Extension of $\mathcal{PT}$-symmetric quantum mechanics to quantum field theory with cubic interaction

Carl M. Bender,* Dorje C. Brody, and Hugh F. Jones
Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom
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It has recently been shown that a non-Hermitian Hamiltonian H possessing an unbroken $\mathcal{PT}$ symmetry (i) has a real spectrum that is bounded below, and (ii) defines a unitary theory of quantum mechanics with positive norm. The proof of unitarity requires a linear operator $\mathcal{C}$, which was originally defined as a sum over the eigenfunctions of H. However, using this definition to calculate $\mathcal{C}$ is cumbersome in quantum mechanics and impossible in quantum field theory. An alternative method is devised here for calculating $\mathcal{C}$ directly in terms of the operator dynamical variables of the quantum theory. This method is general and applies to a variety of quantum mechanical systems having several degrees of freedom. More importantly, this method is used to calculate the $\mathcal{C}$ operator in quantum field theory. The $\mathcal{C}$ operator is a time-independent observable in $\mathcal{PT}$-symmetric quantum field theory.

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I. INTRODUCTION

In 1998 numerical and perturbative methods were used to establish the surprising result that the class of non-Hermitian Hamiltonians

$$H=p^2+x^2(ix)^\epsilon \quad (\epsilon>0) \tag{1}$$

has a positive real spectrum [1]. In Ref. [1] it was conjectured that the spectral positivity was associated with the space-time reflection symmetry ($\mathcal{PT}$ symmetry) of the Hamiltonian. The Hamiltonian H in Eq. (1) is $\mathcal{PT}$ symmetric because under parity reflection $\mathcal{P}$ we have $x\to -x$ and $p\to -p$ and under time reversal $\mathcal{T}$ we have $x\to x$, $p\to -p$, and $i\to -i$. Other $\mathcal{PT}$-symmetric quantum mechanical models have been examined [2–6], and a proof of the positivity of the spectrum of H in Eq. (1) was subsequently given by Dorey *et al.* [7].

The discovery that the spectra of many $\mathcal{PT}$-symmetric Hamiltonians are real and positive raised a fundamental question: Does a non-Hermitian Hamiltonian such as H in Eq. (1) define a consistent unitary theory of quantum mechanics, or is the positivity of the spectrum merely an intriguing mathematical property of special classes of complex Sturm-Liouville eigenvalue problems? To answer this question it is necessary to know whether the Hilbert space on which the Hamiltonian acts has an inner product associated with a positive norm. Furthermore, it is necessary to determine whether the dynamical time evolution induced by such a Hamiltonian is unitary; that is, whether the norm is preserved in time.

Recently, a definitive answer to this question was found [8,9]. For a complex non-Hermitian Hamiltonian having an *unbroken* $\mathcal{PT}$ symmetry, a linear operator $\mathcal{C}$ that commutes with both H and $\mathcal{PT}$ can be constructed. The inner product with respect to $\mathcal{CPT}$ conjugation,

$$\langle\psi|\chi\rangle_{\mathcal{CPT}}=\int dx\,\psi^{\mathcal{CPT}}(x)\chi(x),$$

where $\psi^{\mathcal{CPT}}(x)=\int dy\,\mathcal{C}(x,y)\psi^*(-y)$, satisfies the requirements for the theory defined by H to have a Hilbert space with a positive norm and to be a consistent unitary theory of quantum mechanics. [The term *unbroken* $\mathcal{PT}$ symmetry means that every eigenfunction of H is also an eigenfunction of the $\mathcal{PT}$ operator. This condition guarantees that the eigenvalues of H are real. The Hamiltonian in Eq. (1) has an unbroken $\mathcal{PT}$ symmetry for all real $\epsilon\geq 0$.]

We emphasize that in a conventional quantum theory the inner product is formulated with respect to ordinary Dirac Hermitian conjugation (complex conjugate and transpose). Unlike conventional quantum theory, the inner product for a quantum theory defined by a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian depends on the Hamiltonian itself and is thus determined dynamically. One can view this new kind of quantum theory as a "bootstrap" theory because one must solve for the eigenstates of H before knowing what the Hilbert space and the associated inner product of the theory are. The Hilbert space and inner product are then determined by these eigenstates.

The key breakthrough in understanding these novel non-Hermitian quantum theories was the discovery of the operator $\mathcal{C}$ [9]. This operator possesses three crucial properties. First, it commutes with the space-time reflection operator $\mathcal{PT}$,

$$[\mathcal{C},\mathcal{PT}]=0, \tag{2}$$

although $\mathcal{C}$ does not commute with $\mathcal{P}$ or $\mathcal{T}$ separately. Second, the square of $\mathcal{C}$ is the identity,

$$\mathcal{C}^2=\mathbf{1}, \tag{3}$$

which allows us to interpret $\mathcal{C}$ as a reflection operator. Third, $\mathcal{C}$ commutes with H,

$$[\mathcal{C},H]=0, \tag{4}$$

*Permanent address: Department of Physics, Washington University, St. Louis, MO 63130, USA.

and thus is time independent. To summarize, $\mathcal{C}$ is a new time-independent $\mathcal{PT}$-symmetric reflection operator.

The question now is how to construct $\mathcal{C}$ for a given H. In Refs. [9,10] it was shown how to express the $\mathcal{C}$ operator in coordinate space as a sum over the appropriately normalized eigenfunctions $\phi_n(x)$ of the Hamiltonian H. These eigenfunctions satisfy

$$H\phi_n(x)=E_n\phi_n(x), \tag{5}$$

and, without loss of generality, their overall phases are chosen so that

$$\mathcal{PT}\phi_n(x)=\phi_n(x). \tag{6}$$

With this choice of phase, the eigenfunctions are then normalized according to

$$\int_C dx[\phi_n(x)]^2=(-1)^n. \tag{7}$$

The contour of integration C is described in detail in Ref. [9]. For the quantum mechanical theories discussed in this paper, all of which have a cubic interaction term, the contour C can be taken to lie along the real-x axis.

In terms of the eigenfunctions defined above, the statement of completeness for a theory described by a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian reads [9]

$$\sum_n (-1)^n\phi_n(x)\phi_n(y)=\delta(x-y) \tag{8}$$

for real x and y. The coordinate-space representation of $\mathcal{C}$ is [9]

$$\mathcal{C}(x,y)=\sum_n \phi_n(x)\phi_n(y). \tag{9}$$

Only a *non-Hermitian* $\mathcal{PT}$-symmetric Hamiltonian possesses a $\mathcal{C}$ operator distinct from the parity operator $\mathcal{P}$. Indeed, if one evaluates the summation (9) for a $\mathcal{PT}$-symmetric Hamiltonian that is also Hermitian, the result is $\mathcal{P}$, which in coordinate space is $\delta(x+y)$.

The coordinate-space formalism using Eq. (9) has been applied successfully to

$$H=\frac{1}{2}p^2+\frac{1}{2}\mu^2x^2+i\epsilon x^3, \tag{10}$$

and $\mathcal{C}$ was constructed perturbatively to order ϵ^3 [11]. This formalism has also been applied to calculate $\mathcal{C}$ to order ϵ for the complex Hénon-Heiles Hamiltonian [12]

$$H=\frac{1}{2}(p_x^2+p_y^2)+\frac{1}{2}(x^2+y^2)+i\epsilon x^2y, \tag{11}$$

which has two degrees of freedom, and for the Hamiltonian

$$H=\frac{1}{2}(px^2+p_y^2+p_z^2)+\frac{1}{2}(x^2+y^2+z^2)+i\epsilon xyz, \tag{12}$$

which has three degrees of freedom [13].

Calculating the operator $\mathcal{C}$ by direct evaluation of the sum in Eq. (9) is difficult in quantum mechanics because it is necessary to determine all the eigenfunctions of H. Such a procedure cannot be used at all in quantum field theory because there is no simple analogue of the Schrödinger eigenvalue problem (5) and its associated coordinate-space eigenfunctions.

In this paper we devise an elementary operator technique for calculating $\mathcal{C}$ for the important case of quantum theories having *cubic* interactions, and we demonstrate that our new method readily generalizes from quantum mechanics to quantum field theory. In Sec. II we introduce a general operator representation for $\mathcal{C}$ of the form $e^{Q(x,p)}\mathcal{P}$, where x and p are the dynamical variables. This representation is especially convenient for incorporating the three requirements (2)–(4). In Sec. III we calculate $\mathcal{C}$ to seventh order in powers of ϵ for the Hamiltonian (10) using this operator technique. Then, in Sec. IV we calculate $\mathcal{C}$ for the Hamiltonians (11) and (12) to order ϵ^3. In Sec. V we apply operator methods to calculate $\mathcal{C}$ for the massless Hamiltonian $H=\frac{1}{2}p^2+ix^3$. We derive recursion relations for the operator representation of $\mathcal{C}$ in Sec. VI. In Sec. VII we calculate $\mathcal{C}$ to order ϵ^2 for the self-interacting scalar quantum field theory described by the Hamiltonian

$$H=\int d^Dx\left\{\frac{1}{2}\pi^2(\mathbf{x},t)+\frac{1}{2}[\nabla_{\mathbf{x}}\varphi(\mathbf{x},t)]^2+\frac{1}{2}\mu^2\varphi^2(\mathbf{x},t)+i\epsilon\varphi^3(\mathbf{x},t)\right\} \tag{13}$$

in $(D+1)$-dimensional Minkowski spacetime. In Sec. VIII we calculate $\mathcal{C}$ for cubic scalar quantum field theories with interactions of the form $i\epsilon\varphi_1^2\varphi_2$ and $i\epsilon\varphi_1\varphi_2\varphi_3$. An alternative perturbative calculation of $\mathcal{C}$ for an $i\epsilon\varphi^3$ quantum field theory using diagrammatic and combinatoric methods is given in the Appendix. Some concluding remarks are in Sec. IX.

The principal accomplishment of this paper is the derivation in Secs. VII and VIII and the Appendix of the $\mathcal{C}$ operator for cubic quantum field theories. Cubic quantum field theories, such as that in Eq. (13), are not just of mathematical interest. Such theories emerge in the study of Reggeon field theory [14] and in the analysis of the Lee-Yang edge singularity [15]. For these quantum field theories the operator $\mathcal{C}$ is a new conserved quantity. Knowing how to calculate this operator is crucial because it is necessary to have $\mathcal{C}$ in order to construct observables and to evaluate matrix elements of field operators. Our calculation of $\mathcal{C}$ is a major step in our ongoing program to obtain new physical models by extending conventional quantum mechanics and quantum field theory into the complex domain.

II. GENERAL FORM FOR THE OPERATOR $\mathcal{C}$

To prepare for calculating $\mathcal{C}$ we show in this section that it is advantageous to represent $\mathcal{C}$ as a product of the exponential of a Hermitian operator Q and the parity operator $\mathcal{P}$:

$$\mathcal{C}=e^{Q(x,p)}\mathcal{P}. \tag{14}$$

This representation was first noticed in Ref. [11].

A. Previous work on calculating $\mathcal{C}$

The objective of the investigation in Ref. [11] was to use perturbative methods to calculate $\mathcal{C}$ for the Hamiltonian $H=\frac{1}{2}p^2+\frac{1}{2}x^2+i\epsilon x^3$, where ϵ is treated as a small parameter. In Ref. [11] the operator $\mathcal{C}$ was obtained to third order in ϵ in coordinate space. The procedure was as follows. First, the Schrödinger equation

$$-\frac{1}{2}\phi_n''(x)+\frac{1}{2}x^2\phi_n(x)+i\epsilon x^3\phi_n(x)=E_n\phi_n(x) \tag{15}$$

was solved for the energies E_n and for the wave functions $\phi_n(x)$ as Rayleigh-Schrödinger perturbation series in powers of ϵ. The series for $\phi_n(x)$ has the form

$$\phi_n(x)=i^n\frac{a_n}{\pi^{1/4}2^{n/2}\sqrt{n!}}e^{-x^2/2}[H_n(x)-iA_n(x)\epsilon-B_n(x)\epsilon^2+iC_n(x)\epsilon^3+\cdots], \tag{16}$$

where $H_n(x)$ is the nth Hermite polynomial and $A_n(x)$, $B_n(x)$, and $C_n(x)$ are polynomials in x of degree $n+3$, $n+6$, and $n+9$, respectively. These polynomials were expressed as linear combinations of Hermite polynomials. The value of a_n,

$$a_n=1+\frac{1}{144}(2n+1)(82n^2+82n+87)\epsilon^2+\mathcal{O}(\epsilon^4), \tag{17}$$

ensures that the eigenfunctions are normalized according to $\int_{-\infty}^{\infty}dx\,\phi_n^2(x)=(-1)^n$, as in Eq. (7). The factor i^n in Eq. (16) is included to satisfy the requirement in Eq. (6) that $\mathcal{PT}\phi_n(x)=\phi_n(x)$.

Finally, Eq. (16) was substituted into Eq. (9) and the summation over n was performed to obtain the operator $\mathcal{C}(x,y)$ to order ϵ^3:

$$\mathcal{C}(x,y)=\Bigg\{1-\epsilon\left(\frac{4}{3}p^3-2xyp\right)+\epsilon^2\left[\frac{8}{9}p^6-\frac{8}{3}xyp^4+(2x^2y^2-12)p^2\right]-\epsilon^3\left[\frac{32}{81}p^9-\frac{16}{9}xyp^7+\left(\frac{8}{3}x^2y^2-\frac{176}{5}\right)p^5-\left(\frac{4}{3}x^3y^3-48xy\right)p^3-(8x^2y^2-64)p\right]+\mathcal{O}(\epsilon^4)\Bigg\}\delta(x+y), \tag{18}$$

where $p=-id/dx$. This expression is complicated, but it was observed that it simplifies considerably when the expression in curly brackets is rewritten in exponential form:

$$\mathcal{C}(x,y)=e^{\epsilon Q_1+\epsilon^3Q_3+\cdots}\delta(x+y)+\mathcal{O}(\epsilon^5). \tag{19}$$

In this form the differential operators Q_1 and Q_3 are simply

$$Q_1=-\frac{4}{3}p^3-2xpx,$$

$$Q_3=\frac{128}{15}p^5+\frac{40}{3}xp^3x+8x^2px^2-12p. \tag{20}$$

The main features of the exponential representation (19) are that only odd powers of ϵ appear in the exponent, the coefficients are all real, and the derivative operators act on the parity operator $\delta(x+y)$. Also, $e^{\epsilon Q_1+\epsilon^3Q_3}$ is Hermitian.

B. A new approach to calculating $\mathcal{C}$

The perturbative calculation described above suggests that a simpler and more direct way to calculate $\mathcal{C}$ is to seek an operator representation of it in the form $e^{Q(x,p)}\mathcal{P}$, where $Q(x,p)$ is a Hermitian function of the operators x and p. We will show that $Q(x,p)$ can be found by solving elementary operator equations and that it is not necessary to find the eigenfunctions to determine Q. Thus, the technique introduced in this paper immediately generalizes to quantum field theory. To find the operator equations satisfied by Q we substitute $\mathcal{C}=e^Q\mathcal{P}$ into the three equations (2)–(4). The details are described in Sec. III.

We claim that the representation $\mathcal{C}=e^Q\mathcal{P}$ is general. Let us illustrate this simple representation for $\mathcal{C}$ in two elementary cases: First, consider the shifted harmonic oscillator

$$H=\frac{1}{2}p^2+\frac{1}{2}x^2+i\epsilon x. \tag{21}$$

This Hamiltonian has an unbroken $\mathcal{PT}$ symmetry for all real ϵ. Its eigenvalues $E_n=n+\frac{1}{2}+\frac{1}{2}\epsilon^2$ are all real. The $\mathcal{C}$ operator for this theory is given exactly by $\mathcal{C}=e^Q\mathcal{P}$, where $Q=-\epsilon p$. Note that in the limit $\epsilon\to 0$, where the Hamiltonian becomes Hermitian, $\mathcal{C}$ becomes identical with $\mathcal{P}$.

As a second example, consider the non-Hermitian 2×2 matrix Hamiltonian

$$H=\begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta}\end{pmatrix}, \tag{22}$$

which was discussed in Ref. [10]. This Hamiltonian is $\mathcal{PT}$ symmetric, where $\mathcal{P}$ is the Pauli matrix $\sigma_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\mathcal{T}$ is complex conjugation. This Hamiltonian has an unbroken $\mathcal{PT}$ symmetry when $s^2\geq r^2\sin^2\theta$. The $\mathcal{C}$ operator in the unbroken region is

$$\mathcal{C}=\frac{1}{\cos\alpha}\begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha\end{pmatrix}, \tag{23}$$

where $\sin\alpha=(r/s)\sin\theta$. Our new way to express $\mathcal{C}$ is to rewrite it in the form $\mathcal{C}=e^Q\mathcal{P}$. Thus, the Hermitian operator Q has the form

$$Q=\frac{1}{2}\sigma_2\ln\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right), \tag{24}$$

where $\sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$. Again, observe that in the limit $\theta\to 0$, where the Hamiltonian becomes Hermitian, the $\mathcal{C}$ operator becomes identical with $\mathcal{P}$.

Note that $\mathcal{P}$, which is given in coordinate space as $\delta(x+y)$, can be expressed in terms of the fundamental operators x and p as

$$\mathcal{P}(x,p)=\exp\left[\frac{1}{2}i\pi(p^2+x^2-1)\right]. \tag{25}$$

To show that the parity operator satisfies $\mathcal{P}x\mathcal{P}^{-1}=-x$ and $\mathcal{P}p\mathcal{P}^{-1}=-p$, we define the operator-valued functions $f(\tau)$ and $g(\tau)$ as

$$f(\tau)=e^{i\tau(p^2+x^2)}xe^{-i\tau(p^2+x^2)},$$

$$g(\tau)=e^{i\tau(p^2+x^2)}pe^{-i\tau(p^2+x^2)}. \tag{26}$$

Differentiating $f(\tau)$ and $g(\tau)$ once gives $f'(\tau)=2g(\tau)$ and $g'(\tau)=-2f(\tau)$. A second differentiation then leads to the differential equations $f''(\tau)=4f(\tau)$ and $g''(\tau)=4g(\tau)$. The solutions to these equations satisfying the initial conditions $f(0)=x$ and $g(0)=p$ are

$$f(\tau)=x\cos(2\tau)-ip\sin(2\tau),$$

$$g(\tau)=p\cos(2\tau)-ix\sin(2\tau). \tag{27}$$

Setting $\tau=\frac{1}{2}\pi$, we get $f(\tau)=-x$ and $g(\tau)=-p$, which establishes that the operator $\mathcal{P}$ defined in Eq. (25) indeed has the properties of a parity reflection operator. Specifically, $\mathcal{P}$ is a unitary operator that generates a rotation by π in the (x,p) plane. Another application of $\mathcal{P}$ gives a rotation by 2π in the (x,p) plane. Hence $\mathcal{P}^2=1$. This procedure determines $\mathcal{P}$ up to an additive phase. It is conventional to choose the phase to be $-\frac{1}{2}\pi$, as in Eq. (25).

III. CUBIC OSCILLATOR WITH ONE DEGREE OF FREEDOM

Having shown that Eq. (14) is a natural way to represent the operator $\mathcal{C}$, we now demonstrate how to use this ansatz to calculate $\mathcal{C}$ for the Hamiltonian (10). The procedure is to impose the three conditions (2)–(4) in turn on $\mathcal{C}=e^{Q(x,p)}\mathcal{P}$ and thereby to determine the operator-valued function $Q(x,p)$.

First, we substitute Eq. (14) into the condition (2) to obtain

$$e^{Q(x,p)}=\mathcal{PT}e^{Q(x,p)}\mathcal{PT}=e^{Q(-x,p)},$$

from which we conclude that $Q(x,p)$ is an *even* function of x. Second, we substitute Eq. (14) into the condition (3) and find that

$$e^{Q(x,p)}\mathcal{P}e^{Q(x,p)}\mathcal{P}=e^{Q(x,p)}e^{Q(-x,-p)}=1,$$

which implies that $Q(x,p)=-Q(-x,-p)$. Since we already know that $Q(x,p)$ is an even function of x, we conclude that it is also an *odd* function of p.

The remaining condition (4) to be imposed is that the operator $\mathcal{C}$ commutes with H. Substituting $\mathcal{C}=e^{Q(x,p)}\mathcal{P}$ into Eq. (4), we get $e^{Q(x,p)}[\mathcal{P},H]+[e^{Q(x,p)},H]\mathcal{P}=0$. All of the Hamiltonians H considered in this paper can be expressed in the form $H=H_0+\epsilon H_1$, where H_0 is a free field theory (harmonic oscillator) Hamiltonian that commutes with the parity operator $\mathcal{P}$, and H_1 represents the interaction. For example, for the Hamiltonian $H=\frac{1}{2}p^2+\frac{1}{2}\mu^2x^2+i\epsilon x^3$, $H_0=\frac{1}{2}p^2+\frac{1}{2}\mu^2x^2$ and $H_1=ix^3$. Then this condition reads

$$\epsilon e^{Q(x,p)}[\mathcal{P},H_1]+[e^{Q(x,p)},H]\mathcal{P}=0. \tag{28}$$

Next, we observe that if the interaction is *cubic*, then H_1 is *odd* under parity reflection; that is, H_1 *anticommutes* with $\mathcal{P}$. Hence, for quantum theories with cubic interaction Hamiltonians, Eq. (28) reduces to

$$2\epsilon e^{Q(x,p)}H_1=[e^{Q(x,p)},H]. \tag{29}$$

We note that the structure in Eq. (19) is quite general; for all cubic Hamiltonians, $Q(x,p)$ may be expanded as a series in odd powers of ϵ:

$$Q(x,p)=\epsilon Q_1(x,p)+\epsilon^3Q_3(x,p)+\epsilon^5Q_5(x,p)+\cdots. \tag{30}$$

In quantum field theory we will interpret the series coefficients Q_{2n+1} as interaction vertices (form factors) of $2n+3$ powers of the quantum fields.

Substituting the expansion in Eq. (30) into the exponential $e^{Q(x,p)}$, we get

$$e^{Q(x,p)}\equiv R(x,p)=1+R_1(x,p)\epsilon+R_2(x,p)\epsilon^2+R_3(x,p)\epsilon^3+R_4(x,p)\epsilon^4+\cdots, \tag{31}$$

where

$$R_1=Q_1,$$

$$R_2=\frac{1}{2}Q_1^2,$$

$$R_3=Q_3+\frac{1}{6}Q_1^3,$$

$$R_4=\frac{1}{2}\{Q_1,Q_3\}+\frac{1}{24}Q_1^4,$$

$$R_5=Q_5+\frac{1}{6}(\{Q_1^2,Q_3\}+Q_1Q_3Q_1)+\frac{1}{120}Q_1^5,$$

$$R_6=\frac{1}{2}(Q_3^2+\{Q_1,Q_5\})+\frac{1}{24}(\{Q_1^3,Q_3\}+\{Q_1,Q_1Q_3Q_1\})+\frac{1}{720}Q_1^6, \tag{32}$$

and so on. Here, $\{X,Y\}=XY+YX$ denotes the anticommutator.

We now substitute Eq. (32) into Eq. (29), collect the coefficients of like powers of ϵ^n for $n=1,2,3,\ldots$, and obtain a sequence of equations of the general form

$$\epsilon^n:\ \ [H_0,R_n]=-\{H_1,R_{n-1}\}\ \ (n\geqslant 1), \tag{33}$$

where $R_0\equiv 1$.

The equations in (33) can be solved systematically for the operator-valued functions $Q_n(x,p)$ $(n=1,3,5,\ldots)$ subject to the symmetry constraints that ensure the conditions (2) and (3). Note that the coefficients of even powers of ϵ contain no additional information because the equation arising from the coefficient of ϵ^{2n} can be derived from the equations arising from the coefficients of $\epsilon^{2n-1},\epsilon^{2n-3},\ldots,\epsilon$. This observation leads to a more effective way to express the conditions in Eq. (33). The first four equations read

$$\begin{aligned}
&[H_0,Q_1]=-2H_1,\\
&[H_0,Q_3]=-\frac{1}{6}[Q_1,[Q_1,H_1]],\\
&[H_0,Q_5]=\frac{1}{360}[Q_1,[Q_1,[Q_1,[Q_1,H_1]]]]\\
&\qquad -\frac{1}{6}([Q_1,[Q_3,H_1]]+[Q_3,[Q_1,H_1]]),\\
&[H_0,Q_7]=\frac{1}{15120}[Q_1,[Q_1,[Q_1,[Q_1,[Q_1,[Q_1,H_1]]]]]]\\
&\qquad -\frac{1}{360}([Q_1,[Q_1,[Q_1,[Q_3,H_1]]]]\\
&\qquad +[Q_1,[Q_1[Q_3,[Q_1,H_1]]]]\\
&\qquad +[Q_1,[Q_3,[Q_1,[Q_1,H_1]]]]\\
&\qquad +[Q_3,[Q_1[Q_1,[Q_1,H_1]]]])\\
&\qquad +\frac{1}{6}([Q_1,[Q_5,H_1]]\\
&\qquad +[Q_5,[Q_1,H_1]])+\frac{1}{6}[Q_3,[Q_3,H_1]].
\end{aligned} \tag{34}$$

We now show how to solve these equations for the Hamiltonian in Eq. (10), for which $H_0=\frac{1}{2}p^2+\frac{1}{2}\mu^2x^2$ and $H_1=ix^3$. The procedure is to substitute the most general polynomial form for Q_n using arbitrary coefficients and then to solve for these coefficients. For example, to solve the first of the equations in (34), $[H_0,Q_1]=-2ix^3$, we take as an ansatz for Q_1 the most general Hermitian cubic polynomial that is even in x and odd in p:

$$Q_1(x,p)=Mp^3+Nxpx, \tag{35}$$

where M and N are undetermined coefficients. The operator equation for Q_1 is satisfied if

$$M=-\frac{4}{3}\mu^{-4}\quad\text{and}\quad N=-2\mu^{-2}. \tag{36}$$

It is straightforward, though somewhat tedious, to continue this process. In order to present the solutions for $Q_n(x,p)$ $(n>1)$, it is convenient to introduce the following notation: Let $S_{m,n}$ represent the *totally symmetrized* sum over all terms containing m factors of p and n factors of x. For example,

$$\begin{aligned}
&S_{0,0}=1,\\
&S_{0,3}=x^3,\\
&S_{1,1}=\frac{1}{2}(xp+px),\\
&S_{1,2}=\frac{1}{3}(x^2p+xpx+px^2),\\
&S_{3,1}=\frac{1}{4}(xp^3+pxp^2+p^2xp+p^3x),\\
&S_{2,2}=\frac{1}{6}(p^2x^2+x^2p^2+pxpx+xpxp+px^2p+xp^2x).
\end{aligned} \tag{37}$$

The properties of the operators $S_{m,n}$ are summarized in Ref. [17]. One useful property is that $S_{m,n}$ can be expressed in Weyl-ordered form in two ways:

$$S_{m,n}=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}x^kp^mx^{n-k}=\frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}p^kx^np^{m-k}. \tag{38}$$

We have solved the equations in (34) and have found Q_1, Q_3, Q_5, and Q_7 in closed form. In terms of the symmetrized operators $S_{m,n}$ the functions Q_n are

$$\begin{aligned}
&Q_1=-\frac{4}{3}\mu^{-4}p^3-2\mu^{-2}S_{1,2},\\
&Q_3=\frac{128}{15}\mu^{-10}p^5+\frac{40}{3}\mu^{-8}S_{3,2}+8\mu^{-6}S_{1,4}-12\mu^{-8}p,\\
&Q_5=-\frac{320}{3}\mu^{-16}p^7-\frac{544}{3}\mu^{-14}S_{5,2}-\frac{512}{3}\mu^{-12}S_{3,4}\\
&\qquad -64\mu^{-10}S_{1,6}+\frac{24\,736}{45}\mu^{-14}p^3+\frac{6\ \ 368}{15}\mu^{-12}S_{1,2},
\end{aligned}$$

$$Q_7=\frac{553\,984}{315}\mu^{-22}p^9+\frac{97\,792}{35}\mu^{-20}S_{7,2}+\frac{377\,344}{105}\mu^{-18}S_{5,4}+\frac{721\,024}{315}\mu^{-16}S_{3,6}+\frac{1792}{3}\mu^{-14}S_{1,8}-\frac{2\,209\,024}{105}\mu^{-20}p^5-\frac{2\,875\,648}{105}\mu^{-18}S_{3,2}-\frac{390\,336}{35}\mu^{-16}S_{1,4}+\frac{46\,976}{5}\mu^{-18}p. \tag{39}$$

Combining Eqs. (14), (30), and (39), we obtain an explicit perturbative expansion of $\mathcal{C}$ in terms of the fundamental operators x and p, correct to order ϵ^7.

To summarize, using the ansatz (14) we are able to calculate the $\mathcal{C}$ operator to very high order in perturbation theory. We are able to perform this calculation because this ansatz obviates the necessity of calculating the wave functions $\phi_n(x)$. The calculation bears a strong resemblance to WKB theory. The ansatz used in performing a semiclassical calculation is also an exponential of a power series. The advantage of using WKB theory to calculate the energy eigenvalues is that *to all orders* in powers of $\hbar$ it is possible to construct a system of equations like those in (34) that determine the energies, and it is never necessary to calculate the wave function [18]. Furthermore, only the even terms in the WKB series are needed to determine the energy eigenvalues. The odd terms in the series drop out of the calculation and provide no information about the eigenvalues [18]. The difference between a conventional WKB series and the series representation for Q is that the first term in a WKB series is proportional to $\hbar^{-1}$ while the series expansion for $Q(x,p)$ contains only positive powers of ϵ. Based on the results in Ref. [11], however, we believe that for a $\mathcal{PT}$-symmetric $-\epsilon x^4$ theory, the first term in the expansion of $Q(x,p)$ is proportional to ϵ^{-1}. We plan to discuss quartic $\mathcal{PT}$-symmetric theories in a future paper.

IV. CUBIC OSCILLATORS WITH SEVERAL DEGREES OF FREEDOM

In this section we extend the operator techniques used in Sec. III to systems having two and three dynamical degrees of freedom. Specifically, we generalize the perturbative procedure for calculating the $\mathcal{C}$ operator for the Hamiltonian in Eq. (10) and use it to calculate $\mathcal{C}$ for the Hamiltonians in Eqs. (11) and (12).

Let us first consider H in Eq. (11), which has two degrees of freedom. We write this Hamiltonian in the form $H=H_0+\epsilon H_1$, where $H_0=\frac{1}{2}p^2+\frac{1}{2}q^2+\frac{1}{2}x^2+\frac{1}{2}y^2$ and $H_1=ix^2y$. For these operators we need to solve the system of equations in (34) for the unknown operators Q_1, Q_3, and so on. To simplify the calculation we generalize slightly the notation in Eq. (37) for totally symmetric operators; to wit, we continue to use $S_{m,n}$ to represent a totally symmetric product of m factors of p and n factors of x but we use $T_{m,n}$ to represent a totally symmetric product of m factors of q and n factors of y. For example,

$$T_{1,1}=\frac{1}{2}(qy+yq),$$

$$T_{1,2}=\frac{1}{3}(y^2q+yqy+qy^2)=yqy,$$

$$T_{3,1}=\frac{1}{4}(yq^3+qyq^2+q^2yq+q^3y),$$

$$T_{2,1}=\frac{1}{3}(q^2y+qyq+yq^2)=qyq. \tag{40}$$

To solve $[H_0,Q_1]=-2H_1$, the first equation in (34), we seek a Hermitian cubic polynomial in the variables x, y, p, and q. This polynomial must be even in the coordinate variables and odd in the momentum variables, and it must be able to yield H_1 when commuted with H_0. We therefore introduce the ansatz

$$Q_1(x,y,p,q)=Mp^2q+N_1S_{1,1}y+N_2x^2q. \tag{41}$$

We substitute this ansatz into the commutator and determine the unknown constants M, N_1, and N_2 by solving three linear equations. The result is

$$M=-\frac{4}{3},\quad N_1=-\frac{2}{3},\quad N_2=-\frac{2}{3}. \tag{42}$$

Next, we turn to the second of the equations in (34), $[H_0,Q_3]=-\frac{1}{6}[Q_1,[Q_1,H_1]]$, and evaluate its right side. The resulting equation for Q_3 then reads

$$[H_0,Q_3]=i\frac{8}{27}(4x^4y+4x^2y^3+8p^2T_{2,1}-4x^2T_{2,1}+4S_{1,3}q+8S_{1,1}T_{1,2}+8S_{3,1}q-3y). \tag{43}$$

We now must construct the most general Hermitian fifth-degree polynomial in the variables x, y, p, and q that is even in the coordinate variables, odd in the momentum variables, and has the terms needed to produce the right side of this commutation relation:

$$Q_3(x,y,p,q)=a_1p^2q^3+a_2p^4q+a_3S_{1,1}T_{2,1}+a_4p^2T_{1,2}+a_5S_{3,1}y+a_6S_{2,2}q+a_7x^2q^3+a_8x^2T_{1,2}+a_9S_{1,1}y^3+a_{10}S_{1,3}y+a_{11}x^4q+a_{12}q. \tag{44}$$

Substituting this ansatz into Eq. (43), we obtain twelve simultaneous linear equations for the unknown coefficients a_n, whose solution is

$$a_1=\frac{512}{405},\quad a_2=\frac{512}{405},\quad a_3=\frac{1088}{405},$$

$$a_4=-\frac{256}{405},\quad a_5=\frac{512}{405},\quad a_6=\frac{288}{405},$$

$$a_7=-\frac{32}{405},\quad a_8=\frac{736}{405},\quad a_9=-\frac{256}{405},$$

$$a_{10}=\frac{608}{405},\quad a_{11}=-\frac{128}{405},\quad a_{12}=-\frac{8}{9}. \tag{45}$$

This completes the calculation of the operator $\mathcal{C}$ to third order in ϵ for H in Eq. (11).

The attractive feature of the calculational procedure described here is that it is utterly routine and works in every order of perturbation theory. In contrast, the technique used in Ref. [13] to calculate $\mathcal{C}$ becomes hopelessly difficult beyond first order in powers of ϵ because the technique used earlier requires that one calculate all of the energy eigenstates $\phi_n(x,y)$ perturbatively for all n. These eigenstates must then be substituted directly into the summation in Eq. (9) that defines the operator $\mathcal{C}$. This calculation is difficult because beyond leading order in ϵ one encounters the challenging problems associated with degenerate energy levels. (There are no degenerate energy levels for Hamiltonians having just one degree of freedom.) Of course, for any given n, there is a well-defined procedure for calculating the eigenstate to any order in powers of ϵ. However, this procedure depends on the value of n and, as a result, the calculation becomes extremely complicated. The method of calculation presented here works because it is no longer necessary to calculate the eigenfunctions. Thus, the difficulties associated with degeneracy are circumvented.

We turn next to the case of a cubic oscillator having three degrees of freedom. We express the Hamiltonian in Eq. (12) in the form $H=H_0+\epsilon H_1$, where $H_0=\frac{1}{2}p^2+\frac{1}{2}q^2+\frac{1}{2}r^2+\frac{1}{2}x^2+\frac{1}{2}y^2+\frac{1}{2}z^2$ represents a harmonic oscillator Hamiltonian having three degrees of freedom and $H_1=ixyz$ is a non-Hermitian $\mathcal{PT}$-symmetric interaction term.

To solve $[H_0,Q_1]=-2H_1$, the first of the equations in (34), we must construct the most general Hermitian cubic polynomial in the variables x, y, z, p, q, and r that is even in the coordinate variables, odd in the momentum variables, and has the terms needed to yield H_1 on the right side of this commutation relation:

$$Q_1(x,y,z,p,q,r)=Mpqr+N(yzp+xzq+xyr). \tag{46}$$

We then substitute this ansatz into the commutator and determine the unknown constants M and N by solving two linear equations. The result is

$$M=-\frac{4}{3}\quad\text{and}\quad N=-\frac{2}{3}.$$

To solve the second of the equations in (34), $[H_0,Q_3]=-\frac{1}{6}[Q_1,[Q_1,H_1]]$, we evaluate its right side. The resulting equation for Q_3 then reads

$$\begin{aligned}[H_0,Q_3]=i\frac{8}{27}[&xyz(x^2+y^2+z^2)+(pz+rx)yqy\\&+(py+qx)zrz+(qz+ry)xpx\\&-(pxpyz+qyqxz+rzrxy)\\&+2(rqpxp+rpqyq+pqrzr)].\end{aligned} \tag{47}$$

The most general Hermitian fifth-degree polynomial in the variables x, y, z, p, q, and r that is even in the coordinate variables, odd in the momentum variables, and has the terms needed to produce the right side of this commutation relation is

$$\begin{aligned}&Q_3(x,y,z,p,q,r)\\&\quad=d_1(p^3qr+q^3pr+r^3qp)+d_2[pxp(yr+zq)\\&\quad+qyq(xr+zp)+rzr(xq+yp)]+d_3(xpxqr\\&\quad+ypypr+zrzpq)+d_4(xpxyz+yqyxz+zrzxy)\\&\quad+d_5[x^3(yr+zq)+y^3(xr+zp)+z^3(xq+yp)]\\&\quad+d_6(p^3yz+q^3xz+r^3xy).\end{aligned} \tag{48}$$

Substituting this ansatz into Eq. (43), we obtain six simultaneous equations for the unknown coefficients d_n, whose solution is

$$d_1=\frac{128}{405},\quad d_2=\frac{136}{405},\quad d_3=-\frac{64}{405},$$

$$d_4=\frac{184}{405},\quad d_5=-\frac{32}{405},\quad d_6=-\frac{8}{405}. \tag{49}$$

This completes the calculation of the operator $\mathcal{C}$ to third order in ϵ for H in Eq. (12). This calculation is simpler than that for the Hamiltonian in Eq. (11) because there is symmetry under the interchange of pairs of dynamical variables, such as $(x,p)\leftrightarrow(y,q)$.

V. REPRESENTATION OF $\mathcal{C}$ FOR THE MASSLESS CASE

In this section we examine the massless (strong-coupling) limit of the operator $\mathcal{C}$ for H in Eq. (10). The massless limit $\mu\to 0$ of the massive theory is especially interesting because this limiting case is *singular*. We will see that as the mass parameter μ tends to zero, the perturbation series representation for Q in $\mathcal{C}=e^Q\mathcal{P}$ ceases to exist and an entirely nonpolynomial representation for Q emerges.

Negative powers of μ in Eq. (39) are required for dimensional consistency. As a result, each of the terms in these perturbation series coefficients becomes singular in the massless limit $\mu\to 0$. (Note that the dimensionless perturbation expansion parameter is $\epsilon\mu^{-5/2}$, and thus the massless limit of the theory is equivalent to the strong-coupling limit $\epsilon \to\infty$.)

To find the $\mathcal{C}$ operator for the massless theory it is necessary to return to the sequence of operator equations in (34) and to look for solutions for the special case in which $H_0 =\frac{1}{2}p^2$. The first of these equations reads

$$\left[\frac{1}{2}p^2,Q_1\right]=-2ix^3. \tag{50}$$

However, an examination of Eq. (50) reveals that it is no longer possible to find a solution in the form of a polynomial in the operators p and x. The situation here is quite similar to that considered in Ref. [17], in which the objective was to calculate the time operator in quantum mechanics. In the case of the time operator it was shown that one must generalize the symmetric operators $S_{m,n}$ from the positive integers to the negative integers. Specifically, if m is nonnegative, then n may be negative, and if n is nonnegative, then m may be negative. (It is not possible for *both* m and n to be negative.) We can display these generalized symmetric operators in Weyl-ordered form [see Eq. (38)]. For example,

$$\begin{aligned}
S_{-1,1}&=\frac{1}{2}\left(x\frac{1}{p}+\frac{1}{p}x\right),\\
S_{-3,0}&=\frac{1}{p^3},\\
S_{-2,2}&=\frac{1}{4}\left(\frac{1}{p^2}x^2+2x\frac{1}{p^2}x+x^2\frac{1}{p^2}\right),\\
S_{-2,3}&=\frac{1}{8}\left(x^3\frac{1}{p^2}+3x^2\frac{1}{p^2}x+3x\frac{1}{p^2}x^2+\frac{1}{p^2}x^3\right).
\end{aligned} \tag{51}$$

An exact, dimensionally consistent operator solution to Eq. (50) is

$$\begin{aligned}
Q_1=\frac{1}{2}S_{-1,4}+\alpha S_{-5,0}=\frac{1}{32}\bigg(&x^4\frac{1}{p}+4x^3\frac{1}{p}x+6x^2\frac{1}{p}x^2\\
&+4x\frac{1}{p}x^3+\frac{1}{p}x^4\bigg)+\alpha\frac{1}{p^5},
\end{aligned} \tag{52}$$

where α is an arbitrary number. This solution has the required symmetry properties; to wit, it is odd in p and even in x. Also, it has the same dimensions as Q_1 in Eq. (39).

The solution to the second operator equation in (34), $[\frac{1}{2}p^2,Q_3]=-\frac{1}{6}[Q_1,[Q_1,ix^3]]$, is

$$\begin{aligned}
Q_3=&\frac{1}{40}S_{-5,10}-\frac{3}{32}S_{-7,8}\\
&+\left(\frac{7}{16}+20\alpha\right)S_{-9,6}+\left(\frac{3}{32}+\frac{305}{8}\alpha\right)S_{-11,4}\\
&+\left(-\frac{135}{16}-\frac{5773}{8}\alpha+\frac{75}{12}\alpha^2\right)S_{-13,2}+\beta S_{-15,0},
\end{aligned} \tag{53}$$

where β is a new arbitrary constant. Again, observe that this solution exhibits the required symmetry properties.

A notable feature of the solutions for Q_1 and Q_3 in Eqs. (52) and (53) is that they are *not* unique. Each of these solutions contains an arbitrary constant multiplying a negative odd-integer power of p. There is no obvious way to determine the values of the constants α and β. These terms arise because in the massless case the Hamiltonian H_0 is a function of p only. In general, one can add an arbitrary multiple of p^{-10n-5} to the solution for Q_{2n+1} because it is odd in p and is dimensionally consistent. In the massive case, where $H_0=\frac{1}{2}p^2+\frac{1}{2}\mu^2x^2$, there is no such ambiguity because adding an arbitrary function of H_0 to Q_n would violate the symmetry requirement that Q_n be odd in p.

VI. PRODUCT REPRESENTATION OF $\mathcal{C}$ AND DERIVATION OF RECURSION RELATIONS

In this section we investigate the *product* representation of the operator $\mathcal{C}$ that was defined in Eq. (31); namely $\mathcal{C}(x,p) =R(x,p)\mathcal{P}$. At first, it may not seem worthwhile to reconsider the product representation because it lacks the advantages of the *exponential* representation $\mathcal{C}=e^Q\mathcal{P}$ introduced in Eq. (14). Recall that we argued in Sec. II that the exponential representation is convenient because it incorporates the requirements (2) and (3) as elementary symmetry conditions on $Q(x,p)$: $Q(x,p)=Q(-x,p)$ and $Q(x,p)=-Q(x,-p)$. Furthermore, we showed that the exponential representation of $\mathcal{C}$ in Eqs. (19) and (20) is much simpler than the product representation $\mathcal{C}(x,p)=R(x,p)\mathcal{P}$ in Eq. (18). However, as we demonstrate here, the product representation has the advantage that it can be used to construct a recursive formula for the perturbation coefficients.

The function $R(x,p)$ in the product representation $\mathcal{C}(x,p)=R(x,p)\mathcal{P}$ incorporates the requirement in Eq. (2) as

$$R(x,p)=R(-x,p). \tag{54}$$

Thus, $R(x,p)$ is an even function of x. However, the requirement in Eq. (3) translates into a complicated *nonlinear* condition on R:

$$R(x,p)R(x,-p)=1. \tag{55}$$

We will return to this condition later.

The advantage of the product representation is that it translates the requirement in Eq. (4) into a *linear* difference equation. To obtain this difference equation we use the operators $S_{m,n}$ in Eq. (37). (Recall that $S_{m,n}$ is a totally symmetric combination of products of m factors of p and n factors of x.) It was shown in Ref. [17] that the operators $S_{m,n}$ are *complete* in the sense that any operator may be represented as a linear combination of these symmetric operators. This allows us to represent $R(x,p)$ as the infinite linear combination

$$R(x,p)=\sum_m \sum_n \alpha_{m,n}S_{m,n}, \tag{56}$$

where $\alpha_{m,n}$ are numerical coefficients to be determined. Substituting $\mathcal{C}(x,p)=R(x,p)\mathcal{P}$ into Eq. (4) then gives the condition

$$[R,H_0]=\epsilon\{R,H_1\}. \tag{57}$$

We now substitute R in Eq. (56) into the condition (57) and use the commutation and anticommutation relations [17]

$$[S_{m,n},x^2]=-2imS_{m-1,n+1},$$
$$[S_{m,n},p^2]=2inS_{m+1,n-1},$$
$$\{S_{m,n},x^3\}=-\frac{3}{2}m(m-1)S_{m-1,n+1}. \tag{58}$$

For the Hamiltonian in Eq. (10) we obtain the linear recursion relation

$$n\alpha_{m-2,n}-\mu^2 m\alpha_{m,n-2}=\epsilon\left[-\frac{3}{2}m(m+1)\alpha_{m+1,n-2}+2\alpha_{m-1,n-4}\right]. \tag{59}$$

The boundary conditions on this partial difference equation must be chosen so that the nonlinear constraint (55) is satisfied. In the massive case $\alpha_{0,0}=1$ and $\alpha_{m,n}$ vanishes if either $m<0$ or $n<0$. In the massless case we again have $\alpha_{0,0}=1$, but now $\alpha_{m,n}$ vanishes if either $m>0$ or $n<0$.

One approach to solving this equation is to introduce a generating function $g(s,t)$:

$$g(s,t)\equiv\sum_m\sum_n \alpha_{m,n}s^m t^n. \tag{60}$$

For the massive case [the Hamiltonian in Eq. (10) with $\mu\neq 0$] the summation is taken over nonnegative values of m and n. However, for the massless case ($\mu=0$) the summation must be taken over nonpositive values of m and nonnegative values of n.

We then multiply Eq. (59) by $s^{m-1}t^{m-1}$ and rewrite the result in the form

$$s\frac{\partial}{\partial t}(\alpha_{m-2,n}s^{m-2}t^n)-\mu^2 t\frac{\partial}{\partial s}(\alpha_{m,n-2}s^m t^{n-2})=\epsilon\left[-\frac{3}{2}t\frac{\partial^2}{\partial s^2}(\alpha_{m+1,n-2}s^{m+1}t^{n-2})+2t^3\alpha_{m-1,n-4}s^{m-1}t^{n-4}\right]. \tag{61}$$

Summing over m and n and using Eq. (60), we obtain the partial differential equation

$$sg_t-\mu^2 tg_s=\epsilon\left(-\frac{3}{2}tg_{ss}+2t^3 g\right), \tag{62}$$

where subscripts indicate partial differentiation. A Fourier transform from the s variable to the r variable converts this differential equation into the *Goursat problem* (recall that a Goursat problem involves a wave equation written in light-cone variables)

$$\tilde{g}_{rt}=\left(\mu^2 rt+\frac{3}{2}i\epsilon r^2 t+2i\epsilon t^3\right)\tilde{g}. \tag{63}$$

This is an extremely interesting equation that merits further study. We plan to give a detailed analysis of this partial differential equation and of the partial difference equation (59) in a future paper.

VII. SCALAR QUANTUM FIELD THEORY WITH CUBIC SELF-INTERACTION

This section extends the operator techniques introduced in Sec. III to quantum field theory. Consider the quantum field theory described by the Hamiltonian (13) in $(D+1)$-dimensional Minkowski space-time. This Hamiltonian has the form $H=H_0+\epsilon H_1$, where

$$H_0=\int d^Dx\left\{\frac{1}{2}\pi^2(\mathbf{x},t)+\frac{1}{2}[\nabla_{\mathbf{x}}\varphi(\mathbf{x},t)]^2+\frac{1}{2}\mu^2\varphi^2(\mathbf{x},t)\right\},$$
$$H_1=i\int d^Dx\,\varphi^3(\mathbf{x},t). \tag{64}$$

The integrals above are performed in the spatial variable $\mathbf{x}$, which lies in $\mathbb{R}^D$. In the following we use $\int d\mathbf{x}=\int d^Dx$ to represent the integration in $\mathbb{R}^D$. The field variables satisfy the equal-time canonical commutation relation $[\varphi(\mathbf{x},t),\pi(\mathbf{y},t)]=i\delta(\mathbf{x}-\mathbf{y})$.

The parity operator is given formally by $\mathcal{P}=\exp\{\frac{1}{2}i\pi\int d\mathbf{x}[\varphi^2(\mathbf{x},t)+\pi^2(\mathbf{x},t)-1]\}$. As in quantum mechanics, where the operators x and p change sign under parity reflection, we assume that the fields are *pseudoscalars* and that they also change sign under $\mathcal{P}$:

$$\mathcal{P}\varphi(\mathbf{x},t)\mathcal{P}=-\varphi(-\mathbf{x},t),\quad \mathcal{P}\pi(\mathbf{x},t)\mathcal{P}=-\pi(-\mathbf{x},t). \tag{65}$$

Following the approach in Sec. III, we express $\mathcal{C}$ in the form $\mathcal{C}=e^{\epsilon Q_1+\epsilon^3 Q_3+\cdots}\mathcal{P}$, where Q_{2n+1} ($n=0,1,2,\ldots$) are real functionals of the field variables $\varphi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$. To find Q_1 it is necessary to solve the first of the operator equations in (34):

$$\left[\int d\mathbf{x}\left(\frac{1}{2}\pi^2(\mathbf{x},t)+\frac{1}{2}\mu^2\varphi^2(\mathbf{x},t)-\frac{1}{2}\varphi(\mathbf{x},t)\nabla^2_{\mathbf{x}}\varphi(\mathbf{x},t)\right),Q_1\right]=-2i\int d\mathbf{x}\varphi^3(\mathbf{x},t), \tag{66}$$

where we have integrated by parts: $\int d\mathbf{x}(\nabla_{\mathbf{x}}\varphi)^2=-\int d\mathbf{x}\varphi\nabla^2_{\mathbf{x}}\varphi$. We define the inverse Green's function $G^{-1}_{\mathbf{xy}}$ by $G^{-1}_{\mathbf{xy}}\equiv(\mu^2-\nabla^2_{\mathbf{x}})\delta(\mathbf{x}-\mathbf{y})$, so that $G_{\mathbf{xy}}=(\mu^2-\nabla^2_{\mathbf{x}})^{-1}\delta(\mathbf{x}-\mathbf{y})$ and $\int d\mathbf{z}G^{-1}_{\mathbf{xz}}G_{\mathbf{zy}}=\delta(\mathbf{x}-\mathbf{y})$. Thus, the commutator condition (66) reads

$$\left[\frac{1}{2}\int d\mathbf{x}\pi^2(\mathbf{x},t)+\frac{1}{2}\int d\mathbf{x}d\mathbf{y}\varphi(\mathbf{x},t)G^{-1}_{\mathbf{xy}}\varphi(\mathbf{y},t),Q_1\right]$$
$$=-2i\int d\mathbf{x}\varphi^3(\mathbf{x},t). \tag{67}$$

Equation (67) states that when the operator Q_1 is commuted with quadratic structures of the form $\pi^2(\mathbf{x},t)$ and $\varphi(\mathbf{x},t)\varphi(\mathbf{y},t)$, it must produce the cubic term $\varphi^3(\mathbf{x},t)$. Furthermore, the symmetry requirements on Q_1 that arise from Eqs. (2) and (3) imply that Q_1 is an even functional of $\varphi(\mathbf{x},t)$ and an odd functional of $\pi(\mathbf{x},t)$. These observations allow us to deduce an ansatz for Q_1 that has the same structure as that in Eq. (35):

$$Q_1=\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}M_{(\mathbf{xyz})}\pi_{\mathbf{x}}\pi_{\mathbf{y}}\pi_{\mathbf{z}}$$
$$+\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}N_{\mathbf{x}(\mathbf{yz})}\varphi_{\mathbf{y}}\pi_{\mathbf{x}}\varphi_{\mathbf{z}}, \tag{68}$$

where we have suppressed the time variable t in the fields and for brevity have indicated spatial dependences with subscripts. In Eq. (68) the unknown functions M and N have three arguments each. The function M is totally symmetric in its three arguments, and to emphasize this symmetry we use the notation $M_{(\mathbf{xyz})}$; N is symmetric under the interchange of the second and third arguments, and to emphasize this symmetry we write $N_{\mathbf{x}(\mathbf{yz})}$. The functions M and N are like form factors because they describe the spatial distribution of the three-point interactions of the fields in Q_1. We will see that the interaction of the fields is spatially nonlocal; this nonlocality is an intrinsic property of the operator $\mathcal{C}$.

We now proceed to determine M and N. We substitute the ansatz (68) into the commutator (67) and find after some algebra that two operator identities must hold:

$$\int\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}d\mathbf{w}N_{\mathbf{x}(\mathbf{yz})}G^{-1}_{\mathbf{wx}}\varphi_{\mathbf{y}}\varphi_{\mathbf{w}}\varphi_{\mathbf{z}}=-2\int d\mathbf{w}\varphi^3_{\mathbf{w}}, \tag{69}$$

$$\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}N_{\mathbf{x}(\mathbf{yz})}(\pi_{\mathbf{x}}\pi_{\mathbf{y}}\varphi_{\mathbf{z}}+\varphi_{\mathbf{z}}\pi_{\mathbf{x}}\pi_{\mathbf{y}})$$
$$=3\int\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}d\mathbf{w}M_{(\mathbf{xyz})}G^{-1}_{\mathbf{xw}}\pi_{\mathbf{y}}\varphi_{\mathbf{w}}\pi_{\mathbf{z}}. \tag{70}$$

By commuting Eq. (69) three times with π, and Eq. (70) once with π and twice with φ, we translate these two operator identities into two coupled partial differential equations for M and N:

$$(\mu^2-\nabla^2_{\mathbf{x}})N_{\mathbf{x}(\mathbf{yz})}+(\mu^2-\nabla^2_{\mathbf{y}})N_{\mathbf{y}(\mathbf{xz})}+(\mu^2-\nabla^2_{\mathbf{z}})N_{\mathbf{z}(\mathbf{xy})}$$
$$=-6\,\delta(\mathbf{x}-\mathbf{y})\,\delta(\mathbf{x}-\mathbf{z}), \tag{71}$$

$$N_{\mathbf{x}(\mathbf{yz})}+N_{\mathbf{y}(\mathbf{xz})}=3(\mu^2-\nabla^2_{\mathbf{z}})M_{(\mathbf{xyz})}. \tag{72}$$

To solve the system of coupled differential equations (71) and (72), we Fourier transform to momentum space, denoting the D-dimensional Fourier transform of a function $f_{\mathbf{x}}$ by $\tilde{f}_{\mathbf{p}}\equiv\int d\mathbf{x}f_{\mathbf{x}}e^{i\mathbf{p}\cdot\mathbf{x}}$. Fourier transformation is effective here because it converts the differential equations (71) and (72) into algebraic equations:

$$\frac{1}{\tilde{G}_{\mathbf{p}}}\tilde{N}_{\mathbf{p}(\mathbf{qr})}+\frac{1}{\tilde{G}_{\mathbf{q}}}\tilde{N}_{\mathbf{q}(\mathbf{pr})}+\frac{1}{\tilde{G}_{\mathbf{r}}}\tilde{N}_{\mathbf{r}(\mathbf{pq})}=-6(2\pi)^D\delta(\mathbf{p}+\mathbf{q}+\mathbf{r}), \tag{73}$$

$$\tilde{N}_{\mathbf{p}(\mathbf{qr})}+\tilde{N}_{\mathbf{q}(\mathbf{pr})}=\frac{3}{\tilde{G}_{\mathbf{r}}}\tilde{M}_{(\mathbf{pqr})}, \tag{74}$$

where $\tilde{G}_{\mathbf{p}}=(\mathbf{p}^2+\mu^2)^{-1}$.

Note that the right side of Eq. (73) contains the factor $\delta(\mathbf{p}+\mathbf{q}+\mathbf{r})$, which implies that the two three-point functions M and N conserve momentum. We thus introduce *reduced* representations of these vertex functions in which we have factored off the delta function:

$$\tilde{M}_{(\mathbf{pqr})}=(2\pi)^D\tilde{m}_{(\mathbf{pqr})}\delta(\mathbf{p}+\mathbf{q}+\mathbf{r}),$$
$$\tilde{N}_{\mathbf{p}(\mathbf{qr})}=(2\pi)^D\tilde{n}_{\mathbf{p}(\mathbf{qr})}\delta(\mathbf{p}+\mathbf{q}+\mathbf{r}). \tag{75}$$

The functions $\tilde{m}$ and $\tilde{n}$ satisfy the following algebraic equations:

$$\frac{1}{\tilde{G}_{\mathbf{p}}}\tilde{n}_{\mathbf{p}(\mathbf{qr})}+\frac{1}{\tilde{G}_{\mathbf{q}}}\tilde{n}_{\mathbf{q}(\mathbf{pr})}+\frac{1}{\tilde{G}_{\mathbf{r}}}\tilde{n}_{\mathbf{r}(\mathbf{pq})}=-6, \tag{76}$$

$$\tilde{G}_{\mathbf{r}}\tilde{n}_{\mathbf{p}(\mathbf{qr})}+\tilde{G}_{\mathbf{r}}\tilde{n}_{\mathbf{q}(\mathbf{pr})}=3\tilde{m}_{(\mathbf{pqr})}. \tag{77}$$

There are two ways to solve these equations. A physically transparent but longer procedure making use of tree graphs is given in the Appendix. A shorter analytical approach is presented here. We begin by noting that the right side of Eq. (77) is totally symmetric in its indices. Thus, we can obtain two new equations by permuting the indices:

$$\tilde{G}_{\mathbf{q}}\tilde{n}_{\mathbf{p}(\mathbf{qr})}+\tilde{G}_{\mathbf{q}}\tilde{n}_{\mathbf{r}(\mathbf{pq})}=3\tilde{m}_{(\mathbf{pqr})},$$
$$\tilde{G}_{\mathbf{p}}\tilde{n}_{\mathbf{q}(\mathbf{pr})}+\tilde{G}_{\mathbf{p}}\tilde{n}_{\mathbf{r}(\mathbf{pq})}=3\tilde{m}_{(\mathbf{pqr})}. \tag{78}$$

We now have a sufficient number of algebraic equations, namely (76)–(78), to solve for $\tilde{m}$ and $\tilde{n}$. The final results for M and N are

$$\tilde{M}_{(\mathbf{pqr})}=\frac{4\tilde{G}^2_{\mathbf{p}}\tilde{G}^2_{\mathbf{q}}\tilde{G}^2_{\mathbf{r}}}{\tilde{G}^2_{\mathbf{p}}\tilde{G}^2_{\mathbf{q}}+\tilde{G}^2_{\mathbf{p}}\tilde{G}^2_{\mathbf{r}}+\tilde{G}^2_{\mathbf{q}}\tilde{G}^2_{\mathbf{r}}-2\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}}(\tilde{G}_{\mathbf{p}}+\tilde{G}_{\mathbf{q}}+\tilde{G}_{\mathbf{r}})}$$
$$\times(2\pi)^D\delta(\mathbf{p}+\mathbf{q}+\mathbf{r}), \tag{79}$$

$$\tilde{N}_{\mathbf{p}(\mathbf{qr})}=\frac{6\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}}(\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{r}}+\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}}-\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}})}{\tilde{G}^2_{\mathbf{p}}\tilde{G}^2_{\mathbf{q}}+\tilde{G}^2_{\mathbf{p}}\tilde{G}^2_{\mathbf{r}}+\tilde{G}^2_{\mathbf{q}}\tilde{G}^2_{\mathbf{r}}-2\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}}(\tilde{G}_{\mathbf{p}}+\tilde{G}_{\mathbf{q}}+\tilde{G}_{\mathbf{r}})}$$
$$\times(2\pi)^D\delta(\mathbf{p}+\mathbf{q}+\mathbf{r}). \tag{80}$$

As a check of these results we compare Eqs. (79) and (80) with Eq. (36), which describes the case $D=0$ (quantum mechanics). When $D=0$, we have just $\tilde{G}_{\mathbf{p}}=\mu^{-2}$. Substituting this expression for $\tilde{G}$ into Eqs. (79) and (80), we find that these equations reduce exactly to Eq. (36).

Next, we substitute $\tilde{G}_{\mathbf{p}}=(\mathbf{p}^2+\mu^2)^{-1}$ into Eqs. (79) and (80) and use the inverse Fourier transform $f_{\mathbf{x}}=(2\pi)^{-D}\int d\mathbf{p}\tilde{f}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}$ to express M and N in coordinate space:

$$M_{(\mathbf{xyz})}=\int\int\int \frac{d\mathbf{p}d\mathbf{q}d\mathbf{r}}{(2\pi)^{3D}}\frac{4e^{-i\mathbf{x}\cdot\mathbf{p}-i\mathbf{y}\cdot\mathbf{q}-i\mathbf{z}\cdot\mathbf{r}}(2\pi)^D\delta(\mathbf{p}+\mathbf{q}+\mathbf{r})}{\mathcal{D}(\mathbf{p},\mathbf{q},\mathbf{r})}, \tag{81}$$

$$N_{\mathbf{x}(\mathbf{yz})}=\int\int\int \frac{d\mathbf{p}d\mathbf{q}d\mathbf{r}}{(2\pi)^{3D}}\frac{6e^{-i\mathbf{x}\cdot\mathbf{p}-i\mathbf{y}\cdot\mathbf{q}-i\mathbf{z}\cdot\mathbf{r}}(2\pi)^D\delta(\mathbf{p}+\mathbf{q}+\mathbf{r})(\mathbf{q}^2+\mathbf{r}^2-\mathbf{p}^2+\mu^2)}{\mathcal{D}(\mathbf{p},\mathbf{q},\mathbf{r})}, \tag{82}$$

where $\mathcal{D}(\mathbf{p},\mathbf{q},\mathbf{r})=\mathbf{p}^4+\mathbf{q}^4+\mathbf{r}^4-2(\mathbf{p}^2\mathbf{q}^2+\mathbf{q}^2\mathbf{r}^2+\mathbf{r}^2\mathbf{p}^2)-2\mu^2(\mathbf{p}^2+\mathbf{q}^2+\mathbf{r}^2)-3\mu^4$. We perform the $\mathbf{r}$ integral in Eqs. (81) and (82) using the delta function and obtain

$$M_{(\mathbf{xyz})}=-\frac{4}{(2\pi)^{2D}}\int\int d\mathbf{p}d\mathbf{q}\frac{e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{p}+i(\mathbf{x}-\mathbf{z})\cdot\mathbf{q}}}{\mathcal{D}(\mathbf{p},\mathbf{q})}, \tag{83}$$

where $\mathcal{D}(\mathbf{p},\mathbf{q})=4[\mathbf{p}^2\mathbf{q}^2-(\mathbf{p}\cdot\mathbf{q})^2]+4\mu^2(\mathbf{p}^2+\mathbf{p}\cdot\mathbf{q}+\mathbf{q}^2)+3\mu^4$ is positive, and

$$N_{\mathbf{x}(\mathbf{yz})}=3\left(\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{z}}+\frac{1}{2}\mu^2\right)M_{(\mathbf{xyz})}. \tag{84}$$

A. The (1+1)-dimensional case

For general D it is difficult to evaluate the double integral (83) in closed form. However, when $D=1$ [(1+1)-dimensional quantum field theory] we can evaluate the integral because the quartic terms in $\mathcal{D}(\mathbf{p},\mathbf{q})$ cancel. The evaluation procedure exploits the strict positivity of the denominator $\mathcal{D}(p,q)$,

$$\mathcal{D}(p,q)=4\mu^2(p^2+pq+q^2)+3\mu^4$$
$$=2\mu^2[p^2+q^2+(p+q)^2]+3\mu^4>0,$$

to construct the one-dimensional integral identity $\mathcal{D}^{-1}=\int_0^\infty dt e^{-\mathcal{D}t}$ $(\mathcal{D}>0)$. This identity allows us to rewrite $M_{(xyz)}$ as the triple integral

$$M_{(xyz)}=-\frac{4}{\mu^2(2\pi)^2}\int\int dpdq\int_{t=0}^\infty dt e^{-3\mu^2 t}\times e^{i(x-y)p+i(x-z)q-4t(p^2+pq+q^2)}. \tag{85}$$

Note that when $D=1$ the variables x, y, z, p, q, and so on, are scalars and not vectors, so we no longer use boldface notation.

To evaluate this integral we first complete the square in the q variable in the exponent and translate the q integration variable by $q\to q-p/2+i(x-z)/8t$. We then complete the square in the p integration variable and translate p by $p\to p-i(x-z)/12t+i(x-y)/6t$. This gives

$$M_{(xyz)}=-\frac{4}{\mu^2(2\pi)^2}\int\int dpdq\int_{t=0}^\infty dt\times e^{-3\mu^2 t-3tp^2-4tq^2-\rho^2/(12t)}, \tag{86}$$

where ρ, which is totally symmetric in x, y, and z, is the positive square root of

$$\rho^2=\frac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2]. \tag{87}$$

We now perform the scalings $p\to p/\sqrt{3t}$ and $q\to q/\sqrt{4t}$. The result is that the integral (86) representing $M_{(xyz)}$ factors into three one-dimensional integrals:

$$M_{(xyz)}=-4\mu^{-2}(2\pi)^{-2}12^{-1/2}I^2 J, \tag{88}$$

where I is the Gaussian integral $I=\int dq e^{-q^2}=\sqrt{\pi}$ and $J=\int_{t=0}^\infty dt t^{-1}e^{-3\mu^2 t-\rho^2/(12t)}$. Finally, we use the integral representation [19]

$$\int_{t=0}^\infty dt e^{-t-a^2/t}t^{-1}=2\mathrm{K}_0(2a),$$

where K_0 is the associated Bessel function. Thus, $J=2\mathrm{K}_0(\mu\rho)$. Combining the factors in Eq. (88), we find that for a (1+1)-dimensional quantum field theory Eq. (83) evaluates to

$$M_{(xyz)}=-\frac{1}{\pi\sqrt{3}\mu^2}\mathrm{K}_0(\mu\rho). \tag{89}$$

Next we calculate N using Eq. (84). The result is

$$N_{x(yz)}=-\frac{3\sqrt{3}}{4\pi}\left[1-\frac{(y-z)^2}{\rho^2}\right]\mathrm{K}_0(\mu\rho)+\frac{\sqrt{3}}{\pi}\left[1-\frac{3(y-z)^2}{2\rho^2}\right]\frac{\mathrm{K}_0'(\mu\rho)}{\mu\rho}+\frac{1}{\mu^2}\left[1-\frac{3(y-z)^2}{\rho^2}\right]\delta(x-y)\delta(x-z). \quad (90)$$

The mathematics underlying the solutions in Eqs. (89) and (90) is rather subtle and bears further discussion. First, it is important to mention that while we have expressed M and N as functions of the three variables x, y, and z, translation invariance implies that these functions really depend on only two variables, say, the differences $x-y$ and $x-z$. We therefore define the two variables η and ζ by

$$\eta=x-\frac{1}{2}(y+z) \quad \text{and} \quad \zeta=\frac{\sqrt{3}}{2}(y-z). \quad (91)$$

In terms of these new variables we have

$$\rho^2=\eta^2+\zeta^2,$$

$$\partial_x^2+\partial_y^2+\partial_z^2=\frac{3}{2}(\partial_\eta^2+\partial_\zeta^2)=\frac{3}{2}\nabla^2_{\eta,\zeta},$$

$$\frac{2}{\sqrt{3}}\delta(x-y)\delta(x-z)=\delta(\eta)\delta(\zeta)=\frac{1}{2\pi\rho}\delta(\rho). \quad (92)$$

The reason for introducing new variables and for emphasizing that we are working in the two-dimensional (η,ζ) space is that in *two-dimensional* space the associated Bessel function $\frac{1}{2\pi}\mathrm{K}_0(\mu\rho)$ is the Green's function:

$$(\mu^2-\nabla^2_{\eta,\zeta})\frac{1}{2\pi}\mathrm{K}_0(\mu\rho)=\delta(\eta)\delta(\zeta)=\frac{1}{2\pi\rho}\delta(\rho). \quad (93)$$

This equation explains the appearance of the contact term (delta-function term) in the expression for N in Eq. (90). The delta function is rotationally symmetric because the Green's function is rotationally symmetric and thus we can replace $\nabla^2_{\eta,\zeta}$ in Eq. (93) by $d^2/d\rho^2+d/d\rho$. Hence, we see that two derivatives of $\mathrm{K}_0(\mu\rho)$ give rise to a delta function:

$$\mathrm{K}_0''(\mu\rho)=\mathrm{K}_0(\mu\rho)-\frac{1}{\mu\rho}\mathrm{K}_0'(\mu\rho)-\frac{1}{\mu^2\rho}\delta(\rho). \quad (94)$$

We have checked that M and N in Eqs. (89) and (90) satisfy the partial differential equations (71) and (72). We can verify Eq. (72) by direct differentiation. To verify Eq. (71) we use the variables η and ζ and take the indicated derivatives. The result is a combination of $\mathrm{K}_0(\mu\rho)$, $\mathrm{K}_0'(\mu\rho)$, $\delta(\rho)$, $\delta'(\rho)$, and $\delta''(\rho)$ terms. The coefficients of all of these terms vanish except for the coefficient $\delta(\rho)$ term, and this coefficient reproduces exactly the right side of Eq. (71).

Because our formulas for M and N involve Bessel functions, we see clearly that Q_1 represents a *nonlocal* interaction of three fields. However, as the associated Bessel functions decrease exponentially rapidly for large argument, the degree of nonlocality is small.

VIII. QUANTUM FIELD THEORY WITH SEVERAL INTERACTING FIELDS

The field theoretic calculations in Sec. VII can be extended to cubic quantum field theories having two and three interacting scalar fields.

A. $\varphi_1\varphi_2^2$ theory

We consider first the case of two scalar fields $\varphi_{\mathbf{x}}^{(1)}$ and $\varphi_{\mathbf{x}}^{(2)}$ whose dynamics is described by the Hamiltonian $H=H_0^{(1)}+H_0^{(2)}+\epsilon H_1$, where

$$H_0^{(j)}=\frac{1}{2}\int d\mathbf{x}(\pi_{\mathbf{x}}^{(j)})^2+\frac{1}{2}\int\int d\mathbf{x}d\mathbf{y}(G_{\mathbf{xy}}^{(j)})^{-1}\varphi_{\mathbf{x}}^{(j)}\varphi_{\mathbf{y}}^{(j)} \quad (j=1,2) \quad (95)$$

and $H_1=i\int d\mathbf{x}(\varphi_{\mathbf{x}}^{(1)})^2\varphi_{\mathbf{x}}^{(2)}$. The Green's function $G_{\mathbf{xy}}^{(j)}$ is the solution to the equation

$$(\mu_j^2-\nabla_{\mathbf{x}}^2)G_{\mathbf{xy}}^{(j)}=\delta(\mathbf{x}-\mathbf{y}) \quad (j=1,2).$$

This quantum field theory is the analogue of the quantum mechanical theory in Eq. (11).

To determine $\mathcal{C}$ to order ϵ we need to solve the operator equation

$$[H_0^{(1)}+H_0^{(2)},Q_1]=-2H_1, \quad (96)$$

which is the two-field generalization of Eq. (66). To find the solution to this equation we make an ansatz analogous to that in Eq. (41) for the operator Q_1:

$$Q_1=\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}[M_{\mathbf{x(yz)}}\pi_{\mathbf{x}}^{(2)}\pi_{\mathbf{y}}^{(1)}\pi_{\mathbf{z}}^{(1)}+N_{\mathbf{xyz}}^{(1)}(\pi_{\mathbf{z}}^{(1)}\varphi_{\mathbf{y}}^{(1)}+\varphi_{\mathbf{y}}^{(1)}\pi_{\mathbf{z}}^{(1)})\varphi_{\mathbf{x}}^{(2)}+N_{\mathbf{x(yz)}}^{(2)}\pi_{\mathbf{x}}^{(2)}\varphi_{\mathbf{y}}^{(1)}\varphi_{\mathbf{z}}^{(1)}],$$

where $M_{\mathbf{x(yz)}}$, $N_{\mathbf{xyz}}^{(1)}$, and $N_{\mathbf{x(yz)}}^{(2)}$ are unknown functions of three arguments each. As indicated by the parentheses, M and $N^{(2)}$ are symmetric in their last two arguments.

We then substitute Q_1 into Eq. (96) and use the following identities to perform the algebra:

$$[H_0^{(1)},Q_1]=-i\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}[2N^{(1)}_{\mathbf{xyz}}\pi^{(1)}_{\mathbf{y}}\pi^{(1)}_{\mathbf{z}}\varphi^{(2)}_{\mathbf{x}}+N^{(2)}_{\mathbf{x(yz)}}\pi^{(2)}_{\mathbf{x}}(\varphi^{(1)}_{\mathbf{y}}\pi^{(1)}_{\mathbf{z}}+\pi^{(1)}_{\mathbf{z}}\varphi^{(1)}_{\mathbf{y}})]$$
$$+i\int\int\int\int d\mathbf{w}d\mathbf{x}d\mathbf{y}d\mathbf{z}(G^{(1)}_{\mathbf{wy}})^{-1}M_{\mathbf{x(yz)}}\pi^{(2)}_{\mathbf{x}}(\varphi^{(1)}_{\mathbf{w}}\pi^{(1)}_{\mathbf{z}}+\pi^{(1)}_{\mathbf{z}}\varphi^{(1)}_{\mathbf{w}})$$
$$+2i\int\int\int\int d\mathbf{w}d\mathbf{x}d\mathbf{y}d\mathbf{z}(G^{(1)}_{\mathbf{wz}})^{-1}N^{(1)}_{\mathbf{xyz}}\varphi^{(2)}_{\mathbf{x}}\varphi^{(1)}_{\mathbf{w}}\varphi^{(1)}_{\mathbf{y}},$$
$$[H_0^{(2)},Q_1]=-i\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}N^{(1)}_{\mathbf{xyz}}(\pi^{(1)}_{\mathbf{z}}\varphi^{(1)}_{\mathbf{y}}+\varphi^{(1)}_{\mathbf{y}}\pi^{(1)}_{\mathbf{z}})\pi^{(2)}_{\mathbf{x}}+i\int\int\int\int d\mathbf{w}d\mathbf{x}d\mathbf{y}d\mathbf{z}(G^{(2)}_{\mathbf{wx}})^{-1}N^{(2)}_{\mathbf{x(yz)}}\varphi^{(1)}_{\mathbf{y}}\varphi^{(1)}_{\mathbf{z}}\varphi^{(2)}_{\mathbf{w}}$$
$$+i\int\int\int\int d\mathbf{w}d\mathbf{x}d\mathbf{y}d\mathbf{z}(G^{(2)}_{\mathbf{wx}})^{-1}M_{\mathbf{x(yz)}}\pi^{(1)}_{\mathbf{y}}\pi^{(1)}_{\mathbf{z}}\varphi^{(2)}_{\mathbf{w}}. \tag{97}$$

Substituting Eq. (97) into Eq. (96) gives an operator equation involving the unknown functions M, $N^{(1)}$, and $N^{(2)}$. We then convert these operator equations into a system of three coupled partial differential equations by commuting with products of three fields:

$$(\mu_1^2-\nabla_{\mathbf{z}}^2)N^{(1)}_{\mathbf{xyz}}+(\mu_1^2-\nabla_{\mathbf{y}}^2)N^{(1)}_{\mathbf{xyz}}+(\mu_2^2-\nabla_{\mathbf{x}}^2)N^{(2)}_{\mathbf{x(yz)}}=-2\,\delta(\mathbf{x}-\mathbf{y})\,\delta(\mathbf{x}-\mathbf{z}),$$

$$(\mu_1^2-\nabla_{\mathbf{y}}^2)M_{\mathbf{x(yz)}}=N^{(1)}_{\mathbf{xyz}}+N^{(2)}_{\mathbf{x(yz)}},$$

$$(\mu_2^2-\nabla_{\mathbf{x}}^2)M_{\mathbf{x(yz)}}=N^{(1)}_{\mathbf{xyz}}+N^{(1)}_{\mathbf{xzy}}. \tag{98}$$

To solve this system of partial differential equations we perform Fourier transforms in the variables **x**, **y**, and **z** and obtain

$$\frac{1}{\tilde{G}^{(1)}_{\mathbf{r}}}\tilde{N}^{(1)}_{\mathbf{pqr}}+\frac{1}{\tilde{G}^{(1)}_{\mathbf{q}}}\tilde{N}^{(1)}_{\mathbf{prq}}+\frac{1}{\tilde{G}^{(2)}_{\mathbf{p}}}\tilde{N}^{(2)}_{\mathbf{p(qr)}}=-2(2\pi)^D\delta(\mathbf{p}+\mathbf{q}+\mathbf{r}),$$

$$\frac{1}{\tilde{G}^{(1)}_{\mathbf{q}}}\tilde{M}_{\mathbf{p(qr)}}=\tilde{N}^{(1)}_{\mathbf{pqr}}+\tilde{N}^{(2)}_{\mathbf{p(qr)}},$$

$$\frac{1}{\tilde{G}^{(2)}_{\mathbf{p}}}\tilde{M}_{\mathbf{p(qr)}}=\tilde{N}^{(1)}_{\mathbf{pqr}}+\tilde{N}^{(1)}_{\mathbf{prq}}. \tag{99}$$

Note that in Eq. (99) we have three linear algebraic equations in the four unknowns $M_{\mathbf{p(qr)}}$, $N^{(1)}_{\mathbf{pqr}}$, $N^{(1)}_{\mathbf{prq}}$, and $N^{(2)}_{\mathbf{p(qr)}}$. Thus, we construct another equation from the second of the three equations in (99) by interchanging the momenta **q** and **r**:

$$\frac{1}{\tilde{G}^{(1)}_{\mathbf{r}}}\tilde{M}_{\mathbf{p(qr)}}=\tilde{N}^{(1)}_{\mathbf{prq}}+\tilde{N}^{(2)}_{\mathbf{p(qr)}}. \tag{100}$$

Solving the algebraic equations (99) and (100), we obtain the following expressions for $\tilde{M}$, $\tilde{N}^{(1)}$, and $\tilde{N}^{(2)}$:

$$\tilde{M}_{\mathbf{p(qr)}}=\frac{4}{\mathcal{D}(\mathbf{p},\mathbf{q},\mathbf{r})}(2\pi)^D\delta(\mathbf{p}+\mathbf{q}+\mathbf{r}),$$

$$\tilde{N}^{(1)}_{\mathbf{pqr}}=\frac{2(\mathbf{p}^2+\mathbf{q}^2-\mathbf{r}^2+\mu_2^2)}{\mathcal{D}(\mathbf{p},\mathbf{q},\mathbf{r})}(2\pi)^D\delta(\mathbf{p}+\mathbf{q}+\mathbf{r}),$$

$$\tilde{N}^{(2)}_{\mathbf{p(qr)}}=\frac{2[\mathbf{q}^2+\mathbf{r}^2-\mathbf{p}^2+2\mu_1^2-\mu_2^2]}{\mathcal{D}(\mathbf{p},\mathbf{q},\mathbf{r})}(2\pi)^D\delta(\mathbf{p}+\mathbf{q}+\mathbf{r}), \tag{101}$$

where the denominator $\mathcal{D}(\mathbf{p},\mathbf{q},\mathbf{r})$ is given by

$$\mathcal{D}(\mathbf{p},\mathbf{q},\mathbf{r})=\mathbf{p}^4+\mathbf{q}^4+\mathbf{r}^4-2(\mathbf{p}^2\mathbf{q}^2+\mathbf{p}^2\mathbf{r}^2+\mathbf{q}^2\mathbf{r}^2)+2\mu_2^2(\mathbf{p}^2-\mathbf{q}^2-\mathbf{r}^2)-4\mu_1^2\mathbf{p}^2+\mu_2^4-4\mu_1^2\mu_2^2.$$

Observe that if we set $\mu_1=\mu_2=1$ and take the quantum mechanical limit $D\to 0$, Eq. (101) reduces to Eq. (42).

Finally, we transform Eq. (101) back to coordinate space by calculating the inverse Fourier transforms. The integrals to be performed are triple D-dimensional integrals, but we can perform the integral over **p** by using the delta function. We get

$$M_{\mathbf{x(yz)}}=-\frac{4}{(2\pi)^{2D}}\int\int d\mathbf{q}d\mathbf{r}\frac{e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{q}+i(\mathbf{x}-\mathbf{z})\cdot\mathbf{r}}}{\mathcal{D}(\mathbf{q},\mathbf{r})}, \tag{102}$$

where $\mathcal{D}(\mathbf{q},\mathbf{r})=4[\mathbf{q}^2\mathbf{r}^2-(\mathbf{q}\cdot\mathbf{r})^2]+4\mu_1^2(\mathbf{q}+\mathbf{r})^2-4\mu_2^2\mathbf{q}\cdot\mathbf{r}-\mu_2^4+4\mu_1^2\mu_2^2$, and

$$N^{(1)}_{\mathbf{xyz}}=\left[-\nabla_{\mathbf{y}}^2-\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{z}}+\frac{1}{2}\mu_2^2\right]M_{\mathbf{x(yz)}},$$

$$N^{(2)}_{\mathbf{x(yz)}}=\left[\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{z}}+\mu_1^2-\frac{1}{2}\mu_2^2\right]M_{\mathbf{x(yz)}}. \tag{103}$$

We mention that for the special case $D=1$ [quantum field theory in (1+1)-dimensional space-time], the quartic terms in the denominator $\mathcal{D}(\mathbf{q},\mathbf{r})$ again drop out and it is possible to evaluate the integral in Eq. (102) in terms of Bessel functions using the integration techniques described in Sec. VII. For this special case we obtain

$$M_{x(yz)}=-\frac{1}{\pi\mu_2\sqrt{4\mu_1^2-\mu_2^2}}\mathrm{K}_0(\mu_2\rho), \tag{104}$$

where $4\mu_2^2\rho^2=\mu_2^2(2x-y-z)^2+(4\mu_1^2-\mu_2^2)(y-z)^2$. The result in Eq. (104) reduces to that in Eq. (89) in the equal-mass case $\mu_1=\mu_2=\mu$. Also, our results in $D=1$ for $N_{xyz}^{(1)}$ and $N_{x(yz)}^{(2)}$ are

$$N_{xyz}^{(1)}=-\frac{\sqrt{4\mu_1^2-\mu_2^2}}{4\pi\mu_2}\frac{(x-z)(y-z)}{\rho^2}\mathrm{K}_0(\mu_2\rho)-\frac{1}{2\pi\rho\mu_2^2\sqrt{4\mu_1^2-\mu_2^2}}\left[\mu_2^2-(4\mu_1^2-\mu_2^2)\times\frac{(x-z)(y-z)}{\rho^2}\right]\mathrm{K}_0'(\mu_2\rho)-\frac{1}{\mu_2^2(4\mu_1^2-\mu_2^2)}\times\left[2\mu_2^2-(4\mu_1^2-\mu_2^2)\frac{(x-z)(y-z)}{\rho^2}\right]\delta(x-y)\,\delta(x-z),$$

$$N_{x(yz)}^{(2)}=\frac{\sqrt{4\mu_1^2-\mu_2^2}}{4\pi\mu_2}\frac{(x-z)(y-z)}{\rho^2}\mathrm{K}_0(\mu_2\rho)-\frac{1}{2\pi\rho\mu_2^2\sqrt{4\mu_1^2-\mu_2^2}}\left[2\mu_1^2-\mu_2^2-(4\mu_1^2-\mu_2^2)\times\frac{(x-z)(y-z)}{\rho^2}\right]\mathrm{K}_0'(\mu_2\rho)-\frac{1}{\mu_2^2(4\mu_1^2-\mu_2^2)}\times\left[4\mu_1^2-2\mu_2^2-(4\mu_1^2-\mu_2^2)\frac{(x-z)(y-z)}{\rho^2}\right]\times\delta(x-y)\,\delta(x-z). \tag{105}$$

B. $\varphi_1\varphi_2\varphi_3$ theory

We now consider the case of *three* interacting scalar fields whose dynamics is described by the Hamiltonian

$$H=H_0^{(1)}+H_0^{(2)}+H_0^{(3)}+\epsilon H_1, \tag{106}$$

where $H_0^{(j)}$ is given in Eq. (95) and $H_1=i\int d\mathbf{x}\varphi_{\mathbf{x}}^{(1)}\varphi_{\mathbf{x}}^{(2)}\varphi_{\mathbf{x}}^{(3)}$. This quantum field theory has the interesting property that its perturbative solution is finite for $D<3$; there are no divergent graphs in less than three space-time dimensions.

To find the operator $\mathcal{C}$ to leading order in ϵ, we need to solve the operator equation

$$[H_0^{(1)}+H_0^{(2)}+H_0^{(3)},Q_1]=-2H_1. \tag{107}$$

We introduce the ansatz

$$Q_1=\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}N_{\mathbf{xyz}}^{(1)}\pi_{\mathbf{x}}^{(1)}\varphi_{\mathbf{y}}^{(2)}\varphi_{\mathbf{z}}^{(3)}+\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}N_{\mathbf{xyz}}^{(2)}\pi_{\mathbf{x}}^{(2)}\varphi_{\mathbf{y}}^{(3)}\varphi_{\mathbf{z}}^{(1)}+\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}N_{\mathbf{xyz}}^{(3)}\pi_{\mathbf{x}}^{(3)}\varphi_{\mathbf{y}}^{(1)}\varphi_{\mathbf{z}}^{(2)}+\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}M_{\mathbf{xyz}}\pi_{\mathbf{x}}^{(1)}\pi_{\mathbf{y}}^{(2)}\pi_{\mathbf{z}}^{(3)}. \tag{108}$$

We then establish the following three results:

$$[H_0^{(1)},Q_1]=-i\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}(N_{\mathbf{xyz}}^{(2)}\pi_{\mathbf{z}}^{(1)}\pi_{\mathbf{x}}^{(2)}\varphi_{\mathbf{y}}^{(3)}+N_{\mathbf{xyz}}^{(3)}\pi_{\mathbf{y}}^{(1)}\pi_{\mathbf{x}}^{(3)}\varphi_{\mathbf{z}}^{(2)})+i\int\int\int\int d\mathbf{w}d\mathbf{x}d\mathbf{y}d\mathbf{z}(G_{\mathbf{wx}}^{(1)})^{-1}N_{\mathbf{xyz}}^{(1)}\varphi_{\mathbf{w}}^{(1)}\varphi_{\mathbf{y}}^{(2)}\varphi_{\mathbf{z}}^{(3)}+i\int\int\int\int d\mathbf{w}d\mathbf{x}d\mathbf{y}d\mathbf{z}(G_{\mathbf{wx}}^{(1)})^{-1}M_{\mathbf{xyz}}\pi_{\mathbf{y}}^{(2)}\pi_{\mathbf{z}}^{(3)}\varphi_{\mathbf{w}}^{(1)},$$

$$[H_0^{(2)},Q_1]=-i\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}(N_{\mathbf{xyz}}^{(1)}\pi_{\mathbf{y}}^{(2)}\pi_{\mathbf{x}}^{(1)}\varphi_{\mathbf{z}}^{(3)}+N_{\mathbf{xyz}}^{(3)}\pi_{\mathbf{x}}^{(3)}\pi_{\mathbf{z}}^{(2)}\varphi_{\mathbf{y}}^{(1)})+i\int\int\int\int d\mathbf{w}d\mathbf{x}d\mathbf{y}d\mathbf{z}(G_{\mathbf{wx}}^{(2)})^{-1}N_{\mathbf{xyz}}^{(2)}\varphi_{\mathbf{w}}^{(2)}\varphi_{\mathbf{y}}^{(3)}\varphi_{\mathbf{z}}^{(1)}+i\int\int\int\int d\mathbf{w}d\mathbf{x}d\mathbf{y}d\mathbf{z}(G_{\mathbf{wy}}^{(2)})^{-1}M_{\mathbf{xyz}}\pi_{\mathbf{x}}^{(1)}\pi_{\mathbf{z}}^{(3)}\varphi_{\mathbf{w}}^{(2)},$$

$$[H_0^{(3)},Q_1]=-i\int\int\int d\mathbf{x}d\mathbf{y}d\mathbf{z}(N_{\mathbf{xyz}}^{(1)}\pi_{\mathbf{x}}^{(1)}\pi_{\mathbf{z}}^{(3)}\varphi_{\mathbf{y}}^{(2)}+N_{\mathbf{xyz}}^{(2)}\pi_{\mathbf{x}}^{(2)}\pi_{\mathbf{y}}^{(3)}\varphi_{\mathbf{z}}^{(1)})+i\int\int\int\int d\mathbf{w}d\mathbf{x}d\mathbf{y}d\mathbf{z}(G_{\mathbf{wx}}^{(3)})^{-1}N_{\mathbf{xyz}}^{(3)}\varphi_{\mathbf{y}}^{(1)}\varphi_{\mathbf{z}}^{(2)}\varphi_{\mathbf{w}}^{(3)}+i\int\int\int\int d\mathbf{w}d\mathbf{x}d\mathbf{y}d\mathbf{z}(G_{\mathbf{wz}}^{(3)})^{-1}M_{\mathbf{xyz}}\pi_{\mathbf{x}}^{(1)}\pi_{\mathbf{y}}^{(2)}\varphi_{\mathbf{w}}^{(3)}. \tag{109}$$

From these equations we deduce the following system of differential equations:

$$(\mu_1^2-\nabla_{\mathbf{x}}^2)N_{\mathbf{xyz}}^{(1)}+(\mu_2^2-\nabla_{\mathbf{y}}^2)N_{\mathbf{yzx}}^{(2)}+(\mu_3^2-\nabla_{\mathbf{z}}^2)N_{\mathbf{zxy}}^{(3)}=-2\,\delta(\mathbf{x}-\mathbf{y})\,\delta(\mathbf{x}-\mathbf{z}),$$

$$(\mu_3^2-\nabla_{\mathbf{z}}^2)M_{\mathbf{xyz}}=N_{\mathbf{xyz}}^{(1)}+N_{\mathbf{yzx}}^{(2)},$$

$$(\mu_1^2-\nabla_{\mathbf{x}}^2)M_{\mathbf{xyz}}=N_{\mathbf{yzx}}^{(2)}+N_{\mathbf{zxy}}^{(3)},$$

$$(\mu_2^2-\nabla_{\mathbf{y}}^2)M_{\mathbf{xyz}}=N_{\mathbf{zxy}}^{(3)}+N_{\mathbf{xyz}}^{(1)}. \tag{110}$$

The solutions for the unknown functions are as follows: $M_{\mathbf{xyz}}$ is given by the integral (83) with the more general formula $\mathcal{D}(\mathbf{p},\mathbf{q})=4[\mathbf{p}^2\mathbf{q}^2-(\mathbf{p}\cdot\mathbf{q})^2]+4[\mu_1^2(\mathbf{q}^2+\mathbf{p}\cdot\mathbf{q})+\mu_2^2(\mathbf{p}^2+\mathbf{p}\cdot\mathbf{q})-\mu_3^2\mathbf{p}\cdot\mathbf{q}]+m^4$ with $3m^4=2\mu_1^2\mu_2^2+2\mu_1^2\mu_3^2+2\mu_2^2\mu_3^2-\mu_1^4-\mu_2^4-\mu_3^4$. The N coefficients are expressed as derivatives acting on M:

$$N_{\mathbf{xyz}}^{(1)}=[\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{z}}+\tfrac{1}{2}(\mu_2^2+\mu_3^2-\mu_1^2)]M_{\mathbf{xyz}},$$

$$N_{\mathbf{xyz}}^{(2)}=[-\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{z}}-\nabla_{\mathbf{z}}^2+\tfrac{1}{2}(\mu_1^2+\mu_3^2-\mu_2^2)]M_{\mathbf{xyz}},$$

$$N_{\mathbf{xyz}}^{(3)}=[-\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{z}}-\nabla_{\mathbf{y}}^2+\tfrac{1}{2}(\mu_1^2+\mu_2^2-\mu_3^2)]M_{\mathbf{xyz}}. \tag{111}$$

For the case $D=1$ we have

$$M_{(xyz)}=-\frac{1}{\pi\sqrt{3}m^2}\mathrm{K}_0(m\rho), \tag{112}$$

where $2m^2\rho^2=(\mu_1^2+\mu_2^2-\mu_3^2)(x-y)^2+(\mu_2^2+\mu_3^2-\mu_1^2)(y-z)^2+(\mu_3^2+\mu_1^2-\mu_2^2)(z-x)^2$. We also have

$$N_{xyz}^{(1)}=-\frac{\sqrt{3}(x-y)(x-z)}{4\pi\rho^2}\mathrm{K}_0(m\rho)-\frac{1}{2\pi\rho m}\left[\frac{\mu_2^2+\mu_3^2-\mu_1^2}{\sqrt{3}m^2}-\frac{\sqrt{3}(x-y)(x-z)}{\rho^2}\right]\mathrm{K}_0'(m\rho)-\frac{2}{m^2}\left[\frac{\mu_2^2+\mu_3^2-\mu_1^2}{3m^2}-\frac{(x-y)(x-z)}{2\rho^2}\right]\delta(x-y)\,\delta(x-z), \tag{113}$$

and analogous expressions for $N_{xyz}^{(2)}$ and $N_{xyz}^{(3)}$.

IX. FINAL REMARKS

We have introduced an algebraic technique for constructing the operator $\mathcal{C}$, which is required to define the positive-definite inner product of the Hilbert space in $\mathcal{PT}$-symmetric quantum theories. Unlike the previously used analytical procedure for constructing $\mathcal{C}$, which relies on the determination of all the energy eigenstates, the algebraic approach introduced here allows us to determine $\mathcal{C}$ directly from the operator form of the Hamiltonian. As a consequence, the approach extends naturally to quantum field theory. We have explicitly demonstrated the perturbative derivation of $\mathcal{C}$ both in quantum mechanics and in quantum field theory for the case of cubic interactions.

In the case of quantum field theory we point out that all of the field theories discussed in this paper are Lorentz covariant: They are expressed in terms of covariant fields, which transform as pseudoscalars. For each of these field theories the conventional construction of the generators of the Poincaré group can be carried out, and these generators satisfy the usual commutation relations. Furthermore, the $\mathcal{C}$ operator is a Lorentz scalar. By construction, $\mathcal{C}$ does not depend on the spatial coordinate, which is integrated out [see, for example, Eq. (68)] and it does not depend on time because $\mathcal{C}$ commutes with the Hamiltonian. Thus, $\mathcal{C}$ is like a scalar charge operator, $\int d\mathbf{x}J^0(\mathbf{x},t)$, which is the spatial integral of a locally conserved current satisfying $\partial_\mu J^\mu=0$.

We hope to generalize the breakthrough reported in this paper to noncubic $\mathcal{PT}$-symmetric quantum field theories, such as a $-g\varphi^4$ theory. A $-g\varphi^4$ quantum field theory in four-dimensional space-time is a remarkable model because it has a positive spectrum, is renormalizable, is asymptotically free [16], and has a nonzero one-point Green's function $G_1=\langle\varphi\rangle$. Consequently, this theory may ultimately be useful in elucidating the dynamics of the Higgs sector of the standard model.

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APPENDIX: GRAPHICAL SOLUTION TO EQUATIONS (76) AND (77)

Graphical methods can be used to solve the simultaneous equations (76) and (77) for $\tilde{m}_{(\mathbf{pqr})}$ and $\tilde{n}_{\mathbf{p(qr)}}$. We begin by defining the unknown function $\tilde{F}_{\mathbf{p(qr)}}$ by

$$\tilde{n}_{\mathbf{p(qr)}}=\tilde{G}_{\mathbf{p}}(\tilde{F}_{\mathbf{p(qr)}}-2). \tag{A1}$$

In terms of $\tilde{F}$, Eq. (76) becomes

$$\tilde{F}_{\mathbf{p(qr)}}+\tilde{F}_{\mathbf{q(pr)}}+\tilde{F}_{\mathbf{r(pq)}}=0 \tag{A2}$$

and Eq. (77) becomes

$$\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}}\tilde{F}_{\mathbf{p(qr)}}+\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}}\tilde{F}_{\mathbf{r(pq)}}-\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{r}}\tilde{F}_{\mathbf{p(qr)}}-\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}}\tilde{F}_{\mathbf{q(pr)}}=2\tilde{G}_{\mathbf{p}}(\tilde{G}_{\mathbf{q}}-\tilde{G}_{\mathbf{r}}), \tag{A3}$$

where we have made use of Eq. (74) to eliminate $\tilde{m}$.

We can find an exact solution to Eqs. (A2) and (A3) by using an iterative method. First, we construct a function that

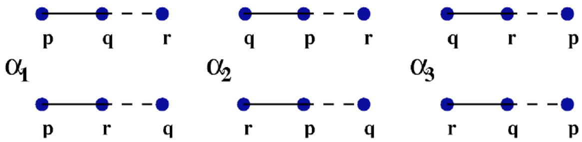

FIG. 1. Representation of $\tilde{F}^{(1)}_{\mathbf{p(qr)}}$ in Eq. (A4) in terms of tree graphs. The solid lines represent factors of $\tilde{G}$ and the dashed lines represent factors of $\tilde{G}^{-1}$. The dots indicate points where momentum is flowing into the graph. The momentum in the graph is conserved and obeys the constraint $\mathbf{p}+\mathbf{q}+\mathbf{r}=0$. The numerical coefficients α_1, α_2, and α_3 multiply the graphs as shown in Eq. (A4).

we call $\tilde{F}^{(1)}_{\mathbf{p(qr)}}$, which exactly solves Eq. (A2) and reproduces the structure on the right side of Eq. (A3). An ansatz that works is

$$\tilde{F}^{(1)}_{\mathbf{p(qr)}}=\alpha_1\left(\frac{\tilde{G}_{\mathbf{q}}}{\tilde{G}_{\mathbf{r}}}+\frac{\tilde{G}_{\mathbf{r}}}{\tilde{G}_{\mathbf{q}}}\right)+\alpha_2\left(\frac{\tilde{G}_{\mathbf{p}}}{\tilde{G}_{\mathbf{q}}}+\frac{\tilde{G}_{\mathbf{p}}}{\tilde{G}_{\mathbf{r}}}\right)+\alpha_3\left(\frac{\tilde{G}_{\mathbf{r}}}{\tilde{G}_{\mathbf{p}}}+\frac{\tilde{G}_{\mathbf{q}}}{\tilde{G}_{\mathbf{p}}}\right), \tag{A4}$$

where the coefficients α_1, α_2, and α_3 are numerical parameters to be determined.

The six terms in Eq. (A4) have a graphical representation (see Fig. 1) in which we use a solid line to represent the Green's function $\tilde{G}_{\mathbf{p}}$ and a dashed line to represent its inverse $\tilde{G}_{\mathbf{p}}^{-1}$. The dots indicate momentum flowing into the graph. Note that the three momenta $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ must satisfy a momentum conservation equation $\mathbf{p}+\mathbf{q}+\mathbf{r}=0$.

The function $\tilde{F}^{(1)}$ exactly solves Eq. (A2) if

$$\alpha_1+\alpha_2+\alpha_3=0, \tag{A5}$$

and $\tilde{F}^{(1)}$ exactly reproduces the right side of Eq. (A3) if

$$\alpha_3-2\alpha_2=2. \tag{A6}$$

However, this choice for $\tilde{F}^{(1)}$ also creates new terms on the right side of Eq. (A3) that must be eliminated. To eliminate these additional terms we add to $\tilde{F}^{(1)}$ a new ansatz, designated $\tilde{F}^{(2)}$. The form of $\tilde{F}^{(2)}$ is

$$\tilde{F}^{(2)}_{\mathbf{p(qr)}}=\beta_1\left(\frac{\tilde{G}^2_{\mathbf{q}}}{\tilde{G}^2_{\mathbf{r}}}+\frac{\tilde{G}^2_{\mathbf{r}}}{\tilde{G}^2_{\mathbf{q}}}\right)+\beta_2\left(\frac{\tilde{G}^2_{\mathbf{p}}}{\tilde{G}^2_{\mathbf{q}}}+\frac{\tilde{G}^2_{\mathbf{p}}}{\tilde{G}^2_{\mathbf{r}}}\right)+\beta_3\left(\frac{\tilde{G}^2_{\mathbf{q}}}{\tilde{G}^2_{\mathbf{p}}}+\frac{\tilde{G}^2_{\mathbf{r}}}{\tilde{G}^2_{\mathbf{p}}}\right)$$
$$+\beta_4\frac{\tilde{G}^2_{\mathbf{p}}}{\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}}}+\beta_5\left(\frac{\tilde{G}^2_{\mathbf{q}}}{\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{r}}}+\frac{\tilde{G}^2_{\mathbf{r}}}{\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}}}\right)+\beta_6\frac{\tilde{G}_{\mathbf{q}}\tilde{G}_{\mathbf{r}}}{\tilde{G}^2_{\mathbf{p}}}$$
$$+\beta_7\left(\frac{\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{q}}}{\tilde{G}^2_{\mathbf{r}}}+\frac{\tilde{G}_{\mathbf{p}}\tilde{G}_{\mathbf{r}}}{\tilde{G}^2_{\mathbf{q}}}\right), \tag{A7}$$

where the coefficients $\beta_1,\beta_2,\ldots,\beta_7$ must be determined. The twelve terms in Eq. (A7) are represented graphically in Fig. 2.

To determine the coefficients in $\tilde{F}^{(2)}$ we require that the right side of Eq. (A2) vanish. This gives the three conditions

$$\beta_1+\beta_2+\beta_3=0,$$
$$\beta_4+2\beta_5=0,$$
$$\beta_6+2\beta_7=0. \tag{A8}$$

In addition, we require that the terms arising on the right side of Eq. (A3) as a result of the ansatz $\tilde{F}^{(1)}$ must be eliminated. This requirement gives the equations

$$2\alpha_3-\alpha_1-\beta_5=0,$$
$$\alpha_1-\beta_2+\beta_3-\beta_7=0,$$
$$\alpha_2-\beta_2+\beta_6-\beta_7=0,$$
$$\alpha_1-\alpha_2+\beta_3-\beta_6=0. \tag{A9}$$

One of these equations is redundant because if we subtract the third equation from the second, we obtain the fourth.

We now continue the process. It is necessary to introduce a third ansatz in order to eliminate the new terms that appear on the right side of Eq. (A3) as a consequence of the ansatz $\tilde{F}^{(2)}$. This ansatz is given by

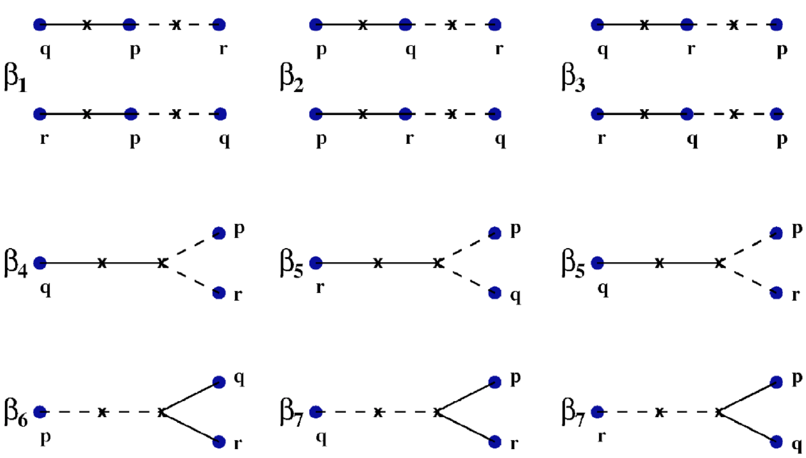

FIG. 2. Representation of the ansatz $\tilde{F}^{(2)}_{\mathbf{p(qr)}}$ in Eq. (A7) in terms of tree graphs. The notation used in this figure is the same as that used in Fig. 1. The crosses in the graphs are vertices at which no momentum is flowing into or out of the graph.

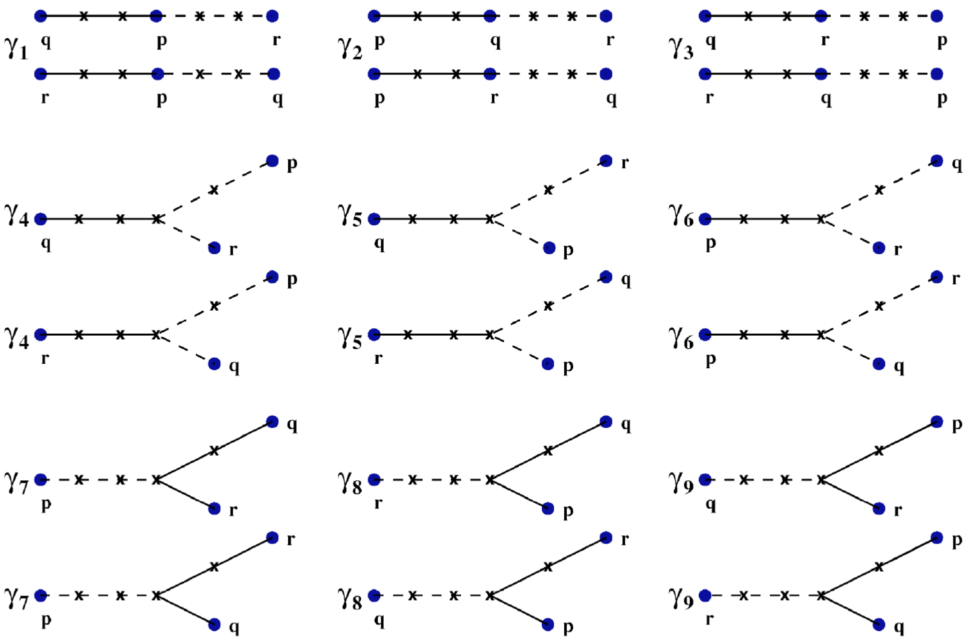

FIG. 3. Representation of the ansatz $\tilde{F}^{(3)}_{\mathbf{p(qr)}}$ in Eq. (A10) in terms of tree graphs. The notation used in this figure is the same as that used in Fig. 2.

$$\tilde{F}^{(3)}_{\mathbf{p(qr)}}=\gamma_1\left(\frac{\tilde{G}^3_{\mathbf{q}}}{\tilde{G}^3_{\mathbf{r}}}+\frac{\tilde{G}^3_{\mathbf{r}}}{\tilde{G}^3_{\mathbf{q}}}\right)+\gamma_2\left(\frac{\tilde{G}^3_{\mathbf{p}}}{\tilde{G}^3_{\mathbf{q}}}+\frac{\tilde{G}^3_{\mathbf{p}}}{\tilde{G}^3_{\mathbf{r}}}\right)+\gamma_3\left(\frac{\tilde{G}^3_{\mathbf{q}}}{\tilde{G}^3_{\mathbf{p}}}+\frac{\tilde{G}^3_{\mathbf{r}}}{\tilde{G}^3_{\mathbf{p}}}\right)$$
$$+\gamma_4\left(\frac{\tilde{G}^3_{\mathbf{q}}}{\tilde{G}^2_{\mathbf{p}}\tilde{G}_{\mathbf{r}}}+\frac{\tilde{G}^3_{\mathbf{r}}}{\tilde{G}^2_{\mathbf{p}}\tilde{G}_{\mathbf{q}}}\right)+\gamma_5\left(\frac{\tilde{G}^3_{\mathbf{q}}}{\tilde{G}_{\mathbf{p}}\tilde{G}^2_{\mathbf{r}}}+\frac{\tilde{G}^3_{\mathbf{r}}}{\tilde{G}_{\mathbf{p}}\tilde{G}^2_{\mathbf{q}}}\right)$$
$$+\gamma_6\left(\frac{\tilde{G}^3_{\mathbf{p}}}{\tilde{G}_{\mathbf{r}}\tilde{G}^2_{\mathbf{q}}}+\frac{\tilde{G}^3_{\mathbf{p}}}{\tilde{G}_{\mathbf{q}}\tilde{G}^2_{\mathbf{r}}}\right)+\gamma_7\left(\frac{\tilde{G}_{\mathbf{q}}\tilde{G}^2_{\mathbf{r}}}{\tilde{G}^3_{\mathbf{p}}}+\frac{\tilde{G}_{\mathbf{r}}\tilde{G}^2_{\mathbf{q}}}{\tilde{G}^3_{\mathbf{p}}}\right)$$
$$+\gamma_8\left(\frac{\tilde{G}_{\mathbf{p}}\tilde{G}^2_{\mathbf{r}}}{\tilde{G}^3_{\mathbf{q}}}+\frac{\tilde{G}_{\mathbf{p}}\tilde{G}^2_{\mathbf{q}}}{\tilde{G}^3_{\mathbf{r}}}\right)+\gamma_9\left(\frac{\tilde{G}^2_{\mathbf{p}}\tilde{G}_{\mathbf{r}}}{\tilde{G}^3_{\mathbf{q}}}+\frac{\tilde{G}^2_{\mathbf{p}}\tilde{G}_{\mathbf{q}}}{\tilde{G}^3_{\mathbf{r}}}\right) \tag{A10}$$

and it is represented graphically in Fig. 3.

To determine the nine numerical coefficients in $\tilde{F}^{(3)}$ we require that the right side of Eq. (A2) vanish. This gives the three conditions

$$\gamma_1+\gamma_2+\gamma_3=0,$$
$$\gamma_4+\gamma_5+\gamma_6=0,$$
$$\gamma_7+\gamma_8+\gamma_9=0. \tag{A11}$$

Also, we require that the terms arising on the right side of Eq. (A3) as a result of the ansatz $\tilde{F}^{(2)}$ must be eliminated. This requirement gives the equations

$$\beta_1-\beta_4+\gamma_5-\gamma_6=0,$$
$$\beta_3-\beta_4+\beta_5-\gamma_6=0,$$
$$\beta_3+\beta_5-\beta_1-\gamma_5=0,$$
$$\beta_1-\beta_2+\gamma_3-\gamma_7=0,$$
$$\beta_1-\gamma_2+\gamma_3-\gamma_9=0,$$
$$\beta_2-\gamma_2+\gamma_7-\gamma_9=0,$$
$$\beta_7+\gamma_7-2\gamma_8=0. \tag{A12}$$

Note that two of these equations are redundant; if we subtract the first from the second we obtain the third, and if we subtract the fourth from the fifth we obtain the sixth.

It is clear that if we continue this process, we obtain more and more linear equations to solve. However, with each new ansatz the number of unknowns always exceeds the number of equations by one. For example, there are three α's but only two equations (A5) and (A6); there are seven β's but only six independent equations (A8) and (A9); there are nine γ's but only eight independent equations (A11) and (A12). Hence, with each new ansatz we have a kind of gauge freedom and we find it convenient to use this freedom to eliminate all graphs having split legs consisting of solid lines. That is, we choose $\beta_6=\beta_7=0$, $\gamma_7=\gamma_8=\gamma_9=0$, and so on. We can also choose $\alpha_1=0$. Thus, for the next iteration our ansatz reads

$$\tilde{F}^{(4)}_{\mathbf{p(qr)}}=\delta_1\left(\frac{\tilde{G}^4_{\mathbf{q}}}{\tilde{G}^4_{\mathbf{r}}}+\frac{\tilde{G}^4_{\mathbf{r}}}{\tilde{G}^4_{\mathbf{q}}}\right)+\delta_2\left(\frac{\tilde{G}^4_{\mathbf{p}}}{\tilde{G}^4_{\mathbf{q}}}+\frac{\tilde{G}^4_{\mathbf{p}}}{\tilde{G}^4_{\mathbf{r}}}\right)+\delta_3\left(\frac{\tilde{G}^4_{\mathbf{q}}}{\tilde{G}^4_{\mathbf{p}}}+\frac{\tilde{G}^4_{\mathbf{r}}}{\tilde{G}^4_{\mathbf{p}}}\right)$$
$$+\delta_4\left(\frac{\tilde{G}^4_{\mathbf{q}}}{\tilde{G}^3_{\mathbf{p}}\tilde{G}_{\mathbf{r}}}+\frac{\tilde{G}^4_{\mathbf{r}}}{\tilde{G}^3_{\mathbf{p}}\tilde{G}_{\mathbf{q}}}\right)+\delta_5\left(\frac{\tilde{G}^4_{\mathbf{q}}}{\tilde{G}_{\mathbf{p}}\tilde{G}^3_{\mathbf{r}}}+\frac{\tilde{G}^4_{\mathbf{r}}}{\tilde{G}_{\mathbf{p}}\tilde{G}^3_{\mathbf{q}}}\right)$$
$$+\delta_6\left(\frac{\tilde{G}^4_{\mathbf{p}}}{\tilde{G}_{\mathbf{r}}\tilde{G}^3_{\mathbf{q}}}+\frac{\tilde{G}^4_{\mathbf{p}}}{\tilde{G}_{\mathbf{q}}\tilde{G}^3_{\mathbf{r}}}\right)+\delta_7\left(\frac{\tilde{G}^4_{\mathbf{q}}}{\tilde{G}^2_{\mathbf{p}}\tilde{G}^2_{\mathbf{r}}}+\frac{\tilde{G}^4_{\mathbf{r}}}{\tilde{G}^2_{\mathbf{p}}\tilde{G}^2_{\mathbf{q}}}\right)$$
$$+\delta_8\frac{\tilde{G}^4_{\mathbf{p}}}{\tilde{G}^2_{\mathbf{q}}\tilde{G}^2_{\mathbf{r}}}. \tag{A13}$$

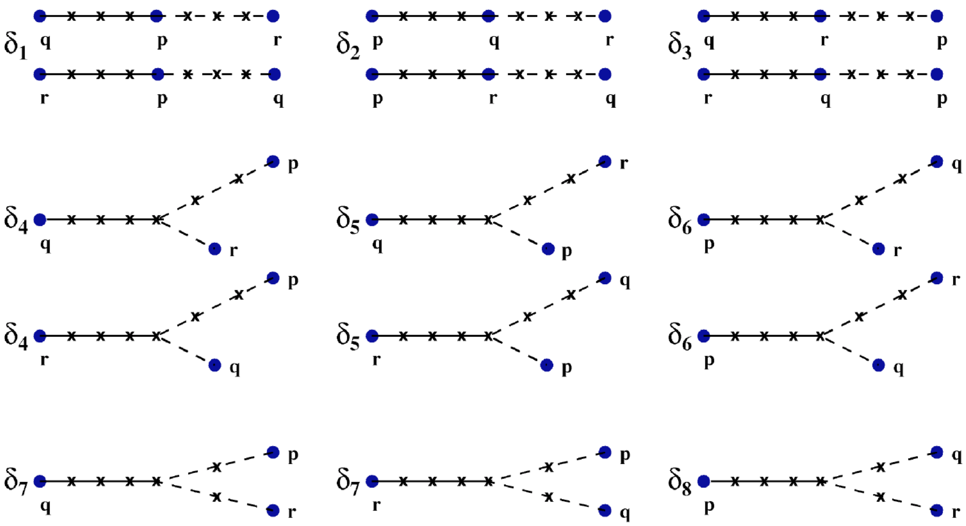

FIG. 4. Representation of the ansatz $\tilde{F}^{(4)}_{\mathbf{p(qr)}}$ in Eq. (A13) in terms of tree graphs. The notation used in this figure is the same as that used in Fig. 2. Note that the number of linear equations that determine the numerical coefficients is less than the number of coefficients. This allows us the freedom to exclude *a priori* all graphs having split legs consisting of solid lines.

The graphical representation of $\tilde{F}^{(4)}$ is shown in Fig. 4.

Requiring that the right side of Eq. (A2) continue to vanish gives the three conditions

$$\delta_1+\delta_2+\delta_3=0,$$
$$\delta_4+\delta_5+\delta_6=0,$$
$$2\delta_7+\delta_8=0. \tag{A14}$$

Also, to eliminate the terms arising on the right side of Eq. (A3) as a result of the ansatz $\tilde{F}^{(3)}$ we must require that

$$\gamma_2-\gamma_6-\delta_4+\delta_5=0,$$
$$\gamma_3+\gamma_5-\gamma_6-\delta_4=0,$$
$$\gamma_3+\gamma_5-\gamma_2-\delta_5=0,$$
$$\gamma_6-2\gamma_4+\delta_7=0,$$
$$\gamma_1-\gamma_2-\delta_3=0,$$
$$\gamma_1-\delta_1=0,$$
$$\gamma_2-\delta_1+\delta_3=0. \tag{A15}$$

Note that two of these equations are redundant; if we subtract the first from the second we obtain the third, and if we subtract the fifth from the sixth we obtain the seventh. Thus, in total there are eight δ's to be determined by eight independent equations.

Let us now examine the solutions for the undetermined numerical coefficients. Solving the system of linear equations for the first five groups of these coefficients, we find that

$$\alpha_1=0,\quad \alpha_2=-2,\quad \alpha_3=2,$$
$$\beta_1=0,\quad \beta_2=-2,\quad \beta_3=2,\quad \beta_4=6,\quad \beta_5=-12,$$
$$\gamma_1=0,\quad \gamma_2=-2,\quad \gamma_3=2,\quad \gamma_4=20,$$
$$\gamma_5=10,\quad \gamma_6=-30,$$
$$\delta_1=0,\quad \delta_2=-2,\quad \delta_3=2,\quad \delta_4=42,\quad \delta_5=14,$$
$$\delta_6=-56,\quad \delta_7=70,\quad \delta_8=-140,$$
$$\epsilon_1=0,\quad \epsilon_2=-2,\quad \epsilon_3=2,\quad \epsilon_4=72,\quad \epsilon_5=18,$$
$$\epsilon_6=-90,\quad \epsilon_7=252,\quad \epsilon_8=168, \epsilon_9=-420. \tag{A16}$$

A brief examination of these coefficients shows that these numbers are all binomial coefficients. In fact, by inspection we can now write down explicitly all terms in $\tilde{F}^{(1)}$, $\tilde{F}^{(2)}$, $\tilde{F}^{(3)}$, and so on, as the following double sum:

$$\tilde{F}_{\mathbf{p(qr)}}=-4\sum_{n=1}^{\infty}\frac{\tilde{G}^n_{\mathbf{p}}}{\tilde{G}^n_{\mathbf{r}}}\sum_{k=0}^{n}\binom{2n}{2k}\frac{\tilde{G}^k_{\mathbf{r}}}{\tilde{G}^k_{\mathbf{q}}}+4\sum_{n=0}^{\infty}\frac{\tilde{G}^{n+1}_{\mathbf{r}}}{\tilde{G}^{n+1}_{\mathbf{p}}}\sum_{k=0}^{n}\binom{2n+1}{2n+1-2k}\frac{\tilde{G}^k_{\mathbf{p}}}{\tilde{G}^k_{\mathbf{q}}}. \tag{A17}$$

It is easy to perform these sums, and the result for $\tilde{N}$ is given in Eq. (80). It is remarkable that the solution for $\tilde{N}$ is *unique* despite the arbitrary choice of "gauge" that we have made in solving the systems of algebraic equations.

We emphasize that all of the tree graphs that appear in this analysis have three external lines. This reflects the fact that the form factors M and N that we are calculating represent the contribution of tree graphs to the three-point vertices. The next order in perturbation theory is proportional to ϵ^3, and the graphical representation of the result consists of all tree graphs having five external legs.

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