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Electronic contribution to the oscillations of a gravitational antenna

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We carefully analyze the contribution to the oscillations of a metallic gravitational antenna due to the interaction between the electrons of the bar and the incoming gravitational wave. To this end, we first derive the total microscopic Hamiltonian of the wave-antenna system and then compute the contribution to the attenuation factor due to the electron-graviton interaction. As compared to the ordinary damping factor, which is due to the electron viscosity, this term turns out to be totally negligible. This result confirms that the only relevant mechanism for the interaction of a gravitational wave with a metallic antenna is its direct coupling with the bar normal modes.

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I. INTRODUCTION

The detection of gravitational waves represents nowadays one of the most challenging and stimulating problems in experimental and theoretical physics.

At the beginning of the 1960s, Weber pioneered a research program based on the construction of metallic antennae (aluminum bars), and computed the cross section for the absorption of gravitational waves [1] (see also [2,3]) under the assumption that the relevant mechanism that governs the transfer of energy is a direct resonant coupling between the wave and the bar normal modes, the so called resonant assumption [3]. The detector is so built that a signal is observed when the frequency of the incoming wave is close to the bar fundamental mode, the cross section taking a Breit-Wigner shape around this frequency.

Weber successively proposed a correction to his own computation. Arguing that the bar has very many resonant frequencies within the tiny detection range, he found an enormous enhancement of the cross section [4]. Later Preparata [5] considered a different mechanism that again lead to an enhancement of the cross section. The problems with both of these mechanisms have been clearly identified [6–8]. It is by now largely accepted that, within the framework of the resonant assumption, the correct result is the old one [1].

However, in addition to the direct graviton-phonon coupling considered in [1–3], which at the microscopic level is due to a modification of the geodesic distance between the ions of the lattice in the presence of the gravitational wave, it is clear that the gravitons also couple to the electrons of the metal. Due to the electron-phonon interaction, an indirect coupling between gravitons and phonons is then generated.[1]

By considering the microscopic Hamiltonian for the interaction of a gravitational wave with the bar, in this paper we carefully analyze the electronic contribution to the absorption of gravitational waves by the antenna. More specifically, we compute the contribution (renormalization) to the attenuation factor, i.e., the imaginary part of the frequency of the propagating acoustic wave, due to the interaction of the electrons with the gravitational wave. Depending on its sign, this term could lead to an attenuation or an amplification of the bar oscillation amplitude. This is a correction to the "ordinary" (positive) factor, called Γ^0 from now on, which is the bar attenuation factor in the absence of gravitational waves (see Sec. V below). Therefore, irrespectively of its sign, the actual relevance of such a term depends on its magnitude relative to Γ^0.

A similar phenomenon has already been considered for the propagation of an acoustic wave in a semiconductor in the presence of an electromagnetic wave [10,11]. Here the electron-phonon interaction generates, via electron-hole loops, an indirect photon-phonon coupling. Although this electron viscosity generally causes an attenuation of the acoustic wave, under certain circumstances the wave can be amplified. We shall see in the following that, while this is possible for the electromagnetic case, for a gravitational wave hitting a metallic antenna such an amplification cannot occur.

Present bars are aluminum detectors operating at a temperature of about 2 K. To avoid uncontrolled complications due to superconductivity effects, the antennae operate at tem-

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[1] It has been recently claimed that this interaction is responsible for an enormous enhancement of the cross section [9]. Our results (see Sec. V) strongly disagree with this conclusion.

peratures $T>T_s$, where T_s is the transition temperature to the superconducting state. We also restrict ourselves to this case.

Let us focus now our attention on the graviton-phonon interaction, which is the only interaction mechanism usually considered, and suppose that a gravitational burst hits perpendicularly the antenna. If the bar lies along, say, the z axis, the wave excites the modes $\omega_{\vec{q}}=(\pi v_s/L)(2n+1)$, where $\vec{q}=\vec{q}_n=[0,0,(\pi/L)(2n+1)]$, L is the length of the bar, v_s is the sound speed in aluminum, and n a positive integer. If, as we assume, the incoming wave h_{ij} contains a superposition of frequencies having a cutoff which does not greatly exceed the kHz, the only excited mode is the fundamental one.

As is well known, the detector measures the energy stored in the bar, E_s, which is proportional to the square of the amplitude of the bar oscillations. This quantity is related to the Fourier transform $\tilde{h}_{zz}$ of the signal in the following way [12]:

$$E_s=\frac{ML^2}{\pi^2}\omega_{\vec{q}_0}^4|\tilde{h}_{zz}(\omega_{\vec{q}_0})|^2. \tag{1}$$

Here $\tilde{h}_{zz}$ is the zz component of the Fourier transform of the gravitational wave h_{ij}, M is the mass of the bar, $\omega_{\vec{q}_0}$ its fundamental frequency, and L, as before, the length of the bar. Note that, for notational convenience, but without loss of generality, we have chosen the axes so that the antenna lies along the z direction. Moreover we have considered a gravitational wave perpendicular to the direction of the antenna. This is why in Eq. (1) only the zz component of the Fourier transform of the gravitational wave appears.

Equation (1) is obtained by considering the graviton-phonon interaction only, and our problem amounts to the question of whether the indirect graviton-phonon interaction induced by the electrons can lead to a significant modification of this equation. In fact, would this additional interaction generate a value of the attenuation factor significantly lower than Γ^0, the bar oscillation amplitude would turn out to be greater than expected (and E_s higher). Therefore, taking into account only the direct interaction, we would be lead to an overestimate of the signal.

The oscillations induced in the bar by the gravitational wave have very small amplitudes ($\Delta L/L\sim 10^{-21}$), and they have to be amplified by an electronic system. The noise in transducing these bar oscillations into electronic signals allows for the amplification only of those amplitudes that are larger than a given threshold. Therefore, in the case of resonance, the detector is practically blind to frequencies other than those within a narrow band around the fundamental one. In the following we consider an incoming plane wave with frequency $\Omega\simeq\omega_{\vec{q}_0}$.

The rest of the paper is organized as follows. In Sec. II we briefly sketch the derivation of the attenuation factor for the electromagnetic case mentioned above. In Sec. III we present the derivation of the microscopic Hamiltonian for the interaction of the gravitational wave with the antenna. In Sec. IV the renormalization of $\omega_{\vec{q}_0}$ in the presence of the gravitational wave is discussed, and in Sec. V we compute the attenuation factor. Section VI contains the summary and conclusions.

II. SOUND WAVES IN THE FIELD OF AN ELECTROMAGNETIC WAVE

To set up the tools for our investigation, and to introduce some notations, in this section we very briefly review the derivation of the correction to the attenuation factor for an acoustic wave that propagates in a semiconductor in the field of an electromagnetic wave, referring to [10,11] for details.

The microscopic Hamiltonian of the system (external electromagnetic wave + electrons + phonons) is

$$H=\frac{1}{2m}\sum_{\vec{p}}\left(\vec{p}-\frac{e}{c}\vec{A}(t)\right)^2 a_{\vec{p}}^\dagger a_{\vec{p}}+\sum_{\vec{k}}\omega_{\vec{k}}b_{\vec{k}}^\dagger b_{\vec{k}}$$
$$+\sum_{\vec{p},\vec{k}}C_{\vec{k}}a_{\vec{p}+\vec{k}}^\dagger a_{\vec{p}}(b_{\vec{k}}+b_{-\vec{k}}^\dagger), \tag{2}$$

where $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$ are the annihilation and creation operators for the electrons, $b_{\vec{k}}$ and $b_{\vec{k}}^\dagger$ the annihilation and creation operators for the phonons, m the electron mass, $\vec{A}(t)$ is the vector potential of the incoming electromagnetic wave, $\vec{A}(t)=\vec{A}_0\cos(\Omega t)$ [corresponding to $\vec{E}(t)=\vec{E}_0\sin(\Omega t)$], and $C_{\vec{k}}$ the coupling constants of the electron-phonon interaction. Note that we are considering an electromagnetic wave of wavelength large with respect to the linear dimension of the system, so that its spatial dependence, within the semiconductor itself, can be neglected.

Following the standard procedure, we assume that in the infinite past, i.e., at $t=-\infty$, the electrons and the phonons do not interact and that the external field is absent. If $\rho(t)$ is the density matrix of the system, and we denote quantum-statistical averages with standard notation (i.e., for a generic operator $\mathcal{O}$ we write $\mathrm{Tr}[\rho(t)\mathcal{O}]=\langle\mathcal{O}\rangle_t$), the above condition for the phonons reads $\langle b_{\vec{k}}\rangle_{-\infty}=0$ and $\langle b_{\vec{k}}^\dagger\rangle_{-\infty}=0$. The external electromagnetic field and the electron-phonon interaction are then adiabatically switched on. At a certain time, with the help of an external source, phonons of given momentum, say $\vec{q}$, are excited. Starting from this time: $\langle b_{\vec{q}}\rangle_t\neq 0$ and $\langle b_{\vec{q}}^\dagger\rangle_t\neq 0$.

The renormalization of the phonon attenuation factor, $\Gamma_{\vec{q}}$, and more generally the renormalization of $\omega_{\vec{q}}$, can be obtained by considering the time evolution equation for $\langle b_{\vec{q}}\rangle_t$.[2]

[2]Alternatively, we could compute the renormalized frequency by considering the Feynman diagrams for the phonon propagator. Note that, due to the presence of the external electromagnetic field, see the first term in Eq. (2), there are additional loop corrections containing insertion of external electromagnetic lines. Equation (7) below gives an example of these additional terms. As it is immediately clear from its form, this is the contribution to the renormalization of $\omega_{\vec{q}}$ where the fermion loop contains the insertion of an external photon line (absorption and emission of a photon with energy $\hbar\Omega$).

This equation is easily obtained with the help of the Liouville equation for the density matrix:

$$i\partial_t\rho=[H,\rho]. \tag{3}$$

For the Hamiltonian (2), we have from Eq. (3)

$$i\partial_t\langle b_{\vec{q}}\rangle_t-\omega_{\vec{q}}\langle b_{\vec{q}}\rangle_t=C_{\vec{q}}\sum_{\vec{p}}\ \langle a^{\dagger}_{\vec{p}-\vec{q}}a_{\vec{p}}\rangle_{t.} \tag{4}$$

The right-hand side (RHS) of Eq. (4) provides a small correction, due to the electron-phonon interaction, to the free equation, $i\partial_t\langle b_{\vec{q}}\rangle_t-\omega_{\vec{q}}\langle b_{\vec{q}}\rangle_t=0$, whose elementary solution is $\langle b_{\vec{q}}\rangle_t=\mathrm{A}e^{-i\omega_{\vec{q}}t}$, with A an integration constant.

Going to Fourier space, i.e., writing $\langle b_{\vec{q}}\rangle_t=\int d\omega B(\omega)e^{-i\omega t}$, we derive from Eq. (4) an equation for $B(\omega)$. Finally, by considering a value of ω close to $\omega_{\vec{q}}$, it is not difficult to see that from this last equation we can obtain the value of the phonon frequency, $\overline{\omega_{\vec{q}}}$, renormalized by the electron-phonon interaction [11]. Moreover, by simple inspection of the free solution, we see that the attenuation factor for the acoustic wave is given by the negative of the imaginary part of $\overline{\omega_{\vec{q}}}$.

In order to extract $\overline{\omega_{\vec{q}}}$ from Eq. (4), a certain number of steps are needed (see [11] for the details). First, again with the help of the Liouville equation, an evolution equation for the correlator $\langle a^{\dagger}_{\vec{p}-\vec{q}}a_{\vec{p}}\rangle_t$, which appears in the RHS of Eq. (4), is derived. As the RHS of Eq. (4) already contains one power of the small electron-phonon coupling, we can limit ourselves to consider the lowest order solution in $C_{\vec{q}}$ to this new equation. This approximated solution contains the factor $\langle b_{\vec{q}}\rangle_t$. Therefore, from Eq. (4), going again to Fourier space and dropping the factor $B(\omega)$, we get (to the lowest order in $C_{\vec{q}}$) [11]:

$$\overline{\omega_{\vec{q}}}=\omega_{\vec{q}}+C^2_{\vec{q}}\Sigma(\omega_{\vec{q}}), \tag{5}$$

where $\Sigma(\omega_{\vec{q}})$ contains the "ordinary" contribution to the renormalization of $\omega_{\vec{q}}$, Σ^0, as well as the additional term due to the presence of the electromagnetic wave (Σ^{el}):

$$\Sigma(\omega_{\vec{q}})=\Sigma^0(\omega_{\vec{q}})+\Sigma^{el}(\omega_{\vec{q}}). \tag{6}$$

For our purposes, we are mainly interested in $\Sigma^{el}(\omega_{\vec{q}})$. To the lowest order in the dimensionless parameter $a=e\vec{E}_0\cdot\vec{q}/m\Omega^2$, it is [11]:

$$\Sigma^{el}(\omega_{\vec{q}})=\frac{a^2}{4}\sum_{\vec{p}}\left(\frac{n_{\vec{p}-\vec{q}}-n_{\vec{p}}}{\epsilon_{\vec{p}-\vec{q}}-\epsilon_{\vec{p}}+\hbar\overline{\omega_{\vec{q}}}+\hbar\Omega+i\epsilon}+\frac{n_{\vec{p}-\vec{q}}-n_{\vec{p}}}{\epsilon_{\vec{p}-\vec{q}}-\epsilon_{\vec{p}}+\hbar\overline{\omega_{\vec{q}}}-\hbar\Omega+i\epsilon}\right), \tag{7}$$

where $n_{\vec{p}}$ the Fermi distribution function, $\epsilon_{\vec{p}}=\vec{p}^2/2m$, and $i\varepsilon$, as usual, implements the appropriate boundary conditions.

From Eqs. (5) and (7), it is not difficult to see that, if $\hbar\Omega\gg p_F^2/2m$ and $\hbar q\gg p_F$ (p_F is the Fermi momentum), the electromagnetic contribution to the attenuation factor (which is nothing but the $\vec{E}_0$ dependent part of $-\mathrm{Im}\,\overline{\omega_{\vec{q}}}$), $\gamma_{\vec{q}}$, vanishes when $p_F<mv_s-|\hbar q/2-m\Omega/q|$ (v_s is the sound velocity in the semiconductor), while it is

$$\gamma_{\vec{q}}=\frac{VC^2_{\vec{q}}a^2}{4\pi\hbar^4}\,\frac{m^2v_s}{\hbar q}\left(\frac{\hbar q}{2}-\frac{m\Omega}{q}\right)s^{-1} \tag{8}$$

for $p_F>mv_s+|\hbar q/2-m\Omega/q|$.

As is clear from Eq. (6), the total attenuation factor is decomposed as $\Gamma_{\vec{q}}=\Gamma^0_{\vec{q}}+\gamma_{\vec{q}}$, where $\Gamma^0_{\vec{q}}$ (>0) is the "ordinary" (i.e., in the absence of an electromagnetic wave) attenuation factor.

From Eq. (8) we see that when the phonon frequency is small enough with respect to the frequency of the incoming photon, more precisely when $q\leqslant\sqrt{2m\Omega/\hbar}$, $\gamma_{\vec{q}}$ becomes negative. Moreover, as it depends quadratically on $\vec{E}_0$, for sufficiently large values of the external field, it can give a significant contribution to $\Gamma_{\vec{q}}$. Under these conditions, an amplification of the acoustic wave is observed [11,13].

As we shall see in the following, in the case of a gravitational wave hitting a metallic bar these conditions are not met.

III. HAMILTONIAN OF THE GW-ANTENNA SYSTEM

In this section we present the derivation of the microscopic Hamiltonian for a gravitational wave interacting with a metallic antenna.

The total energy-momentum of this system is $T_{\mu\nu}=T^{\mathrm{mat}}_{\mu\nu}+T^{\mathrm{gw}}_{\mu\nu}$, where $T^{\mathrm{gw}}_{\mu\nu}$ is the pure gravitational contribution, while $T^{\mathrm{mat}}_{\mu\nu}$ is the matter one, including its interaction with the gravitational wave. Being the metric just the flat background plus a small deviation, $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, we can consider the linearized expression for $T^{\mathrm{gw}}_{\mu\nu}$. Inserting this approximation in the energy-momentum conservation equation, $\partial^\nu T_{\mu\nu}=0$, we get [we use $\eta_{\alpha\beta}=\mathrm{diag}(+1,-1,-1,-1)$]

$$\partial^\nu T^{\mathrm{mat}}_{\mu\nu}-\frac{1}{2}\partial_\mu h^{\alpha\beta}T^{\mathrm{mat}}_{\alpha\beta}=0. \tag{9}$$

Clearly $T^{\mathrm{mat}}_{\mu\nu}$ contains the contributions of both the ions and the electrons:

$$T^{\mathrm{mat}}_{\mu\nu}=T^{\mathrm{el}}_{\mu\nu}+T^{\mathrm{ion}}_{\mu\nu}+T^{\mathrm{el\text{-}ion}}_{\mu\nu}, \tag{10}$$

where the last term in the RHS is due to the electron-ion interaction, while $T^{\mathrm{el}}_{\mu\nu}$ and $T^{\mathrm{ion}}_{\mu\nu}$ are the contributions from the noninteracting electron and ion systems in the presence of the external gravitational field.

Since the electron-ion interaction is small, the Hamiltonian for the free electrons and ions can be derived by considering, at first, the electron gas and the ions as noninteracting. Moreover, once we keep only the first two terms in the RHS of Eq. (10), we note that $T^{\mathrm{el}}_{\mu\nu}$ and $T^{\mathrm{ion}}_{\mu\nu}$, in this limit, separately satisfy Eq. (9).

We could derive the complete Hamiltonian of the system from the linearized Eq. (9). In the following, however, we shall follow this pattern only to derive the Hamiltonian for the electrons interacting with the external gravitational field. In fact, by closely paralleling [6], we shall see that the Hamiltonian for the interaction of the phonons with the gravitational wave is easily obtained from the expression for the force coming from the geodesic deviation equation [14]. Finally, the free phonon and the electron-phonon interaction terms are the second and the third term in the RHS of Eq. (2), respectively.[3]

As is clear from the above considerations, the total Hamiltonian of the system has the form

$$H = H_0^{el} + H^{gw-el} + H_0^{ph} + H^{gw-ph} + H^{el-ph}, \qquad (11)$$

where $H_0^{el}+H^{gw-el}$ is the Hamiltonian for the electrons in the presence of the gravitational wave, $H_0^{ph}+H^{gw-ph}$ is the phonons' one, and H^{el-ph} is the electron-phonon interaction term.

Let us consider now Eq. (9) for $T^{\rm el}_{\mu\nu}$. As we are neglecting the mutual interaction between the electrons, it is sufficient to consider a single electron in the field of the gravitational wave. The Hamiltonian of the electron noninteracting gas is simply the sum of the individual terms.

Taking the spatial integral of Eq. (9) for $T^{\rm el}_{\mu\nu}$, we obtain (dropping the superscript ''el'' for simplicity)

$$\frac{d}{dt}p_\mu = \frac{1}{2}\int d^3x\, \partial_\mu h_{\alpha\beta} T^{\alpha\beta}, \qquad (12)$$

where a vanishing total spatial divergence term has been omitted, and $\int d^3x T^0_\mu$ has been written as p_μ. This is clearly the equation of motion for the electron in the external gravitational field. Our goal here is to find the Lagrangian, and then the Hamiltonian, from which this equation can be derived (see for instance [15]).

As we are considering the linearized theory, the electron energy-momentum tensor can contain only the free term plus, possibly, an additional $O(h_{\mu\nu})$ term. We have then

$$T^{\alpha\beta} = m u^\alpha u^\beta \delta^3(\vec{x}-\vec{x}(t)) + O(h_{\mu\nu}), \qquad (13)$$

where m is the electron mass, u^α the electron four-velocity, and $\vec{x}(t)$ the electron trajectory.

Inserting Eq. (13) in Eq. (12), and keeping terms up to $O(h_{\mu\nu})$, we have

$$\frac{d}{dt}p_\mu = \frac{m}{2} u^\alpha u^\beta \partial_\mu h_{\alpha\beta}. \qquad (14)$$

[3]Clearly the Hamiltonian of our metallic antenna should also contain an electron-electron interaction term. In our case, however, we do not need to write this term explicitly. In fact, its main effect is that of screening the electron-phonon interaction and results in a renormalization of the electron-phonon coupling constants $C_{\vec{k}}$'s [see Eq. (2)]. Therefore, once we regard the $C_{\vec{k}}$'s as the renormalized (screened) couplings, the Hamiltonian (11) written below correctly describes our system.

It is now a trivial exercise to show that the above equation of motion can be obtained from the Lagrangian:

$$L = \frac{m}{2}(\eta_{\alpha\beta} + h_{\alpha\beta}) u^\alpha u^\beta \qquad (15)$$

and that

$$p_\mu = \frac{\partial L}{\partial u^\mu} = m(h_{\alpha\mu}u^\alpha + \eta_{\alpha\mu}u^\alpha). \qquad (16)$$

Choosing the transverse traceless (TT) gauge for $h_{\mu\nu}$, and remembering that, as the electron gas is nonrelativistic, the spatial components of the electron four-velocity, u^i, are nothing but the components of electron velocity, v^i, while $u^0 \cong 1$, the above Lagrangian (up to an irrelevant constant term) becomes

$$L = \frac{m}{2}(\delta_{ij} + h_{ij}) v^i v^j. \qquad (17)$$

The corresponding Hamiltonian is

$$H = \frac{\partial L}{\partial u^i}u^i - L = \frac{1}{2m}(\vec{p}^{\;2} - h_{ij}p^i p^j), \qquad (18)$$

and p^i is the momentum canonically conjugate to the electron position. Note that, as the wavelength of the typical incoming gravitational wave is large with respect to the linear dimension of the bar, h_{ij} can be considered as spatial independent.

The classical Hamiltonian of the nonrelativistic, noninteracting, electron gas in the field of the incoming gravitational wave is the sum of terms of the form (18), and the corresponding quantum Hamiltonian is

$$H_0^{el} + H^{gw-el} = \frac{1}{2m}\sum_{\vec{p}} (\vec{p}^{\;2} - h_{ij}p^i p^j) a^\dagger_{\vec{p}} a_{\vec{p}}. \qquad (19)$$

Let us now turn our attention to H^{gw-ph}. As we said before, rather than following the same line of reasoning that lead to Eq. (19), we derive this interaction term from the geodesic deviation expression for the force acting upon a generic point of mass m [14]:

$$F_j = -mR_{j0k0}x_k, \qquad (20)$$

where x_k are the coordinates of the point and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the metric field. In the TT gauge, again within the linear approximation, it is

$$R_{j0k0} = -\frac{1}{2}\ddot{h}_{jk}, \qquad (21)$$

while all the other components vanish.

Obviously the Hamiltonian corresponding to the force (20) is [6]

$$H = \frac{m}{2}R_{j0k0}x_j x_k. \qquad (22)$$

Now we choose the coordinate system so that the bar lies along the z axis and the origin coincides with the center of the bar. Moreover, in order to face the most favorable conditions for the detection, we assume that the incoming wave propagates perpendicularly to the z axis, say along the x axis. Under these conditions, the only nonvanishing components of h_{ij} are

$$h_{yy}=-h_{zz}=-h \quad \text{and} \quad h_{yz}=h_{zy}. \tag{23}$$

For the purposes of our present analysis, we can neglect the xy (circular) section of the bar, so that we can consider it as being essentially a unidimensional chain of coupled harmonic oscillators (the ions). From Eqs. (21) and (22), the interaction Hamiltonian between the ions and the gravitational wave can be written as

$$H^{\text{ion-gw}}=-\frac{m_{\text{ion}}}{4}\ddot{h}\sum_{n}(z_n)^2, \tag{24}$$

where the sum is over the ion sites and z_n is the position of the nth ion, which is given by

$$z_n=na+\xi_n, \tag{25}$$

where a is the lattice spacing, and ξ_n is the displacement from the equilibrium position of the nth ion. To first order in the displacements, we have

$$H^{\text{ion-gw}}=-\frac{m_{\text{ion}}}{2}a\ddot{h}\sum_{n} n\xi_n. \tag{26}$$

This Hamiltonian can be immediately written in terms of phonons if, as usual, we develop ξ_n in normal modes:

$$\xi_n=\frac{1}{\sqrt{N}}\sum_{k}\tilde{\xi}_k e^{ikan}, \tag{27}$$

where $N \gg 1$ is the number of ions, and the operators $\tilde{\xi}_k$ satisfy the relations $\tilde{\xi}_k^{\dagger}=\tilde{\xi}_{-k}$. By considering periodic boundary conditions, we have $k=2\pi n_k/Na$, where n_k is an integer.

The Hamiltonian for the interaction between the gravitational wave and the phonons is now found by replacing Eq. (27) into Eq. (26) and writing $\tilde{\xi}_k$ in terms of creation and annihilation phonon operators [16]:

$$\tilde{\xi}_k=\sqrt{\frac{\hbar}{2m_{ion}\omega_k}}(b_k+b_{-k}^{\dagger}). \tag{28}$$

Performing the sum over n, in the large N limit we have

$$\sum_{n} ne^{ikan}=-iN^2\frac{(-1)^{n_k}}{2\pi n_k}. \tag{29}$$

Inserting now Eq. (29) in Eq. (26), and replacing Na with L, the length of the bar, and Nm_{ion} with M, the mass of the bar, we get

$$H^{\text{gw-ph}}=-\ddot{h}(t)\sum_{k}\frac{\alpha_k}{\sqrt{\omega_k}}(b_k+b_{-k}^{\dagger}), \tag{30}$$

with

$$\alpha_k=-i\frac{(-1)^{n_k}L}{4\pi n_k}\sqrt{\frac{\hbar M}{2}}. \tag{31}$$

Finally, inserting Eqs. (19) and (30) in Eq. (11), and remembering that the terms H_0^{ph} and H^{el-ph} of this equation are the second and the third term in the RHS of Eq. (2), respectively, we can now write the total microscopic Hamiltonian for the interaction between the gravitational wave and the antenna. It is

$$H=\frac{1}{2m}\sum_{\vec{p}}(\vec{p}^{\,2}-h_{ij}(t)p^ip^j)a_{\vec{p}}^{\dagger}a_{\vec{p}}+\sum_{\vec{k}}\omega_{\vec{k}}b_{\vec{k}}^{\dagger}b_{\vec{k}}$$
$$+\sum_{\vec{p},\vec{k}}C_{\vec{k}}a_{\vec{p}+\vec{k}}^{\dagger}a_{\vec{p}}(b_{\vec{k}}+b_{-\vec{k}}^{\dagger})-\ddot{h}(t)\sum_{\vec{k}}\frac{\alpha_k}{\sqrt{\omega_k}}(b_{\vec{k}}+b_{-\vec{k}}^{\dagger}), \tag{32}$$

where h_{ij} and h are given in Eq. (23).

Having at our disposal the microscopic Hamiltonian of the system, we can now move to the central issue of the present work, namely the computation of the contribution to the attenuation factor due to the interaction between the incoming gravitational wave and the electrons of the metallic antenna. To this end, following the pattern illustrated in Sec. II, we shall first consider the time evolution of $\langle b_{\vec{q}_0}\rangle_t$, where $\vec{q}_0$ is the wave number of the fundamental mode, and then compute the corresponding attenuation factor. This is the subject of the two following sections.

IV. TIME EVOLUTION OF $\langle B_{\vec{q}_0}\rangle_T$ AND RENORMALIZATION OF $\omega_{\vec{q}_0}$

Let us consider a gravitational wave, with frequency Ω of the order of the *KHz*, hitting the metallic antenna. The wavelength of such a wave is much larger than the linear dimension of the bar, which is typically of about one meter. Therefore, the spatial dependence of h_{ij} can be neglected and we can write

$$h_{ij}=A_{ij}\cos(\Omega t). \tag{33}$$

As usual (see Sec. II), we assume that in the infinite past the gravitational wave is absent and that the electrons and the phonons do not interact. Therefore, for each value of $\vec{k}$, $\langle b_{\vec{k}}\rangle_{-\infty}=0$. The evolution equation for $\langle b_{\vec{q}}\rangle_t$ (for notational simplicity, from now on we write $\vec{q}$ rather than $\vec{q}_0$ to indicate the fundamental mode) is obtained with the help of the Liouville equation [Eq. (3)]. With the Hamiltonian (32) we obtain

$$i\partial_t\langle b_{\vec{q}}\rangle_t-\omega_{\vec{q}}\langle b_{\vec{q}}\rangle_t=\frac{\alpha_{-\vec{q}}\Omega^2h(t)}{\sqrt{\omega_{\vec{q}}}}+C_{\vec{q}}\sum_{\vec{p}}\langle a_{\vec{p}-\vec{q}}^{\dagger}a_{\vec{p}}\rangle_t. \tag{34}$$

The correlator $\langle a^\dagger_{\vec{p}-\vec{q}} a_{\vec{p}}\rangle_t$ in the RHS of Eq. (34) comes from the electron-phonon interaction term. Again with the help of Eq. (3), an equation for this correlator can be derived:

$$i\partial_t\langle a^\dagger_{\vec{p}-\vec{q}} a_{\vec{p}}\rangle_t-\left(\epsilon_{\vec{p}}-\epsilon_{\vec{p}-\vec{q}}-\frac{h_{ij}}{2m}(2p_iq_j-q_iq_j)\right)\langle a^\dagger_{\vec{p}-\vec{q}} a_{\vec{p}}\rangle_t$$

$$=\sum_k C_k\langle(a^\dagger_{\vec{p}-\vec{q}} a_{\vec{p}-\vec{k}}-a^\dagger_{\vec{p}-\vec{q}+\vec{k}} a_{\vec{p}})(b_{\vec{k}}+b^\dagger_{-\vec{k}})\rangle_t. \qquad (35)$$

As it was expected, the right-hand side of this equation contains correlators of products of three operators, as for instance the term $\langle a^\dagger_{\vec{p}-\vec{q}} a_{\vec{p}-\vec{k}} b_{\vec{k}}\rangle$, and this is due to the electron-phonon coupling. In fact, repeatedly exploiting the Liouville equation, an infinite system of coupled differential equations for the different correlators is generated. Therefore, we have to resort to a suitable truncation of this system. In the following we shall see how such an approximation can be obtained.

The hypothesis that in the infinite past the electron gas is noninteracting yields the boundary condition: $\langle a^\dagger_{\vec{p}-\vec{q}} a_{\vec{p}}\rangle_{-\infty}=0$. Solving (formally) Eq. (35) under this condition we have

$$\langle a^\dagger_{\vec{p}-\vec{q}} a_{\vec{p}}\rangle_t=-i\sum_{\vec{k}} C_{\vec{k}}\int_{-\infty}^t dt'\langle(a^\dagger_{\vec{p}-\vec{q}} a_{\vec{p}-\vec{k}}-a^\dagger_{\vec{p}-\vec{q}+\vec{k}} a_{\vec{p}})$$

$$\times(b_{\vec{k}}+b^\dagger_{-\vec{k}})\rangle_{t'}\exp\bigg[-i(\epsilon_{\vec{p}}-\epsilon_{\vec{p}-\vec{q}})(t-t')$$

$$-\frac{i(q^iq^j-2p^iq^j)}{2m}\int_{t'}^t dt''h_{ij}(t'')\bigg]. \qquad (36)$$

Finally, inserting Eq. (36) in Eq. (34), the time evolution equation for $\langle b_{\vec{q}}\rangle$ becomes

$$\frac{\partial}{\partial t}\langle b_{\vec{q}}\rangle_t+i\omega_{\vec{q}}\langle b_{\vec{q}}\rangle_t$$

$$=-i\frac{\alpha_{-\vec{q}}\Omega^2 h(t)}{\sqrt{\omega_{\vec{q}}}}+\sum_{\vec{p},\vec{k}} C_{\vec{q}}C_{\vec{k}}\int_{-\infty}^t dt'[\langle a^\dagger_{\vec{p}-\vec{q}+\vec{k}} a_{\vec{p}}$$

$$\times(b_{\vec{k}}+b^\dagger_{-\vec{k}})\rangle_{t'}-\langle a^\dagger_{\vec{p}-\vec{q}} a_{\vec{p}-\vec{k}}(b_{\vec{k}}+b^\dagger_{-\vec{k}})\rangle_{t'}]$$

$$\times\exp\bigg[-i(\epsilon_{\vec{p}}-\epsilon_{\vec{p}-\vec{q}})(t-t')$$

$$-\frac{i}{2m}(q^iq^j-2p^iq^j)\int_{t'}^t dt''h_{ij}(t'')\bigg]. \qquad (37)$$

If we now neglect the electron-phonon interaction in Eq. (37), i.e., we consider the zeroth order in the C's, we have (for the given boundary conditions)

$$\langle b_{\vec{q}}\rangle_t^{(0)}=\frac{\alpha_{-\vec{q}}\Omega^2 A}{\sqrt{\omega_{\vec{q}}}}\,\frac{\omega_{\vec{q}}\cos(\Omega t)-i\Omega\sin(\Omega t)}{\Omega^2-\omega_{\vec{q}}^2}, \qquad (38)$$

where $A=A_{zz}$ is the amplitude of the component of the field in the longitudinal direction. For $\Omega\sim\omega_{\vec{q}}$, this solution has the expected resonant form, with a very large amplitude. Strictly speaking, for $\Omega=\omega_{\vec{q}}$, the RHS of Eq. (38) is divergent. We note, however, that the well known phenomenological Breit-Wigner shape is obtained adding an imaginary part to $\omega_{\vec{q}}$. As we shall see below, this damping factor, which is always added on phenomenological grounds to the equation that describes the bar oscillations, from a microscopic point of view is mainly due to the electron viscosity, i.e., to the interaction of the electrons with the phonons. In more technical terms, it comes from the renormalization of $\omega_{\vec{q}}$ due to the electron-phonon interaction.

We observe now that for the correlator $\langle a^\dagger_{\vec{p}-\vec{q}+\vec{k}} a_{\vec{p}} b_{\vec{k}}\rangle_t$, as well as for the other similar correlators that appear in the RHS of Eq. (37), the assumption that in the infinite past the electron-phonon interaction is absent clearly amounts to the condition

$$\langle a^\dagger_{\vec{p}-\vec{q}+\vec{k}} a_{\vec{p}} b_{\vec{k}}\rangle_{-\infty}=\langle a^\dagger_{\vec{p}-\vec{q}+\vec{k}} a_{\vec{p}}\rangle_{-\infty}\langle b_{\vec{k}}\rangle_{-\infty}. \qquad (39)$$

At the lowest order in the C's, the above factorization property is also satisfied by $\langle a^\dagger_{\vec{p}-\vec{q}+\vec{k}} a_{\vec{p}} b_{\vec{k}}\rangle_t$ at any t:

$$\langle a^\dagger_{\vec{p}-\vec{q}+\vec{k}} a_{\vec{p}} b_{\vec{k}}\rangle_t=\langle a^\dagger_{\vec{p}-\vec{q}+\vec{k}} a_{\vec{p}}\rangle_t\langle b_{\vec{k}}\rangle_t. \qquad (40)$$

This approximation (and the other similar ones) is crucial to find the desired truncation of the above mentioned infinite system of differential equations.

Inserting Eq. (40) in Eq. (35), we see that, to the lowest order, the correlators $\langle a^\dagger_{\vec{p}-\vec{q}} a_{\vec{p}}\rangle$ are time-independent constants. More precisely, they all vanish unless $\vec{q}=0$, in which case $\langle a^\dagger_{\vec{p}} a_{\vec{p}}\rangle=n_{\vec{p}}$, where $n_{\vec{p}}$ is the occupation number for the states with momentum $\vec{p}$.

Therefore, if we limit ourselves to consider in Eq. (37) only terms up to $O(C^2_{\vec{q}})$, we need to keep only those terms with $\vec{k}=\vec{q}$. Then, inserting the expression (33) for h_{ij} in Eq. (37) we have

$$\frac{\partial}{\partial t}\langle b_{\vec{q}}\rangle_t+i\omega_{\vec{q}}\langle b_{\vec{q}}\rangle_t$$

$$=-i\frac{\alpha_{-\vec{q}}\Omega^2 A\cos(\Omega t)}{\sqrt{\omega_{\vec{q}}}}+C^2_{\vec{q}}\sum_{\vec{p}}(n_{\vec{p}}-n_{\vec{p}-\vec{q}})$$

$$\times\int_{-\infty}^t dt'(\langle b_{\vec{q}}\rangle_{t'}+\langle b^\dagger_{-\vec{q}}\rangle_{t'})$$

$$\times\exp\bigg[-i(\epsilon_{\vec{p}}-\epsilon_{\vec{p}-\vec{q}})(t-t')-\frac{iA_{ij}}{2m}(q^iq^j-2p^iq^j)$$

$$\times\int_{t'}^t dt''\cos(\Omega t'')\bigg]. \qquad (41)$$

It is now convenient to move to Fourier space and write

$$\langle b_{\vec q}\rangle_t=\int_{-\infty}^{\infty}d\omega B_{\vec q}(\omega)e^{-i\omega t}. \tag{42}$$

Note that Eq. (41) also contains the term $\langle b^\dagger_{-\vec q}\rangle_t$ and that, from Eq. (42), we have

$$\langle b^\dagger_{-\vec q}\rangle_t=\int_{-\infty}^{\infty}d\omega B^*_{-\vec q}(\omega)e^{i\omega t}. \tag{43}$$

A relation between $B_{\vec q}(\omega)$ and $B^*_{-\vec q}(\omega)$ is immediately obtained with the help of the time evolution equation for $\langle b^\dagger_{-\vec q}\rangle_t$. In fact, taking into account that the electron-phonon coupling constants are real, and that $C_{\vec q}=C_{-\vec q}$, we have

$$i\frac{\partial}{\partial t}\langle b^\dagger_{-\vec q}\rangle_t+\omega_{\vec q}\langle b^\dagger_{-\vec q}\rangle_t=-i\frac{\partial}{\partial t}\langle b_{\vec q}\rangle_t+\omega_{\vec q}\langle b_{\vec q}\rangle_t \tag{44}$$

which implies the relation

$$B^*_{-\vec q}(-\omega)=\frac{\omega_{\vec q}-\omega}{\omega_{\vec q}+\omega}B_{\vec q}(\omega). \tag{45}$$

Inserting now Eqs. (42), (43) and (45) in Eq. (41), we find

$$\begin{aligned}\int_{-\infty}^{\infty}d\omega e^{-i\omega t}(-i\omega+i\omega_{\vec q})B_{\vec q}(\omega)=C^2_{\vec q}\sum_{\vec p}(n_{\vec p}-n_{\vec p-\vec q})\sum_{n,m=-\infty}^{\infty}J_n(a)J_m(a)\\ \times\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{t}dt'e^{-i\omega t'}e^{-in\Omega t}e^{im\Omega t'}e^{-i(\epsilon_{\vec p}-\epsilon_{\vec p-\vec q})(t-t')}\left(B_{\vec q}(\omega)+\frac{\omega_{\vec q}-\omega}{\omega_{\vec q}+\omega}B_{\vec q}(\omega)\right)\\ -i\frac{\alpha_{-\vec q}\Omega^2A}{2\sqrt{\omega_{\vec q}}}\int_{-\infty}^{\infty}d\omega(\delta(\omega-\Omega)+\delta(\omega+\Omega)),\end{aligned} \tag{46}$$

where $J_n(a)$ are Bessel functions, whose argument a is

$$a=\frac{A_{ij}(q^iq^j-2p^iq^j)}{2m\Omega}, \tag{47}$$

and we have used the relation

$$e^{ia\sin\theta}=\sum_{n=-\infty}^{\infty}J_n(a)e^{in\theta}. \tag{48}$$

The time integral in the RHS of Eq. (46) is easily performed. Isolating the Fourier coefficients, we obtain ($-i\varepsilon$, as usual, is the converging factor for the time integral, i.e., it implements the boundary conditions):

$$\begin{aligned}(\omega-\omega_{\vec q})B_{\vec q}(\omega)=\frac{\alpha_{-\vec q}\Omega^2A}{2\sqrt{\omega_{\vec q}}}(\delta(\omega-\Omega)+\delta(\omega+\Omega))+C^2_{\vec q}\sum_{n,m=-\infty}^{\infty}\sum_{\vec p}J_n(a)J_m(a)\frac{n_{\vec p}-n_{\vec p-\vec q}}{-\omega+n\Omega+\epsilon_{\vec p}-\epsilon_{\vec p-\vec q}-i\varepsilon}\\ \times\left(B_{\vec q}(\omega+(m-n)\Omega)+\frac{\omega_{\vec q}-\omega+(n-m)\Omega}{\omega_{\vec q}+\omega+(m-n)\Omega}B_{\vec q}(\omega+(m-n)\Omega)\right).\end{aligned} \tag{49}$$

From Eq. (49) we can obtain $B_{\vec q}(\omega)$ up to $O(C^2_{\vec q})$. To the lowest order, i.e., by keeping only the first term in the RHS of this equation, we have

$$B^{(0)}_{\vec q}(\omega)=\frac{\alpha_{-\vec q}\Omega^2A}{2\sqrt{\omega_{\vec q}}(\omega-\omega_{\vec q})}(\delta(\omega-\Omega)+\delta(\omega+\Omega)). \tag{50}$$

As it can be immediately verified, this is nothing but the Fourier transform of Eq. (38). To get the $O(C^2_{\vec q})$ correction to Eq. (50), we have to insert in Eq. (49) the series $B_{\vec q}(\omega)=B^{(0)}_{\vec q}(\omega)+C^2_{\vec q}B^{(1)}_{\vec q}(\omega)+O(C^4_{\vec q})$, and keep only terms up to $O(C^2_{\vec q})$. Here we are rather interested in the renormalization of the phonon frequency $\omega_{\vec q}$, let us call $\omega^r_{\vec q}$ the renormalized frequency, due to the electron-phonon interaction. In the following we shall see how $\omega^r_{\vec q}$ can be obtained with the help of Eq. (49).

We could compute $\omega^r_{\vec q}$ directly by considering the renormalization of the phonon propagator. As we are interested in the $O(C^2_{\vec q})$ correction to $\omega_{\vec q}$, we only need to keep diagrams with one electron-hole loop. However, the electrons also interact with the external gravitational field [see the first term on the RHS of Eq. (32)]. Therefore, in addition to the usual fermionic loop, an infinite series of one-loop diagrams with the insertion of one, two, ... gravitational external lines is generated. These diagrams are all of $O(C_{\vec{q}}^2)$.

Equivalently, following a standard procedure [10], the same result can be obtained from Eq. (49) once we perform the following steps. First of all we note that the first term on the RHS of this equation cannot contribute to the renormalization of $\omega_{\vec{q}}$. In fact, as it is due to the resonant coupling between the external gravitational field and the phonons [see the last term on the RHS of Eq. (32)], this term does not enter in the loop correction of the phonon propagator.

If we now (i) ignore this term in Eq. (49), (ii) consider a value of ω such that $\omega=\omega_{\vec{q}}+O(C_{\vec{q}}^2)$, and (iii) keep only terms up to $O(C_{\vec{q}}^2)$, we can write Eq. (49) as

$$(\omega-\omega_{\vec{q}}-\Pi(\omega))B_{\vec{q}}(\omega)=0. \tag{51}$$

The precise form of $\Pi(\omega)$ is given below [see Eqs. (52) and (53)]. Apart from harmless factors, the coefficient in front of $B_{\vec{q}}(\omega)$ is the inverse phonon propagator up to $O(C_{\vec{q}}^2)$. Therefore, the one-loop corrected phonon frequency, which is nothing but the pole of the phonon propagator, is given by that value of ω for which this coefficient vanishes.

To begin with, let us consider the case when the gravitational wave is absent. As in this case $a=0$, all the Bessel functions, with the exception of J_0 [$J_0(0)=1$], vanish. Following the procedure outlined above, neglecting higher order terms in $C_{\vec{q}}^2$ and dropping the factor $B_{\vec{q}}$, from Eq. (49) we get

$$\omega_{\vec{q}}^r=\omega_{\vec{q}}+C_{\vec{q}}^2\sum_{\vec{p}}\frac{n_{\vec{p}}-n_{\vec{p}-\vec{q}}}{-\omega_{\vec{q}}+\epsilon_{\vec{p}}-\epsilon_{\vec{p}-\vec{q}}-i\varepsilon}. \tag{52}$$

This is the well known result for the renormalization of $\omega_{\vec{q}}$ [17] in the absence of external fields (see Sec. V below for the detailed computation), where we easily recognize the one-loop structure of the second term on the RHS of Eq. (52). Note that the negative of the imaginary part of $\omega_{\vec{q}}^r$, $\Gamma_{\vec{q}}=-\mathrm{Im}\,\omega_{\vec{q}}^r$, gives the $O(C_{\vec{q}}^2)$ contribution to the damping factor due to the electron-phonon interaction. This term, as we have anticipated, modifies the deltalike structure of Eq. (38) giving to this amplitude the expected Breit-Wigner shape, the maximum being attained for $\Omega=\mathrm{Re}\,\omega_{\vec{q}}^r$.

The same steps can be repeated for the case when the gravitational wave is present. As we only keep terms up to $O(C_{\vec{q}}^2)$, the functions $B_{\vec{q}}$ in both members of Eq. (49) have to be replaced with their lowest order approximations, $B_{\vec{q}}^{(0)}$. Consequently, the only nonvanishing terms in the RHS of Eq. (49) are those for which $n=m$. Moreover, as it was also the case when the gravitational wave was absent, the last term of Eq. (49), which is of higher order in $C_{\vec{q}}^2$, has to be neglected. Therefore, dropping again $B_{\vec{q}}^{(0)}$ from both sides of Eq. (49), we have

$$\omega_{\vec{q}}^r=\omega_{\vec{q}}+C_{\vec{q}}^2\sum_{n=-\infty}^{\infty}\sum_{\vec{p}}\left[J_n^2(a)\frac{n_{\vec{p}}-n_{\vec{p}-\vec{q}}}{-\omega_{\vec{q}}+n\Omega+\epsilon_{\vec{p}}-\epsilon_{\vec{p}-\vec{q}}-i\varepsilon}\right]. \tag{53}$$

As we have anticipated, in Eq. (53) we see that the $O(C_{\vec{q}}^2)$ correction to $\omega_{\vec{q}}$ contains an infinite series of one-loop terms due to the interaction of the electrons with the gravitational wave. Equation (53) is the result we were looking for, namely, the renormalized $\omega_{\vec{q}}$ "dressed" by the interaction with the external gravitational field. This expression clearly contains also the contribution to $\omega_{\vec{q}}^r$ which does not depend on this interaction, which is nothing but the result in Eq. (52). It is easily found in the $n=0$ term of this series once we consider the first term of the expansion of $J_0^2(a)$ in powers of a [see Eq. (55) below].

Our aim is to investigate the impact of this additional "dressing" of $\omega_{\vec{q}}^r$ on the oscillations of the gravitational antenna. To this end, we compute in the next section the damping factor $\Gamma_{\vec{q}}=-\mathrm{Im}\,\omega_{\vec{q}}^r$. Following the notation introduced in Sec. II, we indicate the attenuation factor in the absence of the gravitational wave with $\Gamma_{\vec{q}}^0$ and the gravitational contribution with $\gamma_{\vec{q}}$, so that the total attenuation factor is $\Gamma_{\vec{q}}=\Gamma_{\vec{q}}^0+\gamma_{\vec{q}}$.

V. THE ATTENUATION FACTOR

This section is devoted to the computation of the attenuation (or damping) factor $\Gamma_{\vec{q}}$. In order to compare our estimates of physical quantities with experimentally known values, in the following we consider the specific example of the antenna EXPLORER [12] operating at CERN. Taking the imaginary part of Eq. (53), we have

$$\Gamma_{\vec{q}}=-\mathrm{Im}\,\omega_{\vec{q}}^r=\Gamma_{\vec{q}}^0+\gamma_{\vec{q}}=\pi C_{\vec{q}}^2\sum_{n=-\infty}^{\infty}\sum_{\vec{p}}J_n^2(a)(n_{\vec{p}-\vec{q}}-n_{\vec{p}})\times\delta(\epsilon_{\vec{p}}-\epsilon_{\vec{p}-\vec{q}}-\omega_{\vec{q}}+n\Omega). \tag{54}$$

Due to the weakness of the gravitational wave, a, the dimensionless argument of the Bessel functions, is a small number. Therefore, we can expand these functions in powers of a and limit ourselves to consider terms up to the lowest nontrivial order in a. This amounts to keep only the Bessel functions with $n=0,\pm1$. In fact, it is

$$J_0^2(a)=1-\frac{a^2}{2}+\cdots,$$

$$J_{\pm1}^2(a)=\frac{a^2}{4}+\cdots, \tag{55}$$

where the dots indicate higher power terms in a, and the expansion of all the other Bessel functions $J_n^2(a)$ with $|n|\geqslant 2$ starts with higher powers of a. We know from Eq. (47)

that the quantity a is a function of $\vec{p}$: $a=a(\vec{p})$. For reasons that will be immediately clear, in the following expression it is convenient to indicate explicitly this dependence. Keeping in Eq. (54) only terms up to $O(a^2)$, we have

$$\Gamma_{\vec{q}}=\pi C_{\vec{q}}^2\Bigg(\sum_{\vec{p}} n_{\vec{p}}[\delta(\epsilon_{\vec{p}+\vec{q}}-\epsilon_{\vec{p}}-\omega_{\vec{q}})-\delta(\epsilon_{\vec{p}}-\epsilon_{\vec{p}-\vec{q}}-\omega_{\vec{q}})] + \sum_{\vec{p}} \frac{n_{\vec{p}}}{4}([\delta(\epsilon_{\vec{p}+\vec{q}}-\epsilon_{\vec{p}}-2\omega_{\vec{q}})+\delta(\epsilon_{\vec{p}+\vec{q}}-\epsilon_{\vec{p}}) -2\delta(\epsilon_{\vec{p}+\vec{q}}-\epsilon_{\vec{p}}-\omega_{\vec{q}})]a^2(\vec{p}+\vec{q})-[\delta(\epsilon_{\vec{p}}-\epsilon_{\vec{p}-\vec{q}}-2\omega_{\vec{q}}) + \delta(\epsilon_{\vec{p}}-\epsilon_{\vec{p}-\vec{q}})-2\delta(\epsilon_{\vec{p}}-\epsilon_{\vec{p}-\vec{q}}-\omega_{\vec{q}})]a^2(\vec{p}))\Bigg). \tag{56}$$

The first term in the RHS of Eq. (56) does not depend on a. It is nothing but $\Gamma_{\vec{q}}^0$. Considering for the time being only this term, and performing the summation over $\vec{p}$, we get

$$\Gamma_{\vec{q}}^0=\frac{8\pi^3 V C_{\vec{q}}^2}{h^4}\frac{\omega_{\vec{q}} m^2}{q}, \tag{57}$$

where V is the volume of the bar and m is the electron mass. Moreover, as we have chosen the bar to lie along the z direction, in the above expression we have $q=q_z$.

Let us focus now our attention on the derivation of $\gamma_{\vec{q}}$, i.e., on the computation of the two remaining terms in the RHS of Eq. (56). As previously said, the bar lies along the z direction, while the gravitational wave (in the TT gauge) propagates perpendicularly to the bar, along the x direction. Therefore, the only nonvanishing components of A_{ij} are $A_{yy}=-A_{zz}$ and $A_{xy}=A_{yx}$, so that

$$p^i q^j A_{ij}=p_y q_z A_{yz}+p_z q_z A_{zz}=pq(A_{yz}\sin\theta\sin\phi+A_{zz}\cos\theta), \tag{58}$$

where, to write the last term, we have introduced spherical coordinates in the $\vec{p}$-space (and noted again that $q_z=q$).

Inserting Eq. (58) in Eq. (56), and proceeding as before, we find

$$\gamma_{\vec{q}}=\frac{V}{h^3}\frac{\pi C_{\vec{q}}^2}{\hbar m^2(\hbar\omega_{\vec{q}})^2}\int_0^{p_F}p^2dp\int_{-1}^{+1}d(\cos\theta)\int_0^{2\pi}d\phi \times([(\hbar q)^4A_{zz}^2+4p^2(\hbar q)^2(A_{yz}^2\sin^2\theta\sin^2\phi+A_{zz}^2\cos^2\theta)] \times[\delta_1+\delta_2-\delta_3-\delta_4-2\delta_5+2\delta_6] +4p(\hbar q)^3A_{zz}^2\cos\theta[\delta_1+\delta_2+\delta_3+\delta_4-2\delta_5-2\delta_6]), \tag{59}$$

where we have introduced the compact notations:

$$\delta_1=\delta\left(\frac{p\hbar q\cos\theta}{m}+\frac{(\hbar q)^2}{2m}-2\hbar\omega_{\vec{q}}\right),$$

$$\delta_2=\delta\left(\frac{p\hbar q\cos\theta}{m}+\frac{(\hbar q)^2}{2m}\right),$$

$$\delta_3=\delta\left(\frac{p\hbar q\cos\theta}{m}-\frac{(\hbar q)^2}{2m}-2\hbar\omega_{\vec{q}}\right),$$

$$\delta_4=\delta\left(\frac{p\hbar q\cos\theta}{m}-\frac{(\hbar q)^2}{2m}\right),$$

$$\delta_5=\delta\left(\frac{p\hbar q\cos\theta}{m}+\frac{(\hbar q)^2}{2m}-\hbar\omega_{\vec{q}}\right),$$

$$\delta_6=\delta\left(\frac{p\hbar q\cos\theta}{m}-\frac{(\hbar q)^2}{2m}-\hbar\omega_{\vec{q}}\right). \tag{60}$$

For a typical gravitational antenna we are under the conditions:[4]

$$\frac{(\hbar q)^2}{2m}<l\hbar\omega_{\vec{q}}-\frac{(\hbar q)^2}{2m}<l\hbar\omega_{\vec{q}}+\frac{(\hbar q)^2}{2m}<\frac{p_F\hbar q}{m}\quad(l=1,2), \tag{61}$$

and the integration of Eq. (59) gives

$$\gamma_{\vec{q}}=\frac{6\pi^3VC_{\vec{q}}^2}{h^4}\frac{m^2\omega_{\vec{q}}}{q}[2A_{zz}^2-A_{yz}^2]. \tag{62}$$

Finally, since the incoming gravitational wave is not expected to be polarized, now we have to perform the average over the polarizations. If we call β the polarization angle, we can write

$$A_{zz}=e_+\cos2\beta-e_\times\sin2\beta,\quad A_{yz}=e_+\sin2\beta+e_\times\cos2\beta. \tag{63}$$

Averaging over β, we arrive at the result

$$\gamma_{\vec{q}}=\frac{3\pi^3VC_{\vec{q}}^2}{h^4}\frac{m^2\omega_{\vec{q}}}{q}(e_+^2+e_\times^2). \tag{64}$$

It is interesting to compare Eq. (64) with the corresponding damping factor for the electromagnetic case [see Eq. (8) of Sec. II]. We note that, differently from this case, the gravitational contribution to $\Gamma_{\vec{q}}$ is always positive. The gravitational correction to $\Gamma_{\vec{q}}^0$ can never produce an amplification of the bar oscillation.

Irrespectively of its sign, however, it is more important to compare the magnitude of $\gamma_{\vec{q}}$, Eq. (64), with $\Gamma_{\vec{q}}^0$, Eq. (57). Taking for $e_+\cong e_\times=e$ the realistic value of $e\cong10^{-21}$, we obtain

[4]For instance for EXPLORER it is $\hbar\omega_{\vec{q}}=6\times10^{-31}$ J, $p_F=1{,}22\times10^{-24}$ Kg m/s, $\hbar q=1{,}1\times10^{-34}$ Kg m/s.

$$\frac{\gamma_{\vec{q}}}{\Gamma^0_{\vec{q}}} \cong 10^{-42}. \tag{65}$$

Clearly, the gravitational correction to $\Gamma^0_{\vec{q}}$ is too small to give any appreciable contribution to $\Gamma_{\vec{q}}$.

A few comments are in order. First of all we note that such a result, which could be expected on the basis of the extreme weakness of the gravitational wave amplitude, entirely justifies the common assumption according to which the only physically relevant mechanism for the transfer of energy between the incoming wave and the gravitational antenna is the direct resonant coupling between the wave and the bar normal modes.

We also note that, as the loop integrals extend up to the Fermi momentum, we could expect that correction to $\Gamma^0_{\vec{q}}$ depends on p_F. Performing the momentum integrals in Eq. (59), however, we see that the terms that individually depend on p_F cancel each others.

One could *a priori* think that, would these cancellations not occur, the correction to $\Gamma^0_{\vec{q}}$ could turn out more significative. However, the value of the Fermi momentum (see footnote 4) is not sufficiently high to compensate for the extremely low value of the gravitational amplitude. In fact, as we can easily see by simple inspection of Eq. (60), there are three typical terms that come out from the integration of Eq. (59) before the cancellations. They are $(p_F \hbar q/m)^2$, $(\hbar \omega_{\vec{q}})^2$ and $(\hbar^2 q^2/2m)^2$. While the third term is negligible with respect to the second one, the first term is six orders of magnitude greater than the second. Therefore, the order of magnitude of the "gravitational damping" $\gamma_{\vec{q}}$, even in the absence of these cancellations, would only change of a factor 10^6, and the resulting value of $\gamma_{\vec{q}}$ ($\sim 10^{-36}$ s^{-1}) would still be far too small if compared with $\Gamma^0_{\vec{q}}$.

VI. SUMMARY AND CONCLUSIONS

In the present work, after deriving the microscopic Hamiltonian for the interaction between a gravitational wave and a metallic bar (the gravitational antenna), we have studied the contribution to the oscillations of the antenna due to the interaction of its electrons with the incoming wave.

More precisely, we have considered the contribution to the damping factor $\Gamma_{\vec{q}}$ (the negative of the imaginary part of the phonon frequency) which comes from the interaction between the electrons of the bar and the gravitational external field. It turns out that this term is several orders of magnitude smaller than the ordinary factor which is due to the electron viscosity (i.e., to the electron-phonon interaction).

As is well known, the gravitational wave interacts directly with the normal modes of the bar. This resonant interaction is due to the modification of the geodesic distance between the ions of the lattice induced by the presence of the gravitational field. The contribution $\gamma_{\vec{q}}$ to the damping factor considered in this paper is the result of an additional "indirect" interaction between the gravitational wave and the phonons. This additional coupling, which is due to the electron-graviton interaction, is induced by electron-hole loops.

However, as the incoming wave is extremely weak and the Fermi momentum (which is the highest value of momentum running in the loops) is not sufficiently high to compensate for this weakness, it should not come as a surprise that $\gamma_{\vec{q}}$ turns out to be a negligibly small contribution to $\Gamma_{\vec{q}}$.

Finally, it is worth mentioning that our findings, while confirming the assumption that the only relevant mechanism of interaction between a gravitational wave and a metallic bar is the direct resonant coupling between the wave and its normal modes, strongly disagree with the recent claim that the interaction of the gravitational wave with the electrons of the bar enhances the resonant cross section of several orders of magnitude, actually of four orders of magnitude [9].

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