From free fields to AdS space. I

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Free $\mathcal{N}=4$ super Yang-Mills theory (in the large-N limit) is dual to an, as yet, intractable closed string theory on $AdS_5 \times S^5$. We aim to implement open-closed string duality in this system and thereby recast the free-field correlation functions as amplitudes in AdS space. The basic strategy is to implement this duality directly on planar field theory correlation functions in the worldline (or first quantized) formulation. The worldline loops (remnants of the worldsheet holes) close to form tree diagrams. These tree diagrams are then to be manifested as tree amplitudes in AdS space by a change of variables on the worldline moduli space (i.e., Schwinger parameter space). Restricting to twist-2 operators, we are able to carry through this program for two- and three-point functions. However, it appears that this strategy can be implemented for four- and higher-point functions as well. An analogy to electrical networks is very useful in this regard.

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I. INTRODUCTION

Over the past few years we have grown used to the idea of large-N gauge theories having a dual description in terms of gravitational theories in higher dimensions [1–3]. However, we need to remind ourselves that getting used to an idea is not the same as understanding it. It is fair to say that we do not really understand why or how (some) field theories reorganize themselves into a higher-dimensional gravitational description.

Open-closed string duality, we believe, is the underlying mechanism that drives these dualities. But except in the context of topological string dualities [4,5], we do not explicitly understand how the holes in an open string description close up to form closed string worldsheets. It is clearly important to understand the nuts and bolts of this mechanism better if we hope to shed further light on the miracles of large-N dualities.

A good idea is to begin with the simplest examples. From the field theory point of view, a free theory is as simple as it gets. In particular, $\mathcal{N}=4$ super Yang-Mills theory at zero coupling is believed to be dual to string theory on a highly curved AdS space (zero radius in string units) [6]. It is a measure of our lack of understanding of large-N dualities that we know so little even in this seemingly tractable limit. Interesting attempts to understand the closed string sigma model, in this limit, have not yet yielded fruit [7–9].

Therefore, as an alternative strategy, we might try to start from the free-field theory, which is completely under control, and try to reconstruct the closed string theory, using as our guide the underlying open-closed string equivalence.

A. Open-closed string duality

Let us take this opportunity to elaborate a bit on our viewpoint on the general open-closed string equivalence. The leading large-N field theory correlation functions (planar diagrams with some number of loops) arise from planar (no handles) open string diagrams with some number of vertex insertions on its boundaries. Viewing this diagram in the closed string channel corresponds to gluing up the holes (while keeping the vertex insertions at finite separation). This is then interpreted as a closed string diagram[1] with the same number of closed string vertex insertions, but with the gluing process having modified the background. We will actually make a stronger working assumption which seems to be indicated by our analysis. We assume that this open-closed string equivalence operates at the level of the worldsheet moduli space. An open string surface with particular locations of insertions and shape gets associated with a particular closed string surface.[2] In other words, the gluing of the open string into the closed string is to be implemented on the *integrand* in moduli space. A change of variables on the moduli space would then show this to be a closed string amplitude.

Actually, this is probably too general a picture to be usefully implemented. What we will exploit is the simplification coming from the fact that we are working in the field theory limit. Since the $\mathcal{N}=4$ Yang-Mills theory is obtained as an $\alpha' \to 0$ limit of open string theory, one should really view the planar open string worldsheets as reducing to planar worldlines. There is a precise sense in which this happens. The worldsheet moduli space integral reduces to a worldline moduli space integral, more familiar as a Schwinger (or

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[1]To visualize the geometry of the gluing, think of the open string surface as a rubber sheet pinned at the locations of the vertex insertions. We can then imagine bringing together the boundaries of the rubber sheet (keeping the locations of the pins intact) and gluing them so as to obtain a genus zero surface with punctures at the locations of the pins.

[2]Of course, the usual counting of moduli for the open and closed string gives different dimensions. In our case, the matching of moduli does not appear to be straightforward, especially since, as we will see, some of the open string moduli turn into parameters of the additional dimension. Presumably, this complication is related to the fact that we do not have a CFT description for the closed string on AdS space. It is important to understand this better.

Feynman) parameter space integral for the Feynman diagrams (see, for instance, [10–12]). The integrand reduces to a correlation function of worldline vertex operators in a first quantized formalism [13,14]. The particular simplification of the free-field theory limit is that we have a Gaussian free-particle action in this first quantized language.

So what we will aim to implement is the process of gluing up of planar worldline loops. It may seem puzzling at first that we have any kind of open-closed equivalence when the worldsheets degenerate to lines. In fact, what we will see for the simplest class of field theory correlation functions is that the glued up version of the worldline is also a degenerate genus zero surface—namely, with the topology of a tree. The change of variables on the moduli space, mentioned above, is the one natural to the description of the tree. The new variables will have the interpretation as Schwinger parameters for the tree amplitude in AdS space. Thus, even though the theory might be expected to be very stringy, we find particle-like amplitudes at least for a class of states in AdS space. It appears that the contributions for these amplitudes seem to come from degenerate Riemann surfaces. We will comment on this further on. We will also make some speculations in the concluding section on the appearance of ‘‘fat’’ closed string surfaces from the glued up worldline.

We could go ahead now and examine arbitrary correlators in the free theory in the worldline formalism, but free Yang-Mills (or super Yang-Mills) theory, in the large-N limit, has an exponentially large number of single-trace gauge-invariant operators for a given dimension [15,16]. This is a reflection of its stringinesss. Implementing our strategy on arbitrary correlation functions of these operators is challenging because the worldlines can have a very complicated topology. Therefore, as a first step, it helps to focus on a subclass of simple operators for which the open-closed duality will be easiest to carry out.

B. Twist-2 operators

A very natural choice is to consider operators which are bilinear in the fields of the theory (but with an arbitrary number of derivatives) transforming as symmetric traceless tensors of arbitrary spin. These twist-2 operators have several nice features. One is that these operators form a set of higher spin-conserved currents of the free theory. Another important feature is that they close among themselves under the OPE (Operator Product Expansion) of the free theory[3] suggesting some kind of consistent truncation to this subsector [17]. Moreover, the leading order in N connected n-point functions of these operators are particularly simple in the free theory, being given by a one-loop diagram. In an open string (or, equivalently, double line) representation, these are annulus diagrams with some number of insertions of gauge-invariant operators (see Fig. 1).

Thus topologically these are the simplest diagrams where we have just two holes (the inner and outer boundary) to close. In the field theory limit, the worldsheet reduces to a circular worldline (with some number of insertions). So we just have to glue together a single worldline loop to obtain trees. This simplifies our technical task.

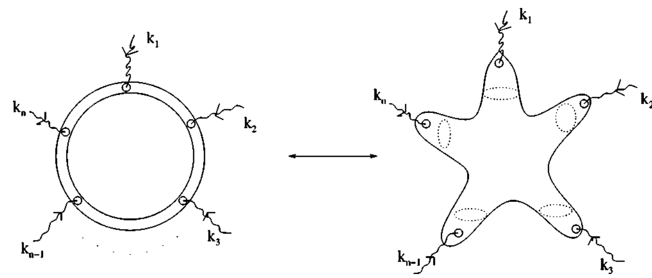

FIG. 1. One-loop open string diagram glued up into a closed string diagram.

The twist-2 operators are also naturally singled out from the closed string point of view. These operators correspond to the leading Regge trajectory of stringy excitations. In the zero radius limit, these are massless higher spin states in AdS space, corresponding to the fact that the currents were conserved in the free theory [15,18]. In fact, classical interacting field theories of exactly these massless higher spin particles in AdS_5 have been studied by Vasiliev and others (see [19–21], for instance). That such classical theories exist at all is some indication that perhaps there is a consistent truncation of the full string theory on $AdS_5 \times S^5$ to this massless Regge trajectory [17]. This also goes with the previous observation of the closure of the OPE in this sector of the gauge theory. It is also perhaps the explanation for why we find a description in terms of particle amplitudes upon implementing open-closed duality on the twist-2 operators. We should add that, even if true, this kind of consistent truncation would probably only hold for classical string theory (i.e., in the large-N limit). In any case, all these facts taken together suggest this sector of the theory is a natural starting point for implementing our strategy.

Klebanov and Polyakov [22] have, in fact, attempted to isolate the dynamics of this sector by pointing out that the ‘‘single-trace’’ singlet operators in the O(N) vector model are all bilinears which have similar features to the gauge theory bilinears above and can thus be placed in exact correspondence with the massless higher spin states mentioned above. They therefore conjectured that the large-N limit of the vector model was exactly dual to the classical Vasiliev theory.[4] Many of our statements, therefore, can be carried over to the vector model at its UV fixed point. But as mentioned earlier, our strategy should enable us to go beyond the bilinears once we have understood sufficiently well the mechanism of the open-closed equivalence in this leading Regge trajectory case. We leave this for the future.

[3]Double trace operators like $(\mathrm{Tr}\,\Phi^2)^2$ are also present, but they correspond to multiparticle states of this sector.

[4]Actually, their conjecture was that the O(N) model in three dimensions, at its interacting (IR) fixed point, is dual to the Vasiliev theory on AdS_4 (see [23–30] for further work). The free-field theory, or the UV fixed point, was also conjectured to be dual to the Vasiliev theory on AdS_4, but with an inequivalent quantization of the spin-0 field, something that is possible in $d=3$ [22].

C. The organization of the paper

Let us now chart out the flow of the paper. In the next section we review the first quantized or worldline formalism. For reasons mentioned above, we will concentrate on the case where the worldline has the topology of a circle. The expressions for a general n-point function on the circle are well known and very similar to those in string theory. As mentioned above, they take the form of an integral over ''moduli space'' with an integrand which is the result of evaluating the correlation function of vertex operators. Section III specializes to the case of three-point functions and shows how the integrand can be viewed as the circle glued into a three-pronged tree. A similar thing happens trivially for the two-point function as well. Section IV connects this tree structure, which emerges from the worldline, to tree amplitudes in AdS space. To that end, we first recast the usual bulk-to-boundary propagators in AdS_{d+1} in a Schwinger representation. The key property that we want to exhibit is the close relationship to the d-dimensional heat kernel. Using this representation, the three-point function in AdS space is seen to be simply related to the three-pronged tree amplitude of Sec. III through a change of variables between the two Schwinger parametrizations. This change of variables is *independent* of the external states and momenta. One interesting feature is that it is essentially the overall proper time modulus that plays the role of the additional dimension in AdS_{d+1}.

Before proceeding further, we devote Sec. V to an old analogy between the Schwinger parametrization of Feynman diagrams and electrical networks. Roughly speaking, for every Feynman graph the Schwinger parameters play the role of resistors while the external momenta play the role of currents. We exploit this analogy to understand how and why the loop got glued into a tree for the case of the one-loop three-point function. It also provides intuition as to why we should expect a generalization of this process to higher-point functions. In fact, these considerations are not special to one-loop correlators. Using the expressions for an arbitrary Feynman diagram in Schwinger parametrization, one might hope to implement the open-closed duality for arbitrary correlators. The intuition behind the gluing of loops into trees is much the same. The expressions themselves are also suggestive in their treelike structure. However, we do not pursue this at the moment. In Sec. VI we look at the four-point function. Guided by the electrical analogy, we describe the equivalent tree diagram. This tree structure turns out to have the right form to be the four-point function in AdS space with a sum over the different channels, including as intermediate states all the particles in the leading Regge trajectory. We have the detailed verification of this for the future. Section VII concludes with a summary, unfinished tasks, and speculations. Appendix A deals with the two-point function. Appendix B gives a convenient Schwinger representation of the scalar bulk-to-bulk propagator in AdS space. Appendix C touches on the relation to the heat kernel expansion and the UV/IR connection.

Finally, a word about our formulas. In order to focus attention on the key physics aspects of various expressions, we have avoided cluttering them with overall factors which are irrelevant to the considerations. Thus the reader would do well to remember that the equality signs in equations are up to various such factors.

II. THE WORLDLINE FORMALISM

Let us consider a Euclidean free-field theory in arbitrary dimension d (though we will mostly have in mind $d=4$ for application to $\mathcal{N}=4$ Yang-Mills theory). For concreteness, take a real scalar field Φ in the adjoint of $\mathrm{SU}(N)$. This will suffice to illustrate the basic procedure. We will remark later on the generalization to the full free super Yang-Mills theory.

Let us consider the connected contribution to an n-point function of the gauge-invariant operator $\mathrm{Tr}\Phi^2$,

$$\Gamma(x_1,x_2,\ldots,x_n) = \frac{1}{N^2}\langle \mathrm{Tr}\,\Phi^2(x_1)\mathrm{Tr}\,\Phi^2(x_2)\cdots\mathrm{Tr}\,\Phi^2(x_n)\rangle_{\mathrm{conn}}. \tag{2.1}$$

The corresponding momentum space correlator $\Gamma(k_1,k_2,\ldots,k_n)$ is given by the double line Feynman diagram in Fig. 1 together with all other possible permutations (of the indices 1,2,...,n) in the external leg insertions. One of the nice things about the worldline formalism is that all these diagrams are captured by a single worldline diagram of a circle with n insertions where the location of each insertion is independent of the others leading to all possible orderings.

The locations (or proper times) τ_i of the insertions are worldline moduli. There is, in addition, an overall modulus, the proper time τ associated with the invariant length of the circle. From the perspective of the first quantized action of a free particle, τ is the remnant of the world line diffeomorphism and plays a similar role to that of the conformal or Liouville mode in a worldsheet action (for a recent discussion, see [31]).

Then, the field theory correlation function takes a form very analogous to that of a string amplitude [13,14],

$$\begin{aligned}\Gamma(k_1,k_2,\ldots,k_n) &= \int [d\mathcal{M}]\langle e^{ik_1\cdot X(\tau_1)}e^{ik_2\cdot X(\tau_2)}\cdots e^{ik_n\cdot X(\tau_n)}\rangle \\ &\equiv \int_0^\infty \frac{d\tau}{\tau}\int_0^\tau \prod_{i=1}^n d\tau_i \langle e^{ik_1\cdot X(\tau_1)}e^{ik_2\cdot X(\tau_2)}\cdots e^{ik_n\cdot X(\tau_n)}\rangle.\end{aligned} \tag{2.2}$$

The measure for τ is consistent with the U(1) invariance along the circle. The correlator of the worldline vertex operators is evaluated with respect to the free-particle action,

$$\langle\cdots\rangle = \int [DX^\mu]\cdots\exp\left(-\frac{1}{4}\int_0^\tau dt(\partial_t X)^2\right). \tag{2.3}$$

A correlation function involving more general twist-2 scalar operators is again given by a one-loop diagram as in Fig. 2. However, the additional derivatives at each vertex are reflected in the fact that the corresponding vertex operators will be linear combinations (depending on the source mo-

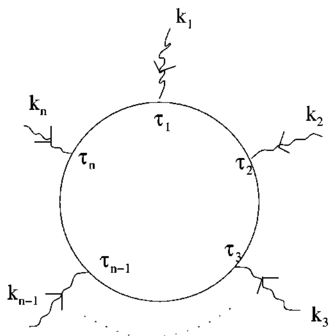

FIG. 2. One-loop worldline diagram with n insertions.

mentum k_i) of $\partial_t X^{\mu_1} \cdots \partial_t X^{\mu_s}(\tau_i) e^{ik_i \cdot X(\tau_i)}$.[5] Unlike the infinitely many oscillators of the string, here the point-particle equation of motion $\partial_t^2 X^\mu = 0$ leads to a restricted class of vertex operators—those of a single Regge trajectory.

In evaluating an arbitrary correlator of bilinears, it is only the vertex operators that are modified as above. The other aspects of the worldline expression remain the same. In particular, we have the same integral over worldline moduli. Though the details of the tensor structure of the more general vertex operators will be important for detailed matching with amplitudes in AdS space, the primary feature of gluing up of the loop will arise from the worldline correlators of $e^{ik_i \cdot X(\tau_i)}$. Therefore, we will mostly concentrate on the n-point function in Eq. (2.2).

The correlation function of vertex operators appearing in Eq. (2.2) is easily evaluated by performing the Gaussian integral of Eq. (2.3). The result is that the integrand in moduli space takes the explicit form (see [13] for instance)

$$\langle e^{ik_1 \cdot X(\tau_1)} e^{ik_2 \cdot X(\tau_2)} \cdots e^{ik_n \cdot X(\tau_n)} \rangle = \delta^{(d)}\left(\sum_i k_i \right) \frac{1}{\tau^{(d/2)}} \exp\left(-\frac{1}{2} \sum_{i \neq j}^{n} k_i \cdot k_j G(\tau_i, \tau_j) \right), \tag{2.4}$$

where

$$G(\tau_i, \tau_j) = -\frac{\tau_{ij}(\tau - \tau_{ij})}{\tau} \quad (\tau_{ij} = |\tau_i - \tau_j|) \tag{2.5}$$

is the appropriate Green's function on the circle. The δ function enforcing momentum conservation comes from the zero-mode integral over the X's. The factor of $1/\tau d/2$ is from the determinant of the nonzero modes.

To evaluate the correlation function of the more general vertex operators, mentioned above, is equally straightforward, involving just keeping track of more indices. As usual, an efficient way to obtain them is to introduce a source term $j \cdot \partial_t X$ in the worldline action and carry out the Gaussian integral and take the required number of functional derivatives. All that essentially changes in Eq. (2.4) is that one has polynomial factors in the external momenta, together with time derivatives of the Green's function, multiplying the Gaussian factor in Eq. (2.4).

Going back to Eq. (2.2), the complete expression for the n-point function is

$$\Gamma(k_1, k_2, \ldots, k_n) = \delta^{(d)}\left(\sum_i k_i \right) \int_0^\infty \frac{d\tau}{\tau^{(d/2)+1}} \times \int_0^\tau \prod_{i=1}^{n} d\tau_i \exp\left(-\frac{1}{2} \sum_{i \neq j}^{n} k_i \cdot k_j G(\tau_i, \tau_j) \right). \tag{2.6}$$

It is this expression that will be our main focus of attention. We will examine it more closely for the three- and (to a lesser extent) the four-point function. In these cases, we will see how this integral over moduli space for a loop can be viewed in terms of contributions from tree graphs.

Finally, a word about further generalizations. First, when there are several species of scalar fields; then the only Wick contractions that survive are ones where the flavor indices contract. This means that only a subset of permutations of the external legs gives a nonzero answer. This effectively translates into a truncation of the regime of integration of the moduli τ_i. This issue only arises for four- and higher-point functions.

Fermionic and gauge bilinears can also be incorporated into the worldline formalism [32]. This is best done by including a worldline Grassmann superpartner ψ^μ to the X^μ and appropriately supersymmetrizing the free worldline action. The natural description for the action and vertex operators is in terms of a worldline superfield which is integrated over a supermoduli space. The expressions for the one-loop correlation functions are reductions of analogous superstring ones. Once again the essential features are captured by Eq. (2.6). We refer the reader to [14] and the review [33] for more details and references.

III. THE THREE-POINT FUNCTION

Having set up the worldline formalism for the n-point function of bilinears, we will now see how the worldline gluing process takes us from the one-loop diagram to a tree-like structure, in the particular case of the three-point function. Logically, one should start with the two-point function. But the gluing process is somewhat trivial there and we shall

[5]For a given symmetric traceless tensor, one way to obtain the corresponding vertex operator is to add a source term ($\propto e^{ik_i x}$) to the free-field Lagrangian, coupling to this operator. Integrating out the fields in this, still quadratic, Lagrangian leads to a determinant which can be written as a proper time Hamiltonian. In obtaining the determinant, one integrates by parts the derivatives, which act on the $e^{ik_i \cdot x}$ and give factors of the momentum k_i. The proper time Hamiltonian now involves a higher power of the derivatives, i.e., proper time momenta p^μ, as well as the factor of $e^{ik_i \cdot x}$, in a definite ordering. The first quantized description involves going to the Lagrangian description. To first order in the source, the additional term in the free-particle Lagrangian is given by replacing the p^μ by $\partial_t X^\mu$. This term, linear in the source, is then the vertex operator corresponding to our original symmetric traceless tensor.

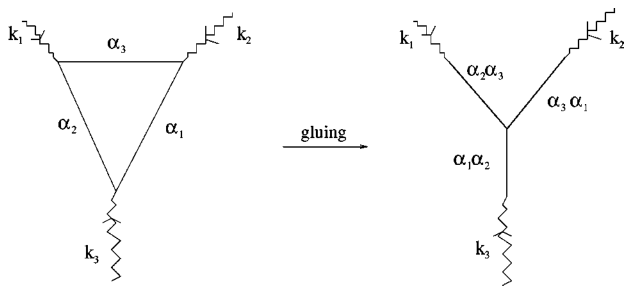

FIG. 3. One-loop three-point function glued up into a three-pronged tree. The relation between the Schwinger parameters of the sides of the loop and the legs of the tree is also shown.

relegate its discussion to Appendix A.

Once again, to keep things uncluttered we will begin with the $n=3$ case of Eq. (2.1), or rather its momentum space version in Eq. (2.2), which was evaluated in Eq. (2.6),

$$\Gamma(k_1,k_2,k_3)=\delta^{(d)}\left(\sum_i k_i\right)\int_0^\infty \frac{d\tau}{\tau^{(d/2)+1}}\int_0^\tau \prod_{i=1}^3 d\tau_i \times \exp\left(-\frac{1}{2}\sum_{i\neq j}^3 k_i\cdot k_j G(\tau_i,\tau_j)\right). \tag{3.1}$$

The momentum-conserving δ function helps in simplifying the kinematic invariants appearing in the exponent, in Eq. (3.1). Thus $2k_1\cdot k_2=k_3^2-k_1^2-k_2^2$, etc. Making the change of variables $\tau_{12}=\tau\alpha_3$, $\tau_{23}=\tau\alpha_1$, $\tau_{31}=\tau\alpha_2$ (with $\Sigma_i\alpha_i=1$) yields the simpler form

$$\Gamma(k_1,k_2,k_3)=\delta^{(d)}\left(\sum_i k_i\right)\int_0^\infty \frac{d\tau}{\tau^{(d/2)+1}}\tau^3\int_0^1\prod_{i=1}^3 d\alpha_i\,\delta\left(\sum_i \alpha_i-1\right)\exp\{-\tau(k_1^2\alpha_2\alpha_3+k_2^2\alpha_3\alpha_1+k_3^2\alpha_1\alpha_2)\}. \tag{3.2}$$

We recognize the integrand to be

$$\exp\left\{-\sum_i \tau k_i^2\alpha_j\alpha_k\right\}=\prod_{i=1}^3\langle k_i|\exp[\tau\alpha_j\alpha_k\Box]|k_i\rangle, \tag{3.3}$$

where the indices i,j,k are a cyclic permutation of $\{1,2,3\}$ and $\Box$ is the d-dimensional Laplacian whose eigenkets are denoted by $|k\rangle$.

That this heat kernel representation is already a glued up version of the original loop diagram can be most clearly exhibited by going to position space. We can easily do the Gaussian integral over momentum (after introducing a center-of-mass variable z as Lagrange multiplier for the δ function),

$$\begin{aligned}\Gamma(x_1,x_2,x_3)&=\int\prod_{i=1}^3 d^dk_i e^{-ik_i\cdot x_i}\Gamma(k_1,k_2,k_3)\\
&=\int_0^\infty\frac{d\tau}{\tau^{(d/2)+1}}\tau^3\int_0^1\prod_{i=1}^3 d\alpha_i\,\delta\\
&\quad\times\left(\sum_i\alpha_i-1\right)\frac{1}{\tau^{3d/2}(\Pi\alpha_i)^d}\\
&\quad\times\int d^dz\exp\left\{-\frac{1}{4}\sum_{i=1}^3\frac{(x_i-z)^2}{\tau\alpha_j\alpha_k}\right\}\\
&=\int_0^\infty\frac{d\tau}{\tau^{(d/2)+1}}\tau^3\int_0^1\prod_{i=1}^3 d\alpha_i\,\delta\left(\sum_i\alpha_i-1\right)\\
&\quad\times\int d^dz\prod_{i=1}^3\langle x_i|\exp[\tau\alpha_j\alpha_k\Box]|z\rangle.\end{aligned} \tag{3.4}$$

The last line exhibits the position space heat kernel representation clearly as a tree amplitude—the product of free-particle amplitudes to propagate from each x_i to a common vertex z (which is then integrated over).[6] Recall that the heat kernel or propagator in position space is given by

$$\langle x|e^{t\Box}|y\rangle=\frac{1}{(4\pi t)^{d/2}}e^{-(x-y)^2/4t}. \tag{3.5}$$

Pictorially, we may depict the process of gluing as in Fig. 3.

In the next section, we will see that this tree is precisely a tree amplitude in AdS_{d+1} once we make a change of variables in Eq. (3.4) into Schwinger parameters for the tree. But before we proceed, let us comment on the three-point correlation function of arbitrary bilinears. As mentioned in the last section, the only changes are multiplicative factors consisting of polynomials in the external momenta (and α_i). Since the crucial Gaussian factor is unchanged, we see that in position space, we continue to have the tree structure of $\Pi_{i=1}^3\langle x_i|\exp[\tau\alpha_j\alpha_k\Box]|z\rangle$. The additional terms are multiplicative polynomials in the $(x_i-z)^\mu$. Similar, slightly generalized remarks apply to the case of fermionic and gauge bilinears. Thus the property of gluing depicted in Fig. 3 is universal to all the one-loop correlation functions. We will better understand the underlying reason for this in Sec. V.

IV. THE THREE-POINT TREE AMPLITUDE IN AdS SPACE

A. The bulk to boundary propagator in AdS space

To compare the tree of the previous section with the tree amplitudes in AdS space, we will find it useful to write the bulk-to-boundary propagators for various fields in a somewhat unconventional manner. We have mostly focused atten-

[6] That the one-loop three-point function can be viewed as a tree in this way was noticed by [34].

tion on correlation functions of $\mathrm{Tr}\,\Phi^2$. The dimension of this operator is $(d-2)$ in the d-dimensional free theory. The AdS/CFT dictionary [2,3] then tells us that it couples to a scalar field in AdS space with $m^2=-2(d-2)$. Hence let us start with the bulk-to-boundary propagator of such a scalar field. This is a solution of the wave equation for a scalar in AdS_{d+1} (where we set the radius of AdS to 1),

$$[-z_0^2\partial_{z_0}^2+(d-1)z_0\partial_{z_0}-z_0^2\Box-2(d-2)]K=0, \quad (4.1)$$

which is proportional to a δ function when the position in the bulk approaches the boundary. Here we are working with Euclidean AdS_{d+1} in the natural Poincare coordinates

$$ds^2=\frac{d\vec{z}^2+dz_0^2}{z_0^2} \quad (4.2)$$

and $\Box$ is the d-dimensional Laplacian in the directions $\vec{z}$ (or simply z when there is no risk of confusion) that parametrize the boundary on which the dual field theory resides.

In terms of the coordinate $t=z_0^2$ and $K=t\tilde{K}$, Eq. (4.1) takes the form

$$\left(2(d-6)\frac{\partial\tilde{K}}{\partial t}-\Box\tilde{K}-4t\frac{\partial^2\tilde{K}}{\partial t^2}\right)=0. \quad (4.3)$$

If the last term were absent, this would have been the heat equation and the solution would have simply been the heat kernel $e^{t\Box}$. However, solutions to Eq. (4.3) can also be expressed in terms of the heat kernel. Thus the usual bulk-to-boundary propagator is given by a solution of the form

$$\tilde{K}(t)=\int_0^\infty d\rho\,\rho^{(d/2)-3}e^{-\rho}e^{(t/\rho)\Box}. \quad (4.4)$$

Thus, $K(t)=t\tilde{K}(t)$ is expressed in terms of a convolution over a heat kernel in terms of the parameter ρ. It is easy to verify that this is just a Schwinger parametrization of the familiar bulk-to-boundary propagator (with $\Delta=d-2$) [3],

$$K(x,z;z_0=t^{1/2})=\left(\frac{t^{1/2}}{t+(x-z)^2}\right)^{d-2}=\langle x|K(t)|z\rangle. \quad (4.5)$$

In momentum space [2], Eq. (4.4) is merely an integral representation of the Bessel function that the bulk-to-boundary propagator is proportional to.

This close relation to the d-dimensional heat kernel is the main reason why the glued up tree of the previous section can be related to a tree amplitude in AdS space. Though we have shown this for the scalar and also for its bulk-to-boundary propagator, it is clear that wave equations for higher spin particles in AdS space can also be put into a similar form as Eq. (4.3), which exhibits the close relation to the heat kernel. Similarly, bulk-to-bulk propagators will also be expressed in terms of d-dimensional heat kernels, as we will see explicitly in Sec. VI.

B. The three-point function

Let us write down the tree amplitude for the three-point function of the above scalar in this Schwinger representation. If there are no derivatives in the cubic couplings of scalars (something that can presumably be achieved by a field redefinition [35]), the point-particle amplitude is [using Eqs. (4.5) and (4.4)]

$$\begin{aligned}\Gamma(x_1,x_2,x_3)&=\int d^dz\int_0^\infty\frac{dz_0}{z_0^{d+1}}\prod_{i=1}^3 K(x_i,z;z_0)\\&=\int_0^\infty\frac{dt}{t^{(d/2)+1}}t^3\int_0^\infty\prod_{i=1}^3 d\rho_i\rho_i^{(d/2)-3}e^{-\rho_i}\\&\quad\times\int d^dz\prod_{i=1}^3\langle x_i|e^{(t/4\rho_i)\Box}|z\rangle. \end{aligned} \quad (4.6)$$

Note the close similarity with the integrand of the worldline expression Eq. (3.4), particularly the striking closeness between the radial coordinate t and the proper time τ. However, here instead of integrating over the worldline moduli of the loop, we have an integral over Schwinger parameters for the tree. There is a simple change of variables between the two which makes the two integrals identical. This is suggested by the relations in Fig. 3 between the loop and the tree. We simply have to set $\rho_i=\rho\alpha_i$, where $\rho=\Sigma_i\rho_i$, which we can implement by introducing $\int_0^\infty\delta\rho\,\delta(\rho-\Sigma_i\rho_i)=1$ into the integral and changing to variables α_i. Finally, we make the change

$$t=4\tau\rho\left(\prod_{i=1}^3\alpha_i\right), \quad (4.7)$$

which relates the proper time τ to the AdS radial coordinate t. The integral over ρ decouples, only contributing to the overall constant which we have dropped all along, and Eq. (4.6) becomes Eq. (3.4).

A number of comments are in order here. The change of variables that we made was independent of the external momenta or positions and even of the number of spacetime dimensions. It is the kind of change of variables one might expect in going between a parametrization of open string moduli space and one of the closed string. In fact, in generalizing to the three-point function of arbitrary bilinears, we expect that the same change of variables will be sufficient. This is essentially because both the exponent and the measure on moduli space continue to be the same for the general three-point function. Of course, it is not guaranteed that the multiplicative tensor structures will work out right.

This is where, we believe, the supersymmetry and the special field content of $d=4$, $\mathcal{N}=4$ Yang-Mills theory will play a special role. After all, what we have done so far works for any free scalar theory in any dimension. It is likely that it is only in the case of $\mathcal{N}=4$ Yang-Mills theory that the tensor structure encoded in the multiplicative factors would also match with that from the bulk-to-boundary propagators for massless higher spin particles.

There is some evidence for this contention. Indeed, in the early days of the AdS/CFT correspondence, a free-field computation of the (two- and) three-point function of R currents (which are bilinears in the fields) in $\mathcal{N}=4$ Yang-Mills theory was compared to supergravity [36]. The authors of [36] employed a Schwinger parametrization of the one-loop diagram and matched the resulting integral expression with that of the supergravity integral, again in a parametrized representation. Making a change of variables, essentially equivalent to that above, they found an explicit agreement of the tensor structures.[7] It was important for them that the contribution from both scalars and fermions to the R-current correlators was taken into account to get the exact matching. We take this as evidence that the special properties of $\mathcal{N}=4$ Yang-Mills theory are likely to play a role in ensuring a detailed matching of tensor structures. A related observation is regarding the conformal anomaly. (Similar considerations apply to the two- and three-point functions of the stress tensor. The detailed matching of these tensor structures in the bulk and the boundary was carried out in [37].) The conformal anomaly in $d=4$, for example, is a linear combination of two independent curvature invariants. The particular combination depends on the field content of the theory. An AdS_5 calculation, on the other hand, gives a definite combination of these two invariants [38]. One needs the full-field content of $\mathcal{N}=4$ Yang-Mills theory to get this particular combination. Thus the bosonic vector model in four dimensions, for example, cannot possibly arise from an AdS_5 calculation since a different linear combination of the curvature invariants arises in the two computations.[8]

One of the very interesting features in making the connection between the worldline picture and amplitudes in AdS space is that the proper time on the worldline is more or less directly related to the radial direction in AdS space, as seen in Eq. (4.7) or in the measures of Eqs. (3.4) and (4.6). This is not altogether unexpected. The proper time τ is a measure of the energy scale in the field theory and it has been seen in various circumstances that the radial coordinate in AdS space plays a very similar role. For instance, a UV cutoff in the field theory can be implemented by cutting off the modulus integral at small τ. From Eq. (4.7), this effectively translates into an IR cutoff in the radial coordinate t in AdS space. This is also apparent from the fact that τ is the remnant of the modulus of the open string annulus, and the small τ regime is where the annulus captures the long-distance (IR) propagation in the closed string channel. Another source of our intuition for why the proper time should play the role of the extra dimension comes from the observation that τ represents the worldline conformal factor, and so in a loose sense it is a Liouville mode.[9] Therefore, it fits in with the idea of the Liouville mode being the origin of the extra dimension in the AdS/CFT correspondence [39,40]. (Note that the idea of the Liouville direction playing the role of an additional spacetime dimension in noncritical string theory goes back to [41–43].) It will be very interesting to flesh this connection out further.

[7]We would like to thank K. Schalm for drawing our attention to [36].

[8]We would like to thank I. Klebanov and K. Skenderis for discussions on this point.

[9]We thank S. Wadia for pointing this out.

V. THE ELECTRICAL NETWORK ANALOGY

At this stage, it might seem that the results of the previous section are due to the special nature of (two- and) three-point functions which are largely constrained by conformal invariance.[10] What we would like to put forward in this section is that the basic mechanism which is operating is indeed open-closed duality (in the limit where the Riemann surfaces on both sides have degenerated to graphs), and that this mechanism can generalize to arbitrary diagrams.

We recall that there were two main steps in the process of going from the field theory three-point function to the AdS amplitude. The first was to argue that the worldline formulation of the field theory loop could be seen in terms of trees involving free-particle heat kernels. The second was to show that these trees were indeed tree diagrams in AdS space. The latter turned out to be true essentially because the wave equation in AdS space implied a close connection between propagators in AdS_{d+1} and free-particle heat kernels in d dimensions. A change of variables on the moduli then demonstrated the identity of the two tree amplitudes. The first step of gluing loops into trees is the one where the geometric mechanism of open-closed duality seems to be operating. To better understand how this operates, and generalizes to arbitrary correlation functions, it will be very useful to revive an old analogy between Feynman diagrams and electrical networks.

The first indication that such an analogy might be present, and important for us, is the observation that the loop-tree duality in Fig. 3 is similar to the standard "star-δ" equivalence in electrical networks.[11] In fact, there is a precise connection. If one views the "δ" (loop) diagram in Fig. 3 as an electrical network with resistances R_i in each of the sides, then this network is exactly equivalent to that of the "star" or tree diagram with resistances $(R_jR_k/\Sigma R_i)$ on the legs as shown in the figure.[12] This may be verified using elementary considerations of Kirchoff's laws. The essential idea involves eliminating the current flowing in the loop from the equations so that we are reduced to an equivalent tree diagram without that loop (see, for instance, [45]).

This is not just a coincidence. There is an analogy between Feynman diagrams and electrical networks going back to the 1960s (see, for instance, Chap. 18 of [46]). An arbitrary Feynman diagram, expressed in Schwinger (or Feynman) parametrization, has a natural interpretation in electrical network terms. The Schwinger moduli can be identified with resistances, and the external, as well as internal, momenta with currents flowing in the respective legs. The process of carrying out the integrals over loop momenta is then equivalent to elimination of the internal currents using Kir-

[10]For correlators of higher spin operators there are a finite number of tensor structures consistent with conformal invariance. The relative coefficients are undetermined.

[11]We thank Justin David for this observation which was instrumental in our pursuing this line of thought.

[12]Star-triangle relations crop up very often in physics and mathematics. In a related context, see [44].

choff's laws.[13] The result is a generalization of our one-loop worldline expressions—an integral over Schwinger moduli space of an integrand that depends on the external momenta. In particular, the crucial piece is a Gaussian exponent as in Eq. (2.4), which is proportional to the power consumed in the equivalent circuit. This is clearly visible in Eq. (3.2), where the exponential factor is the power consumed in the equivalent tree circuit of Fig. 3. For a recent review of (and earlier references to) the expressions for a general Feynman diagram as well as their electrical interpretation, see [47].

What is of interest to us is the tree structure obtained after elimination of the loop momenta/currents. In the language of circuits, it is intuitively plausible that this process of elimination of loop currents results in an equivalent tree structure. The external currents in various linear combinations would flow through the various legs of this tree. For instance, the expression for the power in the equivalent circuit, which appears as the Gaussian exponent in the integral, can be written down explicitly. For a general l loop diagram, it is given in graph theoretic terms [47],

$$P(\alpha,k)=\Delta(\alpha)^{-1}\sum_{T_2}\left(\prod^{l+1}\alpha\right)\left(\sum k\right)^2. \tag{5.1}$$

Here, α_i are the Schwinger parameters for the various internal legs of the loop. The sum is over various 1-trees and 2-trees obtained from the original loop diagram. A 1-tree is obtained by cutting the l loop diagram at l lines so as to make a connected tree, while a 2-tree is obtained by cutting the loop at $l+1$ lines so as to form two disjoint trees. $\Delta(\alpha)$ is then given by a sum, over the set T_1 of all 1-trees, of the product of the α_i of all the cut lines. In the case of a one-loop diagram, this is simply $\Sigma\alpha_i$. The sum over T_2 indicates a sum over the set of all 2-trees, where the product is over the α_i of the $l+1$ cut lines. And (Σk) is understood to be the sum over all those external momenta k_i which flow into (either) one of the 2-trees. It is easy to verify, for example, that in the case of the three-point function, this tallies with the exponent in Eq. (3.2).

This expression can be interpreted as the power dissipated in an equivalent tree circuit in which currents (Σk) flow in legs whose resistances are $\Delta(\alpha)^{-1}(\Pi^{l+1}\alpha)$. The topology of this tree circuit seems somewhat intricate in general. We will examine the case of the one-loop four-point function somewhat more in the next section. However, we will postpone a more general analysis to future work. But hopefully, what should be clear from the above considerations is that the gluing up of loops into trees is not particular to two-or three-point functions. What we are seeing is an implementation of open-closed string duality, in the limit of degenerate worldsheets, and the electrical analogy gives us useful intuition for visualizing this process.

When there are multiple loops, it is likely that the glued up tree is effectively a thick or "fat" worldsheet, at least in the limit where a large number of loops are present. The latter would be true for correlation functions of, say, $\mathrm{Tr}\,\Phi^J$ for large J. It would be nice to make a connection with the BMN picture [48] of a closed-string worldsheet emerging from operators like this carrying a large number of bits.

VI. THE FOUR-POINT FUNCTION

Armed with the intuition from electrical networks, we will take a first look at the four-point function. Here we will only try to convince the reader that the worldline diagram does glue up in the right way as expected from the duality to AdS space. A detailed check will be postponed to the future.

As usual, we will restrict our consideration to the four-point function of $\mathrm{Tr}\,\Phi^2$, only briefly indicating the generalizations. The worldline expression is given from Eq. (2.2) to be

$$\Gamma(k_1,k_2,k_3,k_4)=\int_0^\infty\frac{d\tau}{\tau}\int_0^\tau\prod_{i=1}^4 d\tau_i\langle e^{ik_1\cdot X(\tau_2)}e^{ik_2\cdot X(\tau_2)}\times e^{ik_3\cdot X(\tau_4)}e^{ik_4\cdot X(\tau_4)}\rangle. \tag{6.1}$$

The integral over moduli space can be broken into six cyclically inequivalent orderings of the four insertions. Of these, three are related to the others by a worldline reflection $\tau\to-\tau$. The three inequivalent orderings correspond to three inequivalent Feynman diagrams that can contribute to this amplitude. As mentioned in Sec. II, when there are several flavors of scalars (as in $\mathcal{N}=4$ super Yang-Mills theory), then some of these diagrams might be absent due to the structure of the flavor indices. This can easily be incorporated into the considerations below.

Let us look at a specific time ordering (1234) of the insertions around the circle. The worldline expression is given by Eq. (2.6) with the caveat that the integral over τ_i is restricted to the above time-ordered domain. The all-important Gaussian factor in the integrand can be rewritten using the δ-function constraint on the momentum as

$$e^{-(1/2)\Sigma_{i\neq j}^4 k_i\cdot k_j G(\tau_i,\tau_j)}=e^{-\tau[\alpha_4\alpha_1k_1^2+\alpha_1\alpha_2k_2^2+\alpha_2\alpha_3k_3^2+\alpha_3\alpha_4k_4^2+\alpha_2\alpha_4(k_1+k_2)^2\alpha_1\alpha_3(k_1+k_4)^2]}. \tag{6.2}$$

[13] This can be seen in the Schwinger parametrization, where one is performing Gaussian integrals over the internal momenta. The exponent has the interpretation as being the power consumed in the Feynman diagram. The Gaussian saddle points are precisely the Kirchoff equations for voltages.

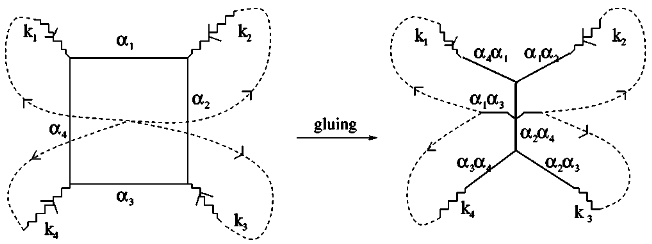

FIG. 4. One-loop four-point function and the equivalent tree. In the language of the electrical network, the thick lines are the equivalent resistors of the tree. The dotted lines complete the rest of the circuit. The whole diagram is best thought of as drawn on a sphere with the dotted lines going behind.

Here $\tau_{41}=\tau\alpha_4$, $\tau_{12}=\tau\alpha_1$, etc. The exponent on the right-hand side tallies with the general Schwinger parametrization expression quoted in Eq. (5.1). As we saw in the previous section, this exponent has the interpretation as the electrical power dissipated in the equivalent circuit obtained after eliminating the loop current.

What is the equivalent circuit in this case? It is as shown on the right side of Fig. 4.[14] Elementary circuit analysis shows that this is the equivalent circuit. One quick way to verify this is to see that the exponent in Eq. (6.2) is proportional to the power dissipated in the thick lines on the right. Note that the topology is now more complicated than in the case of the three-point function. In fact, the set of equivalent resistors, shown with thick lines, is no longer fully connected. The horizontal resistor [which has current (k_1+k_4) flowing through it] is disconnected from the others. Of course, we have drawn the equivalent circuit in the "*s* channel." This is an arbitrary choice. One could equally well have drawn it in the "*t* channel," in which case the vertical line would have been the disconnected one.

We would like to claim that this tree structure, which the loop is glued into, is what is needed for the AdS/CFT duality. If we consider the four-point amplitude in AdS space in a point-particle limit, it is given by a set of tree diagrams. (See [49] and references therein to the large literature on AdS four-point functions. In particular, the four-point functions of the lowest twist-2 operators have been studied in detail, both perturbatively and at strong coupling, in [50].) In the point-particle limit (unlike in a worldsheet description) one separately sums over diagrams in *s,t,u* channels built from three-point vertices. These diagrams are as in the tree of Fig. 4 (minus the horizontal line). In a given channel, say *s*, there can be an infinite set of intermediate states. We will now try to argue that the horizontal line in Fig. 4 captures an infinite summation in the *s* channel.

To interpret the horizontal resistor as a sum over infinitely many states, let us go back to Eq. (6.2). If we expand the corresponding exponential piece $e^{\tau\alpha_1\alpha_3(k_1+k_4)^2}$, we get an infinite series in powers of $t=(k_1+k_4)^2$. A term $\sim t^J$ immediately suggests a contribution from a state of spin J in the intermediate channel. But we could have equally well viewed this diagram in the *t* channel; the *s*-*t* symmetry would imply an infinite summation in this channel too. There are two other inequivalent orderings of insertions, namely (1324) and (1243), which have *t*-*u* and *s*-*u* symmetries, respectively. The sum has complete *s*-*t*-*u* symmetry. We can expand each of these inequivalent diagrams twice,[15] once in each of the respective channels, and sum over all these orderings. We then rearrange the final answer as a sum over all channels (each channel getting infinite summation contributions from two different orderings). As mentioned above, the powers appearing in the infinite sum are suggestive of higher spin intermediate states. Thus the worldline expression *prima facie* has the right structure to assemble itself into an AdS amplitude with different channels, each involving higher spin intermediate states. We must emphasize again that it is because we are viewing the AdS amplitude in a point-particle language that we obtain a sum over different channels, each with an infinite number of intermediate states. This is more like a closed string field theory representation which pieces together different regions of moduli space to achieve a dual answer.[16] In this context, we should mention that there could also be a four-point contact term, in principle. A more careful examination of the amplitude will be necessary to disentangle such a contribution.

Another related viewpoint also indicates an infinite tower of intermediate states in each channel. Since we have taken the ordering (1234), the vertex operators $e^{ik_1\cdot X(\tau_1)}$ and $e^{ik_2\cdot X(\tau_2)}$ are adjacently inserted. Remembering that these vertex operators are normal ordered, we can write the exact worldline operator product,

$$e^{ik_1\cdot X(\tau_1)}e^{ik_2\cdot X(\tau_2)}=e^{ik_1\cdot X(\tau_2)+ik_2\cdot X(\tau_2)}e^{-k_1\cdot k_2 G(\tau_1,\tau_2)}. \tag{6.3}$$

If we expand the nonlocal vertex operator $e^{ik_1\cdot X(\tau_1)+ik_2\cdot X(\tau_2)}$ in powers of the separation τ_{12}, one obtains vertex operators of the form $\partial_t X^{\mu_1}\cdots\partial_t X^{\mu_s}(\tau_0)e^{i(k_1+k_2)\cdot X(\tau_0)}$, where τ_0 is the midpoint of τ_1 and τ_2. We have again used the equation of motion $\partial_t^2 X^\mu=0$. This indicates that as intermediate states in the *s* channel, we will have all the higher spin particles of the leading Regge trajectory. In fact, this actually suggests that we do not have particles from other Regge trajectories appearing as intermediate states. This is in line with the comments in the Introduction about the possibility of a consistent truncation to the leading Regge trajectory.

For four-point functions of more general twist-2 operators, the logic is very similar since we have the same Gaussian factor. There is, in addition, a multiplicative factor in the momenta which contributes to the spin of the exchanged state. The intermediate states are still in the leading Regge trajectory following the arguments of the previous paragraph.

[14]This figure is actually adapted from a textbook on electrical networks (see p. 136 of [45]).

[15]Note that by our reckoning, each ordering appears twice, once in cyclic and again in anticyclic order. So we do not introduce any new factors of 2 in expanding each diagram in two ways.

[16]We thank L. Rastelli, A. Sen, and E. Witten for useful discussions in this regard.

The AdS four-point function

Though we will not make a detailed comparison with the four-point function in AdS space at present, we will look at the glued up worldline answer in position space and exhibit the closeness to AdS amplitudes. Using Eqs. (2.6) and (6.2), the full worldline expression for the four-point function, in the external ordering (1234), is

$$\Gamma(k_1,k_2,k_3,k_4)=\delta^{(d)}\left(\sum_i k_i\right)\int_0^\infty \frac{d\tau}{\tau^{(d/2)+1}}\tau^4\int_0^1\prod_{i=1}^4 d\alpha_i\,\delta\left(\sum_i \alpha_i-1\right)\times e^{-\tau[\alpha_4\alpha_1k_1^2+\alpha_1\alpha_2k_2^2+\alpha_2\alpha_3k_3^2+\alpha_3\alpha_4k_4^2+\alpha_2\alpha_4(k_1+k_2)^2+\alpha_1\alpha_3(k_1+k_4)^2]}. \tag{6.4}$$

We will expand the factor of $e^{-\tau\alpha_1\alpha_3(k_1+k_4)^2}$ and take the Fourier transform term by term.

Let us look at the leading term in this expansion. We can do the Gaussian integral over the momenta after introducing a Lagrange multiplier as in the case of the three-point function. The presence of the term proportional to $(k_1+k_2)^2$ in the exponent of Eq. (6.4) suggests introducing another Lagrange multiplier. That is, we rewrite the momentum-conserving δ function as

$$\delta^{(d)}\left(\sum_i k_i\right)=\int d^dk_s\,\delta^{(d)}(k_1+k_2-k_s)\,\delta^{(d)}(k_3+k_4+k_s)=\int d^dk_s d^dz\,d^dw\,e^{i(k_1+k_2-k_s)\cdot z}e^{i(k_3+k_4+k_s)\cdot w}. \tag{6.5}$$

We now carry out the Fourier transform with respect to the momenta k_i and also perform the integral over k_s. Following steps similar to that in Sec. III, we readily get the position space expression,

$$\Gamma(x_1,x_2,x_3,x_4)=\int_0^\infty \frac{d\tau}{\tau^{(d/2)+1}}\tau^4\int_0^1\prod_{i=1}^4 d\alpha_i\,\delta\left(\sum_i \alpha_i-1\right)\times\int d^dz\,d^dw\langle x_1|e^{\tau\alpha_4\alpha_1\Box}|z\rangle\times\langle x_2|e^{\tau\alpha_1\alpha_2\Box}|z\rangle\times\langle z|e^{\tau\alpha_2\alpha_4\Box}|w\rangle\times\langle w|e^{\tau\alpha_2\alpha_3\Box}|x_3\rangle\langle w|e^{\tau\alpha_3\alpha_4\Box}|x_4\rangle. \tag{6.6}$$

Note that there are two intermediate positions z,w that have to be integrated over. In this form we have clearly exhibited the s-channel tree structure of Fig. 4, in position space. Higher powers in the expansion of $e^{-\tau\alpha_1\alpha_3(k_1+k_4)^2}$ can also be Fourier transformed in a similar way. Since the integrals are still Gaussian, the basic structure of Eq. (6.6) persists. There are now multiplicative tensor structures in x_i^μ,z^μ,w^μ which are necessary for the description of the exchange of higher spin states, as well as conformal descendants.

As we saw in the case of the three-point function, the above heat kernel structure of the tree was important for transforming the integrand into an AdS amplitude. This was essentially because the AdS bulk-to-boundary propagators bore a close relation to the d-dimensional heat kernel. To match terms as in Eq. (6.6) to a contribution to the s-channel four-point amplitude in AdS space, the bulk-to-bulk propagator in AdS space will also have to make an appearance. Since it too is a solution to the wave equation in AdS space (albeit with a δ-function source), it is not surprising that it can also be naturally written in terms of the heat kernel. For instance, the scalar bulk-to-bulk propagator in position space can be written in the Schwinger representation (see Appendix B),

$$G(z,w;t_1,t_2)=\sum_{n=0}^\infty \frac{1}{n!\,\Gamma\left(\frac{d-2}{2}+n\right)}\frac{(4t_1t_2)^{[(d-2)/2]+n}}{(t_1+t_2)^{(d/2)-2+2n}}\times\int_0^\infty d\rho\,\rho^{(d/2)-3+2n}e^{-\rho}\langle z|e^{[(t_1+t_2)/\rho]\Box}|w\rangle, \tag{6.7}$$

where, as before, we have redefined $t_1=z_0^2, t_2=w_0^2$.

This representation is a generalization of Eq. (4.4) and is already in a suggestive form in relation to the worldline expressions. Note that the $n=0$ term in Eq. (6.7) dominates as one of the bulk points approaches the boundary, and is proportional to the bulk-to-boundary propagator in Eq. (4.4). This is identified with the contribution of the conformal primary Tr Φ^2. The higher powers of n can be identified with the contribution of spin zero conformal descendants $\Box^n\mathrm{Tr}\Phi^2$ of this operator [51,52].[17] Similar representations exist for higher spin particles. So all the right ingredients are present for a match with the field theory.

What remains to be seen is that all these ingredients can be put together and an appropriate change of variables be made on the Schwinger parameter space so that the worldline expression goes over into an AdS amplitude. Moreover, the intermediate states exchanged in any channel have to be in the leading Regge trajectory. We hope to verify this conjecture in detail in future work. Our intention here has merely

[17]There are, however, subtleties here involving logarithms [51,52].

been to make it plausible to the reader that our considerations for the three-point function generalize nontrivially to the four-point function.

VII. FINAL REMARKS

We have taken some small steps here in trying to understand how free-field theory could reorganize itself into a theory of closed string modes on a higher-dimensional AdS space. Essentially, we have sought to carry through the logic of open-closed string duality. We hope to have made the case that the worldline representation of the field theory is a natural framework within which this can be done. This was, perhaps, to be expected because it is the appropriate limit of the open string. But, in addition, as we saw in Sec. V, there seems to be a systematic way in which the gluing of loops into tree structures takes place in this Schwinger parametrized representation. The analogy to electrical networks gives us the intuition as to how this happens.

Of course, associating tree structures to loops is just the geometrical aspect of the open-closed string duality. The dynamical aspect consists of understanding how, in this process, the background also changes from flat space to AdS space. Here, we do not yet have any systematic understanding. Nevertheless, the worldline formalism has given some important clues in this direction. The close relation between propagators in AdS space and proper time propagators in the boundary theory is crucial for the transmutation of the field theory amplitude into one on AdS space. As we saw in the case of the two- and three-point functions, a fairly simple change of variables on the Schwinger parameters takes us from one to the other. Though we have not yet worked out such a change of variables for the four-point function, we believe it exists. Relatedly, the close identification of the overall proper time with the radial coordinate appears to be some kind of realization of ideas on the Liouville mode and the extra dimension. Therefore, the worldline formalism seems to also have the power to manifest the change of background in the process of gluing loops into trees. We, however, need to go beyond a case-by-case change of variables and find a way to understand this in more generality. This will require some more insight into the relation between the open and closed string parameters. Rather than working with some particular coordinatization, as we have been doing, we perhaps need a more invariant characterization of the respective moduli spaces.

Actually, the entire discussion of the last paragraph is unavoidably tied up with the issue of the closed string description of AdS space [53,54].[18] We have been trying all along in this paper to bypass this issue by restricting ourselves to the twist-2 operators. The idea, as mentioned in the Introduction, is that the dual description of this sector might conceivably only involve a point-particle-like limit of the string on AdS space. Apart from the existence of a consistent classical theory of the massless higher spin fields, there is another source of intuition for this guess. All attempts that have been made to study closed strings on AdS space in the zero radius limit have found evidence for some kind of bit picture emerging from the closed string worldsheet [7–9]. These bits or partons are to be identified with the Yang-Mills fields. The bilinear operators are then those with the smallest allowed number of bits, namely two. We therefore expect this case to be one where the worldsheet is slimmest and thus closest to that of a point particle. To fatten the worldsheet, we would need a large number of bits, as is familiar from usual light-cone considerations or, more pertinently, in the BMN picture [48]. In any case, our computations, to their limited extent, seem to bear out this working hypothesis for the leading Regge trajectory.

But even here it is clear that, even if the amplitudes can be viewed as those of point particles in AdS space, it is a very cumbersome way of doing things. For instance, in the four-point function, all the infinite number of particles in the leading Regge trajectory should appear as intermediate states. A sum over individual bulk-to-bulk AdS propagators for all these states is not only technically demanding but also ugly. A look at Eq. (6.4) shows that expanding in the s channel (i.e., in powers of t) multilates a nice s-t symmetric expression. It is the analogue of expanding the Veneziano amplitude in the s channel which leads to messy individual terms. Since the worldline expressions are in duality symmetric form, it should be possible to recast them directly into a duality symmetric closed string description. Perhaps the unbroken infinite-dimensional higher spin symmetry on AdS space [19–21,16,18,17] should give us a hint on how to formulate such a description. After all, the free-field Laplacian entering in the worldline formalism also has such a symmetry [55] (see also [56]).

Another clue should come from the generalization to operators with more bits. As mentioned at the end of Sec. V, correlation functions of operators like $\mathrm{Tr}\Phi^J$ for large J will have many worldlines (and loops). It should be possible to examine the Schwinger parametrization of these correlators and see an effective thickening of the worldsheet. It has recently been proposed [57] that there is a huge Yangian symmetry that acts on the set of all free partons which is related to the nonlocal symmetries of the sigma model on AdS space [58–60]. This would be a generalization of the higher spin symmetries of the bilinears.

The idea of seeing an infinite number of unbroken symmetries in string theory in the limit of $\alpha' \to \infty$[19] goes back to Gross [61,62]. Some of the features found in [62], such as the contribution only of special kinds of worldsheets in high-energy amplitudes, seem to reappear in our considerations.[20]

At nonzero coupling (or finite α'), we expect these symmetries to be Higgsed [18,24,63]. The open-closed string duality would, nevertheless, continue to hold. We note, in this context, that the electrical analogy holds for arbitrary Feyn-

[18]It is understood that whenever we talk of a closed-string theory on AdS space, we mainly have in mind the maximally supersymmetric theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

[19]In the zero coupling limit, keeping the radius of AdS fixed, like we have, is equivalent to taking $\alpha' \to \infty$.

[20]We thank David Gross for discussions on these matters.

man diagrams, including that of an interacting gauge theory. Hence we expect the gluing of loops into trees to be implemented in the worldline formalism even at nonzero coupling. The generality of the worldline formalism might also be useful in trying to extend the open-closed string duality to nonsupersymmetric gauge theories. Perhaps this will also enable us to tie these gauge string dualities with the "other" kind of open-closed duality that takes place in the process of tachyon condensation (see [64,65] for recent discussions).

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APPENDIX A: THE TWO-POINT FUNCTION

The two-point function is a simple illustration of the ideas in the main body of the paper. The worldline expression for the two-point correlator is given from Eq. (2.6) to be, after some obvious change of variables,

$$\Gamma(k_1,k_2)=\delta^{(d)}(k_1+k_2)\int_0^\infty \frac{d\tau}{\tau^{(d/2)+1}}\tau^2 \times\int_0^1 d\alpha\, e^{-\tau\alpha(1-\alpha)[\beta k_1^2+(1-\beta)k_2^2]}, \tag{A1}$$

where β is arbitrary since the integrand is actually independent of it. This is more conventionally written in terms of the reduced form

$$\tilde{\Gamma}(k)=\int_0^\infty \frac{d\tau}{\tau^{(d/2)+1}}\tau^2\int_0^1 d\alpha\, e^{-\tau\alpha(1-\alpha)k^2}. \tag{A2}$$

Equation (A2) is a straightforward example of the gluing up process and its interpretation in terms of the electrical analogy. The loop with two insertions is glued up into a tree which is just a line segment in this case. In electrical terms, this is just the elementary fact that two parallel resistors [in this case proportional to the parameters α and $(1-\alpha)$] can be replaced by a single equivalent resistor [proportional to $\alpha(1-\alpha)$ for us]. This is evident from the exponent in Eq. (A2).

As before, the position space expression corresponding to Eqs. (A1) and (A2) is

$$\begin{aligned}\Gamma(x_1,x_2)&=\int_0^\infty \frac{d\tau}{\tau^{(d/2)+1}}\tau^2\int d^dz\int_0^1 d\alpha\langle x_1|e^{\tau\alpha(1-\alpha)\beta\Box}|z\rangle\\ &\quad\times\langle z|e^{\tau\alpha(1-\alpha)(1-\beta)\Box}|x_2\rangle\\ &=\int_0^\infty \frac{d\tau}{\tau^{(d/2)+1}}\tau^2\int_0^1 d\alpha\langle x_1|e^{\tau\alpha(1-\alpha)\Box}|x_2\rangle.\end{aligned} \tag{A3}$$

As in the case of the three-point function, the position space expression clearly exhibits the glued up form of the loop. We will relate Eq. (A3) to the two-point amplitude in AdS space.

The latter is essentially proportional to the convolution of two bulk-to-boundary propagators[21] so that just as in Eq. (4.6) we have

$$\begin{aligned}\Gamma(x_1,x_2)=\int d^dz\int_0^\infty \frac{dt}{t^{(d/2)+1}}t^2\int_0^\infty d\rho_1 d\rho_2(\rho_1\rho_2)^{(d/2)-3}\\ \times e^{-\rho_1-\rho_2}\langle x_1|e^{(t/4\rho_1)\Box}|z\rangle\langle z|e^{(t/4\rho_2)\Box}|x_2\rangle.\end{aligned} \tag{A4}$$

A change of variables to $\rho_1=\rho(1-\beta)$, $\rho_2=\rho\beta$ and introducing a trivial integral over α makes this take the form

$$\begin{aligned}\Gamma(x_1,x_2)=\int d^dz\int_0^\infty \frac{dt}{t^{(d/2)+1}}t^2\int_0^\infty d\rho\,\rho^{d-5}e^{-\rho}\int_0^1 d\beta\\ \times\int_0^1 d\alpha[\beta(1-\beta)]^{(d/2)-3}\times\langle x_1|e^{[t/4\rho(1-\beta)]\Box}|z\rangle\\ \times\langle z|e^{(t/4\rho\beta)\Box}|x_2\rangle.\end{aligned} \tag{A5}$$

We can now relate t to the proper time τ through

$$t=4\tau\rho\beta(1-\beta)\alpha(1-\alpha). \tag{A6}$$

Note the similarity to Eq. (4.7). The integral over ρ decouples and we are left with

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[21]We are being a little cavalier here. Actually, there are contributions from gradient terms as well. However, because of the equation of motion, these are related to each other and one is left with a boundary term which needs to be treated carefully [66]. Since our interest is not in reproducing the right normalization factors, but rather in seeing how loops glue into AdS trees, it suffices to consider the product of two bulk-to-boundary propagators. The price we will pay is that some expressions will be formally divergent.

$$\Gamma(x_1,x_2)=\int_0^1\frac{d\beta}{\beta(1-\beta)}\int_0^\infty\frac{d\tau}{\tau^{(d/2)+1}}\tau^2\int d^dz\int_0^1\frac{d\alpha}{[\alpha(1-\alpha)]^{(d/2)-2}}\langle x_1|e^{\tau\alpha(1-\alpha)\beta\Box}|z\rangle\langle z|e^{\tau\alpha(1-\alpha)(1-\beta)\Box}|x_2\rangle$$

$$=\int_0^1\frac{d\beta}{\beta(1-\beta)}\int_0^\infty\frac{d\tau}{\tau^{(d/2)+1}}\tau^2\int_0^1\frac{d\alpha}{[\alpha(1-\alpha)]^{(d/2)-2}}\langle x_1|e^{\tau\alpha(1-\alpha)\Box}|x_2\rangle. \tag{A7}$$

Modulo the overall divergent factor from the decoupled β integral, Eq. (A7) coincides with Eq. (A3) for $d=4$, the case of interest. The overall divergence is not unexpected given the comments in the previous footnote.

APPENDIX B: THE SCALAR BULK-TO-BULK PROPAGATOR

The position space bulk-to-bulk propagator in AdS space for a scalar field corresponding to an operator of dimension Δ is usually put in the form (see, e.g., [51])

$$G(z,w;z_0,w_0)=\xi^{-\Delta}F\left(\frac{\Delta+1}{2},\frac{\Delta}{2},\Delta-\frac{d}{2}+1;\frac{1}{\xi^2}\right), \tag{B1}$$

where

$$\xi=\frac{z_0^2+w_0^2+|z-w|^2}{2z_0w_0}. \tag{B2}$$

In the case of the free-field operator Tr Φ^2 with $\Delta=d-2$, the hypergeometric function in Eq. (B1) simplifies and

$$G(z,w;z_0,w_0)=\xi^{-\Delta}\frac{1}{\left(1-\frac{1}{\xi^2}\right)^{(d-1)/2}}. \tag{B3}$$

Using the expansion

$$\frac{1}{(1-z)^a}=\sum_{n=0}^{\infty}\frac{\Gamma(n+a)}{\Gamma(a)}\frac{z^n}{n!} \tag{B4}$$

and redefining $z_0^2=t_1$, $w_0^2=t_2$, we can write the bulk-to-bulk propagator in a Schwinger parameter expansion very similar to Eq. (4.4),

$$G(z,w;t_1,t_2)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{d-1}{2}\right)}{n!\Gamma(d+2n-2)}\frac{(4t_1t_2)^{[(d-2)/2]+n}}{(t_1+t_2)^{d-2+2n}}\times\int_0^\infty d\rho\,\rho^{d-3+2n}e^{-\rho}e^{-\rho[|z-w|^2/(t_1+t_2)]}$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!\Gamma\left(\frac{d-2}{2}+n\right)}\frac{(t_1t_2)^{[(d-2)/2]+n}}{(t_1+t_2)^{(d/2)-2+2n}}\times\int_0^\infty d\rho\,\rho^{(d/2)-3+2n}e^{-\rho}\langle z|e^{[(t_1+t_2)/4\rho]\Box}|w\rangle. \tag{B5}$$

Here we have used the Schwinger representation

$$\frac{1}{\xi^\lambda}=\frac{1}{\Gamma(\lambda)}\int_0^\infty d\rho\,\rho^{\lambda-1}e^{-\rho\xi} \tag{B6}$$

as well as the identity

$$\Gamma(\Delta+2n)=\frac{1}{2\pi^{1/2}}2^{\Delta+2n}\Gamma\left(\frac{\Delta+1}{2}+n\right)\Gamma\left(\frac{\Delta}{2}+n\right). \tag{B7}$$

APPENDIX C: DIVERGENCES AND THE UV-IR RELATION

Let us consider the free-field theory in an arbitrary curved background (or more generally one in which some of the higher spin fields also have an expectation value). The action is still quadratic but the effective action is now a complicated nonlocal functional of the metric (and other fields). There are some UV divergences in this effective action, but they are local in the background fields. This is familiar from the study of quantum fields in curved space. The conventional way to isolate these divergences is, in fact, the heat kernel or proper time expansion. This involves studying the proper time representation of the effective action in the background and putting a UV cutoff ϵ at small proper times to regularize the expression. We then make a small time expansion and isolate the leading divergent pieces.

What we would like to remark here is that the same structure of divergences is present in the effective action on AdS space as a functional of the boundary values of the metric (and other fields). The difference is that these divergences are now in the IR and can be regularized by an IR cutoff ϵ' in the radial coordinate of AdS space. This fits in well with our picture where the proper time essentially transmutes itself into the radial coordinate of AdS space.

1. The heat kernel expansion

The effective action for, say, the free adjoint scalar field in a curved background $h_{\mu\nu}$ is given in a heat kernel representation,

$$\frac{1}{N^2}\Gamma(h_{\mu\nu})=\tfrac{1}{2}\ln\det(-\Box_h)=\frac{1}{2}\int_\epsilon^\infty\frac{d\tau}{\tau}\mathrm{Tr}[e^{\tau\Box_h}], \tag{C1}$$

where we have put in the UV cutoff ϵ and indicated the curved background in the subscript for the Laplacian. For the fermions and gauge fields, there are analogous representations of the corresponding kinetic operators. For small proper times, the trace of the heat kernel has the well-known Schwinger-DeWitt expansion in terms of local functionals (see [67] for instance),

$$\mathrm{Tr}[e^{\tau\Box_h}]\sim\int\frac{d^dz\sqrt{h}}{\tau^{d/2}}\left(\sum_{j=0}^{[d/2]}a_j(z)\tau^j+\cdots\right), \tag{C2}$$

where the ellipsis indicates terms which give UV finite contributions. The sum over j is a derivative expansion. The $a_j(z)$ are the familiar Schwinger-DeWitt coefficents which are curvature invariants built from $h_{\mu\nu}$ and having a total of $2j$ derivatives of the metric. Thus (with the appropriate normalization which we have omitted) $a_0=1$, $a_1(x)=\frac{1}{6}R$, etc.

Therefore, the effective action has the expansion

$$\frac{1}{N^2}\Gamma(h_{\mu\nu})=\int_\epsilon^\infty\frac{d\tau}{\tau^{(d/2)+1}}\int d^dz\sqrt{h}\times\left(1+\frac{\tau}{6}R+\tau^2O(R^2)+\cdots\right). \tag{C3}$$

Not accidentally, this is like the worldline expressions we had for correlation functions. In even dimensions the term with $j=d/2$ in Eq. (C2) has a logarithmic dependence on τ and gives rise to a logarithmically divergent term which is responsible for the conformal anomaly.

2. Comparison to AdS space

The measure in Eq. (C3) has the right structure to be that of AdS_{d+1}. In fact, the form of the integrand is also what one would have for a classical action on AdS space evaluated on-shell. When we speak here of a classical action on AdS space, we do not have in mind some kind of Einstein-Hilbert action or supergravity variant. It could be more like a string field action involving an infinite number of derivatives as in the Vasiliev theories.

But already at the level of the Einstein-Hilbert action, one sees a very similar structure, on-shell, to Eq. (C3). One solves Einstein's equations on AdS_{d+1} with an asymptotic boundary metric $h_{\mu\nu}(z)$ by parametrizing the bulk metric to be

$$ds^2=\frac{dt^2}{t^2}+\frac{h_{\mu\nu}(z,t)dz^\mu dz^\nu}{t}, \tag{C4}$$

where, as $t\to 0$,

$$h_{\mu\nu}(z,t)\to h^{(0)}_{\mu\nu}(z)+th^{(2)}_{\mu\nu}(z)+t^2h^{(4)}_{\mu\nu}(z)+\cdots+t^{[d/2]}h^{(d)}_{\mu\nu}(z)+t^{[d/2]}\ln t\tilde{h}^{(d)}_{\mu\nu}(z)+\cdots. \tag{C5}$$

Here $h^{(0)}_{\mu\nu}(z)=h_{\mu\nu}(z)$. The observation of Fefferman and Graham [68] was that one can solve for $h^{(2)}_{\mu\nu}(z)$, $h^{(4)}_{\mu\nu}(z),\ldots,\tilde{h}^{(d)}_{\mu\nu}(z)$ [but not $h^{(d)}_{\mu\nu}(z)$] algebraically in terms of $h^{(0)}_{\mu\nu}(z)$. Putting this back into Einstein's equations gives [38]

$$\frac{1}{N^2}\Gamma(h_{\mu\nu})=\int_{\epsilon'}^\infty\frac{dt}{t^{(d/2)+1}}\int d^dz\sqrt{h}\left(\sum_{j=0}^{[d/2]}\tilde{a}_j(z)t^j+\cdots\right). \tag{C6}$$

Here $\tilde{a}_j(z)$ are local curvature invariants of dimension $2j$ built from $h_{\mu\nu}(x)$ just as $a_j(z)$ in Eq. (C2). In general, $\tilde{a}_j(z)$ and $a_j(z)$ are distinct linear combinations of the finite number of curvature invariants of dimension $2j$. But Henningson and Skenderis [38] could compare the conformal anomaly of the field theory [$j=d/2$ piece of Eq. (C2)] with the similarly logarithmically divergent ($j=d/2$) piece of Eq. (C6). For the full $\mathcal{N}=4$ Yang-Mills multiplet, they found precise agreement. See also [69].

Our purpose here is just to point out that the similarity between Eqs. (C2) and (C6) is another signature of the role of the proper time representation of the field theory in reconstructing the bulk description. The connection between the radial coordinate and the proper time shows up clearly over here.[22] In the examples that we have seen, such as Eqs. (4.7) and (A6), the two have been multiplicatively related to each other. Thus, though the cutoffs ϵ and ϵ' cannot be directly identified with each other, because of their multiplicative relation the logarithmically divergent pieces can be compared. This is also related to the fact that the power-law-divergent terms for $j<d/2$ are prescription-dependent in the field theory whereas the logarithmically divergent one is not.

Similar calculations including backgrounds for scalar fields in AdS space have been carried out in [70,71] with very similar results to the corresponding heat kernel expansion in field theory. Perhaps utilizing the higher spin symmetries of the free Laplacian [55] one might be able to relate the heat kernel in an arbitrary quadratic background to on-shell actions of the Vasiliev type, generalizing the remarks in this section.

[22] It is interesting that the natural parametrization for the AdS metric (C4) in [68,38] employs t rather than the more common $z_0^2=t$.

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