Fermionic and bosonic stabilizing effects for type I and type II dimension bubbles

J. R. Morris*
Physics Department, Indiana University Northwest, 3400 Broadway, Gary, Indiana 46408, USA

(Received 24 March 2004; published 23 July 2004)

We consider two types of "dimension bubbles," which are viewed as 4D nontopological solitons that emerge from a 5D theory with a compact extra dimension. The size of the extra dimension varies rapidly within the domain wall of the soliton. We consider the cases of type I (II) bubbles where the size of the extra dimension inside the bubble is much larger (smaller) than outside. Type I bubbles with thin domain walls can be stabilized by the entrapment of various particle modes whose masses become much smaller inside than outside the bubble. This is demonstrated here for the cases of scalar bosons, fermions, and massive vector bosons, including both Kaluza-Klein zero modes and Kaluza-Klein excitation modes. Type II bubbles expel massive particle modes but both types can be stabilized by photons. Plasma filled bubbles containing a variety of massless or nearly massless radiation modes may exist as long-lived metastable states. Furthermore, in contrast to the case with a "gravitational bag," the metric for a fluid-filled dimension bubble does not exhibit a naked singularity at the bubble's center.

DOI: 10.1103/PhysRevD.70.025007 PACS number(s): 11.27.+d, 04.50.+h, 98.80.Cq

I. INTRODUCTION

An inhomogeneous compactification of a higher dimensional spacetime to four dimensions may result in the formation of "dimension bubbles," where the sizes of the extra dimensions inside such a bubble are much different than those outside the bubble [1–3]. A dimension bubble, from a four dimensional viewpoint, is a nontopological soliton consisting of a closed domain wall that entraps particles and/or radiation, which help to stabilize the bubble against collapse. Since the existence of dimension bubbles depends only upon a dramatic change in the sizes of extra dimensions inside and outside the bubble, the detections of such objects could provide evidence for the existence of extra dimensions, regardless of how large or small their ambient sizes. For simplicity and specificity, we will, as in Refs. [2,3], consider the case where there is one toroidally compactified extra space dimension, so that the 5D spacetime has the topology of $M_4 \times S^1$.

Attention is focused here on several new features of dimension bubbles. First, it is pointed out that two different types of dimension bubbles are possible, and the size of the extra dimension, while differing dramatically in the interior and exterior regions of the bubble, may remain microscopic in both regions. Let us simply label these two bubble types as types I and II. A type I (II) bubble is one for which the size of the extra dimension is larger (smaller) inside the bubble than outside. A type I bubble can be stabilized by massive particles that are trapped inside the bubble, where the particle masses become much smaller than on the outside of the bubble. This stabilization mechanism for type I bubbles was demonstrated for the case of scalar bosons in Ref. [2]. Here we extend these results to include scalar bosons, fermions, and massive vector bosons—both Kaluza-Klein (KK) zero modes and KK excitation modes. A type II bubble expels such massive modes from its interior. The dependence of the particle masses upon the extra dimensional scale factor $|\tilde{g}_{55}|^{1/2}=B(x)$ for the 4D Einstein frame is obtained in each case, for both the KK zero modes and the KK excitations. Both type I and II bubbles can be stabilized by photons, however, due to the high reflectivity of the bubble wall [3].

Finally, we investigate the behavior of the 4D metric near the center of a dimension bubble and compare it with that for a "gravitational bag" [4,5], which can be thought of as a static idealization of an empty type I dimension bubble. (The gravitational bag solution is static, but the scalar field of the bag exhibits a singular behavior at the bag's center.) More specifically, we view a fluid-filled dimension bubble as a *cosmic balloon* [6,7] and investigate the behavior of the interior metric near the bubble's center. Although the interior metric of a "gravitational bag" has a naked singularity near the geometric center [4,5], it is seen that a fluid stabilized dimension bubble has a well behaved, finite metric at the bubble's center. Therefore the naked singularity of a gravitational bag is avoided in a fluid-filled dimension bubble.

A brief summary of the dimension bubble model is presented in Sec. II and conditions on the 4D effective potential are presented for the formation of either type I or type II bubbles. We can consider as an example the case where the extra dimension is, say, TeV sized (i.e., $l_{TeV}\sim \mathrm{TeV}^{-1}$) in one region of space and Planck sized ($l_P \sim M_P^{-1}$) in another region of space, so that the size of the extra dimension may change by roughly 16 orders of magnitude between these two regions while remaining microscopic in both regions. The mass dependence upon the extra dimensional scale factor $|\tilde{g}_{55}|^{1/2}=B$ for KK zero modes is exhibited in Sec. III for scalar bosons, fermions, and massive vector bosons. The resulting expressions make clear how these particles can help stabilize a type I bubble by getting trapped inside, as with an "ordinary" nontopological soliton of the type previously studied by Frieman, Gleiser, Gelmini, and Kolb [8], and how they must be expelled from a type II bubble. These results are then extended in Sec. IV to include Kaluza-Klein excitation modes. Radiation filled metastable bubbles are then con-

*Electronic address: jmorris@iun.edu

templated in Sec. V, where estimates or rough bounds are obtained for bubble mass, radius, and lifetime. Finally, gravitational aspects are addressed in Sec. VI, where it is pointed out that by viewing a fluid filled dimension bubble as a *cosmic balloon* [6] and using the results for a cosmic balloon metric [7], the 4D metric of a fluid-filled dimension bubble exhibits a nonsingular behavior at the bubble's center. This suggests that the undesirable feature of a naked singularity, which appears at the core of a "gravitational bag" [4,5], does not appear at the center of a fluid filled dimension bubble. Section VII consists of a brief summary.

II. THE DIMENSION BUBBLE MODEL

A. Metric ansatz

A five-dimensional (5D) spacetime is assumed to be endowed with a metric $\tilde{g}_{MN}$:

$$ds^2=\tilde{g}_{MN}dx^M dx^N=\tilde{g}_{\mu\nu}dx^\mu dx^\nu - B^2 dy^2 \tag{1}$$

where $x^M=(x^\mu,y)$, with $M,N=0,\ldots,3,5$, $\mu,\nu=0,\ldots,3$, and $B=\sqrt{-\tilde{g}_{55}}$ is the dimensionless scale factor for the extra dimension. We assume an ansatz where the metric $\tilde{g}_{MN}$ is independent of the extra dimension y, i.e., $\tilde{g}_{MN}=\tilde{g}_{MN}(x^\mu)$, $\partial_5\tilde{g}_{MN}=0$, and the metric factorizes with $\tilde{g}_{\mu 5}=0$. The extra dimension, characterized by the coordinate $x^5=y$, with $0\leqslant y\leqslant 2\pi R$, is taken to be toroidally compact, so that the 5D spacetime has a topology of $M_4\times S^1$. We allow for the possibility that the scale factor B has a spatial dependence, i.e., $B=B(x^\mu)$. In the dimensionally reduced effective 4D theory the scale factor B can be associated with a scalar field φ through the relation

$$\varphi=\frac{1}{\kappa_N}\sqrt{\frac{3}{2}}\ln B, \tag{2}$$

where κ_N is related to the 4D Planck mass M_P by $\kappa_N=\sqrt{8\pi G}=\sqrt{8\pi}M_P^{-1}$, so that the scale factor can be written as $B=e^{\sqrt{2/3}\kappa_N\varphi}$. We further consider the situation wherein the scalar φ is governed by a 4D effective potential $U(\varphi)\geqslant 0$, which arises from a Rubin-Roth potential for bosonic and fermionic degrees of freedom [9] (see also Ref. [2]), along with a 5D cosmological constant Λ. When $U(\varphi)$ assumes a "semi-vacuumless" form characterized by the existence of a local minimum at some finite value $\varphi=\varphi_0$, a local maximum at some finite value $\varphi=\varphi_{\max}>\varphi_0$, and an asymptotic form $U(\varphi)\to U_\infty=\text{const}$ as $\varphi\to\infty$, "dimension bubbles" can arise [2–4] as solutions of the 4D theory where the scalar φ (and therefore the scale factor B) can vary rapidly across a region of space. It is this rapid variation of φ that is associated with the domain wall bounding the soliton. We will be interested in the cases where B differs dramatically in the interior and exterior of the bubble. We note that even though the extra dimension may remain microscopic in both regions, there can still be an enormous variation in B across the domain wall. If, for example, the extra dimension varies across the bubble wall from a Planck size where $(BR)_P\sim M_P^{-1}$, characteristic of a "small" extra dimension, to a "TeV size" where $(BR)_{TeV}\sim\text{TeV}^{-1}$, which may be characteristic of a "large" extra dimension, B can change by 16 orders of magnitude.

B. Dimensional reduction of the 5D action

We take the 5-dimensional action to include the 5D Einstein action, cosmological constant Λ, and a source Lagrangian $\mathcal{L}_5$:

$$S_5=\frac{1}{2\kappa_{N(5)}^2}\int d^5x\sqrt{\tilde{g}_5}\{\tilde{R}_5-2\Lambda+2\kappa_{N(5)}^2\mathcal{L}_5\} \tag{3}$$

where $\kappa_{N(5)}^2=8\pi G_5=(2\pi R)\kappa_N^2$, $\tilde{g}_5=|\det\tilde{g}_{MN}|$ and $\tilde{R}_5=\tilde{g}^{MN}\tilde{R}_{MN}$ denotes the 5-dimensional Ricci scalar built from $\tilde{g}_{MN}$. A 4D Einstein Frame metric $g_{\mu\nu}$ can be defined in terms of the 4D Jordan Frame metric $\tilde{g}_{\mu\nu}$ by $g_{\mu\nu}=B\tilde{g}_{\mu\nu}=e^{\sqrt{2/3}\kappa_N\varphi}\tilde{g}_{\mu\nu}$, and the line element in Eq. (1) then takes the Kaluza-Klein form

$$\begin{aligned} ds^2&=B^{-1}g_{\mu\nu}dx^\mu dx^\nu-B^2dy^2\\ &=e^{-\sqrt{2/3}\kappa_N\varphi}g_{\mu\nu}dx^\mu dx^\nu-e^{2\sqrt{2/3}\kappa_N\varphi}dy^2. \end{aligned}\tag{4}$$

Using Eqs. (3) and (4), the 5D action is dimensionally reduced to the effective 4D Einstein frame action [2,3]

$$S=\int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa_N^2}R+\frac{1}{2}(\nabla\varphi)^2+e^{-\sqrt{2/3}\kappa_N\varphi}\left[\mathcal{L}-\frac{1}{\kappa_N^2}\Lambda\right]\right\} \tag{5}$$

where $R=g^{\mu\nu}R_{\mu\nu}$ is the 4D Ricci scalar built from the 4D Einstein frame metric $g_{\mu\nu}$ and $g=\det g_{\mu\nu}$ and $\mathcal{L}=(2\pi R)\mathcal{L}_5$.

The 4D effective Lagrangian that is generated by $\mathcal{L}$ is $\mathcal{L}_4=B^{-1}\mathcal{L}$. The Lagrangian $\mathcal{L}_4$, the effective potential $U(\varphi)$, the φ kinetic term $\frac{1}{2}(\partial\varphi)^2$, and the gravitational term $(1/2\kappa_N^2)R$ produce a total 4D effective Lagrangian

$$\mathcal{L}_{eff}=\frac{1}{2\kappa_N^2}R+\frac{1}{2}(\partial\varphi)^2-U(\varphi)+\mathcal{L}_4. \tag{6}$$

The "semi-vacuumless" potential $U(\varphi)$ admits a domain wall solution separating a region where B becomes very "large" (where φ assumes a value $\varphi_1>\varphi_{\max}$) from a region where B is relatively "small" (at the local minimum of U, where $\varphi=\varphi_0$). In general, $U(\varphi_0)\neq U(\varphi_1)$ and the wall is unstable against bending toward the region of higher energy density [2], and we expect the formation of a network of bubbles to result. A dimension bubble encloses a region of higher vacuum energy density and is surrounded by a region of lower vacuum energy density. For simplicity we consider a spherical thin walled bubble of radius R_B, with wall thickness $\delta\ll R_B$, so that in a simplifying limit we may take the inner radius R_- and outer radius R_+ of the wall to coincide, R_-, $R_+\to R_B$. It is within the wall that the scale factor $B(x)$

varies rapidly, and we are interested in the case where B takes on vastly different values in the interior and exterior regions.

C. Type I and type II dimension bubbles

We can envision two distinct possibilities, corresponding to different sets of model parameters, which can result in two different types of bubbles. Either (i) $U(\varphi_1)>U(\varphi_0)$, giving rise to what we will refer to as ‘‘type I’’ bubbles, or (ii) $U(\varphi_1)<U(\varphi_0)$ associated with ‘‘type II’’ bubbles. In the first case (type I) the bubble interior contains a ‘‘vacuum’’ characterized by $\varphi\approx\varphi_1$, a vacuum energy density $U(\varphi_1)$, and a relatively large scale factor $B\approx B_1=e^{\sqrt{2/3}\kappa_N\varphi_1}$ and in the bubble's exterior region where $\varphi=\varphi_0$ there is a relatively small scale factor $B_0=e^{\sqrt{2/3}\kappa_N\varphi_0}$. In the second case (type II) we have the opposite situation. Therefore, a type I bubble encloses a ‘‘large’’ extra dimension and is surrounded by a ‘‘small’’ extra dimension. A type II bubble encloses a ‘‘small’’ extra dimension and is surrounded by a ‘‘large’’ extra dimension. Again, the extra dimension may remain microscopic in all regions, but we entertain the possibility that its size, characterized by BR, may be extremely different inside and outside of a bubble. For example, we might consider the range $M_P^{-1}\lesssim BR\lesssim \mathrm{TeV}^{-1}$, in which case the values of B inside and outside the bubble would be related by $B_{in}/B_{out}\sim 10^{\pm 16}$.

III. EFFECTS OF KALUZA-KLEIN ZERO MODES

In this section we consider contributions to the 4D effective Lagrangian $\mathcal{L}_4$ from Kaluza-Klein (KK) zero modes of scalar bosons, fermions, and vector bosons which acquire mass through the Higgs mechanism. Each type of zero mode particle field Φ is x^5-independent, i.e., $\Phi=\Phi(x^\mu)$, $\partial_5\Phi=0$. We later consider Kaluza-Klein excitations where the fields have a y-dependence from the cylinder condition. The difference in size of the extra dimensional scale factor B in the interior and exterior regions of the bubble results in a difference in the effective particle mass in these regions. Specifically, the particle mass becomes smaller in a region where B is larger. This results in particles getting trapped inside of type I bubbles and being expelled from type II bubbles. Therefore the KK zero modes have a stabilizing influence on type I bubbles, where the particle pressure can help to support the bubble against collapse due to the wall tension. The dependence of the particle mass m upon B is isolated for each type of particle.

A. Scalar bosons

Consider a contribution to the Lagrangian $\mathcal{L}$ from a scalar boson ϕ,

$$\begin{aligned}\mathcal{L}_S&=\tilde{\partial}^M\phi^*\tilde{\partial}_M\phi-\mu_0^2\phi^*\phi=\tilde{g}^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi-\mu_0^2|\phi|^2\\&=B|\partial\phi|^2-\mu_0^2|\phi|^2\end{aligned}\tag{7}$$

where $\partial_5\phi=0$ and $\tilde{g}^{\mu\nu}=Bg^{\mu\nu}$ and $|\partial\phi|^2=\partial^\mu\phi^*\partial_\mu\phi$. This Lagrangian gives rise to the 4D effective Lagrangian

$$\mathcal{L}_{4,S}=B^{-1}\mathcal{L}_S=|\partial\phi|^2-B^{-1}\mu_0^2|\phi|^2.\tag{8}$$

The scalar boson mass in the effective 4D theory is therefore identified as

$$m_S=B^{-1/2}\mu_0\tag{9}$$

where μ_0 is the mass parameter in the original 5D theory. We see that since the value of B inside of a type I bubble is assumed to be much bigger than that outside the bubble, i.e., $B_{in}\gg B_{out}$, the effective boson mass inside is relatively small, $m_{S,in}\ll m_{S,out}$. The scalar boson is effectively trapped inside the type I bubble since there is an enormous inward force $\vec{F}\approx-\nabla m_S=-\mu_0\nabla(B^{-1/2})$ acting on the particle. The kinetic energies of the light trapped particles exert an outward pressure on the bubble wall to help stabilize it against collapse. However, for a type II bubble the particle mass becomes much smaller outside the bubble, so that massive particles that are initially present inside the bubble are expelled from it.

B. Fermions

Now consider a fermionic contribution to the Lagrangian $\mathcal{L}$ in the form

$$\mathcal{L}_F=\overline{\psi'}(i\Gamma^M\partial_M-\mu_0)\psi=\overline{\psi'}(i\Gamma^\mu\partial_\mu-\mu_0)\psi\tag{10}$$

where $\psi=\psi(x)$, $\partial_5\psi=0$, and $\overline{\psi'}=\psi^\dagger\Gamma^0$. The Γ^M matrices are taken to be normalized according to

$$\{\Gamma^M,\Gamma^N\}=-2\tilde{g}^{MN}.\tag{11}$$

The 5D metric $\tilde{g}_{MN}$, written in terms of the 4D Einstein frame metric $g_{\mu\nu}$ and the scale factor B, is

$$\tilde{g}_{MN}=\begin{pmatrix}B^{-1}g_{\mu\nu} & \\ & -B^2\end{pmatrix}.\tag{12}$$

Equation (11) then implies that $\{\Gamma^\mu,\Gamma^\nu\}=-2Bg^{\mu\nu}$.

For the effective 4D Einstein frame theory we define the new matrices γ^μ related to the original Γ^μ matrices by

$$\Gamma^\mu=B^{1/2}\gamma^\mu,\quad \Gamma^5=B^{-1}\gamma^5\tag{13}$$

with a normalization given by

$$\{\gamma^\mu,\gamma^\nu\}=-2g^{\mu\nu},\quad (\gamma^5)^2=1.\tag{14}$$

Upon defining $\bar{\psi}=\psi^\dagger\gamma^0$ we have $\overline{\psi'}=\psi^\dagger\Gamma^0=\psi^\dagger B^{1/2}\gamma^0=B^{1/2}\bar{\psi}$, and the Lagrangian $\mathcal{L}_F$ can be rewritten as

$$\mathcal{L}_F=B\bar{\psi}(i\gamma^\mu\partial_\mu-B^{-1/2}\mu_0)\psi.\tag{15}$$

This Lagrangian gives rise to an effective 4D fermion Lagrangian

$$\mathcal{L}_{4,F}=B^{-1}\mathcal{L}_F=\bar{\psi}(i\gamma^\mu\partial_\mu-B^{-1/2}\mu_0)\psi.\tag{16}$$

The fermion mass in the effective 4D theory is therefore identified as

$$m_F = B^{-1/2}\mu_0 \tag{17}$$

which resembles the result obtained for scalar bosons.

C. Massive vector bosons

Let us consider the case of vector gauge bosons which acquire mass by the Higgs mechanism, through the interaction with a scalar field χ. There is then a contribution to the Lagrangian given by

$$\mathcal{L}_G = -\frac{1}{4}\tilde{F}'^{MN}\tilde{F}'_{MN} + (\tilde{D}^M\chi)^*(\tilde{D}_M\chi)|_{\chi=\eta} \tag{18}$$

where the tildes remind us that the metric $\tilde{g}_{MN}$ is used to construct 5D scalars, so that, for instance, $(\tilde{D}^M\chi)^*(\tilde{D}_M\chi) = \tilde{g}^{MN}(D_N\chi)^*(D_M\chi)$. The field strength and gauge covariant derivative terms are given by

$$F'_{MN} = \partial_M A'_N - \partial_N A'_M, \quad D_M\chi = (\nabla_M + ie_0 A'_M)\chi. \tag{19}$$

We choose $A'_5 = 0$ and a vacuum state characterized by $\chi = \eta = \text{const}$. In the vacuum state we then have

$$(\tilde{D}^M\chi)^*(\tilde{D}_M\chi)|_{\chi=\eta} = \tilde{g}^{MN}e_0^2\eta^2 A'_M A'_N = \frac{1}{2}B\mu_0^2 A'^{\mu}A'_{\mu} \tag{20}$$

where we have defined $\mu_0 = \sqrt{2}e_0\eta$.

In order to obtain a canonical gauge field term in the 4D theory, we introduce the gauge field $A_\mu = B^{1/2}A'_\mu$. In terms of the metric $g_{\mu\nu}$ and the gauge field A_μ, we can rewrite $\mathcal{L}_G$ in the form

$$\mathcal{L}_G = -\frac{1}{4}BF^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\mu_0^2 A^\mu A_\mu - \frac{1}{2}B^{3/2}F^{\mu\nu}H_{\mu\nu} - \frac{1}{4}B^2 H^{\mu\nu}H_{\mu\nu} \tag{21}$$

where

$$H_{\mu\nu} = A_\nu\partial_\mu(B^{-1/2}) - A_\mu\partial_\nu(B^{-1/2}) \tag{22}$$

which becomes nonzero in regions where the scale factor B changes with position or time. (In the interior and exterior regions of the bubble B is taken to be approximately constant, but B varies rapidly with position within the bubble wall.) The effective 4D gauge field Lagrangian $\mathcal{L}_{4,G} = B^{-1}\mathcal{L}_G$ is then given by

$$\mathcal{L}_{4,G} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}B^{-1}\mu_0^2 A^\mu A_\mu - \frac{1}{2}B^{1/2}F^{\mu\nu}H_{\mu\nu} - \frac{1}{4}BH^{\mu\nu}H_{\mu\nu}. \tag{23}$$

The gauge boson mass in the effective 4D theory is therefore identified as

$$m_G = B^{-1/2}\mu_0. \tag{24}$$

D. Zero mode masses

For the cases of scalar bosons, spin 1/2 Dirac fermions, and massive vector bosons, it is found that the particle mass in the effective 4D theory is of the form $m = B^{-1/2}\mu_0$, where μ_0 is the mass parameter appearing in the Lagrangian $\mathcal{L}$ of the original 5D theory. Since the scale factor is assumed to be much larger inside a type I dimension bubble than outside, i.e., $B_{in} \gg B_{out}$, we have that $m_{in} \ll m_{out}$ for the x^5-independent Kaluza-Klein zero modes. Particles experience a strong, short ranged force within the (thin) wall of a type I bubble toward the interior. Therefore, particles having nonzero mass tend to get trapped inside the type I bubble. As the bubble adjusts its size during equilibration the outward particle pressure on the bubble wall has a tendency to help stabilize the bubble against total collapse. Just the opposite holds for a type II bubble, for which $m_{in} \gg m_{out}$. Particles with nonzero mass are expelled from these bubbles.

IV. EFFECTS OF KALUZA-KLEIN EXCITATIONS

A. Effective 4D Einstein frame Lagrangian

Let us now consider the Kaluza-Klein (KK) excitation ($n \neq 0$) modes of scalar bosons, fermions, and massive vector bosons. Each particle mode contributes a piece to the original 5D Lagrangian of the form

$$S_5 = \int d^5x\sqrt{\tilde{g}_5}\mathcal{L}_5 = \int d^4x\sqrt{-g}B^{-1}\int_0^{2\pi R}dy\,\mathcal{L}_5 \tag{25}$$

where now $\mathcal{L}_5 = \mathcal{L}_5(x,y)$ and the cylinder condition is imposed upon the periodic field $\Phi(x,y) = \Phi(x,y+2\pi R)$ allowing the mode expansion

$$\Phi(x,y) = \sum_{n=-\infty}^{\infty}\Phi_n(x)e^{iny/R}. \tag{26}$$

Defining $\mathcal{L} = (2\pi R)\mathcal{L}_5$ as before, we can integrate out the y dependence and define

$$\langle\mathcal{L}\rangle = \frac{1}{2\pi R}\int_0^{2\pi R}dy\,\mathcal{L} = \int_0^{2\pi R}dy\,\mathcal{L}_5. \tag{27}$$

The effective 4D Einstein frame action then emerges as

$$S = \int d^4x\sqrt{-g}B^{-1}\langle\mathcal{L}\rangle = \int d^4x\sqrt{-g}\mathcal{L}_4 \tag{28}$$

where, as with the case of zero modes, we define the effective 4D Einstein Frame Lagrangian

$$\mathcal{L}_4 = B^{-1}\langle\mathcal{L}\rangle. \tag{29}$$

From an expression for the effective 4D Lagrangian for a field we can identify the effective 4D masses of the KK excitation modes. We again take a zero mode metric $\tilde{g}_{MN} = \tilde{g}_{MN}(x)$ and examine the scale factor B dependence of the masses of KK excitations of scalars, spinors, and vectors.

B. Scalar bosons

Consider the scalar boson contribution to the Lagrangian $\mathcal{L}$ given by

$$\mathcal{L}_S = \tilde{\partial}^M \phi^* \tilde{\partial}_M \phi - \mu_0^2 \phi^* \phi \tag{30}$$

with

$$\phi(x,y) = \sum_n \phi_n(x) e^{iny/R}. \tag{31}$$

The Lagrangian $\mathcal{L}_S$ can then be written in terms of the KK modes as

$$\mathcal{L}_S = \sum_{m,n} \left\{ (\tilde{\partial}^\mu \phi_m^*)(\tilde{\partial}_\mu \phi_n) - \left(\frac{mn}{B^2 R^2} + \mu_0^2 \right) \phi_m^* \phi_n \right\} e^{i(n-m)y/R}. \tag{32}$$

Using

$$\frac{1}{2\pi R}\int_0^{2\pi R} dy e^{i(n-m)y/R} = \delta_{mn} \tag{33}$$

we obtain

$$\langle \mathcal{L}_S \rangle = \sum_n \left\{ B(\partial^\mu \phi_n^*)(\partial_\mu \phi_n) - \left(\frac{n^2}{B^2 R^2} + \mu_0^2 \right) |\phi_n|^2 \right\}. \tag{34}$$

The effective 4D scalar Lagrangian $\mathcal{L}_{4,S} = B^{-1}\langle \mathcal{L}_S \rangle$ is then

$$\mathcal{L}_{4,S} = \sum_n \left\{ (\partial^\mu \phi_n^*)(\partial_\mu \phi_n) - \left(\frac{n^2}{B^3 R^2} + \frac{\mu_0^2}{B} \right) |\phi_n|^2 \right\} \tag{35}$$

and the mass of the n^{th} KK scalar boson excitation in the effective 4D theory is

$$m_{S,n} = \left(\frac{\mu_0^2}{B} + \frac{n^2}{B^3 R^2} \right)^{1/2}. \tag{36}$$

C. Fermions

Consider the fermionic Lagrangian

$$\mathcal{L}_F = \overline{\Psi}'(i\Gamma^M \partial_M - \mu_0)\Psi \tag{37}$$

where $\Psi = \Psi(x,y)$, $\overline{\Psi}' = \Psi^\dagger \Gamma^0$, and the Γ matrices are normalized according to $\{\Gamma^M, \Gamma^N\} = -2\tilde{g}^{MN}$, so that

$$\{\Gamma^\mu, \Gamma^\nu\} = -2\tilde{g}^{\mu\nu} = -2Bg^{\mu\nu}, \quad (\Gamma^{55})^2 = -(\tilde{g}^{55})^2 = B^{-2}. \tag{38}$$

As before, in order to pass to the effective 4D theory we define a set of γ matrices by

$$\Gamma^\mu = B^{1/2}\gamma^\mu, \quad \Gamma^5 = B^{-1}\gamma^5, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{39}$$

satisfying $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$ and $(\gamma^5)^2 = 1$. In terms of the γ matrices, the Lagrangian becomes

$$\begin{aligned} \mathcal{L}_F &= B^{1/2}\overline{\Psi}(B^{1/2} i\gamma^\mu \partial_\mu + B^{-1} i\gamma^5 \partial_5 - \mu_0)\Psi \\ &= B\overline{\Psi}(i\gamma^\mu \partial_\mu + B^{-3/2} i\gamma^5 \partial_5 - B^{-1/2}\mu_0)\Psi \end{aligned} \tag{40}$$

where $\overline{\Psi} = \Psi^\dagger \gamma^0 = B^{-1/2}\overline{\Psi}'$. In the dimensionally reduced 4D theory the term proportional to $\overline{\Psi}(i\gamma^5\partial_5)\Psi$ corresponds to a mass term, so that we require Ψ to be an eigenfunction of $i\gamma^5\partial_5$.

The field Ψ must therefore satisfy the periodicity condition $\Psi(x,y) = \Psi(x,y+2\pi R)$ and be an eigenfunction of $i\gamma^5\partial_5$. Let us introduce a chiral notation and write the field Ψ in the form

$$\begin{aligned} \Psi(x,y) &= \sum_{n=-\infty}^{\infty} \Psi_n(x,y) \\ &= \sum_{n=-\infty}^{\infty} \begin{pmatrix} \psi_{nL}(x)\xi_{nL}(y) \\ \psi_{nR}(x)\xi_{nR}(y) \end{pmatrix}, \quad \begin{pmatrix} \xi_{nL} = e^{i\alpha_L |n| y/R} \\ \xi_{nR} = e^{i\alpha_R |n| y/R} \end{pmatrix} \end{aligned} \tag{41}$$

where $\alpha_{L,R}$ each take a value of ± 1 in order to satisfy both the periodicity and eigenvalue conditions. Using the form of γ^5 given by Eq. (39), the eigenvalue condition

$$i\gamma^5 \partial_5 \Psi_n = \lambda_n \Psi_n \tag{42}$$

yields $\alpha_L = -1$, $\alpha_R = +1$, and $\lambda_n = -|n|/R$. We therefore have

$$\Psi_n = \begin{pmatrix} \psi_{nL}(x) e^{-i|n|y/R} \\ \psi_{nR}(x) e^{i|n|y/R} \end{pmatrix}. \tag{43}$$

We can perform the integration of $\mathcal{L}_F$ over y to obtain $\langle \mathcal{L}_F \rangle$. The orthogonality of the $\xi_n(y)$ functions can be used, and for a specific representation of γ matrices let us use, e.g.,

$$-i\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}.$$

Then for the effective 4D Lagrangian $\mathcal{L}_{4,F} = B^{-1}\langle \mathcal{L}_F \rangle$, we get

$$\mathcal{L}_{4,F} = \sum_n \overline{\psi}_n (i\gamma^\mu \partial_\mu - m_{F,n})\psi_n \tag{44}$$

where

$$\psi_n(x) = \begin{pmatrix} \psi_{nL}(x) \\ \psi_{nR}(x) \end{pmatrix}. \tag{45}$$

The mass of the nth KK excitation in the effective 4D theory is

$$m_{F,n}=\frac{\mu_0}{B^{1/2}}+\frac{|n|}{B^{3/2}R}. \tag{46}$$

D. Massive vector bosons

As in the zero mode description, the contribution to the Lagrangian from a U(1) gauge field A'_M that acquires a mass described by the parameter μ_0 in the 5D theory is

$$\mathcal{L}_G=-\frac{1}{4}\tilde{F}'^{MN}\tilde{F}'_{MN}+\frac{1}{2}\tilde{g}^{MN}\mu_0^2A'_MA'_N \tag{47}$$

where now $A'_M=A'_M(x,y)$. As before, we set $A'_5=0$ and introduce the field $A_\mu=B^{1/2}A'_\mu$, with the periodic field A_μ satisfying the periodicity requirement with a mode expansion

$$A_\mu(x,y)=\sum_{n=-\infty}^{\infty}A_\mu^n(x)e^{iny/R}. \tag{48}$$

Upon rewriting the Lagrangian in terms of the 4D metric $g_{\mu\nu}$ and the field A_μ, $\mathcal{L}_G$ assumes the form

$$\mathcal{L}_G=-\frac{1}{4}BF^{\mu\nu}F_{\mu\nu}+B^{-2}F_{\mu5}F^{\mu}{}_5-\frac{1}{2}B^{3/2}H_{\mu\nu}F^{\mu\nu}-\frac{1}{4}B^2H_{\mu\nu}H^{\mu\nu}+\frac{1}{2}\mu_0^2A_\mu A^\mu \tag{49}$$

where $H_{\mu\nu}$ is defined in Eq. (22). Inserting the mode expansion and integrating out the y dependence leaves

$$\langle\mathcal{L}_G^{(n)}\rangle=-\frac{1}{4}BF_{\mu\nu}^{*n}F^{n\mu\nu}+\frac{1}{2}\left(\frac{n^2}{B^2R^2}+\mu_0^2\right)A_\mu^{*n}A^{n\mu}-\frac{1}{2}B^{3/2}H_{\mu\nu}^{*n}F^{n\mu\nu}-\frac{1}{4}B^2H_{\mu\nu}^{*n}H^{n\mu\nu} \tag{50}$$

where $F_{\mu\nu}^n=\partial_\mu A_\nu^n-\partial_\nu A_\mu^n$, $A_\mu^{*n}=A_\mu^{-n}$, etc., and $\langle\mathcal{L}_G\rangle=\Sigma_n\langle\mathcal{L}_G^{(n)}\rangle$. The effective 4D Lagrangian $\mathcal{L}_{4,G}=B^{-1}\langle\mathcal{L}_G\rangle$ is therefore given by $\mathcal{L}_{4,G}=\Sigma_n\langle\mathcal{L}_{4,G}^{(n)}\rangle$, with

$$\begin{aligned}\mathcal{L}_{4,G}^{(n)}&=B^{-1}\langle\mathcal{L}_G^{(n)}\rangle\\&=-\frac{1}{4}F_{\mu\nu}^{*n}F^{n\mu\nu}+\frac{1}{2}\left(\frac{n^2}{B^3R^2}+\frac{\mu_0^2}{B}\right)A_\mu^{*n}A^{n\mu}\\&\quad-\frac{1}{2}B^{1/2}H_{\mu\nu}^{*n}F^{n\mu\nu}-\frac{1}{4}BH_{\mu\nu}^{*n}H^{n\mu\nu}.\end{aligned} \tag{51}$$

From this we identify the mass of the nth KK vector boson mode appearing in the effective 4D theory as

$$m_{G,n}=\left(\frac{\mu_0^2}{B}+\frac{n^2}{B^3R^2}\right)^{1/2}. \tag{52}$$

E. Kaluza-Klein excitation masses

We see from the above that the mass of the nth KK excitation has the form

$$m_n=\left\{\begin{array}{ll}\left(\dfrac{\mu_0^2}{B}+\dfrac{n^2}{B^3R^2}\right)^{1/2}, & \text{bosons}\\[2ex] \dfrac{\mu_0}{B^{1/2}}+\dfrac{|n|}{B^{3/2}R}, & \text{fermions}\end{array}\right\} \tag{53}$$

where μ_0 is the mass parameter appearing in the 5D theory, and $m_0=\mu_0/B^{1/2}$ is the zero mode mass. If the zero mode mass vanishes, then $m_n=|n|/B^{3/2}R$, and, in this case, $m_{n,out}/m_{n,in}=(B_{in}/B_{out})^{3/2}$. For a type I bubble where $B_{in}/B_{out}\gg1$, KK modes which may be too massive to be produced outside the bubble may be produced in the bubble's interior and can therefore help to stabilize the bubble against collapse. However, for a type II bubble where $B_{in}/B_{out}\ll1$, the KK modes would be expelled from the bubble's interior.

V. RADIATION STABILIZED BUBBLES

Let us consider a type I bubble with a high temperature interior that contains radiation modes comprised of photons as well as particles with masses $m_{in}\ll|\vec{p}|$. For a type II bubble, we take the limit where there are only photons inside. (In a type I bubble, there may be nonrelativistic heavy KK states as well, for example, but the energy density is assumed to be negligible in comparison to that of the relativistic species.) The mass of a particle outside of the type I bubble is $m_{out}\gg m_{in}$ for $B_{out}\ll B_{in}$, but particles in the bubble interior having energy $\omega\geq m_{out}$ can escape the bubble. The relative number of particles that escape over the lifetime of the bubble depends upon $m_{out}/T=\beta m_{out}$, assuming a thermal distribution of energies. (For $\beta m_{out}\gg1$ the fractional number of particles escaping the bubble will be small.) Let us assume that there is an effective number of spin degrees of freedom $\overline{g}=\overline{g}_B+\frac{7}{8}\overline{g}_F$ associated with radiation modes inside the bubble that tend to stabilize the bubble without quickly escaping, i.e., $\beta m_{out}\gg1$. The photons inside the bubble escape at a very low rate as well [3], due to the high photon opacity of the bubble wall. [The transmission coefficient for photons [3] is roughly $\mathcal{T}\sim O(B_</B_>)\ll1$, where $B_<$ $(B_>)$ is the smaller (larger) of B_{in} or B_{out}.] We therefore expect a bubble with a variety of radiation modes to be a metastable state with a lifetime τ that is at least as long as that of a photon stabilized bubble [3], i.e., $\tau\gtrsim R_{B,0}/\mathcal{T}$, where $R_{B,0}$ is the initial radius of the bubble.

We can estimate the equilibrium radius and mass of a stabilized bubble with an interior temperature T. We take the radiation energy density inside to be[1]

$$\rho_{Rad} = AT^4 = \bar{g}\frac{\pi^2}{30}T^4. \tag{54}$$

In addition, there may be a volume term $\mathcal{E}_V = \frac{4}{3}\pi R_B^3 \lambda$, where the value of the effective potential in the bubble's interior, $U(\varphi) = \lambda$, is taken to be a constant. Using this approximation, along with a thin wall approximation for the bubble wall, the expression for the bubble mass becomes

$$M = \mathcal{E}_{Rad} + \mathcal{E}_V + \mathcal{E}_{Wall} = \frac{4}{3}\pi R_B^3(\rho_{Rad} + \lambda) + 4\pi R_B^2 \sigma \tag{55}$$

where σ is the surface energy density of the wall. We further assume that the bubble, after it forms, will adjust its radius to reach an equilibrium state with an approximately adiabatic (isentropic) expansion or contraction. The radiation entropy density is $s \sim T^3$, so that if the bubble reaches equilibrium on a relatively small time scale, we can take $R_B T = \text{const}$ during the equilibration process. With these assumptions we obtain approximate expressions for an equilibrium radius R_B and an equilibrium mass M given by

$$R_B = \frac{6\sigma}{(AT^4 - 3\lambda)} = \frac{6\sigma}{(\rho_{Rad} - 3\lambda)} \tag{56}$$

and

$$M = 2\pi R_B^3\left(\rho_{Rad} - \frac{1}{3}\lambda\right) = 12\pi\sigma R_B^2\left(\frac{\rho_{Rad} - \frac{1}{3}\lambda}{\rho_{Rad} - 3\lambda}\right). \tag{57}$$

For a bubble that rapidly adjusts its size to reach equilibrium in an adiabatic manner so that, approximately, $R_B T = \text{const}$ during this adjustment period, a collapsing bubble rapidly heats up and an expanding bubble rapidly cools down, but the final temperature of the stabilized bubble must satisfy $\rho_{Rad} = AT^4 > 3\lambda$. On the other hand, it is assumed that the bubble's domain wall forms at some temperature T_c (which will depend upon model parameters) where a barrier forms in the effective potential to separate the two low energy states, so that a bubble at equilibrium should have an interior temperature $T < T_c$. In summary, for a stabilized bubble with radius and mass given by Eqs. (56) and (57), respectively, the interior temperature must lie within the range

$$\left(\frac{3\lambda}{A}\right)^{1/4} < T < T_c. \tag{58}$$

The lifetime of the bubble depends upon the rate at which photons and other high energy particles inside the bubble escape. This, in turn, depends upon the interactions of the particles inside the bubble and the rates at which lighter particles with $\omega > m_{out}$ are produced, and the rates at which they escape. However, we can reason that the lifetime of a bubble will be at least as long as that of a bubble stabilized by photons alone, in which case [3] $\tau \gtrsim R_{B,0}/\mathcal{T} \sim \mathcal{O}(B_>/B_<)R_{B,0}$.

If a bubble continues to shrink and the interior temperature increases above the critical temperature T_c (corresponding to the temperature at which the domain wall forms), the wall disappears, i.e., the bubble explodes. When the bubble explodes, its radiative contents consisting of high temperature photons, bosons, and fermions are suddenly released.

VI. 4D GRAVITATIONAL ASPECTS

We now turn attention to consider 4D gravitational aspects of dimension bubbles and argue that, under rather mild assumptions, the interior metric of a dimension bubble shows that its geometrical center is singularity free. This is in contrast to the situation found for a "gravitational bag" [4], which is basically an "empty," static idealization of a type I dimension bubble. In particular, the gravitational bag possesses a naked singularity at its center [4,5]. [The gravitational bag considered in Ref. [4] arises from a Freund-Rubin compactification of a 6D theory, but has the same essential aspects that we have in our 5D model. The *exact* solution obtained for the gravitational bag metric assumes that the effective potential vanishes in the bag's interior, corresponding to the extra dimension becoming infinitely large ($\varphi, B \to \infty$) there.] The gravitational bag contains only the scalar field φ that gives rise to the bubble wall, without any entrapped particles in the bubble's interior. We argue that, unlike a gravitational bag, the fluidic interior of a dimension bubble does not exhibit a naked singularity at its center.

The interior 4D geometry of the gravitational bag is described by [4,5]

$$ds^2 = C_1\left(\frac{ar-1}{ar+1}\right)^{2p} dt^2 - \frac{C_2}{a^4 r^4}\frac{(ar+1)^{2(p+1)}}{(ar-1)^{2(p-1)}}(dr^2 + r^2 d\Omega^2) \tag{59}$$

where r is an isotropic radial coordinate and C_1, C_2, a, and p are constants, with [4] $\frac{1}{2} < p < 1$. The center of geometry is located at $r = 1/a$, where the coefficient of $d\Omega^2$ goes to zero as $r \to 1/a$ from above, and the metric spawns a singularity there, corresponding to a naked singularity of the exact solution [4]. (See also Ref. [5].) In Ref. [4] it is found that although there is a singularity at the bag's center, the mass of the gravitational bag is finite.

Now, instead of a gravitational bag, consider a dimension bubble that is filled with particles trapped in the interior. We assume that the stress-energy of the φ field in the interior is negligible in comparison to that associated with the particles,

[1] It is assumed here, for simplicity, that not too many Kaluza-Klein modes are excited, so that we can use the familiar 4D expressions for radiation energy densities, etc. This condition is satisfied if, for instance, $T < (BR)^{-1}$, where R is the radius parameter associated with the extra dimension. (See, e.g., [10,11].)

and, furthermore, that we can approximate the bubble's contents by an isothermal fluid with pressure p and energy density ρ connected by $p=w\rho$, with w a constant. (For the case of radiation, $w=1/3$.) In this case, the 4D fluid-filled domain wall bubble can be viewed as a *cosmic balloon* [6,7]. The geometry inside the cosmic balloon is described by

$$ds^2=B(r)dt^2-A(r)dr^2-r^2d\Omega^2 \tag{60}$$

where the coordinate r is now a nonisotropic radial coordinate [the metric coefficient $B(r)$ used here is not to be confused with $|\tilde{g}_{55}|^{1/2}$]. The fluid in the bubble's interior has the important effect of removing the singularity at the bubble's center, $r=0$, provided that the fluid density $\rho(r)$ remains finite at the bubble's center. This can be seen by borrowing some of the results obtained by Wang (see [7] and references therein). Specifically, near the center we have

$$A(r)=[1-x(r)]^{-1}\approx\left[1-\frac{8}{3}\pi G\rho(0)r^2\right]^{-1} \tag{61}$$

where $\rho(0)$ is the central density and $x(r)\approx\frac{8}{3}\pi G\rho(0)r^2$. For $\rho(0)$ finite, we therefore have $A(0)=1$, and a naked singularity at the center is avoided. The coefficient $B(r)$ can be obtained from the equilibrium condition

$$\frac{B'}{B}=-\frac{2p'}{p+\rho}=-\left(\frac{2w}{1+w}\right)\frac{\rho'}{\rho} \tag{62}$$

where $'$ denotes differentiation with respect to r. Defining $\bar{t}=[8\pi G\rho(0)]^{1/2}r$, we have $(1/B)(dB/d\bar{t})\approx[4w/(1+w)]\bar{t}$ for $\bar{t}\ll 1$. Integration gives

$$B(r)\approx B(0)\exp\left\{\left(\frac{2w}{1+w}\right)(8\pi G\rho(0)r^2)\right\}. \tag{63}$$

Fluid-filled dimension bubbles, including radiation stabilized bubbles, therefore seem to have interior gravitational fields that are better behaved near the centers than those of gravitational bags.

VII. SUMMARY

An inhomogeneous compactification of a higher dimensional spacetime to 4D may result in the formation of "dimension bubbles." The sizes of the extra dimensions inside the bubble can be either larger (for type I) or smaller (for type II) than outside the bubble. Whether a dimension bubble is type I or type II depends upon the form of the effective potential $U(\varphi)$ controlling the size of the extra dimension. We can therefore have the situation where a type I dimension bubble encloses a "large" extra dimension and is surrounded by a "small" extra dimension, with the opposite holding true for a type II bubble. For example, if the size of an extra dimension BR varies between a Planck size and a TeV size, the extra dimensional scale factor B can change by 16 orders of magnitude. As a result of a dramatic variation of the size of the extra dimension across the bubble wall, the masses of various particle modes can vary dramatically between the inside and outside of the bubble, leading to an enhanced stabilization of a type I bubble by the entrapment of lighter particles inside or the expulsion of heavier particles from inside a type II bubble. The dependence of the particle mass upon the extra dimensional scale factor B has been demonstrated for bosons and fermions, including both Kaluza-Klein (KK) zero modes and KK excitation modes. [See Eq. (53).]

Both types of bubbles can be stabilized by photons, due to the high reflectivity of the bubble wall, and either type of dimension bubble may exist as a long-lived metastable state. Some basic features of plasma-filled bubbles have been examined and estimates obtained for the equilibrium radius and mass. The lifetime of a bubble with an initial radius $R_{B,0}$ is roughly estimated to be $\tau\gtrsim\mathcal{O}(B_>/B_<)R_{B,0}$, where $B_>(B_<)$ is the larger (smaller) value of B on the inside or outside of the bubble.

Finally, we have considered the 4D gravitational aspects of dimension bubbles to argue that, unlike the case for a "gravitational bag" [4,5] (which may be thought of as an empty, static idealization of a type I dimension bubble) the center of a fluid-filled dimension bubble is singularity free. This is seen by treating the dimension bubble as a *cosmic balloon* [6] and borrowing the results for the interior metric of a cosmic balloon [7]. Fluid-filled dimension bubbles are therefore seen to have better behaved interior gravitational fields than those of gravitational bags.

Since the existence of dimension bubbles depends upon a dramatic *change* in the sizes of extra dimensions across the bubble wall rather than upon the actual size of an extra dimension in any region, the detections of such solitons could provide evidence for the existence of extra dimensions, regardless of their sizes.

ACKNOWLEDGMENT

I wish to thank E.I. Guendelman for valuable comments.

[1] S.K. Blau and E.I. Guendelman, Phys. Rev. D **40**, 1909 (1989).
[2] J.R. Morris, Phys. Rev. D **67**, 025005 (2003).
[3] E.I. Guendelman and J.R. Morris, Phys. Rev. D **68**, 045008 (2003).
[4] A. Davidson and E.I. Guendelman, Phys. Rev. D **34**, 3264 (1986).
[5] E.I. Guendelman, Gen. Relativ. Gravit. **22**, 131 (1990).
[6] B. Holdom, Phys. Rev. D **49**, 3844 (1994).
[7] Y. Wang, Phys. Rev. D **49**, 5063 (1994).
[8] J.A. Frieman, G.B. Gelmini, M. Gleiser, and E.W. Kolb, Phys. Rev. Lett. **60**, 2101 (1988).
[9] M.A. Rubin and B.D. Roth, Phys. Lett. **127B**, 55 (1983).
[10] S. Randjbar-Daemi, A. Salam, and J. Strathdee, Phys. Lett. **135B**, 388 (1984).
[11] E.W. Kolb, "Cosmology and Extra Dimensions," FERMILAB-PUB-86-138-A, 1986.