From free fields to AdS space. II

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We continue with the program of paper I [Phys. Rev. D **70**, 025009 (2004)] to implement open-closed string duality on free gauge field theory (in the large-N limit). In this paper we consider correlators such as $\langle\Pi_{i=1}^{n}\mathrm{Tr}\Phi^{J_i}(x_i)\rangle$. The Schwinger parametrization of this n-point function exhibits a partial gluing up into a set of basic skeleton graphs. We argue that the moduli space of the planar skeleton graphs is exactly the same as the moduli space of genus zero Riemann surfaces with n holes. In other words, we can explicitly rewrite the n-point (planar) free-field correlator as an integral over the moduli space of a sphere with n holes. A preliminary study of the integrand also indicates compatibility with a string theory on AdS space. The details of our argument are quite insensitive to the specific form of the operators and generalize to diagrams of a higher genus as well. We take this as evidence of the field theory's ability to reorganize itself into a string theory.

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I. INTRODUCTION

How exactly does a quantum field theory (in the large-N limit) reassemble itself into a closed string theory? This question lies at the heart of the gauge theory/geometry correspondence. Answering it in its generality is likely to give us valuable clues regarding the string dual to QCD, for instance.

What we have learned in the years since Maldacena's breakthrough is that the answer to this question is tied up with open-closed string duality. The gauge theory arising in an open string description is related by worldsheet duality to a closed string description. The holes of the open string worldsheet get glued up, getting replaced by closed string insertions.

In the case of topological string dualities, it was possible [1,2] to make this intuition precise using a linear sigma model description of the worldsheet (the argument for the corresponding F terms in the superstring was made in [3]). Recently, a very nice illustration of open-closed string duality was given, again in a topological context, for the Kontsevich matrix model [4]. Here again, one could concretely see the process of holes closing up and being replaced by closed string insertions.

Nevertheless, the original AdS conformal field theory (CFT) conjecture [5–7] has not yet been understood in such terms.[1] In [11] we embarked on an effort to implement open-closed string duality in the free-field limit of the $\mathcal{N}=4$ super Yang-Mills theory. The tractability of the limit, from the field theory point of view, makes it a natural starting point.[2] The strategy in [11] was to consider a worldline representation (Schwinger parametrization) of the free-field correlators. This was motivated by the fact that these representations can be viewed as being directly inherited from the relevant open string theory in the $\alpha'\to\infty$ limit. A nice feature of this representation is its correspondence with electrical networks. This correspondence suggested that carrying out the integration over the internal loop momenta (eliminating internal currents) should yield an equivalent network, now with a treelike structure. In other words, the holes would have been closed up. The idea was then, through a change of variables on the Schwinger moduli space, to exhibit the integral as that of a closed string tree amplitude on AdS space.

In [11] we restricted ourselves to bilinear operators (such as $\mathrm{Tr}\Phi^2$). The n-point function of these operators is given by a one-loop diagram. For the case of two- and three-point functions, the equivalent tree diagrams are the expected ones. A simple change of variables on the Schwinger parameters converted the integral to a tree amplitude in AdS space. We further gave arguments for the four-point function that the resulting tree structure is again in line with expectations, though a detailed check was not carried out.

In the present paper, we will consider a much more general class of operators and their correlators, such as[3]

$$G^{\{J_i\}}(x_1,x_2,\dots,x_n)=\left\langle\prod_{i=1}^{n}\mathrm{Tr}\Phi^{J_i}(x_i)\right\rangle_{\mathrm{conn}}. \qquad (1.1)$$

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[1] The recent proposals [8,9] (see also [10]) that $\mathcal{N}=4$ Yang-Mills theory arises as the target space theory of a topological sigma model might, perhaps, enable one to view it in a manner close to the other topological examples.

[2] See [12–21] for various investigations of the free/weakly coupled theory with a view to understanding its stringy dual. Another approach starting from light cone field theory is that of Thorn and collaborators [22] as well as that of Karch and collaborators [23]. There is also a lot of literature on the connection between weakly coupled $\mathcal{N}=4$ Yang-Mills theory and integrable spin chains, since the work of [24].

[3] As in [11] we will be considering a U(N) Euclidean gauge field theory. We will again be dropping factors which are "inessential" in all equations.

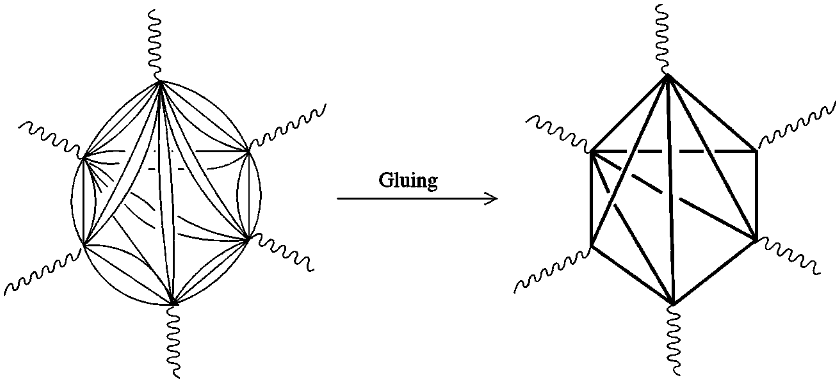

FIG. 1. Gluing up of a planar six-point function into a skeleton graph.

All possible free Wick contractions lead to a large class of diagrams contributing to such a correlator, even if one restricts to planar graphs. These diagrams have n vertices with J_i legs coming out of the ith vertex. We will argue, from the Schwinger parametrized expressions for such diagrams, that they exhibit a (partial) gluing up into a skeleton diagram (with n vertices) which captures the basic connectivity of the original graph. This is illustrated in Fig. 1.

This gluing can be intuitively understood from the electrical analogy since it essentially involves replacing the various parallel resistors (Schwinger parameters), between a pair of vertices, with a single effective resistor. Therefore, any particular contribution to the n-point function (1.1) can be expressed as an integral over a reduced Schwinger parameter space, namely that of the corresponding skeleton graph. The information about the J_i is captured through a specific dependence in the integrand.

However, planar graphs with different connectivities give rise to different skeleton diagrams. All these different skeleton diagram contributions need to be summed over to obtain the complete answer for Eq. (1.1). We will argue that this space of skeleton graphs is in one-to-one correspondence with the familiar cell decomposition of the moduli space $\mathcal{M}_{0,n}$ of a sphere with n holes. This basically follows from considering the graphs which are dual (in the graph theory sense) to the skeleton diagrams.

It is important to stress that this moduli space is *distinct* from that of the string diagrams underlying the original field theory Feynman diagrams. As is evident from the contributions to Eq. (1.1) shown in Fig. 1, these have a large number of loops (the number depending on J_i). In fact, even the skeleton graphs themselves have (generically) $2(n-2)$ faces, whereas the moduli space that we are associating with, namely all n-point correlators such as in Eq. (1.1), is that of a sphere with exactly n holes. Moreover, field theory correlators typically get their contribution from corners of string moduli space, whereas here it is the full moduli space $\mathcal{M}_{0,n}$ which contributes. Thus this stringy representation of field theory is different from that studied by Bern, Kosower [25], and others.

In fact, the emergence of the moduli space of a sphere with n holes is natural from the point of view of the gauge theory/geometry correspondence. The scenario one expects is that the loops of the original field theory planar diagram get glued up to form a surface and one has instead n closed string insertions. The n holes that we see here are to be identified with these closed string insertions. On integrating over the moduli corresponding to the size of these n holes, the holes should effectively pinch off giving rise to n external closed string insertions at punctures. This is indeed the picture that is realized in the topological string dualities of [1,2,4]. The situation is depicted in Fig. 2. We will see evidence that this is realized in our case, both by looking at the three-point function in detail as well as by studying the form of general stringy correlators in AdS space.

It is mainly for the sake of simplicity that we make our arguments for the n-point correlators of scalars. The Schwinger parametrizations for other n-point functions in free-field theory are very similar. In particular, the Feynman graphs get glued up, for exactly the same reasons, into the same skeleton diagrams. And hence replaying the arguments, we can conclude that other n-point correlators in the free theory can also be written in terms of an integral over $\mathcal{M}_{0,n}$.

Moreover, the argument is not restricted to planar diagrams alone. One can generalize to diagrams of arbitrary genus, which also get glued into skeleton diagrams. This time one makes a correspondence with the cell decomposition of the moduli space $\mathcal{M}_{g,n}$. Thus in all cases an n-point function leads to a Riemann surface with n holes. This leads us to believe that what we are seeing is a signature of the string dual of the free-field theory. In fact, an advantage of this procedure is that it is quite likely generalizable to the interacting theory as well.

Having written the field theory expression as an integral over parameters which cover $\mathcal{M}_{g,n}$, the main task then remains to see that the integrand corresponds to that of the appropriate string theory on AdS space. In fact, it is tempting to speculate that the theory on $\mathcal{M}_{g,n}$, which we are seeing here, defines *a consistent open string theory on a zero size AdS space*. The n holes give the contributions of boundary states in this open string theory. As in the picture of tachyon

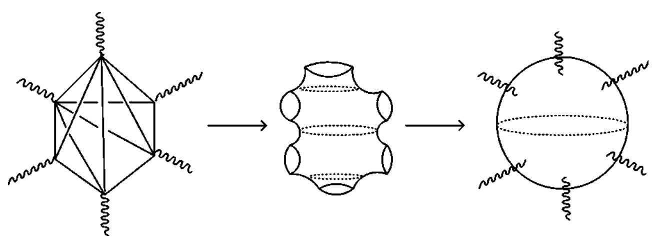

FIG. 2. Skeleton graphs→sphere with holes→sphere with punctures (as the holes go to ∞).

condensation (see, e.g., [26,27]) or the topological duality of [4],[4] integrating the boundary state over the size modulus of the hole would then give rise to a description in terms of closed string vertex operators in AdS space inserted at the n punctures. These issues are under investigation. At present, we will just make a few disparate remarks in support of the above scenario.

First, as we will see in Sec. IV, in the ''critical'' dimension $d=4$, the integrand can be written in a particularly nice form as far as its dependence on the quantum numbers J_i and external momenta k_i go. This form is at least not obviously inconsistent with string theory and in fact shares many structural features consistent with it, as we will also see in Sec. V B. An important check is that the integrand is continuous across the boundaries of the different components in the cell decomposition of the moduli space.

Second, the factorization of field theory correlators following from the spacetime OPE should translate into a factorization of the amplitudes in the closed string channel. It is plausible that the integrand in the Schwinger moduli space should reflect this factorization and thus provide one of the consistency checks for it to be a string amplitude.

Finally, we generalize (in Sec. V) the considerations of [11] for planar three-point functions of bilinears to that for the more general operators $\mathrm{Tr}\Phi^{J_i}$. We see in this case that the three Schwinger parameters that label $\mathcal{M}_{0,3}$ (corresponding to the sizes of the three holes) transmute into parameters for the external legs of AdS propagators. This happens via the same change of variables as in [11]. In a sense, integration over these moduli effectively puts the insertions at the boundary of AdS space. We will also look at the general form of string correlators of scalars in AdS space and argue that they can very naturally be cast in a form compatible with the field theory expressions obtained in Sec. IV. Therefore, together with the emergence of the stringy moduli space, this gives us confidence that we are implementing the expected picture of open-closed string duality in this approach.

There are other issues which we do not address directly in this work. For instance, understanding the role of supersymmetry, if any, will have to await a more detailed study of the properties of the integrand. In any case, our firm belief in the AdS/CFT conjecture tells us that, at least in this case, we are assured of the free-field theory having a closed string dual. But, in our arguments here, we do not really use any aspect of $\mathcal{N}=4$ super Yang-Mills theory. The procedure thus far is quite general. Another related issue is that of the spacetime dimension. The main conclusion of this paper about the emergence of the moduli space $\mathcal{M}_{g,n}$ is valid for any dimension d. But the integrand seems to be particularly nice when $d=4$ (as might be expected of field theories). A closer examination of the integrand should reveal more.

The paper is organized as follows. Section II displays the Schwinger parametrization of correlators such as Eq. (1.1) and exhibits their gluing up into skeleton graphs as in Fig. 1. Section III then makes the correspondence of the parameter space of all planar skeleton diagrams with the moduli space $\mathcal{M}_{0,n}$. It also sketches the generalizations to other correlators as well as to higher genus. Section IV makes some general remarks on the integrand in moduli space as given by field theory. Section V studies the three-point function in some detail. It also gives some evidence for the relation between the field theory integrand and the general n-point stringy correlator on AdS space. Appendix A gives the details associated with a change of variables in Sec. II.

II. SCHWINGER MODULI AND SKELETON GRAPHS

A. A review of the parametric representation

The Schwinger parametric representation of field theory is a well-studied subject. Essentially, one reexpresses the denominator of all propagators in a Feynman diagram via the identity (appropriate for Euclidean space correlators)

$$\frac{1}{p^2+m^2}=\int_0^\infty d\tau \exp\{-\tau(p^2+m^2)\}. \tag{2.1}$$

This has the advantage of converting all the momentum integrals into Gaussian integrals which are then easy to carry out. It is a little intricate to keep track of the details of the momentum flow. But the final expressions for an arbitrary Feynman diagram can be compactly written in graph theoretic terms. For the case of scalar fields, the expressions can be looked up in field theory textbooks such as [28] (Sec. 6-2-3). The expressions involving spinors and gauge fields are more involved. For a recent review containing the general expressions, see [29,30].

Since we will be mostly looking at massless scalar fields, let us consider the expression for an arbitrary Feynman diagram contributing to the momentum space version of Eq. (1.1). The result (in d dimensions) of carrying out the integral over the internal momenta is given in a (deceptively) compact form,[5]

$$G(k_1,k_2,\ldots,k_n)=\int_0^\infty \frac{\Pi_r d\tau_r}{\Delta(\tau)^{d/2}}\exp\{-P(\tau,k)\}. \tag{2.2}$$

Here the product over r goes over all the internal lines in the graph—there being one Schwinger parameter for each such line. The measure factor $\Delta(\tau)$ and the Gaussian exponent $P(\tau,k)$ are given by (see, e.g., [28,29])

$$\Delta(\tau)=\sum_{T_1}\left(\prod^{l}\tau\right). \tag{2.3}$$

$$P(\tau,k)=\Delta(\tau)^{-1}\sum_{T_2}\left(\prod^{l+1}\tau\right)\left(\sum k\right)^2. \tag{2.4}$$

[4]In fact, the authors of [4] speculate on the existence of such an open string theory on AdS space, in analogy with their example.

[5]Here we are suppressing the overall momentum-conserving δ function.

Here we are following the notation of [29]: The sum is over various 1-trees and 2-trees obtained from the original loop diagram. A 1-tree is obtained by cutting l lines of a diagram with l loops so as to make a connected tree, while a 2-tree is obtained by cutting $l+1$ lines of the loop so as to form two disjoint trees. Equation (2.3) indicates a sum over the set T_1 of all 1-trees, with the product over the l Schwinger parameters of all the cut lines. The sum over T_2 in Eq. (2.4) similarly indicates a sum over the set of all 2-trees, where the product is over the τ's of the $l+1$ cut lines. And (Σk) is understood to be the sum over all those external momenta k_i which flow into (either) one of the 2-trees. (Note that because of overall momentum conservation, it does not matter which set of external momenta one chooses.)

A simple illustration of these expressions is for the one-loop diagram with n insertions. There are n Schwinger parameters for each of the n arc segments of this loop. Cutting any of them leads to a 1-tree. Therefore,

$$\Delta(\tau)_{(l=1)}=\sum_{i=1}^{n}\tau_i .$$

Cutting any two distinct ones leads to two disjoint trees and

$$P(\tau,k)_{(l=1)}=\Delta(\tau)^{-1}_{(l=1)}\sum_{i<j}\tau_i\tau_j(k_{i+1}+\cdots k_j)^2,$$

where τ_i is the parameter for the arc joining the i and $(i+1)$th insertion. These expressions naturally agree with those obtained from the worldline formalism of Polyakov, Strassler, etc. [31–33]. In [11] we used these expressions to study the gluing up for bilinears in the free theory.

A beautiful feature of parametric representations is the correspondence with electrical networks, originally discovered in Bjorken's 1958 thesis (see Chap. 18 of [34]). If we identify the external (as well as internal) momenta with currents flowing in the network corresponding to the Feynman diagram, then the Schwinger parameters play the role of resistances. In fact, the Gaussian exponent, before carrying out the momentum integrals, has the interpretation as the power dissipated in the original circuit $(\Sigma_r I_r^2 R_r)$. The process of carrying out the integrals over internal or loop momenta is then equivalent to the standard procedure of elimination of internal currents using Kirchoff's laws. The resulting Gaussian in the external momenta, given in Eq. (2.4), then has the interpretation as the power dissipated in the equivalent circuit after elimination of the internal loops. This gives us a nice source of intuition for the process by which loops can get glued into trees. In [11] we exploited this to understand the gluing of the two-, three-, and four-point functions of bilinears into trees.

As we will now see, the correlators $\langle\Pi_{i=1}^{n}\mathrm{Tr}\Phi^{J_i}(x_i)\rangle_{\mathrm{conn}}$ will exhibit the gluing much more completely. In particular, considering these general correlators will allow us to see all the string moduli, something which was not possible with bilinears alone, for reasons that will become clear as we proceed.

B. Gluing into skeleton graphs

In the free theory, the correlators (1.1) are given by a sum over all possible connected Wick contractions. Let us start by considering the leading large-N contribution. They are given by planar diagrams such as those shown in Fig. 1.[6] We have as many legs coming out of the ith vertex as there are free fields inserted there, namely J_i. These planar diagrams are more easily visualized as spherical diagrams—drawn on a sphere. How do we organize the sum over all the different possible contributions?

First, for a planar graph with a given connectivity [i.e., the set of pairs of vertices (ij) which are linked by at least one contraction compatible with planarity], there can be a multiplicity m_r in the number of lines between each pair. In fact, one can convince oneself that a planar graph, with n vertices, that is maximally connected, has $3(n-2)$ inequivalent connections, where the rth connection is comprised of m_r lines. m_r is only constrained by the fact that there must be a total of J_i lines entering the ith vertex. These n constraints imply that there are $2(n-3)$ undetermined numbers among the m_r. For $n>3$, there is thus a lot of multiplicity for a given connectivity. Second, the above multiplicity was for a fixed connectivity, but it is clear that there are several inequivalent ways (for $n>4$) to connect the vertices themselves, consistent with planarity.

What we will show in this section is that the first set of contributions—from the multiplicity of lines—can all be bunched up in a natural way. For a given connectivity, at first it might seem that the parametric representation (2.2) implies very different contributions for graphs with differing m_r's, since we would have to introduce Schwinger parameters for each internal line. However, we will argue that each of these contributions can be written in terms of a reduced set of Schwinger parameters τ_r^{eff}, where r runs over the edges in the corresponding skeleton graph. This skeleton graph is what we term the graph that captures the connectivity of a given Feynman diagram.[7] In other words, we replace all the m_r lines in a connection by a single edge. In Fig. 1 we have illustrated this for our example. In other words, all contributions of a given connectivity are expressed in terms of an integral over parameters defined on the corresponding skel-

[6]In the figure, the maximal number of connections compatible with planarity have been drawn. Adding a line between two vertices that are not already directly connected will destroy planarity.

[7]*Caveat*: In order that the skeleton graph faithfully capture the color flow of the original diagram, we will only glue together adjacent strips of the underlying double line graph. Lines between the same pair of vertices, but which cannot be deformed into each other without crossing a line between a different pair, will *not* be glued together. Hence the skeleton graph could have *several* edges between a given pair of vertices. Each such edge comes with its own multiplicity. The simplest illustration of such instances is in the four-point function where one can have two contractions along one of the diagonals (on opposite sides of the sphere, so to say), while having none on the other diagonal. Note that such a graph also has six edges just like the tetrahedron, where all pairs of vertices are singly connected.

eton graph. The dependence on the multiplicities m_r is captured by the integrand in a fairly simple manner. The net result is that the skeleton graph and its moduli capture all the contributions of a given connectivity.

We will argue for this result from the explicit form of the parametrization in Eq. (2.2). However, experts might not need much convincing about the truth of this assertion. [They are welcome to skip the technicalities and go to Eq. (2.9).] In the Schwinger parameter representation (2.2), which we are working with, the result can be understood from the electrical network intuition. In this language, all we are doing is to replace all the parallel resistors joining vertices (ij) (subject to the caveat in footnote 7) by an effective resistance given by the usual expression for multiple parallel resistors. In that sense, we are partially gluing up the original Feynman diagram by bunching up various internal lines.

Let us now see how this is reflected in the actual expressions. We start with an n-vertex free-field diagram whose connectivity is specified by a skeleton graph having multiplicity m_r for the rth edge. We will label the Schwinger parameters for the internal lines by $\tau_{r\mu_r}$, where r indexes the edges of the skeleton graph $[r=1,...,3(n-2)]$ and μ_r their multiplicity $(\mu_r=1,...,m_r)$.

Our first claim relates the term $\Delta(\tau)$ of the original graph to that of the skeleton graph,

$$\Delta(\tau)=\frac{\Pi_{r,\mu_r}\tau_{r\mu_r}}{\Pi_r\tau_r^{\mathrm{eff}}}\tilde{\Delta}(\tau^{\mathrm{eff}}). \tag{2.5}$$

Here the effective Schwinger parameter is given by the formula for parallel resistors,

$$\frac{1}{\tau_r^{\mathrm{eff}}}=\sum_{\mu_r=1}^{m_r}\frac{1}{\tau_{r\mu_r}}, \tag{2.6}$$

while $\tilde{\Delta}(\tau^{\mathrm{eff}})$ is given by the same expression as Eq. (2.3) but now the sum over 1-trees is that of the skeleton graph with the effective parameters τ_r^{eff} for the edges. Our claim follows from the definition in Eq. (2.3). We are instructed to take the product of the parameters on the cut lines of the original graph. In the rth bunch, we are forced to cut either (m_r-1) or all m_r of the lines to get a 1-tree. Any fewer cut lines would leave a loop. If we were to cut all of them, then we would get a factor of $\Pi_{\mu_r}\tau_{r\mu_r}$ for that bunch and in the skeleton graph we would have thus cut the corresponding edge. If we were to cut m_r-1 of them, then we would get a factor $(\Pi_{\mu_r}\tau_{r\mu_r})/\tau_r^{\mathrm{eff}}$ corresponding to all the possible ways of cutting (m_r-1) lines in that bunch. In the skeleton graph we would be leaving the rth edge uncut. Now it is clear, from the relative factor of τ_r^{eff} between the two cases, that on summing over all possible 1-trees of the original graph we will end up with a sum over 1-trees of the skeleton graph, obtaining the relation in Eq. (2.5).

The next claim is that the Gaussian exponent in Eq. (2.4) of the original graph can be expressed entirely in terms of the skeleton graph with parameters τ_r^{eff},

$$P(\tau,k)=\tilde{P}(\tau^{\mathrm{eff}},k), \tag{2.7}$$

where $\tilde{P}(\tau^{\mathrm{eff}},k)$ is given by the same expression as in Eq. (2.4), but now for the skeleton graph with its effective Schwinger parameters for its edges. This follows from similar considerations as above. The term in Eq. (2.4) involving the sum over 2-trees is related by a factor of $(\Pi_{r,\mu_r}\tau_{r\mu_r})/\Pi_r\tau_r^{\mathrm{eff}}$ to the corresponding sum over 2-trees of the skeleton graph with τ_r^{eff} for its edges. The reasoning is completely analogous to that of the previous paragraph. Putting this together with the relation (2.5) between the factors of Δ and $\tilde{\Delta}$, we see that the factor of $(\Pi_{r,\mu_r}\tau_{r\mu_r})/\Pi_r\tau_r^{\mathrm{eff}}$ cancels out and we are left with the relation stated in Eq. (2.7).

Putting both these results together, we have, for a diagram of fixed multiplicity and connectivity, the contribution

$$\int_0^\infty\prod_{r,\mu_r}\frac{d\tau_{r\mu_r}}{\tau_{r\mu_r}^{d/2}}\frac{\Pi_r(\tau_r^{\mathrm{eff}})^{d/2}}{\tilde{\Delta}(\tau^{\mathrm{eff}})^{d/2}}\exp\{-\tilde{P}(\tau^{\mathrm{eff}},k)\}. \tag{2.8}$$

The final step is to convert this into an integral over the τ_r^{eff}. Since the nontrivial dependence in the integrand is all on the τ_r^{eff}, the dependence on the $\tau_{r\mu_r}$ can be factored out by a change of variables. The details are worked out in Appendix A. The end result is that the contribution (2.8) to the n-point function (1.1) from a graph with fixed connectivity and multiplicity can be written as

$$\int_0^\infty\prod_{r=1}^{3(n-2)}\frac{C^{(m_r)}d\tau_r}{\tau_r^{(m_r-1)[(d/2)-1]}}\frac{1}{\Delta(\tau)^{d/2}}\exp\{-P(\tau,k)\}. \tag{2.9}$$

Here $C^{(m_r)}$ is a constant, independent of the τ's but depending on m_r, obtained from the change of variables in Appendix A. It is explicitly given by

$$C^{(m_r)}=\int_0^1\prod_{\mu_r=1}^{m_r}dy_{\mu_r}y_{\mu_r}^{(d/2)-2}\delta\left(1-\sum_{\mu_r}y_{\mu_r}\right). \tag{2.10}$$

Note that in the interesting case of $d=4$, $C^{(m_r)}=1/(m_r-1)!$.

We have also dropped the superscript on the τ's as well as the tildes. Hopefully this will not create any confusion, since from now on only the effective Schwinger parameters will play a role. Furthermore, all quantities such as $\Delta(\tau)$ and $P(\tau,k)$ will refer to the skeleton graph.

Therefore, we can write the total planar contribution to the momentum space version of Eq. (1.1) in the form

$$G^{\{J_1\}}(k_1,k_2,\ldots,k_n) = \sum_{\text{skel graphs}} \sum_{\{m_r\}=1}^{\infty} \prod_{i=1}^{n} \delta_{\Sigma m_{r(i)},J_i} \prod_r C^{(m_r)} \times \int_0^{\infty} \prod_{r=1}^{3(n-2)} \frac{d\tau_r}{\tau_r^{(m_r-1)[(d/2)-1]}} \frac{1}{\Delta(\tau)^{d/2}} \exp\{-P(\tau,k)\}. \tag{2.11}$$

The sum is over various inequivalent planar skeleton graphs with n vertices. The sum over multiplicities is constrained by the fact that the net number of legs at the ith vertex is J_i. [$r(i)$ labels an edge which has the ith vertex as one of its end points.]

Thus we see that the planar n-point correlator can be written as an integral over the space of planar skeleton graphs. By this we mean that Eq. (2.11) includes both an integral over the length of the edges (as parametrized by the τ's) of a given skeleton graph as well as a sum over the different ways of joining the n vertices. In the next section, we will show that this space is the same as that of the moduli space of a sphere with n holes. We will also look at various generalizations.

III. FROM SKELETON GRAPHS TO STRING DIAGRAMS

A. Skeleton graphs and the cell decomposition of moduli space

To see the string theory emerge from the field theory, we need to have the space of string diagrams arise from the field theory Feynman graphs. By making a correspondence of the above space of planar skeleton graphs with $\mathcal{M}_{0,n}$ (and more generally $\mathcal{M}_{g,n}$), we will accomplish precisely that.

The correspondence is made by observing first that the space of n-vertex planar skeleton graphs, which we have been considering, is merely the space of all triangulations of the sphere with n vertices. When we say triangulations, we mean that the maximum number of edges, consistent with planarity, namely $3(n-2)$, arises when all the faces of the skeleton graph are triangles. If one of the faces of the discretized sphere were not a triangle, we could always add at least one extra edge without destroying planarity. In other words, the region of parameter space where quadrilaterals and other polygons appear in the faces is codimension 1 or higher in the parameter space. More precisely, quadrilaterals, etc. arise only when one or more of the τ_r go to ∞. That is because the corresponding edges are effectively removed since the resistance in those edges is going to ∞. In sum, associated uniquely, to every discretization of the sphere with n vertices there is a planar skeleton graph arising from a Feynman diagram and vice versa.

Now, to each such discretization of the sphere with n vertices we can uniquely associate a dual graph in the standard manner.[8] Namely, to each edge of the original graph we associate a dual edge which intersects the original one transversally. We will also associate a length $\sigma_r \equiv 1/\tau_r$ ("conductance") to this edge. The length of individual dual edges can then vary in an unconstrained manner from 0 to ∞ as we vary τ_r. In this way, every face of the original graph gives rise to a vertex for the dual and vice versa. The dual graph is thus constrained to have n faces. And corresponding to the triangular faces are now trivalent vertices. But the topology remains that of a sphere. Therefore, as we sum over inequivalent skeleton diagrams, we carry out a sum over the space of dual graphs,[9] that is, over all discretizations of the sphere with n faces formed from graphs with cubic vertices. As mentioned earlier, the lengths σ_r of the edges of the dual graph vary from 0 to ∞.

This can immediately be recognized as the picture of string interactions in Witten's open string field theory [35]. Open string field theory generates string diagrams described by strips of fixed width but varying lengths σ, meeting at cubic vertices. In fact, as shown first by [36] and argued later in full generality by [37], such diagrams of arbitrary genus, with some number of boundaries as well as punctures, precisely generate a single cover of the corresponding moduli space of Riemann surfaces with boundaries and punctures. This "cell (or simplicial) decomposition" of the moduli space was also worked out independently by mathematicians [38].[10] Thus the sum over inequivalent skeleton graphs is the sum over different cells in this decomposition of the moduli space.

An important aspect of the cell decomposition is the way different components in this decomposition of the moduli space connect to each other across boundaries of these cells.[11] It can be verified that the mapping to dual graphs preserves this behavior. For example, in the case of the four-point function, consider an original skeleton graph in the shape of a tetrahedron with all six legs of nonzero length. One can go to a codimension-1 boundary of the cell where the length σ of one of the dual edges goes to zero. This corresponds in the original graph to removing an edge and getting a quadrilateral face. From this boundary one can move to a component in which the edge opposite to it (i.e., having no vertex in common with it) develops a second strand but now traversing the opposite side of the sphere (see footnote 7). Mapping this onto the dual graphs, one exactly

[8]We would like to thank S. Wadia for a helpful remark about the relation between graph duality and open-closed string duality.

[9]The field theory correlators in Eq. (1.1) are usually taken to be those of normal ordered operators. In such a case there are no self-contraction diagrams. In the correspondence to dual graphs, self-contractions lead to tadpole subgraphs. Presumably there exists a redefinition on the AdS side which corresponds to the normal ordering prescription on the field theory side. This would then take care of the tadpole diagram contributions in the cell decomposition of the moduli space.

[10]We would like to thank P. Windey and S. Govindarajan for suggesting early on a possible connection between the approach of [11] and the work of Penner [38]. For a nice introduction to Penner's work, see the recent article [39].

[11]We would like to thank A. Sen for helpful discussions on this point.

gets the matching up of the different codimension-1 components of the cell decomposition of $\mathcal{M}_{0,4}$. Similarly, one can go to codimension-2 boundaries of this codimension-1 cell and see that they also patch together smoothly. In all cases, graph duality faithfully implements the required behavior.

Thus, we can conclude that the space of planar skeleton graphs with n vertices is isomorphic to the moduli space $\mathcal{M}_{0,n}$ of a sphere with n boundaries (faces). Seen in this light, the lengths of the $3(n-2)=n+2(n-3)$ edges of the original graph (and thus of the dual graph) correspond to the number of moduli of a sphere with n holes. In conformal field theory language, one associates n of these to the radii of the holes and $2(n-3)$ to the positions of centers. As described in the Introduction and illustrated in Fig. 2, the appearance of $\mathcal{M}_{0,n}$ is what one might expect from open-closed duality.

Here we should make a remark regarding the J_i that appear in Eq. (1.1). For an n-point function, unless the J_i are greater than a minimum value (set by n), the Feynman graph will not have all the possible contractions. In other words, the corresponding skeleton graph will not have the maximal number of edges, i.e., $3(n-2)$. One concludes that such an amplitude gets its contribution from a lower-dimensional component of the cells of $\mathcal{M}_{0,n}$. In particular, we see that the bilinear operators do not get contributions from the whole of the string moduli space. For example, the Feynman graph for the four-point function of bilinears has only four edges. Thus it gets its support only from a codimension-2 slice of $\mathcal{M}_{0,4}$.

B. Generalizations

We should also remark that the argument of the present section only relied on the existence of skeleton graphs. The procedure by which the skeleton graphs themselves arose from the underlying field theory diagrams also appears to generalize to operators other than the scalars $\mathrm{Tr}\Phi^J$. The parametric representation for diagrams involving more general operators only differs in having additional (momentum- and spin-dependent) polynomial prefactors multiplying the same Gaussian factor $P(\tau,k)$ of Eq. (2.4). General expressions for these prefactors are given in [29,30] [see, in particular, Eqs. (11)–(15) of [29]]. When one takes into account the fermions, gauge fields, and global quantum numbers that a theory like $\mathcal{N}=4$ Yang-Mills possesses, the explicit expressions for general operators become quite cumbersome. However, an examination of the general parametric form in [29] reveals that the gluing arguments of Sec. II generalize for such diagrams as well. In fact, this is to be expected from the correspondence with electrical networks, which holds very generally. The only difference is that the information about the spins and field content of operators now modifies the first term in Eq. (2.9). Therefore, it appears that general (planar) n-point correlators in free-field theory can also be expressed as integrals over $\mathcal{M}_{0,n}$.

Again, the restriction to planar graphs was also not very essential to the whole argument. The gluing into skeleton graphs makes no reference to the underlying genus. It is important, however, that the gluing be carried out compatible with the color flow as outlined in footnote 7. Graphs corresponding to higher genus Feynman diagrams are then glued up into skeleton graphs which are discretizations of Riemann surfaces with more handles. Similarly, the mapping to dual graphs gives rise to string diagrams that cover the moduli space $\mathcal{M}_{g,n}$. As we remarked earlier, the cell decomposition of [36–38] holds for any genus g Riemann surfaces with n holes.

Finally, we should also remark that once we have completely understood the free-field theory (at least in the case of $\mathcal{N}=4$ Yang-Mills theory) as a string theory, we can hope to generalize our approach to the interacting theory. At least, order by order in perturbation theory in the Yang-Mills coupling, the effect of the coupling is through insertions of additional operators in correlators. Since the parametric representation is applicable to the corresponding Feynman diagrams of the interacting theory, we can write it again in terms of an integral over a string moduli space, but now with additional holes for the coupling constant insertions. It should then be possible to view these additional insertions as changing, for instance, the radius of the AdS space. In this way, this procedure may be useful in tackling the AdS/CFT conjecture beyond the free limit as well.

It is satisfying that our arguments are not too tied up either with the specifics of the correlators or of the planar limit (or even too much with the free limit). It suggests a universality that behooves the phenomenon of field theory/string theory duality. Also, the fact that the spacetime dimension does not play a crucial role at this level is also not such a bad thing. It is a feature which we expect will mostly affect the integrand over moduli space. The integrand holds the key to the real dynamics of the string theory which we see emerging from the field theory. In the next two sections, we will make some preliminary attempts at the integrand, leaving a detailed study for later.

IV. REMARKS ON THE INTEGRAND

The primary result of this paper (specializing to the concrete example of scalars) is that we can rewrite field theory correlators, schematically, as

$$G^{\{J_i\}}(k_1,k_2,\ldots,k_n) = \int_{\mathcal{M}_{g,n}} [d\sigma]\rho^{\{J_i\}}(\sigma)\exp\left(-\sum_{i,j=1}^{n} g_{ij}(\sigma)k_i\cdot k_j\right). \tag{4.1}$$

Here we are denoting the coordinates on the moduli space $\mathcal{M}_{g,n}$ collectively by σ. Recall that $\sigma_i=1/\tau_i$ were the natural coordinates in the cell decomposition of $\mathcal{M}_{g,n}$. $\rho^{\{J_i\}}(\sigma)$ is the momentum-independent prefactor which captures the dependence on the J_i, whereas $g_{ij}(\sigma)$ in the exponent is independent of the J_i. We can write down $\rho^{\{J_i\}}(\sigma)$ and $g_{ij}(\sigma)$ in each cell of the moduli space from the expressions at the end of Sec. II.

Thus, for instance in the interesting case of $d=4$, in a particular cell labeled by a given skeleton graph, we can rewrite the contribution in Eq. (2.11) in several equivalent ways,

$$G^{\{J_i\}}_{\rm cell}=\sum_{\{m_r\}=1}^{\infty}\prod_{i=1}^{n}\delta_{\Sigma m_{r(i)},J_i}\int_0^\infty\prod_r\frac{d\sigma_r\sigma_r^{m_r-1}}{(m_r-1)!}\frac{1}{\hat{\Delta}(\sigma)^2}\times\exp\{-\hat{P}(\sigma,k)\}$$
$$=\sum_{\{m_r\}=1}^{\infty}\int_0^\infty\prod_r\frac{d\sigma_r\sigma_r^{m_r-1}}{(m_r-1)!}\int_0^{2\pi}\prod_{i=1}^{n}d\theta_i e^{i\theta_i(\Sigma m_{r(i)}-J_i)}\times\frac{1}{\hat{\Delta}(\sigma)^2}\exp\{-\hat{P}(\sigma,k)\}$$
$$=\int_0^\infty\prod_r d\sigma_r\frac{\exp\{-\hat{P}(\sigma,k)\}}{\hat{\Delta}(\sigma)^2}\int_0^{2\pi}\prod_{i=1}^{n}d\theta_i\times\exp\left(\sum_{r(ij)}\sigma_{r(ij)}e^{i(\theta_i+\theta_j)}\right)e^{-i\Sigma_{i=1}^{n}\theta_i(J_i-N_i)}. \tag{4.2}$$

To obtain the first line, we have changed variables in Eq. (2.11) to $\sigma_r=1/\tau_r$ and reexpressed both $\Delta(\tau)$ and $P(\tau,k)$ in terms of the σ's. In the process, we have defined

$$\hat{\Delta}(\sigma)=\sum_{T_1}\left(\prod\ \sigma\right)=\left(\prod_r\ \sigma_r\right)\Delta(\tau=1/\sigma) \tag{4.3}$$

and

$$\hat{P}(\sigma,k)\equiv\frac{1}{\hat{\Delta}(\sigma)}\sum_{T_2}\left(\prod\ \sigma\right)\left(\sum\ k\right)^2=P(\tau=1/\sigma,k). \tag{4.4}$$

The sum, as before, is over the 1-trees and 2-trees of the skeleton graph but the product in both these definitions is over the lines that are *not* cut.

In the second line of Eq. (4.2) we introduced a Lagrange multiplier for the constraints on the multiplicities. This enables us to carry out the sum over multiplicities in an unconstrained way and obtain the third line. Here $r(ij)$ is an edge that joins vertices i and j; N_i is the number of legs joining at the ith vertex of the skeleton graph. In this last line the cell contribution is clearly in the form (4.1). From Eq. (4.2) it is also not difficult to verify that the integrand is continuous across boundaries of the cells (where at least one of the $\sigma\rightarrow 0$). This is crucial if one wants to interpret the integrand as that of a string theory.

We also notice that the schematic form (4.1) is similar in structure to the expressions for string amplitudes that one is familiar with, such as in flat space. Namely, a prefactor contains the information about the masses/dimensions (and more generally spins), while the Gaussian factor is independent of these details and captures the (worldsheet) correlators of the vertex operators $e^{ikX(\xi)}$. We will see in the next section that one can plausibly argue that this is also the structure one would expect from stringy correlation functions in AdS space.

An important feature of string amplitudes is their factorizability in different channels. This holds at the level of the integrand on moduli space, since it is a consequence of the worldsheet OPE and its associativity. Now, in the AdS/CFT conjecture, the factorizability of AdS amplitudes is reflected in the spacetime OPE relations for the corresponding correlation functions. Associativity of the OPE means that we can factorize it in different channels yielding the same answer. We believe that the above Schwinger parametric representation should reflect the spacetime OPE of the field theory and hence translate into a factorizability of the integrand in the closed string channel. It should be very possible to make this statement precise.

Ultimately, one wants to also demonstrate that the integrand is specifically that of an appropriate string theory on AdS space. We expect that the details of the string theory will depend on the matter content of the field theory. However, any string theory that is dual to a free (and thus conformal) gauge theory should have a background which contains at least an AdS part.

In [11] we pointed out that the appearance of AdS_{d+1} from a free theory in d dimensions could naturally take place in the Schwinger representation that we have been employing. Essentially, propagators in AdS_{d+1} can be parametrically expressed in terms of d-dimensional proper time propagators for free fields. We used this fact, together with the geometric gluing into trees, to argue that the two- and three-point functions of bilinears can be rewritten as tree amplitudes on AdS space. This was accomplished by a simple change of variables on the Schwinger parameter space.

In Sec. V we generalize this to the planar three-point function of $\mathrm{Tr}\Phi^{J_i}$. What will be clear from the details of that calculation is that (as in [11]) the three Schwinger moduli transmute into parameters for the propagators on the external legs of the AdS amplitude. Integrating over these parameters is integrating over the size of the holes of $\mathcal{M}_{0,3}$. It effectively gives rise to punctures in that one gets bulk-to-boundary AdS propagators as a result. This is in line with the intuition, mentioned in the Introduction, of holes closing up as one integrates over their size modulus. Together with the appearance of the string moduli space, this gives us confidence that we are indeed seeing the AdS space emerge from the field theory.

From the form of correlators in AdS space (discussed in Sec. V) we expect this to continue to happen for the n-point function. Namely, one can isolate n size moduli out of the $6g+3(n-2)$ moduli. And these will simply parametrize the n external legs of the corresponding AdS amplitude. The rest of the integral over the moduli space would then give a closed string n-point amplitude on AdS space.

V. THE THREE-POINT FUNCTION AND AdS CORRELATORS

A. From delta to star

We will consider the $n=3$ case of Eq. (1.1) (in the planar sector),

$$G^{\{J_i\}}(k_1,k_2,k_3)=\langle\mathrm{Tr}\Phi^{J_1}(k_1)\mathrm{Tr}\Phi^{J_2}(k_2)\mathrm{Tr}\Phi^{J_3}(k_3)\rangle_{\rm conn}. \tag{5.1}$$

The analysis is a generalization of that in [11] but will be done in a somewhat different way to make things clearer.[12]

The first thing to note, in this case, is that the number of legs m_r in the rth edge is determined completely by the J_i. In fact, we have three equations (from the three vertices),

$$m_{12}+m_{13}=J_1 \tag{5.2}$$

and cyclic permutations of it. Here we are labeling the edges r by the pair of vertices they connect. Equations (5.2) determine the m_{ij} to be

$$m_{12}=\frac{1}{2}\sum_{i=1}^{3} J_k-J_3 \tag{5.3}$$

and cyclic permutations. Thus there is a unique graph contributing to Eq. (5.1) with a fixed number of legs between each pair of vertices. We do not have to carry out any sum over multiplicities.

Now, by the arguments of Sec. II, this graph can be glued up into a skeleton graph, which is just a triangle in this case. And the expression for the amplitude in terms of the effective Schwinger parameters is given by Eq. (2.9). [Since the skeleton graph is unique, up to reflection, this is the same as Eq. (2.11).] Actually, as in Eq. (4.2), we will work with the natural conductance variables $\sigma_r=1/\tau_r$ and rewrite Eq. (2.9), using Eqs. (4.3) and (4.4), as

$$G^{\{J_i\}}(k_1,k_2,k_3)=\int_0^\infty \prod_{r=1}^{3} d\sigma_r \sigma_r^{(m_r-1)[(d/2)-1]+(d/2)-2} \times\frac{1}{\hat{\Delta}(\sigma)^{d/2}}\exp\{-\hat{P}(\sigma,k)\}. \tag{5.4}$$

Here we have relabeled the edges so that $\sigma_1\equiv\sigma_{23}$, etc. and dropped the overall factors of $C^{(m_r)}$. Also, using the expressions (4.3) and (4.4) we have

$$\hat{\Delta}(\sigma)=\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1 \tag{5.5}$$

and

$$\hat{P}(\sigma,k)=\frac{1}{\hat{\Delta}(\sigma)}[\sigma_1 k_1^2+\sigma_2 k_2^2+\sigma_3 k_3^2]. \tag{5.6}$$

We will now reexpress this in terms of new moduli, more appropriate for the tree,

$$\frac{1}{\rho_i}=\frac{\sigma_i}{\hat{\Delta}(\sigma)}\Rightarrow\sigma_i=\frac{\rho_1\rho_2\rho_3}{(\Sigma_k\rho_k)\rho_i}. \tag{5.7}$$

This change of variables is motivated by the star-δ transformation of electrical networks. Namely, if σ_i are the conductances of a δ or triangle network, such as the one we have, then ρ_i are the conductances of the equivalent three-pronged tree or star network (see [44] for example). It can be checked that the Jacobian for this transformation is given by

$$\det\left(\frac{\partial\sigma_i}{\partial\rho_j}\right)=\frac{\rho_1\rho_2\rho_3}{(\Sigma_k\rho_k)^3}. \tag{5.8}$$

We also see that

$$\hat{\Delta}(\sigma)=\frac{\rho_1\rho_2\rho_3}{(\Sigma_k\rho_k)},\quad \hat{P}(\sigma,k)=\sum_{i=1}^{3}\frac{k_i^2}{\rho_i}. \tag{5.9}$$

We can now rewrite the integral in Eq. (5.4), after gathering together various terms,

$$\begin{aligned} G^{\{J_i\}}(k_1,k_2,k_3) &=\int_0^\infty \prod_{i=1}^{3} d\rho_i \rho_i^{(\Sigma_k m_k-m_i)[(d/2)-1]-(d/2)-1} \\ &\quad\times\frac{1}{(\Sigma_k\rho_k)^{\Sigma_k m_k[(d/2)-1]-(d/2)}}e^{-[\Sigma_{i=1}^3(k_i^2/\rho_i)]} \\ &=\int_0^\infty \prod_{i=1}^{3} d\rho_i \rho_i^{\Delta_i-(d/2)-1} \\ &\quad\times\frac{1}{(\Sigma_k\rho_k)^{\Sigma_k(\Delta_k/2)-(d/2)}}e^{-[\Sigma_{i=1}^3(k_i^2/\rho_i)]}. \end{aligned} \tag{5.10}$$

In the second line, we have used Eqs. (5.2) and (5.3) as well as the fact that the operators $\mathrm{Tr}\Phi^{J_i}$ have canonical dimensions $\Delta_i=J_i[(d/2)-1]$ in the free theory.

This last line is close to what one might expect from a string theory on AdS space, as we will shortly see. In any case, it is a short step now to write Eq. (5.10) in terms of the expected bulk-to-boundary propagators in AdS space,

$$G^{\{J_i\}}(k_1,k_2,k_3)=\int_0^\infty\frac{dt}{t^{(d/2)+1}}\int_0^\infty\prod_{i=1}^{3} d\rho_i \times\rho_i^{\Delta_i-(d/2)-1}t^{\Delta_i/2}e^{-t\rho_i}e^{-k_i^2/\rho_i}. \tag{5.11}$$

Here we have used the identity

$$\frac{1}{a^s}=\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}e^{-at}, \tag{5.12}$$

to rewrite the denominator term in Eq. (5.10).

Either in this form or after a Fourier transform to position space, we can recognize this to be the product of three bulk-to-boundary propagators in AdS_{d+1} for the appropriate scalar fields. Thus, for instance in position space (taking into account the overall momentum-conserving δ function), we can write Eq. (5.11) as

[12]See [40–43], etc. for studies of three-point functions of such scalars (chiral primary operators in the $\mathcal{N}=4$ theory) in the context of AdS/CFT.

$$G^{\{J_i\}}(x_1,x_2,x_3)$$
$$=\int_0^\infty \frac{dt}{t^{(d/2)+1}}\int d^dz\int_0^\infty \prod_{i=1}^{3} d\rho_i\rho_i^{\Delta_i-1}t^{\Delta_i/2}e^{-\rho_i[t+(x_i-z)^2]}$$
$$=\int_0^\infty \frac{dt}{t^{(d/2)+1}}\int d^dz\prod_{i=1}^{3} K_{\Delta_i}(x_i,z;t), \tag{5.13}$$

where

$$K_\Delta(x,z;t)=\frac{t^{\Delta/2}}{[t+(x-z)^2]^\Delta} \tag{5.14}$$

is the usual position space bulk-to-boundary propagator for a scalar field corresponding to an operator of dimension Δ. The only slight difference is that we have parametrized the AdS radial coordinate by $z_0^2=t$ as in [11].

What we have thus seen here is that the integral over the moduli space $\mathcal{M}_{0,3}$, which the parametric representation of field theory provided us, is really an integral over AdS space. The original Schwinger parameters σ_i can be traded for the ρ_i which parametrize the propagators for the external legs of the AdS correlator. Integrating over the ρ_i, which correspond to the size of the holes, propagates the AdS scalar field all the way from infinity (the boundary). This corresponds very much to the picture in the Introduction of the holes being replaced by punctures. We will see below how this is likely to be more general than that for the three-point function.

B. Vertex operators in AdS space

We can also understand how Eq. (5.4) or equivalently Eq. (5.10) could arise from a vertex operator calculation in AdS space. Though we do not have a good handle yet on the string theory, we can guess that the n-point correlators are given in terms of vertex operator computations in the worldsheet (WS) theory for AdS space. Thus for scalars we would guess, following [15,45],

$$G^{\{J_i\}}(x_1,\ldots,x_n)=\left\langle \prod_{i=1}^{n} K_{\Delta_i}(x_i,X(\xi_i);t(\xi_i))\right\rangle_{\mathrm{WS}}$$
$$=\left\langle \prod_{i=1}^{n} \frac{t(\xi_i)^{\Delta_i/2}}{\{t(\xi_i)+[x_i-X(\xi_i)]^2\}^{\Delta_i}}\right\rangle_{\mathrm{WS}}. \tag{5.15}$$

Here $X(\xi),t(\xi)$ denote worldsheet fields for the AdS target space. The averaging, as the subscript indicates, is over the worldsheet action for these and other fields (including ghosts). An integral over the moduli space of the Riemann surface with n punctures is also implicit. We can write Eq. (5.15) in the parametric form

$$G^{\{J_i\}}(x_1,\ldots,x_n)$$
$$=\int_0^\infty \prod_{i=1}^{n} d\rho_i\rho_i^{\Delta_i-1}\langle t(\xi_i)^{\Delta_i/2}e^{-t(\xi_i)\rho_i-\rho_i[x_i-X(\xi_i)]^2}\rangle_{\mathrm{WS}}. \tag{5.16}$$

To make a connection with the field theory expressions we go to momentum space, where Eq. (5.16) becomes

$$G^{\{J_i\}}(k_1,\ldots,k_n)=\int_0^\infty \prod_{i=1}^{n} d\rho_i\rho_i^{\Delta_i-(d/2)-1}$$
$$\times e^{-k_i^2/\rho_i}\langle t(\xi_i)^{\Delta_i/2}e^{-t(\xi_i)\rho_i}e^{ik_i\cdot X(\xi_i)}\rangle_{\mathrm{WS}}. \tag{5.17}$$

We believe Eq. (5.17) is the right starting point for a comparison of the (scalar) n-point function in AdS space with the field theory expressions (2.11), etc. But we can already see over here many of the features that we expect. There are n parameters ρ_i which can be identified with the size moduli of holes, as we argued at the end of the last subsection. Then there are the usual $(6g+2n-6)$ moduli for the n-point function. As in the case of the three-point function, we need to find the appropriate change of variables to go from these parameters to the $(6g+3n-6)$ σ_i of the field theory. But it is clear that Eq. (5.17) fits in with the general schematic form of Eq. (4.1).

In the particular case of the three-point function that we studied above, since Eq. (5.17) should be independent of the ξ_i $(i=1,...,3)$ from conformal invariance, it is plausible that only the zero mode of the fields $t(\xi),X(\xi)$ effectively contributes in the worldsheet path integral (after including the contribution of appropriate ghost insertions). The zero mode for t gives the corresponding integral in Eq. (5.11), and that for X just gives the overall momentum-conserving δ function. Thus it is not surprising from this point of view that we could relate the field theory three-point function to the point-particle amplitude (5.13)—only the zero modes contribute. It also suggests that we will really see the stringy structure in the four- and higher-point functions.

Going by the arguments presented in this paper, the field theory expressions such as Eq. (2.11) or Eq. (4.2) are just Eq. (5.17) written in different variables. So we can use this to turn things around and write down the AdS correlators from the field theory (certainly in the case of $\mathcal{N}=4$ super Yang-Mills theory). We would then have reconstructed the string theory on AdS space via all its correlators.

Anyhow, the task now is obviously to make various of these surmises precise and in the process learn about the worldsheet theory for AdS space. In some sense we are in a situation very similar to that in the early days of dual theory when people reconstructed the string picture from the form of the Veneziano-Koba-Nielsen and Virasoro-Shapiro amplitudes.

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APPENDIX A: A CHANGE OF VARIABLES

Here we will see how to effect the change of variables of integration from the $\tau_{r\mu_r}$ in Eq. (2.8) to the effective Schwinger parameters τ_r in Eq. (2.9). First, the relation between τ_r and $\tau_{r\mu_r}$ is given by Eq. (2.6). This can be implemented by inserting into the integral (2.8) the identity

$$\int_0^\infty \frac{d\tau_r}{\tau_r^2}\,\delta\left(\frac{1}{\tau_r}-\sum_{\mu_r=1}^{m_r}\frac{1}{\tau_{r\mu_r}}\right)=1. \tag{A1}$$

The nontrivial dependence on $\tau_{r\mu_r}$ in Eq. (2.8) comes from the first term. So using the above identity, we can write such a contribution as

$$\int_0^\infty \frac{d\tau_r}{\tau_r^2}\int_0^\infty \prod_{\mu_r}\frac{d\tau_{r\mu_r}}{\tau_{r\mu_r}^{d/2}}\,\delta\left(\frac{1}{\tau_r}-\sum_{\mu_r=1}^{m_r}\frac{1}{\tau_{r\mu_r}}\right). \tag{A2}$$

Now define $x_{r\mu_r}=\tau_{r\mu_r}/\tau_r$ and change variables from $\tau_{r\mu_r}$ to $x_{r\mu_r}$. Then Eq. (A2) reads

$$\int_0^\infty \frac{d\tau_r}{\tau_r^{m_r[(d/2)-1]+2}}\int_1^\infty \prod_{\mu_r}\frac{dx_{r\mu_r}}{x_{r\mu_r}^{d/2}}\,\delta\left[\frac{1}{\tau_r}\left(1-\sum_{\mu_r=1}^{m_r}\frac{1}{x_{r\mu_r}}\right)\right]$$
$$=\int_0^\infty \frac{d\tau_r}{\tau_r^{m_r[(d/2)-1]+1}}\int_1^\infty \prod_{\mu_r}\frac{dx_{r\mu_r}}{x_{r\mu_r}^{d/2}}\,\delta\left(1-\sum_{\mu_r=1}^{m_r}\frac{1}{x_{r\mu_r}}\right). \tag{A3}$$

Thus we have factored the integral over $\tau_{r\mu_r}$ into an integral over τ_r times a factor $C^{(m_r)}$ which depends only on m_r, where

$$C^{(m_r)}=\int_1^\infty \prod_{\mu_r}\frac{dx_{r\mu_r}}{x_{r\mu_r}^{d/2}}\,\delta\left(1-\sum_{\mu_r=1}^{m_r}\frac{1}{x_{r\mu_r}}\right)$$
$$=\int_0^1 \prod_{\mu_r=1}^{m_r} dy_{r\mu_r}\,y_{r\mu_r}^{(d/2)-2}\,\delta\left(1-\sum_{\mu_r=1}^{m_r} y_{r\mu_r}\right). \tag{A4}$$

In the second line, we have made the substitution $y_{r\mu_r}=1/x_{r\mu_r}$. In this form, we can do the integral explicitly for general d. But the case of $d=4$ is particularly simple. The δ function over one of the $y_{r\mu_r}$ can be carried out and we are left with an integral over the others (m_r-1) over the region where their sum is less than 1. This is just $1/(m_r-1)!$. In general dimensions, the answer is[13]

$$C^{(m_r)}=\frac{\Gamma\left(\frac{d}{2}-1\right)^{m_r}}{\Gamma\left[m_r\left(\frac{d}{2}-1\right)\right]}. \tag{A5}$$

[13] We would like to thank E. Schreiber for providing us with this expression.

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