\begin{pmatrix}\phi_1(r) & 0\\ 0 & \phi_2(r)\end{pmatrix}U_{C+F}^{-1},$$

$$\frac{\tau^a}{2}A_i^a(x)=\frac{1}{2}n^a\tau^a\epsilon_{ij}\frac{x_j}{r^2}f_3(r),$$

$$A_i(x)=\epsilon_{ij}\frac{x_j}{r^2}f(r), \tag{5.36}$$

where we define

$$U_{C+F}\tau^3U_{C+F}^{-1}=n^a\tau^a. \tag{5.37}$$

Here n^a is a unit vector on S_2, $\vec{n}^2=1$.

It is easy to check that the rotated string (5.36) is a solution to the non-Abelian first-order equations (5.23). Clearly the solution (5.26) interpolates between (1,0) and (0,1) strings. In particular it gives a (1,0)-string for $n=(0,0,1)$ and a (0,1)-string for $n=(0,0,-1)$.

The vector $\vec{n}$ has a clear-cut physical meaning. Its orientation is the orientation of the magnetic flux. The construction above—which was carried out in the singular gauge—shows that the SU(2) flux of the string is directed along the vector n^a. This fact becomes even more transparent, if we examine a gauge-invariant definition of the magnetic flux of the non-Abelian string, which is very instructive. This can be done as follows. Define ''non-Abelian'' field strength (to be denoted by bold letters) as follows:

$$\mathbf{F}_3^{*a}=\frac{1}{\xi}\mathrm{Tr}\left(\Phi^\dagger F_3^{*b}\frac{\tau^b}{2}\Phi\tau^a\right). \tag{5.38}$$

From the very definition it is clear that this field is *gauge invariant*. Moreover, it is clear from Eq. (5.36) that

$$\mathbf{F}_3^{*a}=-n^a\frac{(\phi_1^2+\phi_2^2)}{2\xi}\frac{1}{r}\frac{df_3}{dr}. \tag{5.39}$$

Thus, the physical meaning of these moduli is as follows. The flux of the color-magnetic field in the flux tube is directed along $\vec{n}$. We see that the $SU(2)_{C+F}$ symmetry is physical and does not correspond to any of the gauge rotations which are ''eaten up'' by the Higgs mechanism. At the same time, a non-Abelian gauge group—a ''new color''—is resurrected. For strings in Eq. (5.26) the ''new-color''-magnetic flux is directed along the third axis in the O(3) group space, either upward or downward.

The $SU(2)_{C+F}$ symmetry is exact and the tension of the string solution (5.36) is independent of n^a; see Eq. (5.33). However, an explicit vortex solution breaks the exact $SU(2)_{C+F}$ in the following manner:

$$SU(2)_{C+F}\to U(1). \tag{5.40}$$

Two angles associated with vector n^a becomes two orientational bosonic zero modes of the string. The vector n^a parametrize the quotient space $SU(2)/U(1)\sim CP^1\sim S^2$. This means that, as we have already explained in Sec. V A, the $(1+1)$-dimensional low-energy effective theory for these orientational zero modes is the O(3) sigma model [O(3) sigma model is the same as CP^1 sigma model; if we started from SQCD with the gauge group $SU(N)\times U(1)$, we would instead arrive [5,6] at CP^{N-1}]. Since the string is 1/2-BPS saturated we have four supercharges in the effective world sheet theory. This corresponds to $\mathcal{N}=2$ supersymmetry in $(1+1)$-dimensions.

Classically the O(3) sigma model is characterized by a spontaneous breaking of the O(3) symmetry leading to two massless Goldstone bosons. This is to say that in the quasiclassical treatment the vector n^a points in some particular direction for a given string.

However, quantum physics of $\mathcal{N}=2$ sigma model is quite different. The model is asymptotically free and runs into a strong coupling regime at low energies. This theory has a dynamically generated mass gap

$$\Lambda_{CP^1}\sim\sqrt{\xi}\exp(-2\pi/g_{CP^1}^2)\sim\sqrt{\xi}\exp(-4\pi^2/g_2^2). \tag{5.41}$$

There is no spontaneous breaking of O(3), and no Goldstone bosons are generated. In terms of strings in four dimensions this means that the string orientation vector n^a has no particular direction. The O(3) sigma model has two vacua [48]. In the microscopic four-dimensional picture this means that we have two ''elementary'' non-Abelian strings which form a doublet with respect to $SU(2)_{C+F}$.

Note however, that they are *not* the (1,0) and (0,1) strings of the quasiclassical $U(1)\times U(1)$ theory. In both strings the vector n^a has no particular direction. Still the number of ''elementary'' string states remains the same—two—in the limit of equal quark masses.

The O(3) sigma model has a kink interpolating between the two vacua. In four dimensions this interpolation will be interpreted as a monopole which produces a junction of two ''elementary'' non-Abelian strings [6,7]. This monopole lives on the string world sheet because monopoles are in the confining phase in our theory, and do not exist as free states.

The charge of this monopole lies entirely inside the SU(2) factor of the gauge group. If $\Delta m\neq 0$, its charge is $(1,-1)$. Classically the mass of this monopole is $\Delta m(4\pi/g_2^2)$ and tends to zero when the gauge symmetry is enhanced from $U(1)\times U(1)$ to $SU(2)\times U(1)$ at $\Delta m=0$. Simultaneously its size becomes infinite (cf. [49]). However, in quantum theory the story is different. This monopole has a finite size since there are no massless states in the O(3) sigma model. It is massive but extremely light with a mass determined by the scale of the sigma model Λ_{CP^1}; see Eq. (5.41). The mass of this monopole is lifted from zero and is given by the anomalous term in the central charge (5.18) of the O(3) sigma model. Its charge is no longer $(1,-1)$ because it interpolates now between quantum vacua of the O(3) sigma model for which the vector n^a has no particular direction. Further details are reported in [32].

VI. STRING-WALL JUNCTIONS

In this section we derive the BPS equations and find a 1/4-BPS solutions for string-wall junctions. First we work out the first-order equations for string-wall junctions then find a solution of Abelian string ending on the elementary wall and, finally, discuss a non-Abelian string ending on the composite wall.

A. First-order equations for junctions

In Ref. [4] we found string-wall junction solution picking up two supercharges (1/4 BPS!) which act trivially both on the string and wall solutions. Here we take a slightly different route inspecting the Bogomolny representation for the energy functional. We keep the quark, adjoint and gauge fields in our action because all of these fields play a role in the string-wall junction.

It is natural to assume that at large separations from the string end point at $r=0$, $z=0$, the wall is almost parallel to the (x_1,x_2) plane while the string is stretched along the z axis at negative z. Since both solutions, for the string and the wall, were obtained using the *ansatz* (4.1) we restrict our search for the wall-string junction to the same *ansatz*. As usual, we look for a static solution assuming that all relevant fields can depend only on x_n, $(n=1,2,3)$.

Then we can represent the energy functional of our theory (4.2) as follows:

$$
\begin{aligned}
E=\int d^3x\Bigg\{&\left[\frac{1}{\sqrt{2}g_2}F_3^{*a}+\frac{g_2}{2\sqrt{2}}(\bar{\varphi}_A\tau^a\varphi^A)+\frac{1}{g_2}D_3a^a\right]^2\\
&+\left[\frac{1}{\sqrt{2}g_1}F_3^{*}+\frac{g_1}{2\sqrt{2}}(|\varphi^A|^2-2\xi)+\frac{1}{g_1}\partial_3 a\right]^2\\
&+\frac{1}{g_2^2}\left[\frac{1}{\sqrt{2}}(F_1^{*a}+iF_2^{*a})+(D_1+iD_2)a^a\right]^2\\
&+\frac{1}{g_1^2}\left[\frac{1}{\sqrt{2}}(F_1^{*}+iF_2^{*})+(\partial_1+i\partial_2)a\right]^2+|\nabla_1\varphi^A+i\nabla_2\varphi^A|^2\\
&+\left|\nabla_3\varphi^A+\frac{1}{\sqrt{2}}(a^a\tau^a+a+\sqrt{2}m_A)\varphi^A\right|^2\Bigg\}\\
&+\text{surface terms},
\end{aligned}
\tag{6.1}
$$

where we assume that the quark mass terms and adjoint fields are real. The surface terms are

$$
\begin{aligned}
E_{\text{surface}}=\xi\int d^3xF_3^{*}+\sqrt{2}\xi\int d^2x\langle a\rangle\Big|_{z=-\infty}^{z=\infty}\\
-\sqrt{2}\frac{\langle a^3\rangle}{g_2^2}\int dS_nF_n^{*3},
\end{aligned}
\tag{6.2}
$$

where the integral in the last term runs over a large two-dimensional sphere at $x_n^2\to\infty$, and

$$
F_n^{*3}\equiv\frac{1}{2}\epsilon_{nij}F_{ij}^3,
\tag{6.3}
$$

cf. Eq. (5.22). This is in full accord with the general discussion in Sec. III.

The Bogomolny representation (6.1) leads us to the following first-order equations:

$$
\begin{aligned}
&F_1^{*}+iF_2^{*}+\sqrt{2}(\partial_1+i\partial_2)a=0,\\
&F_1^{*a}+iF_2^{*a}+\sqrt{2}(D_1+iD_2)a^a=0,\\
&F_3^{*}+\frac{g_1^2}{2}(|\varphi^A|^2-2\xi)+\sqrt{2}\partial_3 a=0,\\
&F_3^{*a}+\frac{g_2^2}{2}(\bar{\varphi}_A\tau^a\varphi^A)+\sqrt{2}D_3a^a=0,\\
&\nabla_3\varphi^A=-\frac{1}{\sqrt{2}}(a^a\tau^a+a+\sqrt{2}m_A)\varphi^A,\\
&(\nabla_1+i\nabla_2)\varphi^A=0.
\end{aligned}
\tag{6.4}
$$

These are our *master equations*.

Once these equations are satisfied the energy of the BPS object is given by Eq. (6.2). Please, observe that Eq. (6.2) has three terms corresponding to central charges of the string, domain wall and monopoles of the SU(2) subgroup, respectively. Say, for the string the three-dimensional integral in the first term in Eq. (6.2) gives the string length times its flux. For the wall the two-dimensional integral in the second term in Eq. (6.2) gives the area of a wall times the tension. For the monopole the integral in the last term in Eq. (6.2) gives the monopole flux. This means that our master equations (6.4) can be used to study the BPS strings, domain walls, monopoles, and all their possible junctions.

It is instructive to check that the wall, string and monopole solutions, separately, satisfy these equations. Say, we start from the wall solution. In this case the gauge fields are put to zero, and all fields depend only on z. Thus, the first two and the last two equations in Eq. (6.4) are trivially satisfied. The components of the gauge fields F_3^* and F_3^{*a} vanish in the third and fourth equations; hence these equations reduce to the last two equations in Eq. (4.4). The fifth equation in Eq. (6.4) coincides with the first one in Eq. (4.4).

For the string which lies, say, in the 12-vacuum, all quark fields vanish except q^A, $A=1,2$ while a and a^a are given by their VEV's. The gauge flux is directed along the z axis, so that $F_1^*=F_2^*=F_1^{*a}=F_2^{*a}=0$. All fields depend only on the coordinates x_1 and x_2. Then the first two equations and the fifth one in Eq. (6.4) are trivially satisfied. The third and the fourth equation reduce to the first two ones in Eq. (5.23). The last equation in Eq. (6.4) for $A=1,2$ reduces to the last equation in Eq. (5.23), while for $B=3,4$ these equations are trivially satisfied.

Equations for the monopole arise from the ones in Eq. (6.4) in the limit $\xi=0$. Then all quark fields vanish, and Eq. (6.4) reduces to the standard first-order equation for the monopole in the BPS limit,

$$F_n^{*a}+\sqrt{2}D_n a^a=0, \tag{6.5}$$

while a is given by its VEV and the U(1) gauge field vanishes.

In particular, Eq. (6.2) shows that the central charge of the SU(2) monopole is determined by $\langle a^3\rangle$ which is proportional to the quark mass difference in the given vacuum. Say, for the monopole in the 12-vacuum it gives zero. However, as was mentioned at the end of Sec. V B, the mass of this monopole is lifted from zero at $\xi\neq 0$. In this case this monopole becomes a junction of two ''elementary'' strings of the $SU(2)\times U(1)$ theory and acquires a nonvanishing mass due to nonperturbative effects in the O(3) sigma model on the string world sheet.

Let us note that the Abelian version of the first-order master equations (6.4) was first derived in Ref. [4] and used to find a 1/4-BPS solution for the string-wall junction. Quite recently a non-Abelian version for $SU(2)\times U(1)$ theory was used [7] to study the junction of two ''elementary'' strings via a small-size monopole at $\Delta m\neq 0$ and large.

B. The Abelian string ending on the elementary wall

In this section we consider an Abelian string ending on the elementary wall. The $12\to 14$ wall has a nonvanishing r-component of the first flavor inside the wall; see Sec. IV B. Therefore, only the (0,1)-string whose flux is orthogonal to the r-weight vector can end on this wall.

Needless to say the solution of the first-order equations (6.4) for the string ending on the wall can be found only numerically, especially near the end-point of the string where both the string and the wall profiles are heavily deformed. However, far from the string end point, deformations are weak and we can find the asymptotic behavior analytically.

Let the string be on the $z<0$ side of the wall, inside the 12-vacuum. Consider first the region $z\to -\infty$ far away from the string end-point at $z\sim 0$. Then the solution to Eq. (6.4) is given by an almost unperturbed string. Namely, at $z\to -\infty$ there is no z dependence to the leading order, and, hence, the solution

$$\varphi^{kA}=\begin{pmatrix}\phi_2(r) & 0\\ 0 & \phi_1(r)\end{pmatrix},$$

$$A_i^3(x)=-\epsilon_{ij}\frac{x_j}{r^2}f_3(r),$$

$$A_i(x)=\epsilon_{ij}\frac{x_j}{r^2}f(r) \tag{6.6}$$

[which is a singular-gauge version of the solution (5.26) for $n=0$, $k=1$; cf. Eq. (5.34)] satisfies Eqs. (6.4). We also take the fields $A_3=A_3^a=0$ and φ^B $(B=3,4)$ to be zero, with a's equal to their VEV's (2.12). On the other side of the wall, at $z\to +\infty$, we have an almost unperturbed 14-vacuum with the fields given by their respective VEV's.

Now consider the domain $r\to\infty$ at small z. In this domain the solution to Eq. (6.4) is given by a perturbation of the wall solution. Let us use the *ansatz* in which the solutions for the fields a, a^a and φ^A are given by the same equations (4.9), (4.12), (4.13) and (4.14) in which the size of the wall is still given by Eq. (4.11), and *the only modification* is that the position of the wall z_0 and the phase σ now become slowly-varying functions of r and α [i.e., the polar coordinates on the (x_1,x_2) plane]. It is quite obvious that z_0 will depend only on r.

As long as the third, fourth and fifth equations in Eq. (6.4) do not contain derivatives with respect to x_i, $i=1,2$, they are identically satisfied for any functions $z_0(r,\alpha)$ and $\sigma(r,\alpha)$ (note that $F_3^*=F_3^{*a}=0$, the field strength is almost parallel to the domain wall plane and $A_3=A_3^a=0$).

However, the first two and the last two equations in Eq. (6.4) become nontrivial. Consider the first two. Inside the string the gauge fields are directed along the z axis and their fluxes are 2π for F_3^* and -2π for F_3^{*3} [remember, we treat the (0,1)-string]. This flux is spread out inside the wall and directed almost along x_i in the (x_1,x_2) plane at large r. Since the flux is conserved, we have

$$F_i^*=\frac{1}{R}\,\frac{x_i}{r^2}$$

$$F_i^{*3}=-\frac{1}{R}\,\frac{x_i}{r^2}, \tag{6.7}$$

inside the wall at $|z-z_0(r,\alpha)|<R/2$.

Substituting this in the first two equations in Eq. (6.4) and assuming that z_0 depends only on r we get that the two equations are consistent with each other and

$$\partial_r z_0 = \frac{1}{\Delta m r}. \tag{6.8}$$

Needless to say our "adiabatic" approximation holds only provided the above derivative is small, i.e., sufficiently far from the string end point, $\sqrt{\xi}r \gg 1$.

The solution to this equation is straightforward,

$$z_0 = \frac{1}{\Delta m}\ln r + \text{const.} \tag{6.9}$$

We see that the wall is logarithmically bent according to the Coulomb law in $2+1$ dimensions. Similar to the case considered in [4], one can show that this bending produces a balance of forces between the string and the wall in the z direction so that the whole configuration is static.

Now let us consider the last equation in Eq. (6.4). First, we will dwell on the gauge potential which enters the covariant derivatives in this equation. In order to produce the field strength (6.7), A_μ and A^a_μ in the middle domain should reduce to

$$A_i = \frac{1}{R}\varepsilon_{ij}\frac{x_j}{r^2}\left[z - z_0(r) + \frac{R}{2}\right], \quad i=1,2,$$

$$A^3_i = -\frac{1}{R}\varepsilon_{ij}\frac{x_j}{r^2}\left[z - z_0(r) + \frac{R}{2}\right], \quad i=1,2,$$

$$A_0 = A^3_0 = 0, \quad A_3 = A^3_3 = 0. \tag{6.10}$$

Please, observe that nonvanishing field of the first quark flavor satisfies the equation since it has only the r-component which is not charged with respect to the field (6.10). Consider the second quark flavor whose b-component is given by Eq. (4.13) in the middle domain near the left edge of the wall at $z-z_0 \sim -R/2$. Taking into account the gauge potentials (6.10) and the wall bending (6.8) it is easy to check that the second flavor satisfies the last equation in Eq. (6.4).

Finally, let us consider φ^{b4} in the middle domain near the right edge of the wall, $z-z_0 \sim R/2$. Substituting Eq. (4.14) into the last equation in Eq. (6.4) we get the following equations for the phase σ:

$$\frac{\partial\sigma}{\partial\alpha} = 1, \quad \frac{\partial\sigma}{\partial r} = 0. \tag{6.11}$$

The solution to these equations is

$$\sigma = \alpha. \tag{6.12}$$

In terms of the dual Abelian gauge field localized on the wall, this solution reflects nothing but the unit source charge.

The above relation between the vortex solution and the unit source charge calls for a comment. One can identify the compact scalar field σ with the electric field living on the domain wall world volume via Eq. (4.41). Then the result (6.12) gives for this electric field

$$F^{(2+1)}_{0i} = \frac{e^2_{2+1}}{2\pi}\frac{x_i}{r^2}, \tag{6.13}$$

where the $(2+1)$-dimensional coupling is given in Eq. (4.40).

This is the field of a point-like electric charge in $2+1$ dimensions placed at $x_i=0$. The interpretation of this result is that the string end point on the wall plays a role of the electric charge in the dual U(1) theory on the wall, cf. [4].

C. Non-Abelian string ending on the composite wall

Now we pass to the non-Abelian string ending on the composite wall interpolating between the 12- and 34-vacua. Our strategy is as follows. We start with the (0,1) Abelian string as in Sec. VI B and consider its junction with the composite $12\to 34$ wall of Sec. IV C. We then apply the $SU(2)_{C+F}$ rotation introduced and discussed in Sec. V to this junction. Namely, we write down the first two flavors as a 2×2 matrix φ^{kA} $(A=1,2)$ and the last two flavors as a 2×2 matrix φ^{kB} $(B=3,4)$, and rotate them according to

$$\varphi \to U_{C+F}\varphi U^{-1}_{C+F}, \text{ flavor indices}=1,2,$$

$$\varphi \to U_{C+F}\varphi U^{-1}_{C+F}, \text{ flavor indices}=3,4, \tag{6.14}$$

with one and the same matrix from $SU(2)_{C+F}$.

Note that both the 12- and 34-vacua do not break this symmetry. However, the string and the string-wall junction are not invariant. Therefore, if we apply this rotation to the solution for the (0,1)-string ending on the composite wall we will get the solution of Eq. (6.4) for a non-Abelian string ending on the composite wall, with the same energy.

Having set the general strategy, it is time to proceed to a technical analysis of the junction of the (0,1) string with the composite wall. Assume for simplicity that in the absence of the string, the matrix $\tilde{U}=I$ [see Eq. (4.21)] so that the deviation of $\tilde{U}$ from the unit matrix is due to the string flux. Our composite wall can be considered as a marginally stable bound state of the $12\to 14$ and $14\to 34$ walls. While the solution presented in Sec. IV C has a vanishing separation between the constituents, the two elementary walls in this bound state do not interact and their positions can be shifted to arbitrary separations. As the (0,1)-string can end only on the $12\to 14$ wall, it is clear that it will pick up this wall and pull it out to the left; see Fig. 9. The $(14\to 34)$ constituent stays unbent and does not play a role in the junction solution at hand.[18] We see that the solution for the (0,1)-string ending

[18]Of course, there could be some interaction of the end point of the string on the $12\to 1B$ wall with $14\to 34$ wall but this interaction is short range and dies out at $r \gg 1/\sqrt{\xi}$.

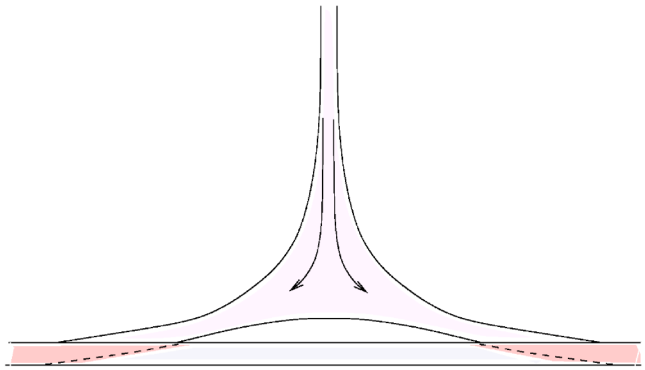

FIG. 9. Junction of the string and the composite wall. The string pulls out one of the components of the composite wall. Arrows show the spread of the magnetic flux inside the wall.

on the composite wall reduces to the solution for the (0,1)-string ending on the $(12\to 14)$ wall considered in Sec. VI B.

The solution for the (0,1)-string ending on the $12\to 14$ wall has the b-component of the fourth flavor multiplied by $\exp(-i\sigma)$ with σ given by Eq. (6.12); see Eq. (4.14). The $14\to 34$ wall has all phases vanishing because there is no flux going inside this wall.

Thus, our junction has the quark matrix of the final vacuum determined by the matrix

$$\tilde{U}=\begin{pmatrix}1 & 0\\ 0 & e^{-i\alpha}\end{pmatrix}, \tag{6.15}$$

see Eq. (4.21). This shows that the junction of the (0,1)-string and the composite wall has the following phase σ_0 and the SU(2) matrix U [see Eq. (4.22)]:

$$\sigma^0=-\frac{\alpha}{2},$$

$$U=\exp\left(i\frac{\tau^3}{2}\alpha\right). \tag{6.16}$$

Now let us apply the $SU(2)_{C+F}$ rotation (6.14) to the whole configuration. The flux of the string is now determined by an arbitrary vector n^a inside the SU(2) subgroup while the quark matrix of the final vacuum gets rotated as

$$\tilde{U}=U_{C+F}\begin{pmatrix}1 & 0\\ 0 & e^{-i\alpha}\end{pmatrix}U_{C+F}^{-1}. \tag{6.17}$$

In terms of the phase σ_0 and the SU(2) matrix U this amounts to

$$\sigma^0=-\frac{\alpha}{2},$$

$$U=\exp\left(i\frac{n^a\tau^a}{2}\alpha\right). \tag{6.18}$$

This result clearly means that the end point of the string is a point-like source of the non-Abelian gauge field on the composite wall. To see this more explicitly let us write down the $(2+1)$-dimensional gauge fields associated with σ_0 and the matrix U using Eq. (4.48). We get

$$F_{0i}^{(2+1)}=-\frac{e_{2+1}^2}{4\pi}\frac{x_i}{r^2},$$

$$F_{0i}^{a(2+1)}=\frac{g_{2+1}^2}{2\pi}\frac{x_i}{r^2}n^a. \tag{6.19}$$

As we see, this is the field of a classical point-like charge in the $SU(2)\times U(1)$ gauge theory on the wall. The direction of the SU(2) field in the color space is determined by the vector n^a associated with the string flux.

D. Gauge symmetry breaking

As was discussed at the end of Sec. IV D 2, if our composite wall is split into elementary components whose separation is larger than their thickness, the non-Abelian gauge symmetry in the world-volume theory (4.49) is broken down to $U(1)\times U(1)$. In particular, the mass of the $(2+1)$-dimensional W-bosons becomes proportional to the separation l between the elementary walls; see Eq. (4.53).

Our analysis demonstrates that localization of a gauge field on a wall and existence of the corresponding string-wall junction are two sides of one and the same phenomenon. In this section we address the question: ‘‘what happens with the string-wall junction in the $(3+1)$-dimensional bulk theory if we split the composite wall and pull the components apart?’’

Consider a string-wall junction for the non-Abelian string ending on the composite wall, as in Sec. VI C. If $\langle l\rangle=0$ two elementary walls which form the $12\to 34$ wall overlap at large r (r is the distance from the string end point along the wall). In fact, the composite $12\to 34$ wall can be represented as a bound state of two elementary walls in many different ways depending on which particular combination of the quark fields is nonzero in the given elementary walls. In particular, the string with flux $\sim n^a$ picks up a particular elementary wall with

$$\varphi^{kA}\sim U_{C+F}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}U_{C+F}^{-1},\quad A=1,2, \tag{6.20}$$

nonvanishing inside the wall. The string pulls it out to the left while the ‘‘other wall’’ is (almost) unbent; see Fig. 9. The string ends on the wall specified by Eq. (6.20) so the string flux spreads inside this wall, and at large r is given by Eq. (6.19). The flux direction in the color space is determined by the string flux vector n^a.

Now suppose that $\langle l\rangle\neq 0$. In other words, the composite wall is split in particular elementary components which do not overlap even at $r\to\infty$; see Fig. 10. Say, if we have the 14-vacuum between the walls the elementary wall on the left has a concrete nonvanishing quark field, with necessity, namely, the quark field proportional to

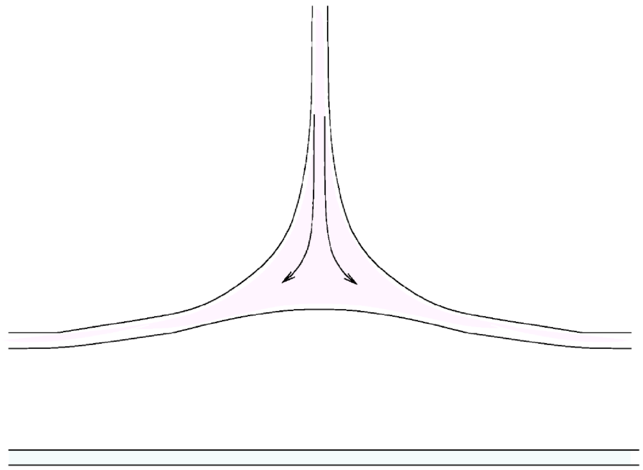

FIG. 10. Junction of the string and the composite wall for $\langle l\rangle\neq 0$.

$$\varphi^{kA}\sim\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad A=1,2. \tag{6.21}$$

This guarantees that only the (0,1)-string can end on the wall. The flux inside the wall is given in this case by Eq. (6.19) with a specific n^a, namely, $n^a=(0,0,1)$.

If, instead, we have the 23-vacuum between the walls the elementary wall on the left has the nonvanishing quark field proportional to

$$\varphi^{kA}\sim\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},\quad A=1,2. \tag{6.22}$$

This means that only the (1,0)-string can end on this wall configuration. The flux inside the wall will be given in this case by Eq. (6.19) with $n^a=(0,0,-1)$. String with generic fluxes determined by an arbitrarily-oriented vector n^a just cannot end on the composite domain wall if $\langle l\rangle > R$.

Of course, this is perfectly consistent with the breaking of the $SU(2)\times U(1)$ gauge group down to $U(1)\times U(1)$, so that the $(2+1)$-dimensional $W^{\pm}$-bosons are heavy, do not propagate, and the massless gauge fields on the wall are by A_n^{2+1} and $A_n^{3(2+1)}$.

VII. DYNAMICS OF THE WORLD-VOLUME THEORY

In this section we briefly discuss dynamics of the world-volume theory emerging on the wall. We will focus on nonperturbative instanton effects which lead to a run-away vacuum in the world-volume theory.

A. BPS saturation of the composite wall

In Sec. IV A we demonstrated that the central charges of the composite wall and its two constituents are aligned, so that the tension of the composite wall equals twice the tension of the elementary wall. This statement is valid to any order in perturbation theory. If so, the composite wall would be marginally stable: there would be no interaction between the constituent walls no matter what the separation between the constituents is.

In terms of the world-volume theory (4.49) this means that the flat direction is not lifted. No superpotential is generated to any finite order in perturbation theory. Phrased this way, the assertion seems almost obvious. From other examples we know, however, that a superpotential might be generated nonperturbatively. An indication that this may be the case comes from the occurrence of the anomalous terms in the central charge (3.3). Some well-known old results will allow us to answer this question quickly.

B. Nonperturbative effects

Dynamics of $\mathcal{N}=2$ $(2+1)$-dimensional gauge theory with the SU(2) gauge group was studied by Affleck, Harvey and Witten [50]. It was shown that instantons [in $(2+1)$ dimensions they are nothing but 't Hooft–Polyakov monopoles, tHP for short] generate a superpotential which produces a run-away vacuum.[19] Classically there is a flat direction in the theory (4.49) so that the scalar field a^3_{2+1} can develop an arbitrary VEV breaking the SU(2) gauge group on the wall down to U(1). Then $A_n^{1,2(2+1)}$ acquire a mass while A_n^{2+1} and $A_n^{3(2+1)}$ remain massless. We can dualize $A_n^{3(2+1)}$ into a compact scalar σ^3 according to Eq. (4.48), to introduce a complex scalar filed $\Phi=a^3_{2+1}+i\sigma^3$. This scalar is the lowest component of a chiral supermultiplet.

It was shown in [50] that instantons-tHP monopoles generate the superpotential[20]

$$W_{2+1}\sim\exp\left(-\frac{\Phi}{g^2_{2+1}}\right). \tag{7.1}$$

The potential arising from this superpotential

$$V_{2+1}\sim\exp\left(-\frac{2a^3_{2+1}}{g^2_{2+1}}\right) \tag{7.2}$$

leads to a run-away vacuum. Using Eq. (4.51) we can reinterpret this potential as an interaction potential between elementary walls which comprise our composite $(12\rightarrow 34)$ wall,

$$V_{int}\sim\exp\left(-\frac{g_2}{g_1}\frac{\Delta m}{\pi}l\right). \tag{7.3}$$

Classically the elementary walls do not interact. Nonperturbative effects on the world volume induce a repulsive interaction between the elementary walls, so that the BPS

[19]Historically this work presented the first example ever in which perturbative nonrenormalization theorem—the absence of the superpotential—was shown to be violated nonperturbatively.

[20]The mechanism is quite similar to the one in the nonsupersymmetric version of the theory studied by Polyakov [36]. Monopoles form a Coulomb gas in $(2+1)$ dimensions which is equivalent to the sine-Gordon theory.

bound state can formally appear only in the limit of infinite separation between walls. However, in fact, the interactions (7.3) become negligibly small already at separations l_* of the order of $(\Delta m)^{-1}$. In other words, the ratio $l_*/R \sim \xi/(\Delta m)^2 \ll 1$. Here R represents the wall thickness.

We observe an interesting interplay between bulk physics and physics on the wall. In particular, above we extracted the interaction potential between the elementary walls from the known results on the effective theory on the wall. This bulk-brane duality is somewhat similar to the AdS-CFT correspondence. A weak coupling regime in the bulk maps onto a strong coupling regime on the wall and *vice versa*. To see that this is indeed the case, suffice it to remember that when the bulk coupling constants g_1^2 and g_2^2 are small the $(2+1)$-dimensional couplings (4.52) are large compared to the characteristic scale of massive excitations on the wall which are of the order of $1/\tilde{R}$ (cf. [4]).

C. Compatibility with the D-brane picture

Returning to the issue of the elementary wall exponential repulsion, one may ask how this can be interpreted in view of the well-known fact that the two-stacks (as well as all other stacks) of D-branes are stable. The answer is quite clear. D-branes have no thickness. Our construction belongs to weak coupling where the walls do have a thickness. The repulsive nonperturbative interaction dies off at distances much less than the wall thickness. Therefore, squeezing the walls to vanishing thickness automatically switches the repulsion off.

VIII. CONCLUSIONS

In this paper we studied localization of non-Abelian gauge fields on domain walls. We showed that although elementary domain walls can localize only Abelian fields the composite domain wall does localize non-Abelian gauge fields. In order to have this localization we considered $\mathcal{N}=2$ QCD with the gauge group $SU(2)\times U(1)$ in a special regime. Although the gauge group is completely Higgsed by the quark VEV's in the bulk it is restored inside the composite domain wall where all quark fields are almost zero. This ensures localization of the non-Abelian gauge field on the wall.

Another side of this phenomenon is the possibility for non-Abelian flux tubes to end on the wall. The non-Abelian flux tubes were recently found in Ref. [6] in four dimensions and in Ref. [5] in three dimensions. They carry additional orientational zero modes corresponding to rotations of the color-magnetic flux inside the SU(2) subgroup of the gauge group. The key ingredient for the existence of such non-Abelian strings is the presence of a diagonal color-flavor group $SU(2)_{C+F}$ unbroken by the vacuum condensates (color-flavor locking). We found a 1/4-BPS solution for such non-Abelian string ending on the composite wall. The end point of the string plays the role of a color charge in the $(2+1)$-dimensional (dual) non-Abelian gauge theory on the wall.

To study the string-wall junctions we use the first-order master equations (6.4) which in the Abelian case were derived in [4]. In fact, the same equations can be used for all possible junctions between domain walls, strings and monopoles. In particular, recently they were used [7] to study the $(1,-1)$ monopole as a junction of the (1,0) and (0,1) strings in the limit of large Δm. We discuss this monopole in the opposite limit of equal quark masses, $\Delta m \to 0$, when it becomes a junction of two strings associated with two quantum vacua of the $(1+1)$-dimensional O(3) sigma model on the string world sheet. We show that the mass of this monopole is lifted from zero by non-perturbative effects in the O(3) sigma model. We will come back to this issue [32].

We also studied the effective $(2+1)$-dimensional non-Abelian theory on the composite domain wall. We found an interesting bulk-brane duality. In particular, the weak coupling regime in the bulk maps onto the strong coupling regime on the wall and *vice versa*. This is quite similar in spirit to the AdS-CFT correspondence.

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