Einstein-aether waves

T. Jacobson*
Insitut d'Astrophysique de Paris, 98 bis Bvd. Arago, 75014 Paris, France

D. Mattingly
Department of Physics, University of California–Davis, Davis, California 95616, USA

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Local Lorentz invariance violation can be realized by introducing extra tensor fields in the action that couple to matter. If the Lorentz violation is rotationally invariant in some frame, then it is characterized by an "aether," i.e., a unit timelike vector field. General covariance requires that the aether field be dynamical. In this paper we study the linearized theory of such an aether coupled to gravity and find the speeds and polarizations of all the wave modes in terms of the four constants appearing in the most general action at second order in derivatives. We find that in addition to the usual two transverse traceless metric modes, there are three coupled aether-metric modes.

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I. INTRODUCTION

Recently there has been an explosion of research on the possibility that Lorentz invariance is violated by quantum gravity effects (see, e.g., [1] and references therein). In a nongravitational setting, it suffices to specify fixed background fields violating Lorentz symmetry in order to formulate the Lorentz violating (LV) matter dynamics. However, fixed background fields break general covariance. If we are to preserve the successes of general relativity in accounting for gravitational phenomena, breaking general covariance is not an option. The obvious alternative is to promote the LV background fields to dynamical fields, governed by a generally covariant action. Virtually any configuration of any matter field breaks Lorentz invariance, but this differs in an important way from what we have in mind. The LV background fields we are contemplating are constrained either dynamically or kinematically not to vanish, so that every relevant field configuration violates local Lorentz symmetry everywhere, even in the "vacuum."

If the Lorentz violation preserves a three-dimensional rotation subgroup, then the background field must be only a timelike vector, which might be described by the gradient of a scalar, or by a vector field. In this paper we consider just the case where the LV field is a unit timelike vector u^a, which can be viewed as the minimal structure required to determine a local preferred rest frame. We call this field the "aether," as it is ubiquitous and determines a locally preferred frame at every point of spacetime. Kinetic terms in the action couple the aether directly to the spacetime metric, in addition to any couplings that might be present between the aether and the matter fields. We refer to the system of the metric coupled to the aether as "Einstein-aether theory."

Here we investigate the linearized wave spectrum of this theory, and determine the complete set of mode speeds and polarizations for generic values of the free parameters in the action. (Results for different special cases were previously published in Refs. [2,3].) These results identify the choices of constants in the action for which the linearized field equations are hyperbolic (and hence admit an initial value formulation), and they will be useful in extracting the observable consequences of such an aether field.

Related work goes back at least to the 1970s, when Nordtvedt and Will began a study of vector-tensor theories of gravity [4–7], which differed from the present work primarily in the fact that the norm of the vector was not constrained. Gasperini, using a tetrad formalism, studied in a series of papers [8] an equivalent formulation of the Einstein-aether theory studied here. Further related work has been done by Kostelecky and Samuel [9] and Jacobson and Mattingly [2] in the special case where the aether dynamics is Maxwell-like. The spherically symmetric weak field solutions were found for the general Einstein-aether theory by Eling and Jacobson [10]. Vector-tensor theories have been studied in a cosmological context by Clayton and Moffatt [11,12] and Bassett *et al.* [13]. The issues of causality and shocks in vector-tensor theories were studied by Clayton [14]. Further discussion on previous work can be found in [2,10]. A proposal for Lorentz symmetry breaking via a scalar field with unusual kinetic term that makes the gradient tend to a timelike vector of constant norm has recently been investigated by Arkani-Hamed *et al.* [15,16]. Most recently, the issue of Lorentz violation in a gravitational setting has been examined in a systematic way by Kostelecky [17].

II. EINSTEIN-AETHER THEORY

In the spirit of effective field theory, we consider a derivative expansion of the action for the metric g_{ab} and aether u^a. The most general action that is diffeomorphism-invariant and quadratic in derivatives is

$$S=\frac{1}{16\pi G}\int d^4x\sqrt{-g}[-R+\mathcal{L}_u-\lambda(u^a u_a-1)] \qquad (1)$$

where

*Permanent address: Department of Physics, University of Maryland, College Park, MD 20742-4111, USA.

$$\mathcal{L}_u = -K^{ab}{}_{mn}\nabla_a u^m \nabla_b u^n \tag{2}$$

with

$$K^{ab}{}_{mn} = c_1 g^{ab} g_{mn} + c_2 \delta^a_m \delta^b_n + c_3 \delta^a_n \delta^b_m + c_4 u^a u^b g_{mn}, \tag{3}$$

R is the Ricci scalar, and λ is a Lagrange multiplier that enforces the unit constraint. The metric signature is $(+---)$, units are chosen such that $c=1$, and other than the signature choice we use the conventions of [18].

The presence of the Lagrange multiplier and c_4 terms differentiates this theory from the vector-tensor theories considered in [7]. The possible term $R_{ab}u^a u^b$ is proportional to the difference of the c_2 and c_3 terms via integration by parts and hence has been omitted. We have also omitted any matter coupling since we are interested here in the dynamics of the metric-aether sector in vacuum.

Varying the action (1) with respect to u^a, g^{ab}, and λ yields the field equations

$$\nabla_a J^a{}_m - c_4 \dot{u}_a \nabla_m u^a = \lambda u_m \tag{4}$$

$$G_{ab} = T_{ab} \tag{5}$$

$$g_{ab} u^a u^b = 1 \tag{6}$$

where to compactify the notation we have defined

$$J^a{}_m = K^{ab}{}_{mn} \nabla_b u^n \tag{7}$$

and

$$\dot{u}^m = u^a \nabla_a u^m, \tag{8}$$

and the aether stress tensor is

$$\begin{aligned} T_{ab} = {} & \nabla_m (J_{(a}{}^m u_{b)} - J^m{}_{(a} u_{b)} - J_{(ab)} u^m) + c_1[(\nabla_m u_a)(\nabla^m u_b) \\ & - (\nabla_a u_m)(\nabla_b u^m)] + c_4 \dot{u}_a \dot{u}_b + [u_n(\nabla_m J^{mn}) - c_4 \dot{u}^2] u_a u_b \\ & - \tfrac{1}{2} g_{ab} \mathcal{L}_u. \end{aligned} \tag{9}$$

In the above expression the constraint has been used to eliminate the term that arises from varying $\sqrt{-g}$ in the constraint term in Eq. (1), and in the fourth line λ has been eliminated using the aether field equation.

A. Linearized field equations

The first step in finding the wave modes is to linearize the field equations about the flat background solution with Minkowski metric η_{ab} and constant unit vector $\underline{u}^a$. The fields are expanded as

$$g_{ab} = \eta_{ab} + \gamma_{ab} \tag{10}$$

$$u^a = \underline{u}^a + v^a. \tag{11}$$

The Lagrange multiplier λ vanishes in the background, so we use the same notation for the linearized version. Indices are raised and lowered with η_{ab}. We adopt Minkowski coordinates (x^0, x^i) aligned with $\underline{u}^a$, i.e., for which $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$ and $\underline{u}^a = (1,0,0,0)$. The letters i,j,k,l are reserved for spatial coordinate indices, and repeated spatial indices are summed with the Kronecker delta.

Keeping only first order terms in v^a and γ_{ab}, the field equations become

$$\partial_a J^{(1)a}{}_m = \lambda \underline{u}_m \tag{12}$$

$$G^{(1)}_{ab} = T^{(1)}_{ab} \tag{13}$$

$$v^0 + \tfrac{1}{2}\gamma_{00} = 0 \tag{14}$$

where the superscript (1) denotes the first order part of the corresponding quantity. The linearized Einstein tensor is

$$G^{(1)}_{ab} = -\tfrac{1}{2}\Box\gamma_{ab} - \tfrac{1}{2}\gamma_{,ab} + \gamma_{m(a,b)}{}^m + \tfrac{1}{2}\eta_{ab}(\Box\gamma - \gamma_{mn,}{}^{mn}), \tag{15}$$

where $\gamma = \gamma_m{}^m$ is the trace, while the linearized aether stress tensor is

$$\begin{aligned} T^{(1)}_{ab} = {} & \partial_m [J^{(1)}_{(a}{}^m \underline{u}_{b)} - J^{(1)m}{}_{(a}\underline{u}_{b)} - J^{(1)}_{(ab)}\underline{u}^m] \\ & + [\underline{u}_n(\partial_m J^{(1)mn})]\underline{u}_a \underline{u}_b. \end{aligned} \tag{16}$$

If we impose the linearized aether field equation (12) then the second and last terms of this expression for $T^{(1)}_{ab}$ cancel, yielding

$$T^{(1)}_{ab} = -\partial_0 J^{(1)}_{(ab)} + \partial_m J^{(1)m}_{(a}\underline{u}_{b)}. \tag{17}$$

The linearized quantity $J^{(1)}_{ab}$ is given by

$$J^{(1)}_{ab} = c_1 \nabla_a u_b + c_2 \eta_{ab} \nabla_m u^m + c_3 \nabla_b u_a + c_4 \underline{u}_a \nabla_0 u_b, \tag{18}$$

where the covariant derivatives of u^a are expanded to linear order, i.e., replaced by

$$(\nabla_a u_b)^{(1)} = (v_b + \tfrac{1}{2}\gamma_{0b})_{,a} + \tfrac{1}{2}\gamma_{ab,0} - \tfrac{1}{2}\gamma_{a0,b}. \tag{19}$$

This completes an explicit display of the linearized field equations.

The aether perturbations are coupled to metric perturbations, due to the presence of the background aether vector $\underline{u}^a$. Were it not for the aether background, the linearized aether stress tensor (16) would vanish, and the metric would drop out of the aether field equation, leaving all modes uncoupled.

B. Gauge choice

Diffeomorphism invariance of the action (1) implies that the field equations are tensorial, hence covariant under diffeomorphisms. The linearized equations inherit the linearized version of this symmetry. To find the independent physical wave modes we must fix the corresponding gauge symmetry.

An infinitesimal diffeomorphism generated by a vector field ξ^a transforms g_{ab} and u^a by

$$\delta g_{ab}=\mathcal{L}_\xi g_{ab}=\nabla_a\xi_b+\nabla_b\xi_a, \tag{20}$$

$$\delta u^a=\mathcal{L}_\xi u^a=\xi^m\nabla_m u^a-u^m\nabla_m\xi^a. \tag{21}$$

In the linearized context, the vector field ξ^a is itself first order in the perturbations, hence the linearized gauge transformations take the form

$$\gamma'_{ab}=\gamma_{ab}+\partial_a\xi_b+\partial_b\xi_a \tag{22}$$

$$v'^a=v^a-\partial_0\xi^a. \tag{23}$$

The usual choice of gauge in vacuum GR is Lorentz gauge $\partial^a\overline{\gamma}_{ab}=0$, where $\overline{\gamma}_{ab}=\gamma_{ab}-\frac{1}{2}\gamma\eta_{ab}$. This gauge is chosen because it simplifies the Einstein tensor. The residual gauge freedom, which exploits the field equations, further allows one to impose $\gamma_{0i}=0$ and $\gamma=0$. In the present case, the aether stress tensor (17) contains multiple terms in the derivatives of the metric perturbation and so the Lorentz gauge is not particularly helpful. Moreover, the residual gauge freedom cannot be used to set γ_{0i} and γ to zero since these do not satisfy the wave equation.

Instead, a convenient choice is to directly impose the four gauge conditions[1]

$$\gamma_{0i}=0 \tag{24}$$

$$v_{i,i}=0. \tag{25}$$

To see that this gauge is accessible, note that the gauge variations of γ_{0i} and $v_{i,i}$ are, according to Eqs. (22) and (23),

$$\delta\gamma_{0i}=\xi_{i,0}+\xi_{0,i} \tag{26}$$

$$\delta v_{i,i}=-\xi_{i,i0}. \tag{27}$$

Thus to achieve the gauge (24), (25) we must choose ξ_0 and ξ_i to satisfy equations of the form

$$\xi_{i,0}+\xi_{0,i}=X_i \tag{28}$$

$$\xi_{i,i0}=Y. \tag{29}$$

Subtracting the second equation from the divergence of the first gives

$$\xi_{0,ii}=X_{i,i}-Y, \tag{30}$$

which determines ξ_0 up to constants of integration by solving Poisson's equation. Then ξ_i can be determined up to a time-independent field by integrating Eq. (28) with respect to time. Having made these choices of ξ_0 and ξ_i, Eq. (28) holds, and the divergence of Eq. (28) implies that Eq. (29) holds.

[1]Alternatively, instead of setting $v_{i,i}$ to zero it is equally convenient for finding the plane wave modes to set the spatial trace γ_{ii} to zero.

In the gauge (24), (25) the tensors in the aether (12) and spatial metric equations (13) take the forms

$$J_{ai,}{}^{a}=c_{14}(v_{i,00}-\tfrac{1}{2}\gamma_{00,i0})-c_1v_{i,kk}-\tfrac{1}{2}c_{13}\gamma_{ik,k0}-\tfrac{1}{2}c_2\gamma_{kk,0i} \tag{31}$$

$$G^{(1)}_{ij}=-\tfrac{1}{2}\Box\gamma_{ij}-\tfrac{1}{2}\gamma_{,ij}-\gamma_{k(i,j)k}-\tfrac{1}{2}\delta_{ij}(\Box\gamma-\gamma_{00,00}-\gamma_{kl,kl}) \tag{32}$$

$$T^{(1)}_{ij}=-c_{13}(v_{(i,j)0}+\tfrac{1}{2}\gamma_{ij,00})-\tfrac{1}{2}c_2\delta_{ij}\gamma_{kk,00} \tag{33}$$

where we use the notation $c_{14}:=c_1+c_4$, etc.

III. WAVE MODES

In general relativity there are just two modes per spatial wave vector. Since v^a has three independent degrees of freedom, we expect that in the Einstein-aether case there will be five modes all together. We now determine the wave modes in the chosen gauge.

We assume a perturbation of the form

$$\gamma_{ab}=\epsilon_{ab}e^{ik_cx^c} \tag{34}$$

$$v^a=\epsilon^a e^{ik_cx^c}, \tag{35}$$

and choose coordinates such that the wave vector is $(k_0,0,0,k_3)$. The gauge conditions (24), (25) then imply

$$\epsilon_{0i}=0 \tag{36}$$

$$\epsilon_3=0. \tag{37}$$

The problem is now to find the set of polarizations $(\epsilon_{ab},\epsilon_a)$ and corresponding wave vectors k_a for which the perturbation is a solution to the field equations (12)–(14).

The 0 component of the aether field equation (12) is solved by definition of λ, while the constraint equation (14) implies the relation

$$\epsilon_0=-\tfrac{1}{2}\epsilon_{00}. \tag{38}$$

This leaves the spatial components of the aether equation, together with the linearized Einstein equation. It suffices to use the spatial components of the Einstein equation, as the other components yield redundant information (although they do provide useful algebraic checks).

Inserting the plane wave ansatz (34), (35) into the field equations yields

$$[A_I]\quad (c_{14}s^2-c_1)\epsilon_I-\tfrac{1}{2}c_{13}s\epsilon_{I3}=0 \tag{39}$$

$$[A_3]\quad c_{14}\epsilon_{00}+c_{123}\epsilon_{33}+c_2\epsilon_{II}=0 \tag{40}$$

$$[E_{II}]\quad \epsilon_{00}+(1+c_2)s^2\epsilon_{33}+\tfrac{1}{2}[(1+c_2+c_{123})s^2-1]\epsilon_{II}=0 \tag{41}$$

TABLE I. Wave mode speeds and polarizations in the gauge $\gamma_{0i}=v_{i,i}=0$.

Wave mode	squared speed $s^2\to$ small c_i limit	polarization
transverse, traceless metric	$1/(1-c_{13})\to 1$	γ_{12}, $\gamma_{11}=-\gamma_{22}$
transverse aether	$(c_1-\frac{1}{2}c_1^2+\frac{1}{2}c_3^2)/c_{14}(1-c_{13})\to c_1/c_{14}$	$\gamma_{I3}=[c_{13}/s(1-c_{13})]v_I$
trace	$(c_{123}/c_{14})(2-c_{14})/[2(1+c_2)^2-c_{123}(1+c_2+c_{123})]\to c_{123}/c_{14}$	$\gamma_{00}=-2v_0$
		$\gamma_{11}=\gamma_{22}=-c_{14}v_0$
		$\gamma_{33}=(2c_{14}/c_{123})(1+c_2)v_0$

$$[E_{11}-E_{22}] \quad [(1-c_{13})s^2-1](\epsilon_{11}-\epsilon_{22})=0 \tag{42}$$

$$[E_{12}] \quad [(1-c_{13})s^2-1]\epsilon_{12}=0 \tag{43}$$

$$[E_{I3}] \quad c_{13}\epsilon_I+(-1+c_{13})s\epsilon_{I3}=0 \tag{44}$$

$$[E_{33}] \quad (1+c_2)\epsilon_{II}+c_{123}\epsilon_{33}=0, \tag{45}$$

where $[A_i]$ and $[E_{ij}]$ indicate the components of the aether and Einstein equations. We use the notation $s=k_0/k_3$ for the wave speed (which will be a true "speed" only when $s^2>0$), and the index I is dedicated to the two transverse spatial directions $I=1,2$, so that $\epsilon_{II}=\epsilon_{11}+\epsilon_{22}$ is the trace of the transverse spatial part of the metric polarization ϵ_{ab}.

We analyze the independent mode solutions assuming generic values of the constants $c_{1,2,3,4}$. There are a total of five modes, two with an unexcited aether which correspond to the usual GR modes, two "transverse" aether-metric modes, and a fifth trace aether-metric mode. The two modes corresponding to the usual gravitational waves in GR are found when all polarization components vanish except ϵ_{11}, ϵ_{22} and ϵ_{12}. To avoid overdetermining the speed s, the trace equation $[E_{II}]$ must be identically satisfied, hence $\epsilon_{II}=0$. Then the $[E_{11}-E_{22}]$ and $[E_{12}]$ equations yield the speed.

The two transverse aether-metric modes have nonzero polarization components ϵ_I and ϵ_{I3}, and the $[A_I]$ and $[E_{I3}]$ equations together yield the speed and the ratio ϵ_{I3}/ϵ_I. The fifth and final mode involves only ϵ_0 and the diagonal polarization components ϵ_{aa} (no sum on $a=0,1,2,3$). To avoid overdetermining the speed, the difference equation $[E_{11}-E_{22}]=0$ must be identically satisfied, hence $\epsilon_{11}=\epsilon_{22}=\epsilon_{II}/2$. Equations $[A_3]$ and $[E_{33}]$ and the constraint (38) then allow all polarization components to be expressed in terms of just one, after which $[E_{II}]$ determines the speed. The resulting mode polarizations and speeds are displayed in Table I.

In the limit $c_i\to 0$ the transverse traceless modes become the usual gravitational waves, with unit speed. Note that these modes are entirely decoupled from the aether perturbations even when $c_i\neq 0$.

The small c_i limits of the transverse aether and trace mode speeds depend on the ratios of the constants. If $c_{2,3,4}$ vanish, both speeds approach unity, but any other value is possible. The wave speeds and nonzero polarization components for the special case $c_{2,3,4}=0$ were previously reported in [3] (the speed for the trace mode is inverted there), and the Maxwell-like case $c_{13}=c_2=c_4=0$ was analyzed in [2] (in both cases using different gauges). In the latter case, the transverse waves all have unit speed, while the trace mode has zero speed, so it does not exist as a propagating wave.

A peculiar special case occurs if $c_{14}=0$, since then the aether wave speeds are generally infinite. This happens because no time derivatives of the aether field then arise in the field equation $(31)=0$. (The more special case $c_{14}=c_{23}=0$ was shown by Barbero and Villaseñor [19] to be equivalent to general relativity via a u^a dependent field redefinition of the metric.)

When the constants c_i are chosen so that s^2 is positive and finite for all modes, the linearized equations are evidently hyperbolic. (It is not known whether this property extends to the nonlinear equations.) In these cases, since the dispersion relation $\omega=sk$ is linear, $|s|$ represents the signal propagation speed of disturbances. It is easily checked from Eqs. (41)–(45) that the Einstein tensor has nonvanishing components for each of the modes. (These equations display components of $G_{ab}-T_{ab}$, so just those of G_{ab} remain when the c_i are set to zero.) Hence the modes all have gauge-invariant, physical significance.

If s^2 is negative for a mode then the corresponding frequency is imaginary, indicating the existence of exponentially growing and decaying solutions. In such a case the theory is unstable and hence presumably unphysical.

IV. OBSERVATIONAL APPLICATIONS

An important open question is the sign of the energy of the various wave modes. To answer this, it is necessary to first determine the expression for energy in the linearized Einstein-aether theory, which has not yet been done.

To compare the wave behavior of the theory with observations, the wave emission from astrophysical sources must be determined. To begin with, the analog of the quadrupole formula would enable the decay of binary pulsar orbits to be computed. Note that the presence of transverse aether and trace modes strongly suggests that dipole and monopole radiation will also exist and contribute to the energy loss.

The wave emission depends of course on how the aether field couples to matter. A direct coupling could lead to local Lorentz violating effects which may exist but are already quite constrained. However, even a small coupling to the matter source might be large enough to produce an observable effect. Even without any direct coupling to matter, the extra modes will still be excited through their coupling to the time dependent metric produced by the moving matter sources.

The results obtained here for the linearized theory should also be useful in computing the PPN parameters.

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[1] *Proceedings of the Second Meeting on CPT and Lorentz Symmetry*, Bloomington, Indiana, 2001, edited by V.A. Kostelecky (World Scientific, Singapore, 2002); S. Sarkar, Mod. Phys. Lett. **17**, 1025 (2002); N.E. Mavromatos, hep-ph/0309221; G. Amelino-Camelia, gr-qc/0309054; T.A. Jacobson, S. Liberati, D. Mattingly, and F.W. Stecker, Phys. Rev. Lett. (to be published), astro-ph/0309681.

[2] T. Jacobson and D. Mattingly, Phys. Rev. D **64**, 024028 (2001).

[3] D. Mattingly and T. Jacobson, *Relativistic Gravity with Dynamical Preferred Frame*, in *CPT and Lorentz Symmetry II*, edited by V.A. Kostelecky (World Scientific, Singapore, 2002).

[4] C.M. Will and K. Nordvedt, Jr., Astrophys. J. **177**, 757 (1972).

[5] K. Nordvedt, Jr. and C.M. Will, Astrophys. J. **177**, 775 (1972).

[6] R.W. Hellings and K. Nordvedt, Jr., Phys. Rev. D **7**, 3593 (1973).

[7] C.M. Will, *Theory and Experiment in Gravitational Physics* (Cambridge University Press, Cambridge, England, 1993).

[8] See, for example, M. Gasperini, Class. Quantum Grav. **4**, 485 (1987); Gen. Relativ. Gravit. **30**, 1703 (1998), and references therein.

[9] V.A. Kostelecký and S. Samuel, Phys. Rev. D **40**, 1886 (1989).

[10] C. Eling and T. Jacobson, Phys. Rev. D **69**, 064005 (2004).

[11] M.A. Clayton and J.W. Moffat, Phys. Lett. B **460**, 263 (1999).

[12] J.W. Moffat, hep-th/0208122.

[13] B.A. Bassett, S. Liberati, C. Molina-Paris, and M. Visser, Phys. Rev. D **62**, 103518 (2000).

[14] M.A. Clayton, gr-qc/0104103.

[15] N. Arkani-Hamed, H.C. Cheng, M.A. Luty, and S. Mukohyama, J. High Energy Phys. **05**, 074 (2004).

[16] N. Arkani-Hamed, P. Creminelli, S. Mukohyama, and M. Zaldarriaga, J. Cosmol. Astropart. Phys. **04**, 001 (2004).

[17] A. Kostelecky, Phys. Rev. D **69**, 105009 (2004).

[18] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[19] J.F. Barbero G. and E.J. Villaseñor, Phys. Rev. D **68**, 087501 (2003).