Near-horizon conformal symmetry and black hole entropy in any dimension

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Recently, Carlip proposed a derivation of the entropy of the two-dimensional dilatonic black hole by investigating the Virasoro algebra associated with a newly introduced near-horizon conformal symmetry. We point out not only that the algebra of these conformal transformations is not well defined on the horizon, but also that the correct use of the eigenvalue of the operator L_0 yields vanishing entropy. It has been shown that these problems can be resolved by choosing a different basis of the conformal transformations which is regular even at the horizon. We also show the generalization of Carlip's derivation to any higher dimensional case in pure Einstein gravity. The entropy obtained is proportional to the area of the event horizon, but it also depends linearly on the product of the surface gravity and the parameter length of a horizon segment in consideration. We finally point out that this derivation of black hole entropy is quite different from the ones proposed so far, and several features of this method and some open issues are also discussed.

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I. INTRODUCTION

In the last several years, much attention has been paid to the challenging idea that the black hole entropy could be derived from some symmetry inherited in general relativity *classically*, without knowing the details of quantum gravity. The essence of this idea is to construct an algebra of generators associated with a certain symmetry inherited classically in the gravity theory considered. If such algebra obtained is a Virasoro algebra with a nonvanishing central charge, it might be true that the degeneracy of a black hole state in the quantum theory of gravity is determined by the central charge appearing in the algebra as in the case of the conformal field theory through the Cardy formula. This idea was initiated by Strominger [1] and Birmingham *et al.* [2]. Based on the work by Brown and Henneaux [3] that the algebra of diffeomorphisms at spatial infinity for configurations of three-dimensional asymptotically anti–de Sitter space (AdS_3) induces a pair of Virasoro algebras with nonvanishing central charge, they showed that the application of the Cardy formula for an AdS_3 black hole exactly yields the Bekenstein-Hawking entropy. Similarly, the entropy of a cosmological horizon in de Sitter spaces has also been reproduced in the context of de Sitter–conformal field theory correspondence (dS/CFT correspondence) [4]. These results are obtained also in the Chern-Simons formulation of the three-dimensional gravity with a cosmological constant [5–7].

Unfortunately, however, these successes described above are still incomplete for the following reasons. First of all, they do not easily extend to black hole horizons in higher dimensional gravitational theories. It has been shown that the algebra of asymptotic symmetries at spatial infinity for asymptotically four-dimensional anti–de Sitter spaces is $SO(3,2)$ and does not admit a nontrivial central extension [8]. In addition the Chern-Simons formulation of the general relativity in dimensions higher than three is not known. Second, the conformal field theory found in Ref. [3] lives at spatial infinity while black hole entropy is expected to be related to physics on the horizon. Recently, Carlip developed the same idea of the algebra of diffeomorphisms, which not only is applicable to a black hole horizon directly but also works in any higher dimensions [9,10]. He analyzed whether the Virasoro algebra with the desirable form of a classical central charge could arise universally on an arbitrary Killing horizon, and also whether the Bekenstein-Hawking entropy could be derived microscopically by the application of Cardy formula.

Much work in this direction appeared subsequently [11–13]. However, nothing is as yet fully satisfactory [14,15]. In particular, in order to obtain the Virasoro algebra with the desirable form of a central charge, which is homomorphic to $\mathrm{Diff}(S^1)$ algebra up to the central term, one has to choose one angular direction on the horizon as in Ref. [10]. Thus, this method clearly violates spherical symmetry in dimensions higher than three for instance, and requires unnatural reduction of the symmetry group on the horizon [15]. A framework without choosing an angular direction is required in order to realize this idea in a satisfactory manner.

In two-dimensional spacetimes, the problem becomes more serious since there is no room for choosing such an angular direction. Recently, by focusing on two-dimensional dilaton gravity, Carlip [16] suggested several new ingredients that might lead to an improved description of the near-horizon symmetries and possibly overcome the problems mentioned above. He claimed that, in the presence of a black hole with a momentarily stationary region near its horizon, the general relativity acquires a new conformal symmetry. Moreover, a new contribution from the horizon is added to the canonical symplectic form of general relativity so that the central term includes an integration along the horizon. With these new ingredients Carlip claimed that the corresponding

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Virasoro algebra acquires a nonvanishing central charge and that the Cardy formula yields the Bekenstein-Hawking entropy of a two-dimensional dilatonic black hole.

From the viewpoint of universality, it is of great interest to see whether or not this new method is applicable to higher dimensional cases. Thus, in this paper, we explore this new method and consider whether it works in higher dimensional Einstein gravity, in particular. As will be shown below explicitly, however, the basis of functions Carlip used to describe such conformal transformations is singular at the horizon, and correspondingly the integral that gives the generators of these conformal transformations and the central charge in the Virasoro algebra are not well defined. Moreover, the correct use of the eigenvalue of the generator for the zero-mode transformation actually yields vanishing entropy. Therefore it is interesting to check if there exists some way to resolve the problems in Carlip's new method, including those mentioned above, while keeping the essential features of Carlip's derivation.

In Sec. II, we briefly summarize Carlip's new derivation for the entropy of the two-dimensional dilatonic black hole and point out the problems mentioned above. By choosing a different basis which is regular even at the horizon, we show that these problems can be resolved. In Sec. III, the generalization of Carlip's derivation to any higher dimensional pure Einstein gravity is given. In Sec. IV, it is shown that our result for the two-dimensional dilatonic black hole entropy is consistent with that in the three-dimensional pure Einstein gravity through dimensional reduction. Open questions and some unsatisfactory features related to this work are finally discussed.

II. THE TWO-DIMENSIONAL BLACK HOLE

In this section we briefly review Carlip's new approach to the derivation of black hole entropy from symmetry for the two-dimensional dilaton gravity [16]:

$$I=\frac{1}{2G}\int d^2x\sqrt{-g}[\phi R+V(\phi)]. \tag{1}$$

He observes that, for field configurations in which a "momentarily stationary" black hole with the Killing generator χ^a is present, conformal transformations in the form of $\delta g_{ab}=\nabla_c(f\chi^c)g_{ab}$ together with $\delta\phi=\nabla_c(h\chi^c)$ leave the action invariant for smooth functions f and h having their support only in a small neighborhood of the horizon. Hence he claimed that this can be regarded as an asymptotic symmetry.[1] Then, by using the fact that the symplectic current density $\boldsymbol{\omega}$ associated with those transformations is closed he also suggests that the symplectic form of general relativity pick up a new contribution from the horizon itself as

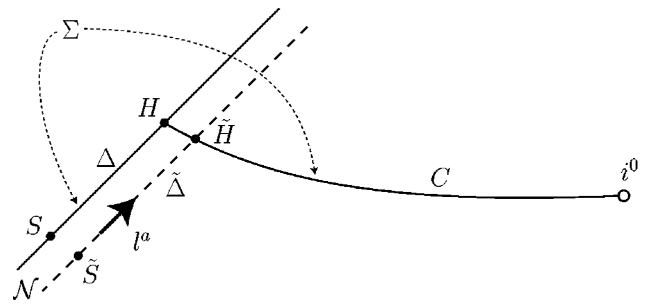

FIG. 1. The Killing horizon Δ, a stretched horizon $\tilde{\Delta}$, and a Cauchy surface C are shown.

$$\hat{\Omega}[\psi;\delta_1\psi,\delta_2\psi]=\int_C\boldsymbol{\omega}[\psi;\delta_1\psi,\delta_2\psi]+\int_\Delta\boldsymbol{\omega}[\psi;\delta_1\psi,\delta_2\psi]. \tag{2}$$

Here the first integral is the standard one integrated over a (partial) Cauchy surface C and the second term is the new contribution introduced by Carlip, which is defined as an integral over the portion Δ of the horizon connecting a reference cross section S and a horizon cross section H under consideration (i.e., the intersection of the Cauchy surface C with the horizon as shown in Fig. 1).

Let l^a be null normal to the stretched horizon $\tilde{\Delta}$. Then the value of the action of two-dimensional gravity with a dilaton field ϕ is invariant under transformations

$$\delta_f g_{ab}=\nabla_c(fl^c)g_{ab}=(l^c\nabla_c f+kf)g_{ab},$$

$$\delta_h\phi=\nabla_c(hl^c)=l^c\nabla_c h+kh \tag{3}$$

in the asymptotic sense that the variation of the action can be made arbitrarily small by restricting smooth functions f and h to have their support only in a small neighborhood $\mathcal{N}$ of the horizon, when we focus only on configurations that possess the horizon. Here $k=\nabla_c l^c$. By rescaling the null vector l^a, he takes furthermore

$$\frac{k}{s}=\text{const on }\tilde{\Delta}, \tag{4}$$

where $s\equiv l^a\nabla_a\phi\equiv\theta\phi$. Thus, this gives that k is proportional to the "expansion" θ, which becomes zero as the stretched horizon approaches the horizon.

Now the variation of the generator $L[f,h]$ associated with the transformations given by Eq. (3) near the horizon is given by[2]

$$\delta L[f,h]=\hat{\Omega}[\delta,\delta_{f,h}]=-\frac{1}{G}\int_{\tilde{\Delta}}[\delta\phi l^a\nabla_a(l^b\nabla_b f+kf)-\delta k(l^b\nabla_b h+kh)]\hat{\boldsymbol{\epsilon}}, \tag{5}$$

[1] Whether or not this is really a symmetry and so whether or not $\boldsymbol{\omega}$ is closed subsequently will be discussed below.

[2] We obtain twice Carlip's expression [16] as shown in the Appendix.

where $\hat{\epsilon}=n$ is the one-dimensional induced volume element on $\tilde{\Delta}$ and n^a is a null vector satisfying $l^a n_a=-1$. Carlip showed that the above equation is integrable if δs is proportional to δk. This integrability condition is satisfied once Eq. (4) holds. In addition, Eq. (4) also implies that

$$h=\frac{s}{k}f. \tag{6}$$

The basis Carlip used is

$$f_n=-\frac{\phi_+}{2\pi s}z^n, \quad z=e^{2\pi i\phi/\phi_+}, \tag{7}$$

where ϕ_+ is the value of ϕ on the horizon.[3] Using $\nabla_a\nabla_b\phi \propto g_{ab}$ on shell [17] and

$$l\cdot\nabla z^n=\frac{2\pi i n}{\phi_+}sz^n, \quad l\cdot\nabla s=ks \tag{8}$$

on the stretched horizon, we have from Eq. (5)

$$\delta_n L[f_m]=-\frac{2\pi i}{G}nm(n-m)\frac{s}{k}\frac{1}{\phi_+}\int_{\tilde{\Delta}}sz^{n+m}\hat{\epsilon}. \tag{9}$$

Note that the integrand of this integration is regular, and vanishes as $\tilde{\Delta}\to\Delta$. However, since the integration limits on a portion of the stretched horizon are chosen from $\phi=\phi_i$ on $\tilde{S}$ to $\phi=\phi_f$ on $\tilde{H}$ keeping $\phi_f-\phi_i=\phi_+$, the relevant integration in Eq. (9) becomes

$$\frac{1}{\phi_+}\int_{\tilde{\Delta}}sz^{n+m}\hat{\epsilon}=-\frac{1}{\phi_+}\int_{\phi_i}^{\phi_f}e^{2\pi i(n+m)\phi/\phi_+}d\phi$$
$$=-\frac{1}{2\pi i}\oint z^{n+m}\frac{dz}{z}=-\delta_{n+m,0}, \tag{10}$$

where $\hat{\epsilon}=-d\phi/s$ and ϕ is assumed to increase along the stretched horizon in the future. Thus we have

$$\delta_{f_n}L[f_m]=\frac{4\pi i}{G}\frac{s}{k}m^3\delta_{m+n,0}. \tag{11}$$

If the generators form an algebra, as is assumed by Carlip, the central charge can be read off by comparing it with the Virasoro algebra

$$\{L[f_m],L[f_n]\}=\delta_{f_n}L[f_m]=-i(m-n)L[f_{m+n}]$$
$$-i\frac{c}{12}m^3\delta_{m+n,0}, \tag{12}$$

resulting in

$$c=-\frac{48\pi}{G}\frac{s}{k}. \tag{13}$$

By integrating δL in Eq. (5), Carlip also obtained separately the eigenvalue of the operator L as

$$L[f_n]=-\frac{1}{G}\int_{\tilde{\Delta}}(2l\cdot\nabla s-ks)f_n\hat{\epsilon}$$
$$=\frac{1}{2\pi G}\frac{k}{s}\phi_+\int_{\tilde{\Delta}}sz^n\hat{\epsilon}=-\frac{1}{2\pi G}\frac{k}{s}\phi_+^2\delta_{n,0}. \tag{14}$$

Finally, the Cardy formula corresponding to Eq. (12),

$$\rho(\Delta)\sim\exp\left[2\pi\sqrt{\frac{c\Delta}{6}}\right], \tag{15}$$

yields the entropy

$$S=\log\rho(\Delta)=\frac{4\pi\phi_+}{G}, \tag{16}$$

where Δ is the eigenvalue of the operator $L_0\equiv L[f_0]$ given from Eq. (14) as

$$\Delta=-\frac{1}{2\pi G}\frac{k}{s}\phi_+^2. \tag{17}$$

This entropy is twice the Bekenstein-Hawking entropy known for two-dimensional dilatonic black holes.

Here we point out several flaws in Carlip's new approach briefly summarized above. First of all, Carlip's identification of the eigenvalue of the operator L_0 is somewhat erroneous, namely, since the L_0 operator is defined up to an arbitrary additive constant in Eq. (5), the direct identification of the eigenvalue of L_0 operator as in Eq. (17) could be incorrect when one applies Cardy's formula. One way to avoid such ambiguity would be to use the result obtained in Eq. (11) based on the uniqueness of the central extension of the Virasoro algebra. By comparing Eq. (11) with Eq. (12), we see that the eigenvalue Δ of the operator L_0 vanishes. Then, Cardy's formula Eq. (15) indicates that the entropy actually vanishes as well.

Second, notice that the base function f_n in Eq. (7) diverges as the horizon is being approached since $s\to 0$, and that z is constant on the horizon (i.e., $z=1$) since ϕ is constant there. It indicates that the integral in Eq. (5) is not well defined on the horizon. Let us consider the derivative $l^a\nabla_a f_n$, for instance. Although the value of this derivative cannot be computed directly on the horizon, it is clear that this quantity is independent of n since f_n does not depend on n along the horizon. On the other hand, let us compute it on the stretched horizon first and take the limit $\tilde{\Delta}\to\Delta$. We have

$$l^a\nabla_a f_n=\left(in-\frac{\phi_+}{2\pi}\frac{k}{s}\right)z^n\to in-\frac{\phi_+}{2\pi}\frac{k}{s}. \tag{18}$$

Thus, this limiting value depends on n on the horizon. Such discrepancy implies that the derivative considered is not con-

[3]The overall sign of f_n here is opposite from Carlip's. However, the above choice of the sign is necessary in order to obtain the standard sign of the first term in the right-hand side of the Virasoro algebra Eq. (12).

tinuous at the horizon and so it is ill defined there. Similarly, one can see that the integral in Eq. (5) is not continuous at the horizon. Therefore, the algebra given by Eq. (11) is actually not well defined on the horizon. It is not convincing at all to expect that such an ill-defined algebra is responsible for physics on the horizon. All these problems described above are seemingly due to the specific choices of the rescaling of l^a (i.e., $k/s=\text{const}$) and the base functions f_n (i.e., the use of a bad coordinate ϕ that does not distinguish points along the event horizon).

Now we show how these flaws mentioned above can be avoided. In Carlip's case, $k\equiv\nabla_a l^a$ vanishes at the horizon since the rescaling freedom of the null vector l^a is used to satisfy Eq. (4). Let us not assume this condition for l^a. Instead we choose the null vector l^a in such a way that, as the horizon is being approached, it becomes the horizon Killing generator χ^a so that the quantity k becomes the surface gravity of the horizon, κ, which is a nonvanishing constant.[4] Defining v as a nonaffine parameter describing the null trajectory on $\tilde{\Delta}$ such that $l^a\nabla_a v=1$, we expand the transformation function f in terms of mode functions given by

$$f_n=-\frac{P}{2\pi}z^n,\quad z=e^{2\pi i v/P},\tag{19}$$

where the periodicity P is assumed to be an arbitrary constant for the moment. Note that the coordinate v varies along the horizon, and these base functions are not singular at the horizon, in contrast to Carlip's basis function. Finally we assume

$$h=\alpha f,\tag{20}$$

where α is constant on $\tilde{\Delta}$. This relation guarantees the variational equation Eq. (5) integrable and $\alpha=\phi_+/2$ as will be shown from dimensional reduction below.

Here we assume that P is the null distance between a reference cross section $\tilde{S}$ and a horizon cross section $\tilde{H}$ measured by the function v on each stretched horizon so that z makes one full turn counterclockwise as v runs from $\tilde{S}$ to $\tilde{H}$. Note that the null distance P is taken to be same as $\tilde{\Delta}\to\Delta$. With these modifications we find

$$\delta_{f_n}L[f_m]=-i(m-n)\frac{(P\kappa)^2}{2\pi G}\frac{\phi_+}{2}\delta_{m+n,0}-i\frac{4\pi}{G}\frac{\phi_+}{2}m^3\delta_{m+n,0}.\tag{21}$$

Accordingly, by comparing it with the Virasoro algebra in Eq. (12), one can read off

$$c=\frac{24\pi\phi_+}{G},\quad \Delta=\frac{\pi\phi_+}{G}\left(\frac{P\kappa}{2\pi}\right)^2.\tag{22}$$

[4]The explicit form of l^a in the four-dimensional Schwarzschild black hole case, for example, is given by Eq. (29).

Thus the Cardy formula in Eq. (15) yields

$$S=P\kappa\frac{2\pi\phi_+}{G}.\tag{23}$$

Notice that this entropy becomes the Bekenstein-Hawking entropy known for the two-dimensional dilaton black hole [17] if the periodicity could be adjusted to $P=\kappa^{-1}$. Note also that in our method the integration of Eq. (5) gives, up to an additive constant,

$$L[f_n]=\frac{1}{G}\int_{\tilde{\Delta}}(2sl\cdot\nabla f_n+\alpha k^2 f_n)\hat{\epsilon}=\frac{(P\kappa)^2}{2\pi G}\frac{\phi_+}{2}\delta_{n,0}.\tag{24}$$

Hence one finds that the eigenvalue of L_0 above coincides with the one obtained from Eq. (21).

III. HIGHER DIMENSIONAL BLACK HOLES

In this section we extend the method described in the previous section to higher dimensional cases. Let us consider the pure Einstein gravity with a cosmological constant in an arbitrary dimension given by

$$I=\frac{1}{16\pi G}\int d^Dx\sqrt{-g}(R-2\Lambda).\tag{25}$$

The symplectic current $(D-1)$-form for this theory may be written as

$$\boldsymbol{\omega}[\psi;\delta_1\psi,\delta_2\psi]=\delta_1\boldsymbol{\Theta}(\psi;\delta_2\psi)-\delta_2\boldsymbol{\Theta}(\psi;\delta_1\psi),\tag{26}$$

where $\Theta_{bcd\ldots}=\epsilon_{abcd\ldots}\Theta^a$ with

$$\Theta^a=\frac{1}{16\pi G}(g_{bc}\nabla^a\delta g^{bc}-\nabla_b\delta g^{ab}).\tag{27}$$

As in the previous section, we consider a symmetry variation given by

$$\delta_f g_{ab}=\nabla_c(fl^c)g_{ab}.\tag{28}$$

Here l^a is null normal to the stretched horizon $\tilde{\Delta}$ that becomes the horizon Killing generator χ^a as $\tilde{\Delta}\to\Delta$. As shall be shown below, $k\equiv\nabla^a l_a$ approaches the surface gravity κ which is defined by $\chi^c\nabla_c\chi_b=\kappa\chi_b$ at the horizon. In the case of the four-dimensional Schwarzschild black hole, for instance, l^a is given by

$$l^a=\frac{1}{2}[(\partial_t)^a+(1-r_h/r)(\partial_r)^a].\tag{29}$$

And at the horizon the function v coincides with the ingoing null coordinate, i.e., $v\sim t+r_*$ where $r_*=r_h\ln(r-r_h)$ is the usual "tortoise" coordinate. The variation of the generator $L[f]$ is now given as

$$\delta L[f] = -\frac{(D-1)(D-2)}{32\pi GD}\int_{\tilde{\Delta}}[g^{ab}\delta g_{ab}l^c\nabla_c(l^d\nabla_d f + kf) - l^c\nabla_c(g^{ab}\delta g_{ab})(l^d\nabla_d f + kf)]\hat{\epsilon} = -\frac{(D-1)(D-2)}{16\pi GD}\int_{\tilde{\Delta}} g^{ab}\delta g_{ab}l^c\nabla_c(l^d\nabla_d f + kf)\hat{\epsilon}, \tag{30}$$

where we have used the integration by parts and $\hat{\epsilon}$ is the $(D-1)$-dimensional induced volume element on $\tilde{\Delta}$. One can check that the variational form of the generator L in Eq. (30) is actually integrable, when we focus on a narrow subspace of the phase space and assume that all relevant variations are described by Eq. (28); namely, when the variation δ is thought of as a derivative on the space of metric fields, an explicit calculation shows that $\delta_1(\delta_2 L) - \delta_2(\delta_1 L) = 0$. Thus, one does not require any further condition for the integrability.

As in Eq. (19), we choose a basis of functions as

$$f_n = -\frac{P}{\pi D}z^n, \quad z = e^{2\pi i v/P}, \tag{31}$$

where the normalization is chosen such that the base function f_n satisfies the commutation relations isomorphic to the Diff(S^1) algebra,

$$\{f_m, f_n\} = i(n-m)f_{m+n}, \tag{32}$$

with the brackets between the basis functions defined through

$$[\delta_{f_m}, \delta_{f_n}]g_{ab} \equiv \delta_{\{f_m, f_n\}}g_{ab}. \tag{33}$$

Now one can explicitly obtain from Eq. (30) that

$$\delta_{f_m}L[f_n] = -\frac{(D-1)(D-2)}{16\pi G}\int_{\tilde{\Delta}}(l^c\nabla_c f_m + kf_m)l^c\nabla_c \times(l^d\nabla_d f_n + kf_n)\hat{\epsilon} = -\frac{(D-1)(D-2)}{D^2}\frac{1}{4\pi G}\int_{\tilde{\Delta}}\left[-mn^2 + n\left(\frac{Pk}{2\pi}\right)^2 + \frac{Pki}{2\pi}n(m+n) + \frac{P^2}{4\pi^2}\left(m - \frac{Pki}{2\pi}\right)l^c\nabla_c k\right] \times z^{m+n-1}dz d\Sigma. \tag{34}$$

Here $d\Sigma$ is the infinitesimal volume element of the spatial cross section of the stretched horizon $\tilde{\Delta}$, and $dv = Pdz/2\pi iz$.

The integrand above is not constant in general. As the stretched horizon approaches the horizon (i.e., $\tilde{\Delta}\to\Delta$), however, one can see that the quantity $k = \nabla^c l_c$ becomes the surface gravity κ. Let the null vector n^a be tangent to the ingoing null trajectory and be scaled such that $n^c l_c = -1$, and let σ_{ab} be the induced metric on the spatial cross section of the stretched horizon as $\sigma_{ab} \equiv g_{ab} + l_a n_b + l_b n_a$. Then

$$k = g^{ab}\nabla_a l_b = (\sigma^{ab} - l^a n^b - l^b n^a)\nabla_a l_b = \theta - n^b l^a \nabla_a l_b \to \kappa. \tag{35}$$

Here we used that $l^b n^a \nabla_a l_b = n^a\nabla_a(l^b l_b)/2 = 0$, that the expansion of the congruence of null geodesics $\theta = \sigma^{ab}\nabla_a k_b = \sigma^{ab}\nabla_a l_b$, where k^a is the geodesic tangent, vanishes at the horizon, and that $l^a \to \chi^a$ and $l^a\nabla_a l_b \to \chi^a\nabla_a\chi_b$ as $\tilde{\Delta}\to\Delta$. The surface gravity of a stationary event horizon may be defined as $\chi^c\nabla_c\chi_b = \kappa\chi_b$, which is constant along the horizon provided that the dominant energy condition is satisfied. Thus, the integrand becomes constant as the stretched horizon approaches the event horizon. Finally, from the same calculation as in the previous section, we have at the horizon

$$\delta_{f_n}L[f_m] = \frac{(D-1)(D-2)}{D^2}\frac{A}{4G} \times\left[-i(m-n)\left(\frac{P\kappa}{2\pi}\right)^2\delta_{m+n,0} - 2im^3\delta_{m+n,0}\right], \tag{36}$$

where $A = \oint d\Sigma$ is the surface area of the $(D-2)$-dimensional horizon cross section.

By comparing it with Eq. (12), therefore, we obtain the nonvanishing central charge given by

$$c = \frac{24(D-1)(D-2)}{D^2}\frac{A}{4G}, \tag{37}$$

and the eigenvalue of the L_0 operator given by

$$\Delta = \left(\frac{P\kappa}{2\pi}\right)^2\frac{(D-1)(D-2)}{D^2}\frac{A}{4G} = \left(\frac{P\kappa}{2\pi}\right)^2\frac{c}{24}. \tag{38}$$

By applying the Cardy formula for the density of states in Eq. (15), the entropy becomes

$$S = \log\rho(\Delta) = \frac{2(D-1)(D-2)P\kappa}{D^2}\frac{A}{4G}. \tag{39}$$

This can be adjusted to the Bekenstein-Hawking entropy if the periodicity can be chosen such that

$$P = \frac{D^2}{2(D-1)(D-2)}\kappa^{-1}. \tag{40}$$

IV. DIMENSIONAL REDUCTION

Since the two-dimensional dilaton gravity can be obtained from a dimensional reduction of a higher dimensional pure Einstein gravity, it is interesting to see whether our results for the entropies obtained above are consistent in this context. If we consider three-dimensional black holes for simplicity, the entropy is given by

$$S=\frac{4}{9}P\kappa\frac{A}{4G} \tag{41}$$

from Eq. (39). We "2+1" decompose the three-dimensional metric g_{ab} as

$$g_{ab}=h_{ab}+m_a m_b, \tag{42}$$

where the unit normal m_a to 2D subspace is given by

$$m_a=r\nabla_a\varphi, \tag{43}$$

in terms of the radius r and the angular coordinate φ of the circles in the third dimension, and h_{ab} is the induced metric of the two-dimensional subspace. Since r plays the role of the "lapse" function for the "evolution" in the φ direction, we can write as

$$\sqrt{-g}\,{}^{(3)}R=\sqrt{-h}r({}^{(2)}R+\mathcal{L}), \tag{44}$$

where $\mathcal{L}=K_{ab}K^{ab}-K^2$ denotes terms consisting of the extrinsic curvature of a $\varphi=$const surface. We rewrite the radius r in terms of the dilaton field ϕ as $r=\phi$. Then the terms of the extrinsic curvature can be considered as matter parts. As has been shown in the Nöther charge method of black hole entropy [18,19], this matter action does not change the entropy result. Accordingly we ignore the terms of the extrinsic curvature and see if this is consistent with the entropies obtained in the previous sections.

By assuming configurations independent of φ, we have

$$I=\frac{1}{16\pi G}\int d^3x\sqrt{-g}({}^{(3)}R-2\Lambda) \tag{45}$$

$$=\frac{1}{16\pi G}\int d\varphi d^2x\sqrt{-h}\phi({}^{(2)}R-2\Lambda+\mathcal{L}[\phi])$$

$$=\frac{1}{8G}\int d^2x\sqrt{-h}\phi({}^{(2)}R-2\Lambda). \tag{46}$$

Notice that Eq. (46) is one-quarter of Eq. (1), and so are the values of c and Δ. Then, from the two-dimensional result Eq. (23) the entropy is given by

$$S=P\kappa\frac{\pi\phi_+}{2G} \tag{47}$$

if l^a is null and approaches the Killing vector also in the two-dimensional subspace. Now we expect that this should be equivalent to the entropy of the three-dimensional black hole. By substituting $\phi_+=r_+$, Eq. (47) becomes

$$S=P\kappa\frac{\pi r_+}{2G}=P\kappa\frac{A}{4G}, \tag{48}$$

where $A=2\pi r_+$ is the "area" (i.e., circumference) of the horizon in three dimensions. One can see that this coincides with the full three-dimensional result in Eq. (41) since the difference by a numerical factor 4/9 simply comes from the dimension dependent normalization of the mode function f_n in Eq. (31). Therefore, our method gives a consistent result under dimensional reduction. Since the additional matter action resulting from the dimensionally reduced theory is ignored, our result also indicates that the matter action does not contribute to the entropy as happens in the usual cases.

Carlip [16] imposed the relationship between f and h given by Eq. (6) in order to make the conformal transformations integrable. In the context of dimensional reduction explained above, it is expected that the variation of the dilaton field ϕ in the two-dimensional theory can be deduced from that of the metric field in the three-dimensional theory. By requiring that the normalization condition $m^a m^b g_{ab}=1$ is preserved under the variations, we first find from Eq. (28) and the variation of Eq. (43) that

$$\nabla_c(fl^c)=2\frac{\delta_h\phi}{\phi}, \tag{49}$$

where the variation induced on ϕ is denoted as $\delta_h\phi$, and $r=\phi$ as well as a relation between f and h, which is to be determined, are understood. On the other hand, from the fact that l^a is null in both three dimensions and two dimensions, we see that l^a should reside in the two-dimensional subspace as $m_a l^a=0$, and then we can show by using it that

$$\nabla_c(fl^c)=\frac{1}{\phi}D_c(\phi f l^c), \tag{50}$$

where D_c is the covariant derivative associated with the induced metric h_{ab}. Since $\delta_h\phi$ given by Eq. (3) is written as $\delta_h\phi=D_c(hl^c)$, we thus find

$$h=\frac{\phi}{2}f, \tag{51}$$

which is consistent with the integrability condition in the two-dimensional case.[5]

V. DISCUSSION

To conclude, we have analyzed whether the entropy of black holes for the pure Einstein gravity in any dimensions can be derived from a Virasoro algebra associated with a specific class of near-horizon conformal transformations given in Eq. (28). We simply extended Carlip's derivation developed for two-dimensional dilaton black holes in Ref. [16]. However, there are some important modifications in choices of the null vector l^a and the base function f_n as in Eqs. (29) and (31). As can be seen in Eq. (39), the entropy obtained is proportional to the area of the event horizon, but it also depends linearly on the product of the surface gravity and the parameter length of a horizon segment in consideration (i.e., $\sim P\kappa$).

The entropy derivation explained above does not depend on the details of black hole solutions. What we actually need

[5]With this relationship one can see that $\delta s=\phi\delta\kappa/2+s\nabla_c(fl^c)$. Thus, δs becomes proportional to $\delta\kappa$ as the horizon is being approached since $s\to 0$ and $\nabla_c(fl^c)$ is regular.

is simply the neighborhood of a Killing horizon; namely, the horizon is a null hypersurface which is generated by a Killing vector field. Therefore it is straightforward to apply the same method to other types of horizons such as a Rindler horizon or a de Sitter horizon. Recently, a generalization of the black hole thermodynamics to any "causal horizon" has been argued in Ref. [20].

Another feature of this derivation is that, as can be seen in Eqs. (30) and (34), the central charge is given by

$$K[f_1,f_2]\sim\int dv\,d\Sigma(Df_1D^2f_2-Df_2D^2f_1), \qquad (52)$$

where $D=l\cdot\nabla=\partial_v$. This becomes the standard form of the central term for 2D conformal field theory. Note that it contains an integration over the "time" parameter v explicitly, in contrast to other approaches as in Refs. [9,10]. In particular, in Ref. [10] an orthogonality condition for base functions was used to recover a conventional Virasoro algebra. Although fixing the average surface gravity on the horizon cross section can naturally lead to such an orthogonality relation, the physical origin of such a boundary condition still remains unclear. The orthogonality relations in Refs. [9,10] were realized by adding extra angular dependence to the r-t diffeomorphisms. As mentioned above, however, this prescription not only requires an unnatural reduction of the symmetry group, but also cannot work for two-dimensional cases. The new entropy derivation described in the present work is free from the problem of adding extra angular dependence. This new feature is essentially because the newly added contribution to the symplectic form in Eq. (2) has an integration along the horizon from the beginning.

However, it is important to note that the conformal transformations we considered cannot be described by diffeomorphisms. One way to see this is the following. Suppose that the transformations are indeed described by diffeomorphisms. Then the integrability condition derived by Wald and Zoupas [21] for the generator L should be satisfied. The condition is

$$\int_{\partial\Sigma}\xi\cdot\boldsymbol{\omega}=0, \qquad (53)$$

where ξ is a vector field generating diffeomorphisms and Σ is the union of C and a portion of the horizon. Since the part defined over the horizon Δ in Eq. (2) was shown to be integrable separately, this condition becomes equivalent to $\int_{\hat{H}}\xi\cdot\boldsymbol{\omega}=0$ where $\hat{H}$ is a horizon cross section which is the inner boundary of C. Hence the contribution to the variation of the generator L arising from the integral along the null direction on the horizon should vanish since

$$\delta L_\xi=\int_\Delta\boldsymbol{\omega}\sim\int_S^H dv\int_{\hat{H}}\xi\cdot\boldsymbol{\omega}. \qquad (54)$$

However, the explicit calculation of the variation of L shows that it does not vanish as in Eq. (36). Thus the conformal transformations considered in this paper cannot be regarded as diffeomorphisms such as "conformal isometries" generated by conformal Killing vectors in the vicinity of the horizon. Therefore, the algebra appeared in the present formulation is quite different from those associated with diffeomorphisms in the literature so far.

Since the near-horizon conformal transformations considered in this paper are not diffeomorphisms as argued above, the standard covariant phase space method [18,22] that we have adopted in this paper might not work. For example, the linearized field equations for δg_{ab} are not ensured to hold in general. Consequently the symplectic current $\boldsymbol{\omega}$ is not necessarily closed. The new idea to add the integral along the null direction to the variation of L as in Eq. (2) might then be unjustifiable. As a related issue, we should point out that there does not exist any convincing reasoning to consider that the transformations proposed by Carlip and us are Nöther's symmetry transformations. Although Nöther's symmetry transformations make the action invariant for arbitrary background configurations, i.e., Nöther's symmetry is a symmetry of the action itself without depending on configurations, neither Carlip's nor our transformation possesses this property. (See Ref. [16] for Carlip's argument, where "invariance of action" is shown only for configurations that have a special property, not for arbitrary configurations.) Then the generators of these transformations are not ensured to be closed, and are hence not ensured to form an algebra as we assumed. Therefore, we should explore the reasoning further to justify the method proposed by Carlip and extended by us in this paper, as Carlip himself mentioned in Ref. [16]. In addition, when we show the integrability of the operator L, we assumed that all variations are described by the conformal transformations given by Eq. (28). Thus, the subspace of the phase space we considered is quite narrow, compared to the covariant phase space in Ref. [22]. It should be clarified whether quantization, or defining eigenstates at least, in this narrow subspace can be carried out in a manner similar to the standard Dirac method.

It should be pointed out that, in addition to the usual Bekenstein-Hawking entropy, the entropy result Eq. (39) obtained in our method has a multiplication factor that is a function of the periodicity P, the surface gravity κ, and the spacetime dimension D. Recall that P is the parameter length between a reference horizon cross section S and the horizon cross section H in consideration. Since the horizon cross section can be chosen arbitrarily on the horizon, such explicit P dependence in the entropy derivation is highly unsatisfactory. In Carlip's method [16], the parameter length between any two points on the horizon vanishes when measured in ϕ, since ϕ is constant on the horizon. Then, in order to keep $P=\phi_f-\phi_i=\phi_+$ fixed in Eq. (10) the position of the reference horizon cross section must move toward the past as the limiting process $\tilde{\Delta}\to\Delta$ is taken. In other words, in Carlip's method [16] it is likely that the reference horizon cross section should be located somewhere in the past "infinity" (or in the future "infinity") on the horizon, while a "momentarily stationary" region, i.e., a finite segment of the horizon, is considered. Note that the smoothness of the Euclidean sector of near-horizon geometry naturally requires a periodicity of $\sim\kappa^{-1}$. Thus, one might expect that it will be possible to choose a specific value of the periodicity P such that the multiplication factor becomes unity as in Eq. (40). It implies that for a given black hole the Bekenstein-Hawking entropy is reproduced in our derivation only when a specific length of the horizon segment, which depends on the surface gravity and the spacetime dimension through Eq. (40), is taken. There is no clear physical explanation at the moment for why such specific value should be chosen for the periodicity though.

We should mention that the method presented in this paper yields vanishing entropy in 2D pure Einstein gravity, as we can see from Eq. (39). Since the field equation is trivially satisfied in this case, it may indicate that thermodynamic entropy in this theory is not well defined. Then it might make no sense to consider the horizon entropy in 2D pure Einstein gravity. On the other hand, however, a horizon is shown to be a thermal object even in 2D Einstein gravity without using the field equation of gravity, and hence vanishing statistical entropy looks inconsistent with this fact since it implies a frozen object. Actually, we considered a sort of statistical entropy in this paper, which has not been shown to coincide with the thermodynamic entropy. The ill-defined classical dynamics may be simply a problem of the thermodynamic entropy, not of the statistical entropy. From this point of view, the vanishing entropy result in the 2D pure Einstein gravity seems to signal failure of the method proposed by Carlip and extended in this paper.

From the physical point of view, it is also of interest to consider what ‘‘microscopic states’’ are responsible for black hole entropy. We see that the zero mode of the conformal transformations, which is generated by L_0, rescales the metric by a constant. Since the entropy derived above is defined as the logarithm of the number of the eigenstates of L_0, this indicates that the ‘‘microscopic states’’ responsible for black hole entropy are the eigenstates of ‘‘rescaling of the metric by a constant.’’ It does not seem to have something to do with properties of a black hole horizon, such as energy and angular momentum.

Therefore, the method analyzed in this paper possesses some unsatisfactory features, and further investigation is necessary in the future to understand whether this method is really correct or not.

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APPENDIX: GENERATOR IN 2D DILATON GRAVITY

Here we explicitly derive Eq. (5), i.e., the variation of the generators $L[f,h]$ in 2D dilaton gravity. From the action in Eq. (1) $\Theta^b(\psi;\delta\psi)$ is calculated as

$$\Theta^b(\psi;\delta\psi)=\frac{1}{4G}[\nabla_a\phi(\delta^a_d\delta^b_c+\delta^b_d\delta^a_c-2g^{ab}g_{cd})\delta g^{cd} - \phi(\delta^a_d\delta^b_c+\delta^b_d\delta^a_c-2g^{ab}g_{cd})\nabla_a\delta g^{cd}]. \tag{A1}$$

The symplectic current density ω_a defined in Eq. (26) can be obtained by using $\Theta_a=\varepsilon_{ba}\Theta^b$ and its variation. Now the substitution of Eq. (3) gives

$$\omega_a(\psi;\delta_1\psi,\delta_2\psi)=\frac{1}{2G}\varepsilon_{ba}[\nabla_c(f_2l^c)\nabla^b\delta_1\phi-\delta_1\phi\nabla^b\nabla_c(f_2l^c) - \nabla_c(f_1l^c)\nabla^b\delta_2\phi+\delta_2\phi\nabla^b\nabla_c(f_1l^c)], \tag{A2}$$

where ε_{ba} is the volume element expressed as

$$\varepsilon_{ba}=-l_bn_a+l_an_b \tag{A3}$$

with n_a being a null vector satisfying $l^cn_c=-1$.

Since l_a is normal to the stretched horizon $\tilde{\Delta}$, its pullback to $\tilde{\Delta}$ vanishes, and hence ω_a above becomes, on $\tilde{\Delta}$,

$$\begin{aligned}\omega_a(\psi;\delta_1\psi,\delta_2\psi) &= -\frac{1}{2G}n_a[\nabla_c(f_2l^c)l_b\nabla^b\delta_1\phi-\delta_1\phi l_b\nabla^b\nabla_c(f_2l^c) \\ &\quad -\nabla_c(f_1l^c)l_b\nabla^b\delta_2\phi+\delta_2\phi l_b\nabla^b\nabla_c(f_1l^c)], \\ &= -\frac{1}{2G}n_a\{-2\delta_1\phi l^b\nabla_b\nabla_c(f_2l^c)+2\delta_2\phi l^b\nabla_b\nabla_c(f_1l^c) \\ &\quad + l^b\nabla_b[\nabla_c(f_2l^c)\delta_1\phi-\nabla_c(f_1l^c)\delta_2\phi]\}.\end{aligned} \tag{A4}$$

The fact that l_a is normal to $\tilde{\Delta}$ gives also $n_al^b\nabla_b=(n_al^b+l_an^b)\nabla_b=-\delta^b_a\nabla_b=-\nabla_a$ on $\tilde{\Delta}$, and hence the last term in the above equation is totally divergent, which we discard.

Now by choosing the same direction of integration (the orientation of $\tilde{\Delta}$) as Carlip [16], relabeling δ_1 as δ and δ_2 as $\delta_{f,h}$, and making use of Eq. (3) and the relation

$$\delta k=l^b\nabla_b\nabla_c(fl^c)=l^b\nabla_b(l^c\nabla_cf+kf), \tag{A5}$$

we finally obtain Eq. (5) given by

$$\begin{aligned}\hat{\Omega}[\delta,\delta_{f,h}] &= \int_{\tilde{\Delta}}\omega_a(\psi;\delta\psi,\delta_{f,h}\psi) \\ &= -\frac{1}{G}\int_{\tilde{\Delta}}n_a[\delta\phi l^b\nabla_b(l^c\nabla_cf+kf) - \delta k(l^c\nabla_ch+kh)].\end{aligned} \tag{A6}$$

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