Bound states in the AdS/CFT correspondence

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We consider a massive scalar field theory in anti–de Sitter space, in both minimally and non-minimally coupled cases. We introduce a relevant double-trace perturbation at the boundary, by carefully identifying the correct source and generating functional for the corresponding conformal operator. We show that such a relevant double-trace perturbation introduces changes in the coefficients in the boundary terms of the action, which in turn govern the existence of a bound state in the bulk. For instance, we show that the usual action, containing no additional boundary terms, gives rise to a bound state, which can be avoided only through the addition of a proper boundary term. Another notorious example is that of a conformally coupled scalar field, supplemented by a Gibbons-Hawking term, for which there is no associated bound state. In general, in both minimally and non-minimally coupled cases, we explicitly compute which boundary terms give rise to a bound state, and which ones do not. In the non-minimally coupled case, and when the action is supplemented by a Gibbons-Hawking term, this also fixes the allowed values of the coupling coefficient to the metric. We interpret our results to indicate that the requirement to satisfy the Breitenlohner-Freedman bound does not suffice to prevent tachyonic behavior from existing in the bulk, as it must be supplemented by additional conditions on the coefficients in the boundary terms of the action.

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I. INTRODUCTION

The AdS conformal field theory (CFT) correspondence [1–3] (see [4,5] for reviews) proposes the existence of a duality between a field theory on $(d+1)$-dimensional anti–de Sitter (AdS_{d+1}) space, and a conformal field theory living at its boundary, and since its formulation a large amount of work has been devoted to exploring different aspects of this conjecture. A prescription for mapping one theory into the other was proposed in [2,3], and it reads

$$\exp(-I_{AdS}[\phi_0]) \equiv \left\langle \exp\left(\int d^d x \mathcal{O}(\vec{x}) \phi_0(\vec{x}) \right) \right\rangle, \qquad (1)$$

where ϕ_0 is the boundary value of the bulk field ϕ, and it couples to the boundary CFT operator $\mathcal{O}$. Throughout this paper, we will be concerned with a scalar field theory in AdS space. In the minimally coupled case, the action reads

$$I_0 = -\frac{1}{2}\int d^{d+1}y \sqrt{g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2), \qquad (2)$$

where m is the mass of the scalar field. The corresponding equation of motion is of the form $(\nabla^2 - m^2)\phi = 0$.

One relevant aspect of the study of the AdS/CFT conjecture is the analysis of perturbations of the boundary CFT by double-trace operators. It was proposed in [6,7] that they give rise to a new perturbation expansion for string theory, based on a non-local world-sheet. Later, it was suggested in [8,9] that multi-trace interactions can be incorporated in the AdS/CFT correspondence by generalizing the boundary conditions that are considered in the usual single-trace case.

In any prescription describing this phenomenon, we should take into account the existence of two normalizable modes for the scalar field on the AdS space [10,11] (see also [12]), namely "regular" and "irregular" ones, which behave close to the border as $\phi_R \sim \epsilon^{\Delta_+}$ for regular modes, and $\phi_I \sim \epsilon^{\Delta_-}$ for irregular ones. Here ϵ is a measure of the distance to the boundary, and

$$\Delta_\pm = \frac{d}{2} \pm \nu, \qquad (3)$$

$$\nu = \sqrt{\frac{d^2}{4} + m^2}, \qquad (4)$$

where m satisfies the Breitenlohner-Freedman bound

$$m^2 \geqslant -\frac{d^2}{4}. \qquad (5)$$

The range $m^2 < -d^2/4$ corresponds to tachyons in AdS space [10,11], and, in fact, if Eq. (5) is not satisfied, the energy is neither conserved nor positive definite [12]. It was also shown in [10,11] that irregular modes are normalizable only for

$$\nu < 1. \qquad (6)$$

In the AdS/CFT picture, the interpretation of the above results was considered in [13] (see also [14] for previous results), which points out that we should find two different CFT's at the boundary, corresponding to both possible quantizations in the bulk. However, the usual prescription Eq. (1) accounts for only the CFT with conformal dimension Δ_+, corresponding to regular modes propagating in the bulk. In order to also reproduce the missing conformal dimension

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Δ_-, corresponding to irregular modes in the bulk, the proposal in [13] was that its generating functional could be found by performing a Legendre transformation to the original one in the theory with conformal dimension Δ_+ (see also [15] for previous results involving group-theoretic analysis). Thus, starting from the generating functional in the theory with conformal dimension Δ_+, as considered in [2,16,17], it was explicitly shown in [13] that its Legendre transform gives rise to the conformal dimension Δ_-, as expected.

Note that, near the boundary, the scalar field behaves as (for $\nu<1$)

$$\phi(\epsilon,\vec{x})=\epsilon^{\Delta_+}\alpha(\vec{x})+\epsilon^{\Delta_-}\beta(\vec{x}), \tag{7}$$

where $\vec{x}$ are coordinates in the boundary. One possibility is to impose the boundary condition $\beta(\vec{x})=0$. In this circumstance, $\alpha(\vec{x})$ is understood as the expectation value of a conformal operator $\mathcal{O}_\beta$ with dimension Δ_+ [13,14]. On the other hand, when we consider the boundary condition

$$\alpha(\vec{x})=0, \tag{8}$$

then irregular modes, instead of regular ones, propagate in the bulk. Now the boundary theory has a conformal operator $\mathcal{O}_\alpha$ of dimension Δ_-, whose expectation value is given by

$$\beta(\vec{x})\equiv\langle\mathcal{O}_\alpha(\vec{x})\rangle. \tag{9}$$

In order to describe the way in which double-trace perturbations are incorporated in the AdS/CFT conjecture, we first note that, since $2\Delta_-<d$, a relevant double-trace deformation should be of the form [8]

$$W[\mathcal{O}_\alpha]=\frac{f}{2}\mathcal{O}_\alpha^2, \tag{10}$$

where f is a coupling constant, and, as pointed out before, $\mathcal{O}_\alpha$ has dimension Δ_-. Then, the prescription in [8] is that the above double-trace perturbation is described by the generalized boundary condition

$$\alpha=f\beta. \tag{11}$$

Note that, for $f=0$, the above boundary condition reduces to Eq. (8), as expected. The above equation describes a renormalization group flow [8], starting from the UV fixed point at $f=0$, and having an end point at an IR fixed point whose generating functional is related to the one of the $f=0$ case by a Legendre transformation, as explained above (see [18–20] for analyses on this subject). Additional references on the topic of double-trace interactions in the AdS/CFT correspondence are [21–30].

In particular, in this work we will be concerned with the recent results in [28], regarding unstable double-trace perturbations. As pointed out in [8], the stability of a double-trace deformation is related to the sign of its coefficient f [see Eq. (10)]. Specifically, stable perturbations correspond to $f>0$, whereas unstable ones exist for $f<0$. The author of [28] asks how the bulk theory detects an unstable theory in the boundary, and, in order to answer such a question, considers a careful analysis of the solution to the radial wave equation for a massive scalar field in AdS space. Calculations are performed in the representation of the AdS_{d+1} space in Lorentzian Poincaré coordinates. The author of [28] points out the existence of a bound state having tachyonic behavior, in the spectrum of the radial equation, when α and β in Eq. (7) satisfy

$$\frac{\alpha}{\beta}<0. \tag{12}$$

But, on the other hand, we note from Eq. (11) that the above condition is equivalent to set $f<0$, and this led the author of [28] to conclude that an unstable double-trace deformation in the boundary corresponds to the existence of a solution to the bulk wave equation with tachyonic behavior in a Minkowski slice. Note that the tachyonic behavior appears even when the Breitenlohner-Freedman bound Eq. (5) is satisfied. This result adds a new entry to the AdS/CFT dictionary, and, as pointed out in [28], could be relevant to the analysis of causality and Lorentzian aspects of the AdS/CFT correspondence.

One of the purposes of this paper is to propose a deeper insight into the above detailed results. In particular, we will be concerned with the role of the action in the phenomenon of the existence of bound states for the scalar field in AdS space. Note that, as emphasized in [28], a bulk theory is specified not only by the background geometry, but also by the boundary conditions that are imposed on the bulk field. But boundary conditions are governed by the action, and this suggests that a careful analysis, in this context, of the action of the bulk theory could shed some new light into the phenomenon of the existence of bound states.

There have been previous situations where a careful study of the action, and not only of its corresponding equation of motion, has proven to be fruitful in the context of the AdS/CFT correspondence. An example of this is the case of the spinor field, whose action contains, at most, first order derivatives, and vanishes on shell. Because of this, it was pointed out in [31] that, in order to compute the generating functional for the corresponding dual CFT, a boundary term should be added to the bulk action. Such a boundary term was later computed using the Hamiltonian formalism [32] and the variational principle [33]. Analogous situations were found in the cases of the antisymmetric tensor field [34] and the self-dual model [35].

In the case of the scalar field theory, it was shown in [36] that a boundary CFT with conformal dimension Δ_- could be generated by adding a proper boundary term to the usual action. Later, this result was combined with the Legendre transform prescription in [13] in order to find a generalized AdS/CFT prescription which is able to map to the boundary all constraints arising from the quantization in the bulk. In order to illustrate this, we consider a non-minimally coupled scalar field, where the action Eq. (2) should be replaced by

$$\mathcal{I}_0=-\frac{1}{2}\int d^{d+1}y\sqrt{g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+(m^2+\varrho R)\phi^2]. \tag{13}$$

Here R is the Ricci scalar corresponding to the AdS space [and it is a negative constant, $R=-d(d+1)$] and ϱ is an arbitrary coupling coefficient. The equation of motion generalizes to $[\nabla^2-(m^2+\varrho R)]\phi=0$. In this situation, Eqs. (3), (4) should include a dependence on ϱ, and thus be replaced by

$$\Delta_{\pm}(\varrho)=\frac{d}{2}\pm\nu(\varrho), \tag{14}$$

$$\nu(\varrho)=\sqrt{\frac{d^2}{4}+m^2+\varrho R}. \tag{15}$$

As before, irregular modes propagate for

$$\nu(\varrho)<1. \tag{16}$$

Note that the Breitenlohner-Freedman bound Eq. (5) now reads

$$m^2+\varrho R\geqslant-\frac{d^2}{4}. \tag{17}$$

When performing the quantization of the non-minimally coupled scalar field in AdS space, the energy of such a theory was defined in [10,11] as the conserved charge arising from the current which is obtained by contracting the stress-energy tensor with the Killing vector corresponding to time translations. With this definition, the energy is conserved, positive and finite for irregular modes propagating in the bulk when the following constraint is satisfied [10,11]:

$$\varrho=\frac{1}{2}\,\frac{\Delta_-(\varrho)}{1+2\Delta_-(\varrho)}. \tag{18}$$

Now, the Legendre transformation in [13] can be performed for any values of ϱ, so that the constraint Eq. (18) should be imposed by hand. An additional difficulty was that such a prescription, where the leading non-local term of the action is selected *before* performing the Legendre transformation, does not work for $\nu=0$, due to the presence of a logarithmic term in the generating functional [36].

In order to solve these problems, the proposal in [37] was to introduce a modified formulation both in the bulk and in the AdS/CFT correspondence. From the bulk point of view, a modified quantization was performed where the "canonical" energy, which is constructed out of the Noether current corresponding to time displacements, is employed instead of the usual "metrical" one which is defined through the stress-energy tensor, as in [10,11]. The reason for this is that, as explained in [37], the canonical energy is sensitive to the addition of boundary terms to the action, a property inherited from the Noether current, whereas the usual metrical one is not.[1] This modified quantization gives rise to new constraints for the propagation of irregular modes (which make the canonical energy to be conserved, positive and finite) that come to replace Eq. (18). Such constraints depend on the particular boundary term that is added to the action, and many different examples were considered in [37]. On the other hand, the constraint Eq. (16), together with the Breitenlohner-Freedman bound [see Eqs. (5), (17)], remain unchanged, as expected.

From the AdS/CFT point of view, the proposal in [37] was to consider a modified prescription where the source ϕ_0 in Eq. (1) is replaced by a more general one, which depends on the boundary conditions, or, equivalently, on the boundary term that is added to the action, and could be a combination of both the field and its normal derivative. In addition, it was considered a generalized Legendre transform prescription in which the Legendre transformation is performed on the whole on-shell action, containing all local and non-local terms, instead of only on the leading non-local term, as in the usual prescription. The generalized Legendre transformation contains all the information about the constraints arising from the quantization in the bulk, and it was shown in [37] that it solves all the problems mentioned above regarding the usual formulation. In particular, the main goal in [37] was to show, for many different boundary terms added to the action, that the constraints for which the irregular modes propagate in the bulk when the canonical energy is considered instead of the metrical one, are the same for which the divergent local terms of the on-shell action cancel out, and the generalized Legendre transformation interpolates between different conformal dimensions, namely Δ_+ and Δ_-.

Motivated by the above detailed results in [28], regarding the relation between unstable double-trace perturbations in the boundary, and bound states in the bulk, the purpose of this paper is to show that the action of the bulk theory governs, via the addition of boundary terms, the existence of a bound state in the bulk. We aim at computing, in both minimally and non-minimally coupled cases, which boundary terms give rise to the existence of a bound state in the bulk, and which ones do not. We would also like to incorporate relevant double-trace perturbations, and the description of bound states in the bulk and unstable theories at the boundary, into an extended formalism which is able to describe phenomena involving both the AdS/CFT duality and the quantization of the theory in the bulk. In this respect, the constraints computed in [37] for the propagation of irregular modes in the bulk will appear here again, this time in the role of the points where the relevant double-trace perturbations begin.

In Sec. II, we introduce the background formalism regarding irregular modes, which will be the basis for the rest of this paper. With illustrative purposes, we will add to the action Eq. (2) the simplest possible boundary term, which is quadratic in the field. We will consider the canonical energy of the theory for irregular modes propagating in the bulk, and, in the AdS/CFT context, we will analyze some subtleties regarding the Legendre transformation. Then, we will focus on a relevant double-trace perturbation at the border and, by also making use of the results in [28], we will show that the boundary term in the action governs the existence of a bound state in the bulk, which will be related to an unstable

[1]Recently, another definition of the energy of the scalar field theory in AdS space, which is also sensitive to the boundary conditions, was introduced in [38].

perturbation at the boundary. A particular notorious example will be that of the usual action Eq. (2), which contains no additional boundary term. We will show that it has an associated bound state. In general, in performing calculations we will pay a careful attention to the fact that relevant double-trace perturbations begin at the special points where irregular modes are allowed to propagate, as computed in [37].

In Sec. III, we will consider the case of a non-minimally coupled scalar field, supplemented by a Gibbons-Hawking boundary term [39], plus an additional, optional mass term at the boundary. We will show the existence of allowed values of ϱ [see Eq. (13)] for which there is no bound state in the bulk. In particular, we will show that a conformally coupled scalar field has no associated bound state.

In Sec. IV, we introduce, in both the minimally and non-minimally coupled cases, all remaining boundary terms allowed by the variational principle, whose analysis allows us to perform additional consistency checks on the formalism. In particular, we will reproduce, once again, the result that the usual action Eq. (2), containing no additional boundary term, has an associated bound state.

Finally, in Sec. V, we reconsider the Breitenlohner-Freedman bound, and argue that the requirement for it to be satisfied does not suffice to prevent tachyonic behavior from existing in the bulk. It must be supplemented by additional conditions on the coefficients in the boundary terms of the action, i.e., the ones computed in Secs. II, III and IV. The reason for this is that the Breitenlohner-Freedman bound misses the part of the information in the action which is contained in the boundary terms. We also formulate our concluding remarks.

II. IRREGULAR MODES AND BOUND STATES

In this paper, we will consider double-trace perturbations by a relevant operator $\mathcal{O}_\alpha^2$, where $\mathcal{O}_\alpha$ has dimension Δ_- (see the Introduction for notation and details). But such a conformal operator is associated with irregular modes propagating in the bulk, and this suggests that, in order to get a complete understanding of such relevant double-trace perturbations, we should first carefully analyze the phenomenon of the propagation of irregular modes. In order to do this, we first introduce some background results which will be extensively used throughout this paper. We will closely follow [37], and also write parts of the formalism developed in that reference in a more refined manner, which will be useful for our present purposes. In particular, we will focus on some subtleties regarding the Legendre transformation, which were not considered in [37]. Then, we will introduce a relevant double-trace perturbation at the boundary, and show that boundary terms in the action govern the existence of a bound state in the bulk. The non-minimally coupled case will be analyzed in the following section.

We begin by considering global coordinates in a $(d+1)$-dimensional AdS space. After setting the radius of AdS_{d+1} equal to one, the metric reads

$$ds^2=\frac{1}{\cos^2\rho}(-d\tau^2+d\rho^2+\sin^2\rho\, d\Omega_d^2) \quad (d\geqslant 2), \tag{19}$$

where $d\Omega_d^2$ is the angular element, and ρ and τ are the radial and time coordinates respectively. They satisfy

$$0\leqslant\rho<\frac{\pi}{2} \quad (d\geqslant 2), \tag{20}$$

$$-\infty<\tau<\infty. \tag{21}$$

The above equation (which replaces $-\pi\leqslant\tau<\pi$) indicates that we are considering the universal covering space CAdS. This is done in order to avoid closed timelike curves (see for instance [40]).

We consider the AdS space as foliated by d-dimensional surfaces $\partial\mathcal{M}_\rho$ of fixed radial coordinate ρ. Such surfaces are homeomorphic to the boundary $\partial\mathcal{M}$ at $\rho\to\pi/2$. We refer to $\partial\mathcal{M}_\rho$ as the boundary to the interior region $\mathcal{M}_\rho$. The limit $\rho\to\pi/2$ is to be taken only at the end of the calculations. The surface forming an outer normal vector to $\partial\mathcal{M}_\rho$ is given by

$$n_\mu=\frac{1}{\cos\rho}\delta_\mu^{(\rho)}. \tag{22}$$

We first focus on the case of a minimally coupled scalar field. The non-minimally coupled case will be analyzed in the following section. For reasons to be clarified later, it will be relevant to our work to generalize the usual action Eq. (2) by adding a surface term to it. There are different possible choices for such a surface term (see Sec. IV). The simplest one is as follows:

$$I_1=I_0-\lambda_1\int_{\partial\mathcal{M}_\rho} d^d y\sqrt{h}\phi^2, \tag{23}$$

where λ_1 is a coefficient, and $h_{\mu\nu}$ is the induced metric. The above action was not considered in [37], but the calculations are analogous to the ones involving other surface terms. Some results we will find were already considered in [22], but here we will give a more detailed account of the calculations, as this will be useful for our present purposes.

Following a procedure analogous to that in [37], we compute the Noether current corresponding to time displacements (which are isometries of the background metric), and then, making use of the equation of motion, we find the following expression for the canonical energy:

$$E_1=-\int d^d y\sqrt{g}[\Theta_\tau{}^\tau-\lambda_1\nabla_\mu(n^\mu\phi^2)], \tag{24}$$

where the global minus sign is due to the "mostly plus" signature of the metric, the integration is carried out over the spatial coordinates, and

$$\Theta_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi+m^2\phi^2). \tag{25}$$

It can be shown that E_1 is conserved, positive and finite for irregular modes propagating in the bulk only when Eq. (6), together with the following constraint [22]:

$$\lambda_1=\frac{\Delta_-}{2}, \tag{26}$$

is satisfied. This allows us to perform a quantization of the scalar field in AdS, in the manner described in [37], and which is analogous to the one considered in [10,11]. However, the details of such calculations are not relevant for our present purposes. The constraint Eq. (26), which is intimately related to the propagation of irregular modes in the bulk, will be of fundamental importance in what follows. The reason for this is that we will consider perturbations at the border by relevant conformal operators, which correspond to irregular modes in the bulk [as $2\Delta_-\leqslant d$; see Eq. (3)].

Once we have shown how Eq. (26) arises when working in global coordinates and requiring the canonical energy to be conserved, positive and finite for irregular modes, let us show how the same constraint arises from AdS/CFT calculations. This takes us to a consideration of the Euclidean representation of AdS_{d+1} (with radius equal to one) in Poincaré coordinates described by the half space $x_0>0$, $x_i\in\mathbf{R}$, with metric

$$ds^2=\frac{1}{x_0^2}\sum_{\mu=0}^{d} dx^\mu dx^\mu. \tag{27}$$

The space will be considered as foliated by a family of surfaces $x_0=\epsilon$ where we will formulate a boundary-value problem for the scalar field. As pointed out in [16], the limit $\epsilon\to 0$ is to be taken only at the end of calculations. Note that, in these coordinates, the outward pointing unit normal vector is given by[2]

$$n_\mu=(-\epsilon^{-1},\mathbf{0}). \tag{28}$$

In the Euclidean coordinates Eq. (27), the action Eq. (23) reads

$$I_1=I_0+\lambda_1\int d^dx\sqrt{h}\,\phi_\epsilon^2, \tag{29}$$

where ϕ_ϵ is the value of the field at $x_0=\epsilon$, and

$$I_0=\frac{1}{2}\int d^{d+1}x\sqrt{g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+m^2\phi^2). \tag{30}$$

Here a Wick rotation has been performed.

We wish to consider a boundary-value problem on the scalar field. Note that under an infinitesimal variation

$$\phi\to\phi+\delta\phi, \tag{31}$$

the action Eq. (29) transforms as follows:

$$\delta_\phi I_1=\int d^dx\sqrt{h}\,\psi_\epsilon^{(1)}\delta\phi_\epsilon, \tag{32}$$

where the absence of a bulk contribution is due to the equation of motion. Here $\psi^{(1)}$ is defined through

$$\psi^{(1)}=\partial_n\phi+2\lambda_1\phi, \tag{33}$$

where $\partial_n\phi$ is the normal derivative of ϕ,

$$\partial_n\phi=n^\mu\partial_\mu\phi. \tag{34}$$

From Eq. (32), we conclude that the action is stationary under a Dirichlet boundary condition at $x_0=\epsilon$,

$$\delta\phi_\epsilon=0. \tag{35}$$

Integrating by parts, and making use of the equation of motion, I_1 can be written as the following pure-surface term:

$$I_1=\frac{1}{2}\int d^dx\sqrt{h}\,\phi_\epsilon\psi_\epsilon^{(1)}. \tag{36}$$

We will make use of the solution to the equation of motion which is regular at $x_0\to\infty$. It reads [16]

$$\phi(x)=\int\frac{d^dk}{(2\pi)^d}e^{-i\vec{k}\cdot\vec{x}}x_0^{d/2}a(\vec{k})K_\nu(kx_0), \tag{37}$$

where $\vec{x}=(x^1,\ldots,x^d)$, $k=|\vec{k}|$, K_ν is the modified Bessel function, and ν is given by Eq. (4).

We have just seen that, in the particular case of the action I_1, we are in the presence of a Dirichlet boundary-value problem which fixes ϕ_ϵ at the boundary [see Eq. (35)]. This means that we should write $\psi_\epsilon^{(1)}$ in terms of the boundary data ϕ_ϵ. We find

$$\psi_\epsilon^{(1)}(\vec{k})=-\left[\frac{d}{2}+\nu-2\lambda_1-k\epsilon\frac{K_{\nu+1}(k\epsilon)}{K_\nu(k\epsilon)}\right]\phi_\epsilon(\vec{k}), \tag{38}$$

where $\phi_\epsilon(\vec{k})$ is the Fourier transform of $\phi_\epsilon(\vec{x})$. Inserting the above equation into Eq. (36), we arrive at

$$I_1[\phi_\epsilon]=-\frac{1}{2}\int d^dxd^dy\sqrt{h}\,\phi_\epsilon(\vec{x})\phi_\epsilon(\vec{y})\int\frac{d^dk}{(2\pi)^d}\times e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\left[\frac{d}{2}+\nu-2\lambda_1-k\epsilon\frac{K_{\nu+1}(k\epsilon)}{K_\nu(k\epsilon)}\right]. \tag{39}$$

In order to get the full information about the boundary CFT, we still need to compute the Legendre transform of I_1. We should follow a procedure analogous to that in [37], and perform the generalized Legendre transformation on the expression Eq. (39). Even when such a procedure is correct, a more illuminating point of view will arise by performing the Legendre transformation *before* writing $\psi_\epsilon^{(1)}$ in terms of the boundary data ϕ_ϵ, i.e. on Eq. (36), instead of Eq. (39).

[2]Note that n^μ is not a true vector, as its variation under an infinitesimal diffeomorphism includes an extra deviation term. See for instance the Appendix of [41].

We begin by writing Eq. (36) as

$$I_1[\phi_\epsilon]=\frac{1}{2}\int d^dx\sqrt{h}\,\phi_\epsilon(\psi_\epsilon^{(1)}[\phi_\epsilon]),\qquad(40)$$

where the notation $\psi_\epsilon^{(1)}[\phi_\epsilon]$ explicitly indicates that the boundary data are ϕ_ϵ, and that $\psi_\epsilon^{(1)}$ must be written in terms of it. Note from Eq. (38) the identity

$$\frac{\delta\psi_\epsilon^{(1)}}{\delta\phi_\epsilon}=\frac{\psi_\epsilon^{(1)}}{\phi_\epsilon}.\qquad(41)$$

So let $\tilde{\phi}_\epsilon$ stand for the Legendre conjugate of ϕ_ϵ. The generalized Legendre transformation, including all local and non-local terms in the on-shell action, is performed in momentum space for convenience. It reads

$$\mathcal{J}_1[\phi_\epsilon,\tilde{\phi}_\epsilon]=\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\phi_\epsilon(\vec{k})\psi_\epsilon^{(1)}(-\vec{k})-\int\frac{d^dk}{(2\pi)^d}\phi_\epsilon(\vec{k})\tilde{\phi}_\epsilon(-\vec{k}).\qquad(42)$$

Note that the above generalized Legendre transformation contains all local and non-local terms of the action, via Eq. (38). Now, setting $0=\partial\mathcal{J}_1/\partial\phi_\epsilon$ (for fixed $\tilde{\phi}_\epsilon$), we find

$$0=\frac{1}{2}\psi_\epsilon^{(1)}+\frac{1}{2}\phi_\epsilon\frac{\delta\psi_\epsilon^{(1)}}{\delta\phi_\epsilon}-\tilde{\phi}_\epsilon,\qquad(43)$$

and using Eq. (41) we arrive at

$$\tilde{\phi}_\epsilon=\psi_\epsilon^{(1)}.\qquad(44)$$

This is the demonstration that ϕ_ϵ and $\psi_\epsilon^{(1)}$ are Legendre conjugates, a result to which we will come back later. Note that we could have arrived at Eq. (44) also by performing the Legendre transformation on Eq. (39) instead of Eq. (40), i.e. by following a procedure analogous to the one considered in [37]. But in this case we should follow an indirect path, by first computing the relation between $\tilde{\phi}_\epsilon$ and ϕ_ϵ, and then verifying that it is identical to that between $\psi_\epsilon^{(1)}$ and ϕ_ϵ [see Eq. (38)]. Summarizing, both procedures contain exactly the same information, as expected, but the above considered one is more compact and illuminating, as it contains Eq. (44) as a necessary intermediate result. Due to the relevance of Eq. (44) for our present purposes, the above detailed procedure is the one that we will employ in this paper, in the context of double-trace perturbations.

Now, introducing Eq. (44) into Eq. (42), we find the Legendre transform of I_1 [see Eq. (40)], which reads

$$\tilde{I}_1[\psi_\epsilon^{(1)}]=-\frac{1}{2}\int d^dx\sqrt{h}(\phi_\epsilon[\psi_\epsilon^{(1)}])\psi_\epsilon^{(1)},\qquad(45)$$

where we explicitly indicate through the notation $\phi_\epsilon[\psi_\epsilon^{(1)}]$ that, unlike the original functional Eq. (40), now it is ϕ_ϵ which must be written in terms of $\psi_\epsilon^{(1)}$. Introducing Eq. (38) into the above equation, we find

$$\tilde{I}_1[\psi_\epsilon^{(1)}]=\frac{1}{2}\int d^dxd^dy\sqrt{h}\,\psi_\epsilon^{(1)}(\vec{x})\psi_\epsilon^{(1)}(\vec{y})\int\frac{d^dk}{(2\pi)^d}\times e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\frac{1}{d/2+\nu-2\lambda_1-k\epsilon K_{\nu+1}(k\epsilon)/K_\nu(k\epsilon)},\qquad(46)$$

which, together with Eq. (39), contains the information about the boundary dual theory.

We consider here the case of ν a not integer value satisfying Eq. (6), which is the relevant one for our present purposes, as we are interested in analyzing situations when both regular and irregular modes are allowed to propagate in the bulk.[3] Expanding Eqs. (39), (46) in powers of ϵ, we find

$$I_1[\phi_\epsilon]=-\frac{1}{2}\int d^dxd^dy\,\phi_\epsilon(\vec{x})\phi_\epsilon(\vec{y})\epsilon^{-d}\int\frac{d^dk}{(2\pi)^d}e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\left[(\Delta_--2\lambda_1)-2^{1-2\nu}\frac{\Gamma(1-\nu)}{\Gamma(\nu)}(k\epsilon)^{2\nu}+\cdots\right],\qquad(47)$$

$$\tilde{I}_1[\psi_\epsilon^{(1)}]=\frac{1}{2}\int d^dxd^dy\,\psi_\epsilon^{(1)}(\vec{x})\psi_\epsilon^{(1)}(\vec{y})\epsilon^{-d}\int\frac{d^dk}{(2\pi)^d}e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\frac{1}{(\Delta_--2\lambda_1)-2^{1-2\nu}[\Gamma(1-\nu)/\Gamma(\nu)](k\epsilon)^{2\nu}+\cdots},\qquad(48)$$

where the ellipses stand for higher orders. Note that here the constraint Eq. (26) arises again, this time in a different context, as this is precisely the situation for which the divergent local term in Eqs. (47), (48) vanishes. Let us first consider the case when Eq. (26) is not satisfied, i.e. when only regular modes are allowed to propagate in the bulk. Then, integrating over $\vec{k}$ we get

[3]Other values of ν, such as $\nu>1$, integer or not, and $\nu=0$, can be considered following procedures analogous to the ones in [37], but we will not analyze them here.

$$I_1[\phi_\epsilon]=\text{divergent local terms}-\frac{\nu}{\pi^{d/2}}\frac{\Gamma(\Delta_+)}{\Gamma(\nu)}\times\int d^dx d^dy\,\phi_\epsilon(\vec{x})\,\phi_\epsilon(\vec{y})\,\frac{\epsilon^{-2\Delta_-}}{|\vec{x}-\vec{y}|^{2\Delta_+}}+\cdots, \tag{49}$$

$$\tilde{I}_1[\psi_\epsilon^{(1)}]=\text{divergent local terms}-\frac{\nu}{\pi^{d/2}}\frac{1}{(\Delta_- -2\lambda_1)^2}\frac{\Gamma(\Delta_+)}{\Gamma(\nu)}\times\int d^dx d^dy\,\psi_\epsilon^{(1)}(\vec{x})\,\psi_\epsilon^{(1)}(\vec{y})\,\frac{\epsilon^{-2\Delta_-}}{|\vec{x}-\vec{y}|^{2\Delta_+}}+\cdots, \tag{50}$$

where the ellipses stand for higher orders. The non-local term in Eq. (49) was analyzed in [16,17], and the one in Eq. (50) differs from it only by a normalization coefficient. The limit $\epsilon\to 0$ is taken through

$$\lim_{\epsilon\to 0}\epsilon^{-\Delta_-}\phi_\epsilon(\vec{x})=\phi_0(\vec{x}), \tag{51}$$

which is the usual limit, and

$$\lim_{\epsilon\to 0}\epsilon^{-\Delta_-}\psi_\epsilon^{(1)}(\vec{x})=\psi_0^{(1)}(\vec{x}), \tag{52}$$

which has the same form as Eq. (51). Both fields ϕ_ϵ and $\psi_\epsilon^{(1)}$ exhibit the same behavior, due to the fact that only regular modes propagate, as Eq. (26) is not satisfied. Note that both functionals Eqs. (49), (50) give rise to a boundary conformal operator with dimension Δ_+, as expected.

A different picture emerges when Eq. (26) is satisfied, and irregular modes are allowed to propagate as well [we emphasize that, in this analysis, we are considering the case when Eq. (6) is also satisfied, which is the relevant one for our present purposes]. In such a situation, the locally divergent terms in Eqs. (47), (48) vanish. Note that I_1 still gives rise to the conformal dimension Δ_+, as it reads

$$I_1[\phi_\epsilon]=-\frac{\nu}{\pi^{d/2}}\frac{\Gamma(\Delta_+)}{\Gamma(\nu)}\int d^dx d^dy\,\phi_\epsilon(\vec{x})\,\phi_\epsilon(\vec{y})\,\frac{\epsilon^{-2\Delta_-}}{|\vec{x}-\vec{y}|^{2\Delta_+}}+\cdots. \tag{53}$$

On the other hand, Eq. (48) is written

$$\tilde{I}_1[\psi_\epsilon^{(1)}]=-\frac{1}{4\pi^{d/2}}\frac{\Gamma(\Delta_-)}{\Gamma(1-\nu)}\int d^dx d^dy\,\psi_\epsilon^{(1)}\times(\vec{x})\,\psi_\epsilon^{(1)}(\vec{y})\,\frac{\epsilon^{-2\Delta_+}}{|\vec{x}-\vec{y}|^{2\Delta_-}}+\cdots, \tag{54}$$

and, instead of Eq. (52), we find the behavior

$$\lim_{\epsilon\to 0}\epsilon^{-\Delta_+}\psi_\epsilon^{(1)}(\vec{x})=\psi_0^{(1)}(\vec{x}). \tag{55}$$

Note that, as expected, Eq. (54) gives rise to a conformal operator $\tilde{\mathcal{O}}$ with dimension Δ_-, corresponding to irregular modes propagating in the bulk. The regular modes are accounted for by $I_1[\phi_\epsilon]$.

It will be useful for our present purposes to further elaborate on Eq. (38). Note that, expanding in powers of ϵ, we have

$$\psi_\epsilon^{(1)}=-(\Delta_- -2\lambda_1+\cdots)\,\phi_\epsilon, \tag{56}$$

where the ellipses stand for higher orders. Using Eqs. (51), (52), and taking the limit $\epsilon\to 0$, we get

$$\psi_0^{(1)}=-(\Delta_- -2\lambda_1)\,\phi_0. \tag{57}$$

We will come back to the above equation later in this section.

At this point, it is interesting to note the analogy between Eqs. (51), (55) and Eq. (7). It suggests that, in this formulation, ϕ_0 and $\psi_0^{(1)}$ encode the information on β and α, respectively. We have just shown that ϕ_ϵ and $\psi_\epsilon^{(1)}$ are Legendre conjugates [see Eq. (44)], just as happens to β and α. Note that, for regular modes, ϕ_0 acts as the source [see Eq. (53)], as happens to β in Eq. (7). On the other hand, for irregular modes, it is $\psi_0^{(1)}$ that acts as the source [see Eq. (54)], a role played by α in Eq. (7). A precise description of some aspects of Eq. (7) is perhaps more clearly seen in global coordinates Eq. (19), when the quantization is performed (see, for instance, the discussion on regular and irregular modes in Sec. 3 of Ref. [37], and references therein). But we have just shown that, in the present formulation, the information on the propagation of regular and irregular modes in the bulk is encoded in the Legendre conjugates ϕ_ϵ and $\psi_\epsilon^{(1)}$, and we will make great use of this result in what follows.

Now, we are led to analyze how to describe the perturbation at the boundary CFT by a relevant double-trace perturbation. The first thing to notice is that it should involve a conformal operator of dimension Δ_-, as $2\Delta_- < d$. But we note from Eq. (54) that it corresponds to the conformal operator $\tilde{\mathcal{O}}$, having $\psi_0^{(1)}$ as its source. The double-trace perturbation reads

$$W[\tilde{\mathcal{O}}]=\frac{f}{2}\tilde{\mathcal{O}}^2, \tag{58}$$

which is analogous to Eq. (10). Here f is a coupling coefficient. But, as we have just pointed out, the source to $\tilde{\mathcal{O}}$ is $\psi_0^{(1)}$, and this means that ϕ_0 should be understood as its expectation value; namely

$$\phi_0\equiv\langle\tilde{\mathcal{O}}\rangle, \tag{59}$$

which is analogous to Eq. (9). From Eqs. (58), (59), we can write

$$W[\phi_0]\equiv\frac{f}{2}\phi_0^2. \tag{60}$$

In this process, we have carefully identified the correct source for the conformal operator of dimension Δ_-. But there is still another crucial observation to be made, which is that we have to consider the case when Eq. (26) is satisfied, and irregular modes, which correspond to the conformal dimension Δ_-, are allowed to propagate. Note, also, that we should focus on the functional $\tilde{I}_1$, which is the one that gives rise to the conformal operator $\tilde{\mathcal{O}}$ of dimension Δ_- [see Eq. (54)]. This means that the starting point is [see Eq. (45)]

$$\tilde{I}_1 = -\frac{1}{2}\int d^dx\sqrt{h}\,\phi_\epsilon\psi_\epsilon^{(1)}\bigg|_{\lambda_1=\Delta_-/2}, \tag{61}$$

where we have indicated that we are evaluating at the critical point Eq. (26) where irregular modes propagate, and from Eq. (33) we have

$$\psi^{(1)}|_{\lambda_1=\Delta_-/2} = \partial_n\phi + \Delta_-\phi. \tag{62}$$

We come back to the Legendre transformation Eq. (42), which is schematically written as

$$\mathcal{J}_1 = \int \frac{d^dk}{(2\pi)^d}\left(\frac{1}{2}\phi_\epsilon\psi_\epsilon^{(1)}\bigg|_{\lambda_1=\Delta_-/2} - \phi_\epsilon\tilde{\phi}_\epsilon\right). \tag{63}$$

From Eq. (44), ϕ_ϵ and $\psi_\epsilon^{(1)}|_{\lambda_1=\Delta_-/2}$ are Legendre conjugates

$$\tilde{\phi}_\epsilon = \psi_\epsilon^{(1)}|_{\lambda_1=\Delta_-/2}. \tag{64}$$

Now we perturb the boundary CFT by the relevant double-trace perturbation Eq. (58). From Eqs. (60), (63), this takes $\mathcal{J}_1$ to

$$\mathcal{J}_1 \to \mathcal{J}_1^{(f)} = \int \frac{d^dk}{(2\pi)^d}\phi_\epsilon\left[\frac{1}{2}(\psi_\epsilon^{(1)}|_{\lambda_1=\Delta_-/2} + f\phi_\epsilon) - \tilde{\phi}_\epsilon\right], \tag{65}$$

which can be written [see Eq. (33)]

$$\mathcal{J}_1^{(f)} = \int \frac{d^dk}{(2\pi)^d}\left(\frac{1}{2}\phi_\epsilon\psi_\epsilon^{(1)}\bigg|_{\lambda_1=\Delta_-/2+f/2} - \phi_\epsilon\tilde{\phi}_\epsilon\right), \tag{66}$$

where

$$\psi^{(1)}|_{\lambda_1=\Delta_-/2+f/2} = \partial_n\phi + (\Delta_- + f)\phi. \tag{67}$$

Setting $0 = \partial\mathcal{J}_1^{(f)}/\partial\phi_\epsilon$ (for fixed $\tilde{\phi}_\epsilon$), and using Eq. (41), we get

$$\tilde{\phi}_\epsilon = \psi_\epsilon^{(1)}|_{\lambda_1=\Delta_-/2+f/2}, \tag{68}$$

and inserting the above equation into Eq. (66), we find

$$\tilde{I}_1^{(f)} = -\frac{1}{2}\int d^dx\sqrt{h}\,\phi_\epsilon\psi_\epsilon^{(1)}|_{\lambda_1=\Delta_-/2+f/2}. \tag{69}$$

From the comparison between Eq. (64) and Eq. (68), or between Eq. (61) and Eq. (69), we note that the effect of the relevant double-trace perturbation Eq. (58) has been to introduce the replacement

$$\lambda_1 = \frac{\Delta_-}{2} \to \lambda_1 = \frac{\Delta_-}{2} + \frac{f}{2}. \tag{70}$$

Here is where we should include the sign of the coefficient f in our analysis. We know that positive f corresponds to stable perturbations, whereas negative f corresponds to unstable ones [8]. On the other hand, the results in [28] indicate that the bulk theory detects an unstable theory in the boundary through the existence of a bound state. As pointed out in the Introduction, the results in [28] are based on a careful analysis of the spectrum of the radial wave equation in Lorentzian Poincaré coordinates, and show the existence of a bound state with tachyonic behavior in a Minkowski slice when Eq. (12) is satisfied. Note that this relates the existence of a bound state to negative f via Eq. (11). Thus, the author of [28] concludes that negative f indicates the presence of a bound state in the bulk. But we note in Eq. (70) that negative f is equivalent to the condition

$$\lambda_1 < \frac{\Delta_-}{2}. \tag{71}$$

In other words, given an action such as I_1 in Eq. (23), we conclude that it will be associated with a bound state, which is detected in a Minkowski slice in Lorentzian Poincaré coordinates, provided that Eq. (71) is satisfied. A notorious particular case is that of $\lambda_1 = 0$, which satisfies the condition Eq. (71), and thus has an associated bound state. It corresponds to the usual action I_0 in Eq. (2), containing no additional surface term. Allowed values of λ_1, for which there is no associated bound state in the bulk, are the ones which do not satisfy Eq. (71), i.e.

$$\lambda_1 \geq \frac{\Delta_-}{2}. \tag{72}$$

In this way, we have just demonstrated that the boundary terms in the action govern the existence of a bound state in the bulk, which is detected in a Minkowski slice in Lorentzian Poincaré coordinates, in the manner described in [28]. Such a bound state is present even when the Breitenlohner-Freedman bound Eq. (5) is satisfied, which suggests that the last must be supplemented by Eq. (72). We will come back to this topic in Sec. V. Notice that, in cases where a bound state is present, the addition of a proper boundary term to the action, as above described, should be required.

As a last observation to be made, we point out that, by replacing λ_1 in Eq. (57) by the right-hand side of Eq. (70) (i.e. $\lambda_1 = \Delta_-/2 + f/2$), we find

$$\psi_0^{(1)} = f\phi_0, \tag{73}$$

which is the analogue of Eq. (11).

III. THE NON-MINIMALLY COUPLED CASE

In this section, we focus on the non-minimally coupled case, where we will show the existence of allowed values of the coupling coefficient ϱ [see Eq. (13)] for which there is no bound state in the bulk. Notice that, in this situation, we should employ Eq. (14) instead of Eq. (3).

We begin by considering the following action in global coordinates Eq. (19):

$$\mathcal{I}_1=\mathcal{I}_0+\varrho\int_{\partial\mathcal{M}_\rho}d^dy\sqrt{h}K\phi^2-\sigma\int_{\partial\mathcal{M}_\rho}d^dy\sqrt{h}\phi^2, \quad (74)$$

where $\mathcal{I}_0$ is given by Eq. (13), K is the trace of the extrinsic curvature, and σ is a coefficient. The first boundary term is the natural extension of the Gibbons-Hawking term [39], which is needed in order to have a well-defined variational principle under variations of the metric.[4] The last term in Eq. (74) has the form of a mass term which is added at the boundary. We include it for completeness, as it does not spoil the property of having a well-defined variational principle under variations of the metric. In the particular case $m=\sigma=0$, and when ϱ satisfies

$$\varrho=\frac{d-1}{4d}, \quad (75)$$

both the bulk and boundary terms in Eq. (74) are Weyl invariant (see for instance [42] for a recent treatment).

The canonical energy of the theory Eq. (74) was computed in [37] in the particular case $\sigma=0$. Here we extend the results in [37] to arbitrary σ. It can be shown that the canonical energy is conserved, positive and finite for irregular modes propagating in the bulk only when the following constraint is satisfied:

$$\varrho+\frac{\sigma}{d}=\frac{\Delta_-(\varrho)}{2d}, \quad (76)$$

which comes to replace the usual constraint Eq. (18). It has the solutions

$$\varrho^{\pm}=\frac{d-1}{8d}\left[1-\frac{8\sigma}{d-1}\pm\sqrt{1+\left(\frac{4}{d-1}\right)^2[m^2+(d+1)\sigma]}\right], \quad (77)$$

which should be supplemented by the reality condition

$$m^2+(d+1)\sigma\geqslant-\left(\frac{d-1}{4}\right)^2. \quad (78)$$

Note that, when $m=\sigma=0$, ϱ^- vanishes, whereas ϱ^+ reduces to the conformal value Eq. (75). This could be considered as a check on the formalism.

In the case $\sigma=0$, it was shown in [37] that, as expected, the constraint Eq. (76) arises again from AdS/CFT calculations, this time in the role of the condition for the divergent local terms in the on-shell action to vanish, and the generalized Legendre transformation to interpolate between different conformal dimensions $\Delta_+(\varrho)$ and $\Delta_-(\varrho)$ [see Eq. (14)]. This result is straightforwardly extended to arbitrary σ. Note that, in Euclidean Poincaré coordinates Eq. (27), $\mathcal{I}_1$ reads

$$\mathcal{I}_1=\mathcal{I}_0-\varrho\int d^dx\sqrt{h}K_\epsilon\phi_\epsilon^2+\sigma\int d^dx\sqrt{h}\phi_\epsilon^2, \quad (79)$$

where

$$\mathcal{I}_0=\frac{1}{2}\int d^{d+1}x\sqrt{g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+(m^2+\varrho R)\phi^2]. \quad (80)$$

Since in Euclidean Poincaré coordinates the trace of the extrinsic curvature satisfies $K=-d$, we note that the AdS/CFT calculations for $\mathcal{I}_1$ are analogous to the ones performed in the previous section in the minimally coupled case. All we need to do is to perform the replacements $\lambda_1\to\varrho d+\sigma$, $\nu\to\nu(\varrho)$ and $\Delta_\pm\to\Delta_\pm(\varrho)$ [see Eqs. (14), (15)]. We find that the analogue to Eq. (71) reads

$$\varrho+\frac{\sigma}{d}<\frac{\Delta_-(\varrho)}{2d}, \quad (81)$$

which is the condition for a bound state to exist in the bulk, and is related to the presence of an unstable double-trace perturbation at the boundary.

But the above condition, together with Eq. (16), still needs to be solved for ϱ, a further step that was not needed in the minimally coupled case. We first note that the solution to Eq. (16) reads

$$\varrho>\frac{1}{4d(d+1)}[(d+2)(d-2)+4m^2], \quad (82)$$

which, together with the reality condition Eq. (78), should always be required, as the case Eq. (16) is the relevant one for our analysis. In fact, it will be useful to consider the following condition:

$$\varrho>\frac{1}{16d(d+1)}[4(d+2)(d-2)-(d-1)^2-16(d+1)\sigma], \quad (83)$$

which is obtained from Eqs. (78), (82). Now, solving Eq. (81) and using Eq. (83), we find that, for $d\geqslant3$, a bound state exists in the bulk only when the following conditions are simultaneously satisfied:

$$m^2+(d+1)\sigma<\frac{d}{4} \quad\text{and}\quad \varrho^+<\varrho<\frac{1}{4}-\frac{\sigma}{d} \quad (d\geqslant3). \quad (84)$$

Allowed values of ϱ for which there is no associated bound state are the ones that do not satisfy the above conditions, i.e.

$$m^2+(d+1)\sigma<\frac{d}{4}: \quad \varrho\leqslant\varrho^+ \quad\text{or}\quad \varrho\geqslant\frac{1}{4}-\frac{\sigma}{d},$$

$$m^2+(d+1)\sigma\geqslant\frac{d}{4}: \quad \text{any } \varrho \quad (d\geqslant3). \quad (85)$$

[4] When performing a variation of the metric, the action $\mathcal{I}_0$ in Eq. (13) turns out to be stationary only after the metric and certain of its normal derivatives are fixed at the boundary. It can be shown that the addition of the Gibbons-Hawking term accounts for the terms of the variation containing derivatives of the metric.

We emphasize that the above conditions should be required simultaneously with Eq. (82), together with the reality condition Eq. (78), and the Breitenlohner-Freedman bound Eq. (17). The remarkable result that Eqs. (85) supplement the Breitenlohner-Freedman bound is far from trivial, and we will come back to this topic in Sec. V.

It is interesting to note that, in the notorious example of a conformally coupled scalar field, supplemented by a Gibbons-Hawking term, there is no associated bound state, as in this case ϱ^+ equals the conformal value Eq. (75), and we have just seen that ϱ^+ is in the range of allowed values for ϱ [see Eqs. (85)]. In fact, as pointed out before, the case $m=\sigma=0, \varrho=\varrho^+$ corresponds to an unperturbed situation, where irregular modes are allowed to propagate.

To close this section, we extend the conditions Eqs. (84) for a bound state to exist in the bulk, which hold for $d \geqslant 3$, to the case $d=2$, where a bound state exists in the bulk only when the following conditions are satisfied:

$$m^2+3\sigma<\frac{1}{2}: \quad \varrho<\varrho^- \quad \text{or} \quad \varrho^+<\varrho<\frac{1}{4}-\frac{\sigma}{2},$$

$$m^2+3\sigma\geqslant\frac{1}{2}: \quad \varrho<\varrho^- \quad (d=2). \tag{86}$$

Allowed values of ϱ for which there is no associated bound state are the ones that do not satisfy Eqs. (86), i.e.

$$m^2+3\sigma<\frac{1}{2}: \quad \varrho^-\leqslant\varrho\leqslant\varrho^+ \quad \text{or} \quad \varrho\geqslant\frac{1}{4}-\frac{\sigma}{2},$$

$$m^2+3\sigma\geqslant\frac{1}{2}: \quad \varrho\geqslant\varrho^- \quad (d=2). \tag{87}$$

As in the case $d \geqslant 3$, the above conditions must be supplemented by Eqs. (17), (78), (82).

IV. OTHER BOUNDARY TERMS

In the previous sections, we have illustrated our proposal that boundary terms in the action govern the existence of bound states in the bulk, by considering the examples in Eqs. (29), (79) (in Euclidean coordinates). Now, we would like to analyze how the previous results are modified when other boundary terms are considered. This will also allow us to perform additional checks on the formalism.

We should first notice that it is not possible to add any arbitrary boundary term to the actions Eqs. (30), (80). The reason for this is that we should consider only situations where the variational principle is well defined. In the minimally coupled case, it is possible to verify, by direct inspection, that, apart from Eq. (29), we are left with only two additional possibilities, which in Euclidean coordinates Eq. (27) read [22,37]

$$I_2=I_0+\lambda_2\int d^dx\sqrt{h}(\partial_n\phi_\epsilon)^2 \tag{88}$$

and

$$I_3=I_0+\lambda_3\int d^dx\sqrt{h}\phi_\epsilon^2-\int d^dx\sqrt{h}\phi_\epsilon\partial_n\phi_\epsilon, \tag{89}$$

where λ_2 and λ_3 are arbitrary coefficients, and I_0 is given by Eq. (30).[5] Under the variation Eq. (31) we have[6]

$$\delta_\phi I_2=\int d^dx\sqrt{h}\partial_n\phi_\epsilon\delta\psi_\epsilon^{(2)} \quad \delta_\phi I_3=-\int d^dx\sqrt{h}\phi_\epsilon\delta\psi_\epsilon^{(3)}, \tag{90}$$

where

$$\psi^{(2)}=\phi+2\lambda_2\partial_n\phi, \quad \psi^{(3)}=\partial_n\phi-2\lambda_3\phi. \tag{91}$$

So I_2 and I_3 are stationary under the mixed boundary conditions

$$\delta\psi_\epsilon^{(2)}=0 \tag{92}$$

and

$$\delta\psi_\epsilon^{(3)}=0, \tag{93}$$

[5]There is a curious feature about Eqs. (88), (89), shown in [37], which is the fact that, whereas the on-shell action I_1 gives rise to the boundary CFT of conformal dimension Δ_+, and its Legendre transform $\tilde{I}_1$ corresponds to Δ_- [see Eqs. (53), (54)], when considering I_2 and I_3 we find an "inverted" situation, where the original generating functional corresponds to the conformal dimension Δ_-, and the Legendre transformed one is associated to Δ_+ (this "inversion" phenomenon is also found in the non-minimally coupled case [37]). At this point, we do not know if such an "inversion" phenomenon is, or is not, associated with any property of the boundary CFT.

[6]For illustrative purposes, we give here an example of a one-parameter family of boundary terms which is not allowed. Consider the action $I'=I_0+\gamma\int d^dx\sqrt{h}\phi_\epsilon\partial_n\phi_\epsilon$, where γ is an arbitrary coefficient, and I_0 is given by Eq. (30). Then, under the variation Eq. (31) we have $\delta_\phi I'=\int d^dx\sqrt{h}[(1+\gamma)\partial_n\phi_\epsilon\delta\phi_\epsilon+\gamma\phi_\epsilon\delta(\partial_n\phi_\epsilon)]$. This result is to be contrasted with Eqs. (32), (90). Notice that, in order for the action to be stationary, we should fix both ϕ_ϵ and $\partial_n\phi_\epsilon$ at the border, $\delta\phi_\epsilon=\delta(\partial_n\phi_\epsilon)=0$. From the AdS/CFT point of view, the source for the boundary conformal operator is ill defined, unlike what happens to Eqs. (32), (90), where the sources are ϕ_ϵ, $\psi_\epsilon^{(2)}$ and $\psi_\epsilon^{(3)}$ respectively [22,37]. This situation is analogous to the one found in the case of the Einstein-Hilbert action, whose variation requires both the metric and its derivatives to be fixed at the border. In this case, the requirement to have a well-defined variational principle is satisfied through the addition of the Gibbons-Hawking boundary term [39], which accounts for the derivatives of the metric. Notice that, in the above case, the only possible choice for the coefficient γ is $\gamma=-1$, where the variation reduces to $\delta_\phi I'=-\int d^dx\sqrt{h}\phi_\epsilon\delta(\partial_n\phi_\epsilon)$. Here the action is stationary under a Neumann boundary condition which fixes $\partial_n\phi_\epsilon$ at the boundary, $\delta(\partial_n\phi_\epsilon)=0$. Note that the allowed Neumann situation $\gamma=-1$ is a particular case of Eq. (89) with $\lambda_3=0$.

respectively. In particular, this means that, after taking the limit $\epsilon \to 0$ through a proper rescaling, $\psi_\epsilon^{(2)}$ and $\psi_\epsilon^{(3)}$ become the sources for the corresponding boundary conformal operators [22,37].

In the case of I_2, it can be shown that the corresponding canonical energy is conserved, positive and finite for irregular modes propagating in the bulk, or equivalently, that the divergent local terms of the on-shell action cancel out and the Legendre transformation interpolates between conformal dimensions Δ_+ and Δ_-, only when Eq. (6), together with the constraint [37]

$$\lambda_2 = \frac{1}{2\Delta_-}, \tag{94}$$

is satisfied. Performing calculations analogous to those previously detailed in the case of I_1, we find that $\psi_\epsilon^{(2)}$ and $\partial_n \phi_\epsilon$ are Legendre conjugates, i.e.

$$\tilde{\psi}_\epsilon^{(2)} = \partial_n \phi_\epsilon. \tag{95}$$

Here $\psi_\epsilon^{(2)}$ and $\partial_n \phi_\epsilon$ are the sources for the conformal operators of dimensions Δ_- and Δ_+ respectively, so that a relevant double-trace perturbation at the boundary can be written

$$W[\partial_n \phi_0] \equiv \frac{f}{2}(\partial_n \phi_0)^2, \tag{96}$$

and introduces the replacement

$$\lambda_2 = \frac{1}{2\Delta_-} \to \lambda_2 = \frac{1}{2\Delta_-} + \frac{f}{2}. \tag{97}$$

Bound states exist in the bulk only when the following condition is satisfied:

$$\lambda_2 < \frac{1}{2\Delta_-}. \tag{98}$$

Allowed values of λ_2 for which there is no bound state in the bulk are the ones in the range

$$\lambda_2 \geqslant \frac{1}{2\Delta_-}. \tag{99}$$

A notorious example is that of the usual action I_0 in Eq. (2), containing no additional surface term, which corresponds to the case $\lambda_2 = 0$ [see Eq. (88)], so that it is not in the range Eq. (99), and has an associated bound state. The same result has already been found when considering I_1, and the fact that both analyses, involving I_1 or I_2, give rise to the same result, could be considered as a consistency check on the formalism.

Now, in the case of I_3, the constraint for which the canonical energy is conserved, positive and finite for irregular modes propagating in the bulk, and the divergent local terms in the on-shell action cancel out, causing the Legendre transformation to interpolate between conformal dimensions Δ_+ and Δ_-, is given by [22]

$$\lambda_3 = -\frac{\Delta_-}{2}. \tag{100}$$

It can be shown that $\psi_\epsilon^{(3)}$ and $-\phi_\epsilon$ are Legendre conjugates, i.e.

$$\tilde{\psi}_\epsilon^{(3)} = -\phi_\epsilon. \tag{101}$$

Here $\psi_\epsilon^{(3)}$ and $-\phi_\epsilon$ are sources for the conformal operators of dimensions Δ_- and Δ_+ respectively, so that a relevant double-trace perturbation is of the form

$$W[-\phi_0] \equiv \frac{f}{2}(-\phi_0)^2, \tag{102}$$

and performs the replacement

$$\lambda_3 = -\frac{\Delta_-}{2} \to \lambda_3 = -\frac{\Delta_-}{2} + \frac{f}{2}. \tag{103}$$

Bound states exist in the bulk only for λ_3 satisfying

$$\lambda_3 < -\frac{\Delta_-}{2}. \tag{104}$$

Allowed values of λ_3 for which there is no bound state in the bulk are the ones in the range

$$\lambda_3 \geqslant -\frac{\Delta_-}{2}. \tag{105}$$

Finally, in the non-minimally coupled case, we should take into account that, in order to have a well-defined variational principle under variations of the metric, any expression for the action should contain a Gibbons-Hawking term, as in Eq. (79). It can be verified that, apart from Eq. (79), we are left with only one additional possibility, namely

$$\mathcal{I}_2 = \mathcal{I}_0 - \varrho \int d^d x \sqrt{h} K_\epsilon \phi_\epsilon^2 + \sigma \int d^d x \sqrt{h} \phi_\epsilon^2 - \int d^d x \sqrt{h} \phi_\epsilon \partial_n \phi_\epsilon, \tag{106}$$

where $\mathcal{I}_0$ is given by Eq. (80), and the first two boundary terms are as in Eq. (79). It can be shown that the last surface term does not spoil the property of having a well-defined variational principle under variations of the metric.

The case $\sigma = 0$ was considered in [37], but here we will extend such results to the case of arbitrary σ. It can be shown that the canonical energy is conserved, positive and finite for irregular modes propagating in the bulk only when the following constraint is satisfied

$$\varrho + \frac{\sigma}{d} = -\frac{\Delta_-(\varrho)}{2d}, \tag{107}$$

which is to be contrasted with Eq. (76). It has the solutions

$$\tilde{\varrho}^{\pm}=\frac{3d+1}{8d}\left[-1-\frac{8\sigma}{3d+1}\pm\sqrt{1+\left(\frac{4}{3d+1}\right)^{2}[m^{2}+(d+1)\sigma]}\right]. \tag{108}$$

Notice the reality condition

$$m^{2}+(d+1)\sigma\geqslant-\left(\frac{3d+1}{4}\right)^{2}, \tag{109}$$

which should always be required, together with Eq. (82). As expected, the constraint Eq. (107) arises again from AdS/CFT calculations, in the role of the condition for the divergent local terms in the on-shell action to vanish, and the Legendre transformation to interpolate between different conformal dimensions $\Delta_{+}(\varrho)$ and $\Delta_{-}(\varrho)$.

By performing calculations analogous to the ones in the previous cases, we find that the condition

$$\varrho+\frac{\sigma}{d}<-\frac{\Delta_{-}(\varrho)}{2d}, \tag{110}$$

corresponds to the existence of a bound state in the bulk, in the sense of [28]. Such condition is related to the presence of an unstable double-trace perturbation at the boundary. Solving for ϱ, we find the following solution:

$$m^{2}+(d+1)\sigma<-\frac{1}{16}[(3d+1)^{2}-(d+5)^{2}]:\quad \varrho<\tilde{\varrho}^{+},$$

$$m^{2}+(d+1)\sigma\geqslant-\frac{1}{16}[(3d+1)^{2}-(d+5)^{2}]:$$

$$\varrho<-\frac{d-2}{4d}-\frac{\sigma}{d}. \tag{111}$$

Allowed values of ϱ for which there is no associated bound state are the ones that do not satisfy the above conditions. This gives

$$m^{2}+(d+1)\sigma<-\frac{1}{16}[(3d+1)^{2}-(d+5)^{2}]:\quad \varrho\geqslant\tilde{\varrho}^{+},$$

$$m^{2}+(d+1)\sigma\geqslant-\frac{1}{16}[(3d+1)^{2}-(d+5)^{2}]:$$

$$\varrho\geqslant-\frac{d-2}{4d}-\frac{\sigma}{d}. \tag{112}$$

We emphasize that the above conditions should be required simultaneously with Eqs. (17), (82), (109).

Note that, in the notorious particular case of a conformally coupled scalar field, where $m=\sigma=0$ and ϱ is given by Eq. (75), the conditions Eqs. (112) show that there is no bound state in the bulk, as it happened in the case of $\mathcal{I}_1$ [see Eq. (79)]. However, there is a fundamental difference, because, whereas for $\mathcal{I}_1$ the conformally coupled case corresponds to an unperturbed situation where irregular modes are allowed to propagate in the bulk, in the case of $\mathcal{I}_2$ it is associated to a (stable) non-zero double-trace perturbation at the boundary. This result is not surprising when we note that the last boundary term in Eq. (106) breaks the Weyl invariance of Eq. (79).

V. BREITENLOHNER-FREEDMAN BOUND RECONSIDERED

In this work, we have argued that coefficients in the boundary terms in the action are sensitive to the perturbation at the boundary CFT by a relevant double-trace operator [see for instance Eqs. (70), (97), (103)], and govern the existence of a bound state in the bulk. The relation was made precise by using the proposal in [28] that unstable theories at the boundary are detected by the presence of such a bound state in the bulk. In all calculations, we have also made strong use of the formalism in [8,37]. In particular, we have paid a careful attention to the fact that relevant double-trace perturbations are constructed out of a conformal operator of dimension Δ_{-}. This means that we have to identify the correct source and generating functional for the conformal operator, and introduce the perturbations at the special points at which irregular modes are allowed to propagate [see Eqs. (26), (76), (94), (100), (107)]. From the bulk point of view, such special points arise from the requirement for the canonical energy to be conserved, positive and finite for irregular modes propagating in the bulk [37]. From AdS/CFT calculations, they play the role of the conditions for the divergent local terms in the on-shell action to vanish, and the generalized Legendre transform to interpolate between different conformal dimensions Δ_{+} and Δ_{-} [37].

Throughout this paper, we have considered many different allowed boundary terms in the action [see Eqs. (29), (79), (88), (89), (106)]. By proposing such boundary terms to be the objects involved in the connection between unstable double-trace perturbations at the boundary and bound states in the bulk, we were able to compute explicit conditions on the coefficients of the boundary terms in the action for which we expect a bound state to exist in the bulk [see Eqs. (71), (98), (104)]. In the non-minimally coupled case, and when the action is supplemented by a Gibbons-Hawking term, this also gave rise to ''forbidden'' values of the coupling coefficient to the metric [see Eqs. (84), (86), (111)].

Notorious particular examples were also considered. For instance, we have shown that the usual action Eq. (2), containing no additional boundary terms, is associated with the existence of a bound state in the bulk. This result was found by considering independent analyses involving either actions I_1 or I_2 [see Eqs. (29), (88)]. This could be considered as a consistency check on the formalism. Another notorious example has been that of a conformally coupled scalar field, supplemented by a Gibbons-Hawking term, to which we have shown that there is no associated bound state.

To close this paper, we come back to the result in [28] that tachyonic behavior in the bulk exists even when the Breitenlohner-Freedman bound [see Eqs. (5), (17)] is satisfied. We argue here that this happens so, because the Breitenlohner-Freedman bound should be supplemented by additional conditions involving the coefficients on the boundary terms in the action. Such conditions depend on the particular boundary term which is added to the action, and are given by Eqs. (72), (99), (105) (in the minimally coupled case), and Eqs. (85), (87), (112) [in the non-minimally coupled case, supplemented by a Gibbons-Hawking term, where we should simultaneously require the condition Eq. (82), together with the reality conditions Eq. (78) for $\mathcal{I}_1$ or Eq. (109) for $\mathcal{I}_2$]. The reason why the requirement to satisfy the Breitenlohner-Freedman bound is not enough to prevent tachyonic behavior from existing is that such a bound misses a part of the information contained in the action, namely that included in the boundary terms, and which is related to the boundary conditions on the field.

The fact that some results are modified or generalized when the boundary terms in the action are taken into account is not surprising. See, for instance, the replacement of Eq. (18) by Eq. (76) or Eq. (107).

It would be interesting to investigate if any further information arising from the boundary terms in the action can be obtained by performing additional related calculations in the Lorentzian Poincaré metric.

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