Chern-Simons-like action induced radiatively in general relativity

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The Chern-Simons-like gravitational action is evaluated explicitly in four-dimensional space-time by radiative corrections at the one-loop level. The calculation is performed in the fermionic sector where the Dirac fermions interact with the background gravitational field, including the parity-violating term $\bar{\psi}\not{b}\gamma_5\psi$. The investigation takes into account the weak field approximation and dimensional regularization scheme.

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I. INTRODUCTION

The theory of the electromagnetism was crucial to questioning Galilei invariance and to giving rise to Lorentz symmetry. Nowadays, in string theory one may find a way to question Lorentz invariance, since there are interactions that support spontaneous breaking of Lorentz symmetry [1], one of those interactions being described by the Chern-Simons-like action.

The Chern-Simons term was first introduced in three-dimensional gauge field and gravitational field theories by Deser, Jackiw, and Templeton [2]. In the gauge theories interesting phenomena are exhibited such as exotic statistics, fractional spin, and massive gauge fields. These phenomena are of topological nature and they can be produced when we add the Chern-Simons term to the Lagrangian which describes the system under consideration. Posteriorly, was observed that if one adds the Chern-Simons-like term in four-dimensional space-time to Maxwell's theory, both Lorentz and *CPT* symmetries are violated. The model predicts the rotation of the plane of polarization of radiation from distance galaxies, an effect which has not been observed yet [3]. In recent papers [4,5], modification of general relativity when one adds the Chern-Simons-like gravitational term has been studied. The authors have observed that in this modified theory the Schwarzschild metric is a solution, gravitational waves possess two polarizations which travel with the velocity of light, and polarized waves are suppressed or enhanced.

The induction by radiative correction of the Chern-Simons term has been analyzed in the last 20 years. Redlich [6] in a seminal paper studied the subject in the context of quantum electrodynamics in three dimensions of space-time. Following this paper other models in quantum field theory were investigated [7–9]. Extension to higher odd dimensions was done and also in the case of a gravitational background field [10,11].

Colladay and Kostelecky [12] analyzed the question of whether the Chern-Simons-like term is generated by radiative corrections when the Lorentz and *CPT*-violating term $\bar{\psi}b_\mu\gamma^\mu\gamma_5\psi$ is added to the conventional Lagrangian of quantum electrodynamics in four-dimensional space-time. They have observed that such term is dependent on the regularization scheme. Coleman and Glashow [13] argued that such a term must unambiguously vanish to first order in b_μ for any gauge-invariant *CPT* odd interaction. They considered that the axial current j^5_μ should stay gauge invariant in the quantum theory at any momentum or at any space-time point. Since $\langle j^5_\mu\rangle=\delta L(x)/\delta b_\mu$, this condition is equivalent to the requirement that the Lagrangian density corresponding to the quantum effective action should be gauge invariant. Thus, based on this requirement, the Chern-Simons-like term is not generated since its Lagrangian density is explicitly not gauge invariant. Jackiw and Kostelecký [14] showed that the Chern-Simons-like term is induced. They thought that since j^5_μ only couples with a constant 4-vector b_μ, it is true to require only that j^5_μ with zero-momentum be gauge invariant at the quantum level. Since $\langle\int d^4x j^5_\mu\rangle=\delta S/\delta b_\mu$, this condition is equivalent to the requirement that the quantum effective action should be gauge invariant. This controversy on a possible Chern-Simons-like term generated through radiative corrections was carefully investigated by many authors [15–26]. This phenomenon was analyzed in quantum electrodynamics as a part of the standard model. Our purpose in this paper is to derive the Chern-Simons-like gravitational action induced by Dirac fermions coupled to a background gravitational field. The result shows that the Chern-Simons-like term is generated by radiative fermion loops, under the assumption of the weak field approximation and dimensional regularization scheme.

II. EVALUATING THE CHERN-SIMONS-LIKE GRAVITATIONAL ACTION

The action that we are interested is given by

$$S=\int \mathrm{d}^4x\left(\frac{1}{2}iee^\mu{}_a\bar{\psi}\gamma^a\overleftrightarrow{D}_\mu\psi-ee^\mu{}_a\bar{\psi}b_\mu\gamma^a\gamma_5\psi\right), \qquad (1)$$

where we have included the parity-violating term. Here $e^\mu{}_a$ is the tetrad (vierbein), $e\equiv\det e^\mu{}_a$, and b_μ is a constant 4-vector. The covariant derivative is given by

$$D_\mu\psi=\partial_\mu\psi+\frac{1}{2}w_{\mu cd}\sigma^{cd}\psi, \qquad (2)$$

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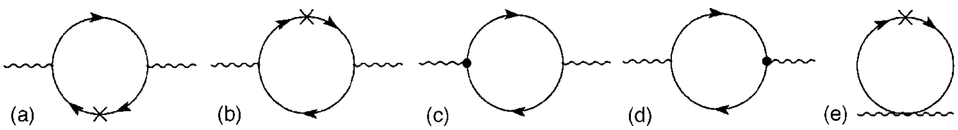


FIG. 1. One-loop contributions.

where $w_\mu{}^{cd}$ is the spin connection and $\sigma^{cd}=\frac{1}{4}[\gamma^c,\gamma^d]$, whereas the covariant derivative on a Dirac-conjugate field $\overline{\psi}$ is

$$D_\mu\overline{\psi}=\partial_\mu\overline{\psi}-\frac{1}{2}w_{\mu cd}\overline{\psi}\sigma^{cd}. \tag{3}$$

Using the expressions above we can rewrite the Eq. (1) as

$$S=\int d^4x\left(\frac{1}{2}iee^\mu{}_a\overline{\psi}\gamma^a\overleftrightarrow{\partial}_\mu\psi+\frac{1}{4}iee^\mu{}_a\overline{\psi}w_{\mu cd}\Gamma^{acd}\psi - ee^\mu{}_a\overline{\psi}b_\mu\gamma^a\gamma_5\psi\right), \tag{4}$$

where $\Gamma^{acd}=\frac{1}{6}(\gamma^a\gamma^c\gamma^d\pm\text{permutations})$—i.e., the antisymmetrized product of three γ matrices.

In the weak field approximation we consider $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ $(g^{\mu\nu}=\eta^{\mu\nu}-h^{\mu\nu})$, which induces an expansion for the vierbein $e_{\mu a}=\eta_{\mu a}+\frac{1}{2}h_{\mu a}$ $(e^\mu_a=\eta^\mu_a-\frac{1}{2}h^\mu_a)$. Then, the linearized Chern-Simons-like action takes the form [4]

$$S_{linear}=\frac{1}{4}\int d^4xh^{\mu\nu}v^\lambda\epsilon_{\alpha\mu\lambda\rho}\partial^\rho(\partial_\gamma\partial^\gamma h^\alpha_\nu-\partial_\nu\partial_\gamma h^{\gamma\alpha}). \tag{5}$$

The main purpose of the present work is to induce this action by radiative correction of fermionic matter field to obtain the relation between v_λ and b_μ. In order to perform this calculation we consider the fermionic model represented by the action

$$e^{i\Gamma[h]}=\int\mathcal{D}\overline{\psi}\mathcal{D}\psi e^{iS[h,\overline{\psi},\psi]}, \tag{6}$$

where the linearized effective action is given by

$$S[h,\overline{\psi},\psi]=\int d^4x\left(\frac{1}{2}i\overline{\psi}\Gamma^\mu\overleftrightarrow{\partial}_\mu\psi+\overline{\psi}h_{\mu\nu}\Gamma^{\mu\nu}\psi - \overline{\psi}b_\mu\gamma^\mu\gamma_5\psi\right), \tag{7}$$

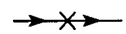
with $\Gamma^\mu=\gamma^\mu-\frac{1}{2}h^{\mu\nu}\gamma_\nu$ and $\Gamma^{\mu\nu}=\frac{1}{2}b^\mu\gamma^\nu\gamma_5-(i/16)(\partial_\rho h_{\alpha\beta})\eta^{\beta\nu}\Gamma^{\rho\mu\alpha}$. In this expression, we neglect the terms proportional to $h=\eta^{\mu\nu}h_{\mu\nu}$ because they do not contribute to generate the Chern-Simons-like action.

The Feynman rules that we obtain from Eq. (7) are the following:

fermion propagator

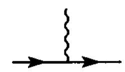
$$=S(p)=\frac{i}{\not{p}-m}, \tag{8}$$

fermion propagator with $\not{b}$ insertion

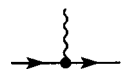


$$=-i\not{b}\gamma_5. \tag{9}$$

The three relevant interaction fermion-graviton vertices are

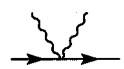


$$=-\frac{i}{4}\gamma_\mu(2p+q)_\nu, \tag{10}$$



$$=i\gamma_\mu b_\nu\gamma_5, \tag{11}$$

and



$$=-\frac{i}{16}\eta^{\beta\nu}\Gamma^{\mu\rho\alpha}(q_1-q_2)_\rho. \tag{12}$$

The one-loop order correction to the effective action are given in Fig. 1.

Graphs (c), (d), and (e) do not contribute to generate Chern-Simons-like action. The only relevant graphs are (a) and (b) whose the Feynman integral are given by

$$\Pi_a^{\mu\nu\alpha\beta}(q)=-\frac{i}{16}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}[\gamma^\mu(2p+q)^\nu S(p) \times\gamma^\alpha(2p+q)^\beta S(p+q)\not{b}\gamma_5S(p+q)] \tag{13}$$

and

$$\Pi_b^{\mu\nu\alpha\beta}(q)=-\frac{i}{16}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}[\gamma^\mu(2p+q)^\nu \times S(p)\not{b}\gamma_5S(p)\gamma^\alpha(2p+q)^\beta S(p+q)]. \tag{14}$$

It is straightforward to see that

$$\Pi^{\mu\nu\alpha\beta}(q)=\Pi_a^{\mu\nu\alpha\beta}(q)=\Pi_b^{\alpha\nu\mu\beta}(-q), \tag{15}$$

which appears when substituting the loop momenta, $p\to p-qx$, and we using the cyclic properties of the trace of a product of γ matrices. So from now on we work only with Eq. (13) which takes the form

$$\Pi^{\mu\nu\alpha\beta}(q)=-\frac{1}{8}\int_0^1 \mathrm{d}x x\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{[2p+q(1-2x)]^\nu[2p+q(1-2x)]^\beta}{[p^2-m^2+x(1-x)q^2]^3} \times \mathrm{tr}\{\gamma^\mu(\not p-\not q x+m)\gamma^\alpha[\not p+\not q(1-x)+m]\not b\gamma_5[\not p+\not q(1-x)+m]\}, \tag{16}$$

where we have used the Feynman parameter to combine the denominator in Eq. (16). First of all, to simplify the numerator of Eq. (16) we take into account that the trace of an odd product of γ matrices times γ_5 is zero. Thus we have

$$\Pi^{\mu\nu\alpha\beta}(q)=-\frac{1}{8}\int_0^1 \mathrm{d}x x\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{[2p+q(1-2x)]^\nu[2p+q(1-2x)]^\beta}{[p^2-m^2+x(1-x)q^2]^3}\mathrm{tr}\{[\not p+\not q(1-x)]\gamma^\mu(\not p-\not q x)\gamma^\alpha \times[\not p+\not q(1-x)]\not b\gamma_5+m^2[\not p+\not q(1-x)]\gamma^\mu\gamma^\alpha\not b\gamma_5+m^2\gamma^\mu(\not p-\not q x)\gamma^\alpha\not b\gamma_5 +m^2\gamma^\mu\gamma^\alpha[\not p+\not q(1-x)]\not b\gamma_5\}. \tag{17}$$

Other algebraic properties of the γ matrices are used to calculate the numerator of Eq. (17). For instance, we can use that the trace of γ_5 times an even number of γ matrices can be reduced: $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma_5]=4i\epsilon^{\mu\nu\alpha\beta}$ and $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_5]=\mathrm{Tr}[\gamma_5]=0$. Also, we drop all terms that are odd in p to get

$$\Pi^{\mu\nu\alpha\beta}(q)=-\frac{1}{8}\int_0^1 dx x\int \frac{d^4p}{(2\pi)^4}\frac{N^{\mu\nu\alpha\beta}(p^0,p^2,p^4)}{[p^2-m^2+x(1-x)q^2]^3}, \tag{18}$$

when the numerator $N^{\mu\nu\alpha\beta}(p^0,p^2,p^4)$ has the form

$$N^{\mu\nu\alpha\beta}(p^0,p^2,p^4)=4p^\nu p^\beta(T_0^{\alpha\mu}+T_{pp}^{\alpha\mu})+2(1-2x) \times(p^\nu q^\beta+p^\beta q^\nu)(T_p^{\alpha\mu}+T_{ppp}^{\alpha\mu}) +(1-2x)^2q^\nu q^\beta(T_0^{\alpha\mu}+T_{pp}^{\alpha\mu}), \tag{19}$$

with

$$T_0^{\alpha\mu}=-4ib_\lambda\epsilon^{\alpha\mu\lambda\theta}q_\theta[x(1-x)^2q^2+(2-x)m^2], \tag{20}$$

$$T_p^{\alpha\mu}=-4ib_\lambda\epsilon^{\alpha\mu\lambda\rho}[m^2+(1-x^2)q^2]p_\rho -8i(1-x)b_\lambda[\epsilon^{\mu\lambda\rho\theta}q^\alpha-\epsilon^{\alpha\lambda\rho\theta}q^\mu -(1-x)\epsilon^{\alpha\mu\lambda\theta}q^\rho]q_\theta p_\rho, \tag{21}$$

$$T_{pp}^{\alpha\mu}=-4ib_\lambda[2(\epsilon^{\mu\lambda\rho\theta}p_\rho p^\alpha-\epsilon^{\alpha\lambda\rho\theta}p_\rho p^\mu +x\epsilon^{\alpha\mu\lambda\rho}p_\rho p^\theta)-(2-x)\epsilon^{\alpha\mu\lambda\theta}p^2]q_\theta, \tag{22}$$

$$T_{ppp}^{\alpha\mu}=4ib_\lambda\epsilon^{\alpha\mu\lambda\rho}p^2p_\rho. \tag{23}$$

The integral (18) is badly divergent. By power counting we note that the kinds of divergences are quadratic and logarithmic. Perhaps the most convenient method for regulating divergent integrals without impairing gauge invariance is the dimensional regularization scheme developed by 't Hooft and Veltman [27] in 1972. Thus, we change dimensions from 4 to D and we change $d^4p/(2\pi)^4$ to $(\mu^2)^{(2-D/2)}[d^Dp/(2\pi)^D]$, where μ^2 is an arbitrary parameter that identifies the mass scale. Thus Eq. (17) takes the form

$$\Pi^{\mu\nu\alpha\beta}(q)=b_\lambda\epsilon^{\alpha\mu\lambda\rho}q_\rho[Aq^2\eta^{\beta\nu}+Bq^\beta q^\nu], \tag{24}$$

where A and B are given by

$$A=\frac{-1}{32\pi^2}\int_0^1 \mathrm{d}x\left[(3-2x)x^2(1-x)\Gamma(\epsilon/2)-3x^2\frac{M^2}{q^2} \times\Gamma(-1+\epsilon/2)\right]\left(\frac{4\pi\mu^2}{-M^2}\right)^{\epsilon/2} \tag{25}$$

and

$$B=\frac{-2}{32\pi^2}\int_0^1 \mathrm{d}x(1-2x)x^2(1-x)\Gamma(\epsilon/2)\left(\frac{4\pi\mu^2}{-M^2}\right)^{\epsilon/2} +\frac{1}{64\pi^2}\int_0^1 \mathrm{d}x(1-2x)^2\left[x(2-3x)+(3-2x)x^2 \times(1-x)\frac{q^2}{M^2}\frac{\epsilon}{2}\right]\Gamma(\epsilon/2)\left(\frac{4\pi\mu^2}{-M^2}\right)^{\epsilon/2}, \tag{26}$$

where $\epsilon=4-D$ and $M^2=m^2-x(1-x)q^2$. The next step concerns expanding the gamma function that appears in A and B around $\epsilon\to 0$; thus, we have

$$A=\frac{1}{32\pi^2}\int_0^1 \mathrm{d}x\left\{3x^3(1-x)+(5x-3)x^2(1-x) \times\left[\frac{2}{\epsilon}+\ln\left(\frac{4\pi\mu^2}{-M^2}\right)-\gamma\right]-3x^2\left[\frac{2}{\epsilon}+\ln\left(\frac{4\pi\mu^2}{-M^2}\right) -\gamma+1\right]\frac{m^2}{q^2}\right\} \tag{27}$$

and

$$B=\frac{1}{64\pi^2}\int_0^1 dx\Bigg\{(1-2x)^2(3-2x)\frac{x^2(1-x)q^2}{M^2}+[(2-3x)(1-2x)-4x(1-x)]x(1-2x)\times\left[\frac{2}{\epsilon}+\ln\left(\frac{4\pi\mu^2}{-M^2}\right)-\gamma\right]\Bigg\}. \tag{28}$$

As one can see $\int_0^1 dx[(5x-3)x^2(1-x)](2/\epsilon-\gamma)=0$ in A and $\int_0^1 dx[(2-3x)(1-2x)-4x(1-x)]x(1-2x)(2/\epsilon-\gamma)=0$ in B; then, A and B take the form

$$A=\frac{1}{32\pi^2}\int_0^1 dx\Bigg\{3x^3(1-x)+\frac{(1-2x)x^3(1-x)^2q^2}{m^2-x(1-x)q^2}-3x^2\left[\frac{2}{\epsilon}+\ln\left(\frac{4\pi\mu^2}{-M^2}\right)-\gamma+1\right]\frac{m^2}{q^2}\Bigg\} \tag{29}$$

and

$$B=\frac{1}{32\pi^2}\int_0^1 dx\frac{(1-2x)^2x^2(1-x)q^2}{m^2-x(1-x)q^2}. \tag{30}$$

Observe that we have performed an integration by parts on x for the logarthmic term in A and B. Note that in A the divergent part is present which will disappear when we consider the limit $m^2\to 0$. Now performing the x integration, we have

$$A\overset{m^2\to 0}{=}-B\overset{m^2\to 0}{=}\frac{1}{192\pi^2}. \tag{31}$$

We use these results in Eq. (24) to obtain the Chern-Simons-like term

$$\Pi^{\mu\nu\alpha\beta}(q)=\frac{1}{192\pi^2}b_\lambda\epsilon^{\alpha\mu\lambda\rho}q_\rho[q^2\eta^{\beta\nu}-q^\beta q^\nu]. \tag{32}$$

Finally, the Chern-Simons-like gravitational action induced by radiative corrections is given by

$$\Gamma_{\rm cs}[h]=\frac{1}{192\pi^2}\int d^4x b^\lambda h^{\mu\nu}\epsilon_{\alpha\mu\lambda\rho}\partial^\rho[\partial_\gamma\partial^\gamma h^\alpha_\nu-\partial_\nu\partial_\gamma h^{\gamma\alpha}]. \tag{33}$$

Comparing to Eq. (5) we obtain the relation between the parameters v_λ and b_μ which is written as

$$v_\lambda=\frac{1}{48\pi^2}b_\lambda. \tag{34}$$

III. CONCLUSIONS

We summarize our work recalling that we have calculated the radiative corrections induced by Dirac fermions coupled to a gravitational background field, including the nonstandard contribution $\bar{\psi}\not{b}\gamma_5\psi$, that violate parity symmetries. In this calculation we have used the weak field approximation and dimensional regularization scheme. The coefficient of the Chern-Simons-like gravitational action obtained in Eq. (33) is in agreement with the result obtained by Avarez-Gaumé and Witten within the context of gravitational anomalies [28].

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