Weak lensing in generalized gravity theories

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We extend the theory of weak gravitational lensing to cosmologies with generalized gravity, described in the Lagrangian by a generic function depending on the Ricci scalar and a nonminimal coupled scalar field. We work out the generalized Poisson equations relating the dynamics of the fluctuating components to the two gauge-invariant scalar gravitational potentials, fixing the contributions from the modified background expansion and fluctuations. We show how the lensing equation gets modified by the cosmic expansion as well as by the presence of anisotropic stress, which is non-null at the linear level both in scalar-tensor gravity and in theories where the gravitational Lagrangian term features a nonminimal dependence on the Ricci scalar. Starting from the geodesic deviation, we derive the generalized expressions for the shear tensor and projected lensing potential, encoding the spacetime variation of the effective gravitational constant and isolating the contribution of the anisotropic stress, which introduces a correction due to the spatial correlation between the gravitational potentials. Finally, we work out the expressions of the lensing convergence power spectrum as well as the correlation between the lensing potential and the integrated Sachs-Wolfe effect affecting cosmic microwave background total intensity and polarization anisotropies. To illustrate phenomenologically the effects, we work out approximate expressions for the quantities above in extended quintessence scenarios where the scalar field coupled to gravity plays the role of the dark energy.

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I. INTRODUCTION

In recent years we reached a remarkable convergence in the most important cosmological parameters describing the cosmic content as well the statistics of perturbations. The Universe is nearly geometrically flat, with an expansion rate of about 70 km/sec/Mpc, and structures grown out of a primordial linear spectrum of nearly Gaussian and scale-invariant perturbations in the distribution of the energy density. About 5% of the critical energy density is made of baryons, while the remaining dark part is supposed to interact at most weakly with the baryons themselves, since we observe it only through its gravitational effects. The dark component appears to be 30% pressureless, like in cold dark matter (CDM) scenarios, dominating the gravitational potentials perturbations that host visible structures such as galaxies or clusters. The remaining 70% should be in some sort of vacuum energy, with negative pressure acting as a repulsive gravity, and responsible for a late time cosmic acceleration era. The case for this ''concordance cosmological model'' is now quite robust, supported by several independent data sets: the distant type Ia supernovae (hereafter SNIa [1,2]), the cosmic microwave background (CMB) anisotropies (see Ref. [3] and references therein), the Large Scale Structure (LSS [4,5]) and the Hubble Space Telescope (HST [6]).

The picture is clearly far from being satisfactory: in particular, without a better insight into the nature of the dark cosmological component, we cannot claim to have a satisfactory physical understanding of cosmology. The simplest description of the vacuum energy responsible for cosmic acceleration is a purely geometric term in the Einstein equations, the cosmological constant. On the other hand, while the CDM has well established support from theories beyond the standard model of particle physics, a cosmological constant providing the 70% of the critical density today raises two problems. The first is the fine tuning required to fix the vacuum energy scale about 120 orders of magnitude less than the Planck energy density, which is supposed to be the unification scale of all forces in the early Universe. The second is a coincidence issue, simply why among all the small nonzero values of the cosmological constant, the value was chosen to be comparable to the critical energy density today. These questions, still largely unsolved, could be answered only if the concept of cosmological constant is extended to a more general one, admitting a dynamics of the vacuum energy, known now as the dark energy (see [7–9] and references therein).

The simplest generalization, already introduced well before the evidence for cosmic acceleration [10,11], is a scalar field, dynamical and fluctuating, with a background evolution slow enough to mimic a constant vacuum energy given by its potential, providing cosmic acceleration. As soon as the latter was discovered, renewed interest in these models appeared immediately [12,13]. In particular, it was demonstrated how the dynamics of this component, under suitable potential shapes inspired by supersymmetry and supergravity theories (see Refs. [14] and [15], respectively, and references therein), can possess attractors in the trajectory space, named tracking solutions, capable of reaching the present dark energy density starting from a wide set of initial conditions in the very early Universe, thus alleviating, at least classically, the problem of fine tuning [16,17]. The scalar field playing the role of the dark energy was named quintessence. Its coherent insertion among the other cosmological components allowed one to constrain it from the existing data [18–25], as well as to investigate the relation of the dark energy with the other cosmological components: the explicit coupling with baryons is severely constrained by observations [26], while

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the possible coupling of the quintessence with the Ricci scalar [27–40] and the dark matter [41–44] as well as the phenomenology arising from generalized kinetic energy terms [45–47] have been extensively studied.

The generalization of cosmology that we consider here concerns the gravitational sector of the fundamental Lagrangian, admitting a general dependence on the Ricci and Brans-Dicke scalar fields. This subject is interesting per se (see Refs. [48,49] and references therein), and is receiving more attention after the discovery of cosmic acceleration, with the attempt to interpret the evidence for dark energy as a manifestation of gravity; this scenario has been recently proved to have relevant consequences for what concerns the dark energy fine-tuning problem mentioned above [40].

The gravitational lensing effect in cosmology is gathering a great interest becoming one of the most promising tools to investigate cosmological structures (see Ref. [50] and references therein), and commonly thought in terms of strong and weak regimes. The first one concerns highly magnified sources, generically through the generation of multiple images of a background object by a single lens with the typical size of a galaxy.

The weak gravitational lensing, which is the subject of the present work, produces weak amplification, generically through the distortion of the pattern of background light; moreover, it is generated by a large set of scales and objects, ranging roughly from nonlinear structures like galaxy clusters to the large scale distribution of matter, still in linear regime. The weak lensing shear was detected recently by independent groups with astonishing agreement [51–55]. Although the precision of such measurements does not allow one to constrain different cosmological models, the planned observations will become certainly a crucial tool to investigate the behavior of dark matter and energy during the structure formation process and at the onset of cosmic acceleration [56].

In particular, several authors considered the possibility of investigating the dark energy component through weak lensing, with different approaches ranging from shear distortion of background galaxies from clusters of galaxies to the weak lensing power causing a non-Gaussian pattern into the CMB anisotropies [57–64]. The reason of this interest is the timing: the structure formation, and the weak lensing carrying its physical information occurs at an epoch that overlaps with the onset of cosmic acceleration; by virtue of this fact, it is reasonable to expect a good sensitivity of the weak lensing effect to the main dark energy properties such as the equation of state and its redshift behavior.

These studies are entering in an higher level of sophistication: the first outcome of N-body simulations in several dark energy scenarios have been published recently [65–67], mainly studying the impact of the background rate of expansion on the internal parameter of structures. The implementation of a light ray tracing technique through those structures allows one to check numerically the weak lensing pattern produced by mildly and full nonlinear density perturbations [68].

The theory of weak gravitational lensing in ordinary cosmologies, i.e., made by radiation, dark matter, and cosmological constant, is known and well established, but a comprehensive treatment in cosmologies with generalized theories of gravity still lacks in literature. The aim of the present work is to fill this gap, by providing the community with the recipe to interpret the weak lensing observations in a more general context. We follow the harmonic approach to weak lensing [69] and the treatment for generalized cosmological scenarios including cosmological perturbation [49], already exploited for investigating the effects of the explicit coupling between dark energy and gravity (see Ref. [40] and references therein).

This work is organized as follows. In Sec. II we describe the cosmological models we deal with, for background and linear perturbations. In Sec. III we write the generalized Poisson equations, relating the fluctuations in metric and cosmological components. The derivation and discussion of the lensing equation and weak lensing potential are in Secs. IV and V, respectively. Finally, in Sec. VI we draw conclusions.

II. GENERALIZED COSMOLOGIES

We shall consider a class of theories of gravity whose action is written in natural units as

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}f(\phi,R)-\frac{1}{2}\omega(\phi)\phi^{;\mu}\phi_{;\mu}-V(\phi)+\mathcal{L}_{\rm fluid}\right], \tag{1}$$

where g is the determinant of the background metric, R is the Ricci scalar, ω generalizes the kinetic term, and $\mathcal{L}_{\rm fluid}$ includes contributions from the matter and radiation cosmological components; $\kappa=8\pi G_*$ plays the role of the "bare" gravitational constant [34]. Here and throughout the paper Greek indices run from 0 to 3, Latin indices from 1 to 3.

The usual gravity term $R/16\pi G$ has been generalized by the general function $f/2\kappa$ [31,49]. Note that the formalism adopted is suitable for describing non-scalar-tensor gravity theories, i.e., without ϕ, where, however, the dependence on R is generic.

A. Einstein equations

By defining $F(\phi,R)=(1/\kappa)\partial f/\partial R$, the Einstein equations $G_{\mu\nu}=\kappa T_{\mu\nu}$ take the following form:

$$G_{\mu\nu}=T_{\mu\nu}=\frac{1}{F}\left[T^{\rm fluid}_{\mu\nu}+\omega\left(\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}g_{\mu\nu}\phi_{,\sigma}\phi^{,\sigma}\right)+g_{\mu\nu}\frac{f/\kappa-RF-2V}{2}+F_{,\mu;\nu}-g_{\mu\nu}F^{;\sigma}_{;\sigma}\right],$$

where again one can recognize a part depending on the fluid variables, and a part relative to the nonminimally coupled scalar field of the theory plus a contribution arising from the generalized gravity coupling represented by the function $1/F$; for practical purposes we will render this splitting explicit rewriting $T_{\mu\nu}$ as

$$T_{\mu\nu}=\frac{1}{F}T^{\rm fluid}_{\mu\nu}+\mathcal{T}^{\rm gc}_{\mu\nu}. \tag{2}$$

Note that in our scheme the generalized cosmology term is active also if gravity is the same as in general relativity, and a minimally coupled scalar field, like in the quintessence models, represents the only ingredient with respect to the ordinary case. This is true also in the limiting case where the scalar field reduces to a cosmological constant. Moreover, note that $\mathcal{T}^{\rm gc}_{\mu\nu}$ is not conserved if gravity differs from general relativity; nonetheless, contracted Bianchi identities still hold and ensure

$$(T^{\rm fluid}_{\mu\nu})^{;\mu}=0, \tag{3}$$

which allows us to derive the equations of motion for the matter variables only, leading to a remarkable simplification [49]. The expression for the stress-energy tensor relative to the scalar field, which is conserved in the generalized scenarios described by the action (1), must include the term accounting for the interaction with the gravitational field [37]. It is also worth noting how Eqs. (2) get simplified if the function f is a product of R times a function of the field only:

$$\frac{f}{\kappa}=RF. \tag{4}$$

In the following, we will refer to this class of cosmologies as nonminimally coupled (NMC) models.

We willingly keep this work as much general as possible, due to the large variety of scenarios covered by the action (1). We shall only consider the extended quintessence (EQ) [33,38] scenario as an example to illustrate the aspects of the weak lensing process with respect to ordinary cosmologies; in that case, the field ϕ, nonminimally coupled to gravity, also represents the dark energy, providing acceleration through its potential V. Specifically, the original works considered a NMC model defined as

$$F(\phi)=\frac{1}{\kappa}+\xi\phi^2, \tag{5}$$

and an inverse power law potential $V(\phi)=M^{4+\alpha}/\phi^\alpha$ providing cosmic acceleration today. The constraints from solar-system experiments force the correction to the gravitational constant to be small in this specific model. Therefore it is suitable to make approximations to illustrate a sort of first order variation of the weak lensing in generalized gravity theories with respect to the case of ordinary cosmology.

B. Background

We will write the unperturbed Friedmann-Robertson-Walker (FRW) metric in spherical coordinates as

$$ds^2=a^2(\tau)\left(-d\tau^2+\frac{1}{1-Kr^2}dr^2+r^2d\Omega\right), \tag{6}$$

where K is the uniform spatial curvature of a spherically symmetric three-space, $d\Omega$ is the metric of the two-sphere, and τ stands for the conformal time variable, related to cosmic time by the usual relation $dt=a(\tau)d\tau$.

The energy-momentum tensor (2) can be recast in a perfect-fluid form:

$$T_{\mu\nu}=(p+\rho)u_\mu u_\nu+pg_{\mu\nu}; \tag{7}$$

the corresponding background energy density and pressure are easily computed to be

$$\rho=\frac{1}{F}\left(\rho_{\rm fluid}+\frac{\omega}{2a^2}\phi'^2+\frac{RF-f/\kappa}{2}+V-\frac{3\mathcal{H}F'}{a^2}\right)$$
$$=\frac{1}{F}\rho_{\rm fluid}+\rho_{\rm gc}, \tag{8}$$

$$p=\frac{1}{F}\left(p_{\rm fluid}+\frac{\omega}{2a^2}\phi'^2-\frac{RF-f/\kappa}{2}-V+\frac{F''}{a^2}+\frac{\mathcal{H}F'}{a^2}\right)$$
$$=\frac{1}{F}p_{\rm fluid}+p_{\rm gc}; \tag{9}$$

the prime denotes differentiation with respect to conformal time and $\mathcal{H}$ is the conformal Hubble factor a'/a. As above, $\rho_{\rm gc}$ and $p_{\rm gc}$ do not obey the conservation law in ordinary cosmologies, $\rho'_{\rm gc}+3\mathcal{H}(\rho_{\rm gc}+p_{\rm gc})\neq 0$.

In FRW cosmologies the expansion equation reads

$$\mathcal{H}^2=a^2\rho-K, \tag{10}$$

and it cannot be solved directly due to the appearance of $\mathcal{H}$ in $\rho_{\rm gc}$, which is explicit in the last term but is also contained in $RF-f/\kappa$ through

$$R=\frac{6}{a^2}(\dot{\mathcal{H}}+\mathcal{H}^2). \tag{11}$$

Note that this is true also in theories where $f\equiv f(R)$ and no scalar field is present.

On the other hand, NMC scenarios admit a formal solution, which is

$$\mathcal{H}=-\frac{3}{2}\frac{F'}{F}+\sqrt{\frac{9}{4}\left(\frac{F'}{F}\right)^2+\frac{1}{F}\left(a^2\rho_{\rm fluid}+\frac{\omega}{2}\phi'^2+a^2V\right)-K}, \tag{12}$$

where we have selected the expansion solution with positive $\mathcal{H}$. Note that the dependence of the comoving distances r on the redshift $z=1/a-1$ also gets modified, according to (12):

$$r=\int_0^z\frac{dz}{H(z)}. \tag{13}$$

We generically indicate the single components in the fluid with x. Since $T^{\mu\nu}_{\rm fluid}$ is conserved, energy density, pressure, equation of state, and sound velocity, defined as

$$\rho_x=-T^0_{0x},\quad p_x=1/3T^i_{ix},\quad w_x=p_x/\rho_x,\quad c_s^2=p'_x/\rho'_x, \tag{14}$$

give rise to conservation equations having the familiar form

$$\rho_x' + 3\mathcal{H}\rho_x(1+w_x) = 0. \tag{15}$$

The last ingredient is the Klein-Gordon equation for the evolution of the field, which is substantially different from the case of ordinary cosmologies:

$$\phi'' + 2\mathcal{H}\phi' + \frac{1}{2\omega}\left(\omega_{,\phi}\phi'^2 - a^2\frac{f_{,\phi}}{\kappa} + 2a^2V_{,\phi}\right) = 0. \tag{16}$$

As we stress in the next section, the relevant changes with respect to the standard picture are represented by the change in time of the function f. In EQ scenarios the dynamics of the field possesses two distinct regimes. At low redshift, the behavior of the energy density coincides with the corresponding one in the tracking trajectories in ordinary quintessence models, linked to the potential exponent as $w_\phi = -2/(2+\alpha)$. At high redshift, generally much earlier than the epoch of structure formation, eventually the effective potential coming from the nonminimal interaction with gravity takes over (R boost), and imprints a behavior $w_\phi = -1/3$ for the quadratic coupling (5) [33,40].

C. Linear cosmological perturbations

We will describe the linear cosmological perturbations in the real as well as in the Fourier space. For this reason, we follow a notation close to that introduced recently by Liddle and Lyth [70], which allows one not to make explicit the Laplace operator eigenfunctions when working in the Fourier space, minimizing the formal changes needed to go from the real to the Fourier space and vice versa. See Ref. [71] for the usual formulation cast in the Fourier space. The general expression for the linear perturbation to the metric (6) can be written as [70]

$$ds^2 = a^2(\tau)\{-(1+2A)d\tau^2 - B_i d\tau dx^i + [(1+2D)\delta_{ij} + 2E_{ij}]dx^i dx^j\}, \tag{17}$$

where the function E_{ij} is chosen to be traceless in order to uniquely identify the nondiagonal spatial perturbation.

It is usual to further decompose the above quantities B_i and E_{ij} into pure scalars (S), scalar-type (S), and vector-type (V) components of vectors, and scalar-type (S), vector-type (V), and tensor-type (T) components of tensors, according to their behavior with respect to a spatial coordinate transformation:

$$A = A^S, \tag{18}$$

$$B_i = B_i^S + B_i^V, \tag{19}$$

$$E_{ij} = E_{ij}^S + E_{ij}^V + E_{ij}^T, \tag{20}$$

where

$$B_i^S = \nabla_i B, \quad E_{ij}^S = \left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E, \tag{21}$$

$$\nabla^i B_i^V = 0, \quad E_{ij}^V = \frac{1}{2}(\nabla_i E_j + \nabla_j E_i), \quad E_i^{iV} = 0, \tag{22}$$

$$E_i^{jT} = 0, \quad \nabla^i E_{ij}^T = 0, \tag{23}$$

and E and B are scalar functions. Our notation slightly differs from that of Liddle and Lyth which is given in the Fourier space: the quantities E, E_i and B we use here correspond to the original ones divided by $k^2 = k_i k^i$ and k, respectively. In the linear theory the different types of perturbations evolve independently of each other and can thus be treated separately. An analogous decomposition can be performed for the stress-energy tensor, whose expression up to the first perturbative order is

$$\tilde{T}^0_0 = -(\rho + \delta\rho), \tag{24}$$

$$\tilde{T}^0_i = (\rho + p)(v_i - B_i), \tag{25}$$

$$\tilde{T}^i_0 = -(\rho + p)v^i, \tag{26}$$

$$\tilde{T}^i_j = (p + \delta p)\delta^i_j + p\Pi^i_j, \tag{27}$$

where the fluid velocity v_i and the anisotropic stress Π_{ij} can be split as

$$v_i = v_i^S + v_i^V, \tag{28}$$

$$\Pi_{ij} = \Pi_{ij}^S + \Pi_{ij}^V + \Pi_{ij}^T, \tag{29}$$

with the same properties of their metric counterparts:

$$v_i^S = \nabla_i v, \quad \Pi_{ij}^S = \left(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\Pi, \tag{30}$$

$$\nabla^i v_i^V = 0, \quad \Pi_{ij}^V = \frac{1}{2}(\nabla_i\Pi_j + \nabla_j\Pi_i), \quad \Pi_i^{iV} = 0, \tag{31}$$

$$\nabla^i\Pi_i = \nabla^i\Pi_{ij}^T = 0, \tag{32}$$

where again Π is a scalar quantity and the same differences of our notation with that by Liddle and Lyth hold here for the stress-energy tensor perturbations.

In this paper we will take into account only density (i.e., scalar-type) perturbations. The reason is that also in the generalized scenarios we consider here they play the dominant role. In fact, as we stress in detail in the following, the scalar-tensor coupling does generate a non-null anisotropic stress already at the linear level, but that is of scalar-type only, and therefore does not act as a source for gravitational waves.

We work in the so called conformal Newtonian gauge, where the nondiagonal perturbations to the metric are set to zero:

$$B = E = 0. \tag{33}$$

Furthermore, we will rename the lapse function A and the spatial diagonal perturbation D after the widely used gauge-invariant potentials [72]:

$$A \to \Psi, \quad D \to \Phi. \tag{34}$$

The line element used through the rest of the paper will therefore be

$$ds^2 = a^2[-(1+2\Psi)d\tau^2 + (1+2\Phi)dl^2], \tag{35}$$

where dl^2 is the unperturbed spatial length element from (6).

We now write down the main equations driving the evolution of the perturbed quantities defined above. For each fluid component, the evolution of the scalar perturbed quantities can be followed through the dynamical variables $\delta_x = \delta\rho_x/\rho_x$, v_x, δp_x, Π_x, defined in terms of the stress energy tensor as

$$\delta\rho_x = -\delta T^0_{0x}, \quad \delta p_x = 1/3\delta T^i_{ix},$$
$$\nabla^i v_x = -\delta T^i_{0x}/(\rho_x + p_x), \quad p_x \nabla_i \nabla^{j\neq i}\Pi_x = \delta T^{j\neq i}_i. \tag{36}$$

Note that from now on we do drop the subscript S, meaning that we always treat scalar cosmological perturbations, unless otherwise specified. In the Fourier space, the equations for δ_x and v_x take the form

$$\delta_x' - 3\mathcal{H}w_x\delta_x = k^2(1+w_x)v_x - 3(1+w_x)\Phi' - 3\mathcal{H}w_x p_x \delta p_x, \tag{37}$$

$$v_x' + \mathcal{H}(1-3w_x)v_x + \frac{w_x'}{1+w_x}v_x = -\frac{\delta p_x}{\rho_x(1+w_x)} - \Psi + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\frac{w_x}{1+w_x}\Pi_x, \tag{38}$$

while δp_x and Π_x depend on the particular species considered. The perturbed Klein-Gordon equation can be written in terms of two equations formally equivalent to (37) and (38) by building the conserved expression for the perturbed energy density, pressure, and anisotropic stress perturbations [37]. Their combination leads to the Klein-Gordon equation at first perturbative order:

$$\delta\phi'' + \left(2\mathcal{H} + \frac{\omega_{,\phi}}{\omega}\phi'\right)\delta\phi' + \left[k^2 + \frac{1}{2}\left(\frac{\omega_{,\phi}}{\omega}\right)_{,\phi}\phi'^2 + \left(\frac{-a^2 f_{,\phi}/\kappa + 2a^2 V_{,\phi}}{2\omega}\right)_{,\phi}\right]\delta\phi = (\Psi' - 3\Phi')\phi' + \left(2\phi'' + 4\mathcal{H}\phi' + \frac{\omega_{,\phi}}{\omega}\phi'^2\right)\Psi + \frac{1}{2\omega\kappa}\frac{\partial^2 f}{\partial\phi\partial R}\delta R. \tag{39}$$

III. GENERALIZED POISSON EQUATIONS

In this section we work out the equations relating the stress-energy tensor perturbations to the scalar metric gravitational potentials. The latter, together with the background cosmic geometry and expansion, entirely determine the lensing process.

We start writing the generalized expression of the density fluctuation [49]:

$$\delta\rho = \rho\cdot\delta = -\delta T^0_0 = \frac{1}{F}\left[\delta\rho_{\rm fluid} + \frac{\omega}{a^2}\phi'\delta\phi' + \frac{1}{2}\left(\frac{\omega_{,\phi}}{a^2}\phi'^2 - \frac{f_{,\phi}}{\kappa} + 2V_{,\phi}\right)\delta\phi - \left(\frac{\rho+3p}{2} - \frac{1}{a^2}\nabla^2\right)\delta F - 3\frac{\mathcal{H}\delta F'}{a^2} + 6\frac{\mathcal{H}\Psi F'}{a^2} - 3\frac{\Phi' F'}{a^2} - \frac{\omega}{a^2}\Psi\phi'^2\right]. \tag{40}$$

We can focus on two main aspects of the generalized expression above, playing the major role of the generalization of the Poisson equation: the contribution from the field fluctuations $\delta\phi$ and the $1/F$ term in front of the expression for $\delta\rho$, which acts as an effective time varying gravitational constant. As we shall see below, the latter is the relevant effect in typical extended quintessence models.

The Poisson equation relates the fluctuations in the time-time component of the metric to the usual combination of density and scalar-type velocity perturbations, named Δ, whose expression in Newtonian gauge is [71]

$$\Delta = \delta - 3\mathcal{H}wv. \tag{41}$$

We follow as much as possible the notation of earlier works [69]. The $\delta G^0_0 = \delta T^0_0$ equation can be cast in such a way that it formally coincides with the case of ordinary cosmologies. In Fourier space it is

$$\frac{2}{a^2}(k^2 - 3K)\Phi = 3\Delta\left(H^2 + \frac{K}{a^2}\right), \tag{42}$$

so that we can exploit our distinction between fluid and generalized cosmology terms. Note that the Hubble expansion rate is evaluated with respect to the ordinary time, $H = \dot{a}/a = a'/a^2 = \mathcal{H}/a$. By using Eqs. (8) and (10), we can write $H^2 = H^2_{\rm fluid} + H^2_{\rm gc}$, where

$$H^2_{\rm fluid} = \frac{1}{F}\frac{\rho_{\rm fluid}}{3} = \frac{F_0}{F}H_0^2[\Omega_{0m}(1+z)^3 + \Omega_{0r}(1+z)^4 + (1-\Omega_0)(1+z)^2], \tag{43}$$

and Ω_{0m}, Ω_{0r}, and Ω_0 are the contribution to the present expansion rate from the matter, radiation, and the total density, respectively, while F_0 is the actual value of the gravitational coupling strength, and can be replaced with $1/8\pi G$. It is important to note that $H^2_{\rm fluid}$ is linked to the energy density of the fluid components, all but the scalar field, but it contains a most important generalization, represented by the F_0/F term, which plays the role of a time dependent gravitational constant into the Friedmann equation. Moreover, using the relation $K/a^2 = H_0^2(\Omega_0 - 1)(1+z)^2$, we included the

effect of the spatial curvature into $H_{\rm fluid}$. The expressions for $H_{\rm gc}^2$ and $\Omega_{\rm gc}$ can be easily obtained by making use of Eqs. (8) and (10):

$$H_{\rm gc}^2=\rho_{\rm gc}, \quad \Omega_{\rm gc}=\frac{H_{\rm gc}^2}{H_0^2}. \tag{44}$$

Starting from (42), let us write down the relation between the power spectra of Φ, $P_\Phi=k^3\Phi^2/2\pi^2$, and that of Δ:

$$P_\Phi=\frac{9}{4}\left(\frac{H_0}{k}\right)^4\left(1-\frac{3(1-\Omega_0)H_0^2}{k^2}\right)^{-2}\left[\frac{F_0}{F}\Omega_{0m}(1+z)+\frac{F_0}{F}\Omega_{0r}(1+z)^2+(1-\Omega_0)\left(\frac{F_0}{F}-1\right)+\frac{1}{(1+z)^2}\Omega_{\rm gc}\right]^2 P_\Delta. \tag{45}$$

This is the equation that generalizes the link between time-time metric fluctuations with density and scalar-type velocity perturbations. The new effects arise from the scalar field contribution, encoded in $H_{\rm gc}^2$, the $1/F$ term behaving as a time-dependent gravitational constant, as well as from the fluctuations of the scalar field, contained in Δ, both in $\delta\rho$ and v.

Let us check the most relevant corrections to the quantities above in the EQ models we take as a reference [33,38], where the scalar field fluctuations $\delta\phi$ play a minor role [37], and the overall geometry is assumed to be flat, $K=0$. Under these conditions, the most important correction is represented by the $1/F$ term, effectively representing the time variation of the gravitational constant. The expression for F in (5) can be conveniently rewritten as

$$F=\frac{1}{8\pi G}+\xi(\phi^2-\phi_0^2), \tag{46}$$

to make explicit that at the present $F=1/8\pi G$. The observational constraints [73,74] usually are expressed as bounds on the quantities

$$\frac{1}{G}\frac{dG}{dt}\simeq-\frac{1}{F}\frac{dF}{dt}=\frac{2}{\phi}\frac{d\phi}{dt}, \quad \omega_{JBD}=\frac{F}{F_\phi^2}=\frac{F}{4\xi^2\phi^2}, \tag{47}$$

calculated at the present time, where the $\simeq$ sign above is due to the slight difference with the gravitational constant measured in Cavendish-like experiments [34] and ω_{JBD} is the usual Jordan-Brans-Dicke parameter, which usually implies the strongest constraint. Typically [31,33] the correction to the $1/8\pi G$ term is small, so that

$$\frac{1}{F}\simeq 8\pi G[1-8\pi G\xi(\phi^2-\phi_0^2)]. \tag{48}$$

Moreover, in tracking trajectories with inverse power law potentials, the field approaches ϕ_0 of the order of $1/\sqrt{G}$ from below, being generally much smaller than that in the past, when $1/F$ freezes to the value $8\pi G(1+8\pi G\xi\phi_0^2)$. The magnitude of the correction is therefore

$$8\pi G\xi\phi_0^2=\frac{1}{4\xi\omega_{JBD}}=\phi_0\sqrt{\frac{2\pi G}{\omega_{JBD}}}; \tag{49}$$

note that as ω_{JBD} approaches infinity, recovering general relativity, the correction may still be relevant depending on the values of ξ or ϕ_0. The gravitational potential receives a contribution

$$\delta\Phi=-8\pi G\xi(\phi^2-\phi_0^2)\Phi_{{\rm fluid}+\phi}. \tag{50}$$

The subscript fluid$+\phi$ represents all the terms coming from the fluid quantities as well as the scalar field ones from the minimal coupling, i.e., not involving F explicitly:

$$k^2\Phi_{{\rm fluid}+\phi}=4\pi G\Delta_{{\rm fluid}+\phi}H_0^2[\Omega_{0m}(1+z)^2+\Omega_{0r}(1+z)^4+\Omega_\phi]. \tag{51}$$

Note also that if the trajectory is tracking, with an almost constant equation of state w_ϕ, the expression above reduces to

$$k^2\Phi_{{\rm fluid}+\phi}=4\pi G\Delta_{{\rm fluid}+\phi}H_0^2[\Omega_{0m}(1+z)^2+\Omega_{0r}(1+z)^4+\Omega_{\phi 0}(1+z)^{3(1+w_\phi)}]; \tag{52}$$

in order to keep the notation simple, we drop such subscripts in the following, always meaning that it is there when discussing approximate expressions in EQ models. Similarly, the gravitational potential power spectrum gets an extra contribution, which at first order in the correction to $1/8\pi G$ is

$$\delta P_\Phi=-16\pi G\xi(\phi^2-\phi_0^2)P_\Phi, \tag{53}$$

where

$$P_\Phi=\frac{9}{4}\left(\frac{H_0}{k}\right)^4[\Omega_{0m}(1+z)+\Omega_{0r}(1+z)^2+\Omega_{\phi 0}(1+z)^{1+3w_\phi}]P_\Delta. \tag{54}$$

In general, another important effect that arises in cosmology from the generalization of the underlying gravity physics is represented by the relation between Ψ and Φ. The difference between f and R, which may arise from a scalar-tensor coupling as well as a nonstandard dependence of f from R itself, gives origin to tidal forces exciting the anisotropic stress of scalar origin [49]; in the Fourier space its simple form is

$$p_\phi\Pi_\phi=\frac{k^2}{a^2}\frac{\delta F}{F}, \tag{55}$$

and implies a shift between the two gauge-independent scalar metric perturbations, which in our gauge takes the form

$$\Psi+\Phi=-\frac{a^2}{k^2}p\Pi=-\frac{\delta F}{F}, \tag{56}$$

where the last equality holds if the anisotropic stress is due to the generalization of gravity and it does not come from matter or radiation. This is an important aspect of generalized cosmologies, which implies a change in almost all the equations describing the weak lensing effect in cosmology, to be discussed next. For this reason, it is convenient to give a name to the $\Psi+\Phi$ combination, valid both for the real and the Fourier space:

$$\Xi=\Psi+\Phi. \tag{57}$$

One has $\Xi=0$ in ordinary cosmology, and $\Xi=-\delta F/F$ in the generalized scenarios of interest here. As we already stressed, Ξ is excited both by a scalar-tensor coupling as in EQ models, and a generalized dependence of the gravitational Lagrangian term on R; its expression in terms of f is

$$\Xi=-\left(\frac{\partial f}{\partial R}\right)^{-1}\left[\frac{\partial^2 f}{\partial\phi\partial R}\delta\phi+\frac{\partial^2 f}{\partial R^2}\delta R\right]. \tag{58}$$

While in the first case the correction is small because of the smallness of the scalar field fluctuations $\delta\phi$ [31,37], the contribution from the second term has not been investigated yet.

IV. LENSING EQUATION

Before working out the generalized expression for the weak lensing power spectrum it is necessary to reconsider the lensing equation to track the effects coming from background dynamics and perturbations. We follow the approach used in Ref. [75], deriving the photon trajectories as solutions with $ds^2=0$ of the geodesic equation for the metric (17); the lensing deflection around a given direction in the sky is described, introducing angular coordinates θ_x and θ_y, defined as

$$\theta_x=\theta\cos\varphi;\quad \theta_y=\theta\sin\varphi, \tag{59}$$

where $\theta=\sqrt{\theta_x^2+\theta_y^2}$ and φ are the polar coordinates in the (θ_x,θ_y) plane.

Furthermore, we perform a change of the radial coordinate,

$$\chi(r)=\frac{1}{\sqrt{K}}\arcsin\sqrt{K}r \tag{60}$$

such that the spatial background metric takes the more readable form:

$$dl^2=d\chi^2+\frac{\sin^2\sqrt{K}\chi}{K}(d\theta_x^2+d\theta_y^2). \tag{61}$$

This notation is convenient because the weak lensing hypothesis immediately reflects in the condition

$$\theta\ll 1, \tag{62}$$

which allows us to write the geodesic equation at first order in the deflection angle, in addition to the usual linear approximation for metric perturbations. The geodesic equation is indeed

$$\frac{d^2r^\alpha}{d\lambda^2}=-g^{\alpha\beta}\left(g_{\beta\nu,\mu}-\frac{1}{2}g_{\mu\nu,\beta}\right)\frac{dr^\mu}{d\lambda}\frac{dr^\nu}{d\lambda}, \tag{63}$$

which for $\alpha=0$ and $\alpha=1$ gives $d\tau=1/a^2d\lambda$ and $dr=1/a^2d\lambda$; substituting these expressions into the perturbed equation for the angular part, we get

$$\frac{d^2\theta_x}{d\tau^2}=-\frac{K}{\sin^2\sqrt{K}\chi}\partial_{\theta_x}(\Psi-\Phi)-2\sqrt{K}\frac{\cos\sqrt{K}\chi}{\sin\sqrt{K}\chi}\frac{d\theta_x}{d\tau} \tag{64}$$

for the angular coordinate θ_x and another one for θ_y, formally equivalent with $x\to y$.

In terms of Ξ, defined in (57), the geodesic equation assumes the familiar form [50,75] plus the perturbation coming from the anisotropic stress (i stands now for x or y):

$$\frac{d^2\theta_i}{d\tau^2}=\frac{K}{\sin^2\sqrt{K}\chi}\partial_{\theta_i}(2\Phi-\Xi)-2\sqrt{K}\frac{\cos\sqrt{K}\chi}{\sin\sqrt{K}\chi}\frac{d\theta_i}{d\tau}. \tag{65}$$

The effects coming from the modified cosmological expansion are encoded in χ, through the modified dependence of the distances r given by (12) and (13) with respect to the redshift z.

In terms of the comoving displacement from the polar axis, $x_i=\theta_i\sin\sqrt{K}\chi/\sqrt{K}$, Eq. (65) simply reads

$$x_i''+Kx_i=-\frac{\partial(\Psi-\Phi)}{\partial x_i}=2\partial_i\Phi-\partial_i\Xi, \tag{66}$$

where the first term on the left-hand side describes the tendency of two nearby rays to converge, diverge, or remain parallel according to the geometry of the Universe, while the right-hand side accounts for the lensing effect due to the metric perturbations. The general solution of this equation is

$$\begin{aligned}x_i+A\frac{\sin\sqrt{K}\chi}{\sqrt{K}}+B\cos\sqrt{K}\chi&\\ =-\int_0^\chi d\chi'\,\partial_i[\Psi(\hat{n},\chi')-\Phi(\hat{n},\chi')]\frac{\sin\sqrt{K}\chi'}{\sqrt{K}}&\\ =\int_0^\chi d\chi'\,\partial_i[2\Phi(\hat{n},\chi')-\Xi(\hat{n},\chi')]\frac{\sin\sqrt{K}\chi'}{\sqrt{K}},&\end{aligned} \tag{67}$$

where A and B are integration constants, and the position $\vec{x}$ on the light cone is completely specified by the line of sight direction $\hat{n}$ and the generalized radial coordinate χ.

V. WEAK LENSING

It is convenient to begin with the comoving separation between two lensed rays, starting from the same point, one in the direction of the polar axis and the other one in a direction $\hat{n}$, on a source plane at distance χ_s; in terms of the angular separation θ_i in the direction i one has

$$x_j(\chi_s)=(\delta_{ij}-\psi_{ij})\frac{\sin\sqrt{K}\chi_s}{\sqrt{K}}\theta_i, \tag{68}$$

where ψ_{ij} is the *distortion tensor*:

$$\begin{aligned}\psi_{ij}(\hat{n},\chi)&=\frac{1}{\sqrt{K}}\int_0^{\chi}d\chi'\frac{\sin\sqrt{K}\chi'\sin\sqrt{K}(\chi-\chi')}{\sin\sqrt{K}\chi}\\&\quad\times\partial_i\partial_j[\Psi(\hat{n},\chi')-\Phi(\hat{n},\chi')]\\&=\frac{1}{\sqrt{K}}\int_0^{\chi}d\chi'\frac{\sin\sqrt{K}\chi'\sin\sqrt{K}(\chi-\chi')}{\sin\sqrt{K}\chi}\\&\quad\times\partial_i\partial_j[-2\Phi(\hat{n},\chi')+\Xi(\hat{n},\chi')].\end{aligned} \tag{69}$$

Note that at first order the effect of cosmological perturbations can be computed along the unperturbed trajectories, which corresponds to neglect the difference in the relative deviation of two lensed and unlensed rays inside the above integral.

The components of ψ_{ij} are usually interpreted in terms of the *shear* $\gamma=\gamma_1+i\gamma_2$ and of the *effective convergence* κ, respectively identified as

$$\gamma_1=\frac{1}{2}(\psi_{11}-\psi_{22}),\quad \gamma_2=\psi_{12},$$

$$\kappa=\frac{1}{2}(\psi_{11}+\psi_{22}). \tag{70}$$

Moreover, as we usually deal with lensing phenomena from a multiplicity of sources, the distortion tensor, and thus the projected potential, are usually meant to be integrated over the possible source distances:

$$\psi_{ij}(\hat{n})=\int d\chi g(\chi)\psi_{ij}(\hat{n},\chi), \tag{71}$$

where $g(\chi)$ is a normalized function describing the distribution of the relevant sources. When one considers the effect of lensing on the CMB, the source distribution may be replaced by a δ function at the last scattering surface.

Let us evaluate the correction to the distortion tensor in EQ models. The contribution from Ξ is negligible in these scenarios, since it arises from the scalar field fluctuations $\delta\phi$, yielding a correction that is small with respect to the one coming from the variation of $1/F$ [37].

Therefore, in flat cosmologies and at a given χ, the correction to $\psi_{ij}(\hat{n},\chi)$ is due only to the shift in the gravitational potential, represented by (50):

$$\begin{aligned}\delta\psi_{ij}(\hat{n},\chi)&=16\pi G\xi\phi_0^2\int_0^{\chi}d\chi'\frac{\chi'(\chi-\chi')}{\chi}\\&\quad\times\left(\frac{\phi^2}{\phi_0^2}-1\right)\partial_i\partial_j\Phi(\hat{n},\chi').\end{aligned} \tag{72}$$

Note, however, that integrating in the χ variable, although convenient in order to minimize the formal corrections to ψ_{ij}, hides the effect of the varying gravitational constant on χ itself; in flat cosmologies the latter coincides with r given by (13), which has to be corrected as

$$\delta r=4\pi G\xi\phi_0^2\int_0^z\frac{dz}{H(z)}\left(\frac{\phi^2}{\phi_0^2}-1\right), \tag{73}$$

as can be easily verified since $H\propto 1/\sqrt{F}$.

A. Generalized lensing potential

The lensing equation is often rewritten in terms of the *projected potential* ϕ, defined through the relation

$$\psi_{ij}=\frac{K}{(\sin\sqrt{K}\chi)^2}\partial_i\partial_j\phi. \tag{74}$$

Indicating the radial coordinate distance with $D(\chi)=\sin\sqrt{K}\chi/\sqrt{K}$, we get

$$\begin{aligned}\phi(\hat{n})&=\int_0^{\chi_\infty}d\chi D(\chi)[\Psi(\hat{n},\chi)-\Phi(\hat{n},\chi)]\\&\quad\times\int_{\chi}^{\chi_\infty}d\chi'\frac{D(\chi'-\chi)}{D(\chi')}g(\chi')\\&=\int_0^{\chi_\infty}d\chi D(\chi)[-2\Phi(\hat{n},\chi)+\Xi(\hat{n},\chi)]\\&\quad\times\int_{\chi}^{\chi_\infty}d\chi'\frac{D(\chi'-\chi)}{D(\chi')}g(\chi'),\end{aligned} \tag{75}$$

where χ_∞ stands for the comoving distance at infinite redshift. By defining the integral involving the source distribution as

$$g'(\chi)=D(\chi)\int_{\chi}^{\chi_\infty}d\chi'\frac{D(\chi'-\chi)}{D(\chi')}g(\chi'), \tag{76}$$

the lensing potential (75) takes the compact form

$$\begin{aligned}\phi(\hat{n})&=\int_0^{\chi_\infty}d\chi g'(\chi)[\Psi(\hat{n},\chi)-\Phi(\hat{n},\chi)]\\&=\int_0^{\chi_\infty}d\chi g'(\chi)[-2\Phi(\hat{n},\chi)+\Xi(\hat{n},\chi)].\end{aligned} \tag{77}$$

The expression above acquires several new contributions in the generalized scenarios of interest here. The modified

background expansion affects the angular diameter distance as well as the effective gravitational constant; the perturbations get contributions from the field fluctuations affecting Φ and exciting the metric fluctuation mode represented by Ξ. In EQ models, if the integration is made on the variable χ, the main correction is due to the time variation of the effective gravitational constant:

$$\delta\phi(\hat{n}) = 16\pi G\xi\phi_0^2 \int_0^{\chi_\infty} d\chi g'(\chi)\left(\frac{\phi^2}{\phi_0^2}-1\right)\Phi(\hat{n},\chi). \tag{78}$$

In this equation, and in Eqs. (86), (91), and (94), the quantity Φ is meant to be $\Phi_{\text{fluid}+\phi}$ as defined in Eq. (50).

We need now to track these effects into the angular power spectrum of the projected lensing potential. This is defined as usual as

$$C_l^{\phi\phi} = \langle|\phi_{lm}|^2\rangle, \quad \phi_{lm} = -2\int d\Omega_{\hat{n}}\phi(\hat{n})Y_{lm}(\hat{n}), \tag{79}$$

where the -2 is purely conventional in order to keep the notation consistent with earlier works [69]. One needs now to expand the metric fluctuations in the Fourier space with respect to the position $\vec{x} = r\cdot\hat{n}$. The expansion functions are just the eigenfunctions $Y_{\vec{k}}(\vec{x})$ of the Laplace operator in curved spacetime, defined in general in curved FRW geometry [71]. Their radial and angular dependences are further expanded in ultraspherical Bessel functions u_l and scalar spherical harmonics, by exploiting the relation

$$Y_{\vec{k}}(\vec{x}) = 4\pi\sum_{l,m} i^l u_l(kx)Y_{lm}(\hat{k})Y_{lm}(\hat{x}), \tag{80}$$

where k and x denote the modulus of the corresponding vectors. By using the completeness of the spherical harmonics, and the fact that x coincides with the radial distance $D(\chi)$, the final expression for ϕ_{lm} is

$$\phi_{lm} = \sqrt{\frac{8}{\pi}}\int_0^{\chi_\infty} d\chi g'(\chi)\int d^3k[2\Phi(\vec{k},\chi)-\Xi(\vec{k},\chi)] \times i^l u_l(kD(\chi))Y_{lm}(\hat{k})Y_{lm}(\hat{n}). \tag{81}$$

The lensing potential angular power spectrum (79) is therefore

$$C_l^{\phi\phi} = \frac{32}{\pi}\int_0^{\chi_\infty} d\chi g'(\chi)\int_0^{\chi_\infty} d\chi' g'(\chi')\int d^3k\int d^3k' u_l \times[kD(\chi)]u_l[k'D(\chi')]Y_{lm}(\hat{k})Y_{lm}^*(\hat{k}') \times\left[\left\langle \Phi(\vec{k},\chi)\Phi(\vec{k}',\chi')^* - \frac{1}{2}\Xi(\vec{k},\chi)\Phi(\vec{k}',\chi')^* - \frac{1}{2}\Xi(\vec{k}',\chi')^*\Phi(\vec{k},\chi) + \frac{1}{4}\Xi(\vec{k},\chi)\Xi(\vec{k}',\chi')^*\right\rangle\right]. \tag{82}$$

Assuming that the statistical average above eliminates the correlation between different Fourier modes, as well as the dependence on the direction of the wave numbers,

$$\langle\mathcal{A}(\vec{k},\chi)\mathcal{B}(\vec{k}',\chi')\rangle = \langle\mathcal{A}(k,\chi)\mathcal{B}(k,\chi')\rangle\delta(\vec{k}-\vec{k}'), \tag{83}$$

where $\mathcal{A}$ and $\mathcal{B}$ represent either Ψ or Ξ and they are meant to be ensemble averaged, one finally gets

$$C_l^{\phi\phi} = \frac{32}{\pi}\int_0^{\chi_\infty} d\chi g'(\chi)\int_0^{\chi_\infty} d\chi' g'(\chi')\int k^2 dk u_l[kD(\chi)]u_l \times[kD(\chi')]\left[\langle\Phi(k,\chi)\Phi(k,\chi')\rangle - \langle\Xi(k,\chi)\Phi(k,\chi')\rangle + \frac{1}{4}\langle\Xi(k,\chi)\Xi(k,\chi')\rangle\right]. \tag{84}$$

It is also useful to write down explicitly the equivalent form of the expression above which contains the gravitational potentials only:

$$C_l^{\phi\phi} = \frac{32}{\pi}\int_0^{\chi_\infty} d\chi g'(\chi)\int_0^{\chi_\infty} d\chi' g'(\chi')\int k^2 dk u_l \times[kD(\chi)]u_l[kD(\chi')]\left[\frac{1}{4}\langle\Psi(k,\chi)\Psi(k,\chi')\rangle + \frac{1}{4}\langle\Phi(k,\chi)\Phi(k,\chi')\rangle - \frac{1}{2}\langle\Psi(k,\chi)\Phi(k,\chi')\rangle\right], \tag{85}$$

putting in evidence the correlation between Ψ and Φ.

From this expression we can easily infer the main correction to the lensing potential angular power spectrum arising in EQ cosmologies, using (53):

$$\delta C_l^{\phi\phi} = -512 G\xi\phi_0^2\int_0^{\chi_\infty} d\chi g'(\chi)\int_0^{\chi_\infty} d\chi' g'(\chi') \times\left(\frac{\phi^2}{\phi_0^2}-1\right)\int k^2 dk u_l[kD(\chi)]u_l[kD(\chi')] \times\langle\Phi(k,\chi)\Phi(k,\chi')\rangle. \tag{86}$$

Note that the numbers here conspire to yield a quite large factor in front of this expression, which may render the correction above relevant even for values of the product $G\xi\phi_0^2$ as small as 10^{-3}. In the following we further specialize our results computing the generalized expression of some quantity particularly relevant for observations, as well as their main corrections in EQ models.

B. Convergence power spectrum

The convergence, represented by the trace of the distortion tensor, is usually used as a main magnitude of the weak lensing distortion. Indeed, in the weak lensing hypotheses, it coincides with the shear power spectrum (see Ref. [50]) which has been recently observed from the distortion induced in the shape of background galaxies in the optical band [51–55].

The expression we need to compute is given by

$$\kappa_{\rm eff}=\frac{1}{2}(\psi_{11}+\psi_{22}), \tag{87}$$

which shall be averaged over the source distribution as usual. We get

$$\begin{aligned}\bar{\kappa}_{\rm eff}&=\frac{1}{2}\int_0^{\chi_\infty}d\chi' g(\chi')\int_\chi^{\chi_\infty}\frac{D(\chi)D(\chi'-\chi)}{D(\chi')}\\&\quad\times\partial^i\partial_i[\Psi(\hat{n},\chi)-\Phi(\hat{n},\chi)]\\&=\frac{1}{2}\int_0^{\chi_\infty}d\chi' g(\chi')\int_\chi^{\chi_\infty}\frac{D(\chi)D(\chi'-\chi)}{D(\chi')}\\&\quad\times\partial^i\partial_i[-2\Phi(\hat{n},\chi)+\Xi(\hat{n},\chi)]\\&=\frac{1}{2}\int_0^{\chi_\infty}d\chi g'(\chi)\partial^i\partial_i[-2\Phi(\hat{n},\chi)+\Xi(\hat{n},\chi)].\end{aligned} \tag{88}$$

The two-dimensional Laplacian appearing in this equation can be safely replaced with its three-dimensional analogue (see Refs. [50] and [76] for a numerical check of this point). Once this substitution has been made, we can expand the generalized gravitational potential in Fourier harmonics transforming with respect to the spatial point $\hat{n}\cdot\chi$, and transform the Laplacian in a multiplication by $(-k^2)$:

$$\begin{aligned}\bar{\kappa}_{\rm eff}&=\frac{1}{2(2\pi)^{3/2}}\int_0^{\chi_\infty}d\chi g'(\chi)\\&\quad\times\int d^3k k^2[2\Phi(\vec{k},\chi)-\Xi(\vec{k},\chi)]Y_{\vec{k}}(\vec{x}).\end{aligned} \tag{89}$$

Comparing this expression with those for the lensing potential power spectrum (84) and (85) we can immediately infer the result

$$\begin{aligned}P_{\bar{\kappa}}(l)&=\frac{8}{\pi}\int_0^{\chi_\infty}d\chi g'(\chi)\int_0^{\chi_\infty}d\chi' g'(\chi')\int dk k^6 u_l\\&\quad\times(kD(\chi))u_l(kD(\chi'))\Big[\langle\Phi(k,\chi)\Phi(k,\chi')\rangle\\&\quad-\langle\Xi(k,\chi)\Phi(k,\chi')\rangle+\frac{1}{4}\langle\Xi(k,\chi)\Xi(k,\chi')\rangle\Big]\\&=\frac{8}{\pi}\int_0^{\chi_\infty}d\chi g'(\chi)\int_0^{\chi_\infty}d\chi' g'(\chi')\int dk k^8 u_l\\&\quad\times(kD(\chi))u_l(kD(\chi'))\Big[\frac{1}{4}\langle\Psi(k,\chi)\Psi(k,\chi')\rangle\\&\quad+\frac{1}{4}\langle\Phi(k,\chi)\Phi(k,\chi')\rangle-\frac{1}{2}\langle\Psi(k,\chi)\Phi(k,\chi')\rangle\Big].\end{aligned} \tag{90}$$

In EQ cosmologies, using again (53), one finds

$$\begin{aligned}\delta P_{\bar{\kappa}}(l)&=-128G\xi\phi_0^2\int_0^{\chi_\infty}d\chi g'(\chi)\int_0^{\chi_\infty}d\chi' g'(\chi')\\&\quad\times\left(\frac{\phi^2}{\phi_0^2}-1\right)\int dk k^6 u_l(kD(\chi))u_l(kD(\chi'))\\&\quad\times\langle\Phi(k,\chi)\Phi(k,\chi')\rangle,\end{aligned} \tag{91}$$

where the correction to χ must be taken into account following (73) if the integration is made on the redshift.

C. Lensing in the CMB signal

The lensing potential correlates significantly with secondary anisotropies of the CMB, because it arises at the epoch of structure formation; here we generalize the lensing cross-correlation with the integrated Sachs-Wolfe effect (ISW, see Ref. [69] for a comparison with the case of ordinary cosmologies).

The latter can be represented in terms of temperature fluctuations as

$$\begin{aligned}\Theta^{ISW}(\hat{n})&=-\int_0^\infty d\chi[\dot{\Phi}(\hat{n},\chi)-\dot{\Psi}(\hat{n},\chi)]\\&=-\int_0^\infty d\chi[2\dot{\Phi}(\hat{n},\chi)-\dot{\Xi}(\hat{n},\chi)].\end{aligned} \tag{92}$$

Note that, in order to avoid confusion with the integration variable χ', in this paragraph only we will denote with an overdot the derivative with respect to conformal time.

Again, using the expressions for the lensing potential power spectra (84) and (85), and making use of the statistical independence of different Fourier modes, we are able to write immediately the cross-correlated spectrum:

$$\begin{aligned} C^{\Theta\phi} &= \frac{8}{\pi}\int_0^{\chi_\infty} d\chi g'(\chi)\int_0^{\chi_\infty} d\chi' \int k^2 dk u_l(kD(\chi))u_l(kD \\ &\quad \times(\chi'))\Big[\langle\Phi(k,\chi)\dot{\Phi}(k,\chi')\rangle - \frac{1}{2}\langle\Xi(k,\chi)\dot{\Phi}(k,\chi')\rangle \\ &\quad - \frac{1}{2}\langle\dot{\Xi}(k,\chi')\Phi(k,\chi)\rangle + \frac{1}{4}\langle\Xi(k,\chi)\dot{\Xi}(k,\chi')\rangle\Big] \\ &= \frac{2}{\pi}\int_0^{\chi_\infty} d\chi g'(\chi)\int_0^{\chi_\infty} d\chi' \int k^2 dk u_l(kD(\chi))u_l \\ &\quad \times(kD(\chi'))[\langle\Psi(k,\chi)\dot{\Psi}(k,\chi')\rangle \\ &\quad + \langle\Phi(k,\chi)\dot{\Phi}(k,\chi')\rangle - \langle\Psi(k,\chi)\dot{\Phi}(k,\chi')\rangle \\ &\quad - \langle\dot{\Psi}(k,\chi')\Phi(k,\chi)\rangle]. \end{aligned} \tag{93}$$

The main correction in EQ cosmologies is obtained again by using (53). Interestingly, the time derivative reintroduces a term proportional to $|\Phi|^2$:

$$\begin{aligned} \delta C^{\Theta\phi} &= -128 G\xi\phi_0^2\int_0^{\chi_\infty} d\chi g'(\chi)\int_0^{\chi_\infty} d\chi'\left(\frac{\phi^2}{\phi_0^2}-1\right) \\ &\quad \times\int k^2 dk u_l(kD(\chi))u_l(kD(\chi'))\langle\Phi(k,\chi)\dot{\Phi}(k,\chi')\rangle \\ &\quad - 256 G\xi\phi_0^2\int_0^{\chi_\infty} d\chi g'(\chi)\int_0^{\chi_\infty} d\chi'\frac{\phi\dot{\phi}}{\phi_0^2}\int k^2 dk u_l \\ &\quad \times(kD(\chi))u_l(kD(\chi'))\langle\Phi(k,\chi)\Phi(k,\chi')\rangle. \end{aligned} \tag{94}$$

These expressions can be further simplified noticing that for lensing on the CMB signal the source distribution is well represented by a δ function at the last scattering (LS) surface; thus the averaging function $g'(\chi)$ can be written as

$$\begin{aligned} g'(\chi) &= D(\chi)\int_\chi^{\chi_\infty} d\chi'\frac{D(\chi'-\chi)}{D(\chi')}\delta(\chi'-\chi_{\rm LS}) \\ &= \frac{D(\chi)D(\chi_{\rm LS}-\chi)}{D(\chi_{\rm LS})}. \end{aligned} \tag{95}$$

VI. CONCLUSIONS

The weak lensing in cosmology is one of the most important tools to investigate the structure formation process, and in particular the mechanics of the dark cosmological component, which represents almost 95% of the cosmic budget according to the recent measurements. Since the onset of cosmic acceleration occurs at the same epoch of structure formation, the weak lensing looks most promising in order to gain insight into the nature of the dark energy. The candidates which have been proposed for explaining the dark energy are suitably described in a cosmological context that is generalized with respect to the ordinary one, admitting dark-matter–energy couplings as well as generalized theories of gravity.

In particular, for the latter class of theories, a systematic treatment of the weak lensing process is lacking in the literature, and this work aims at filling this gap. We considered a Lagrangian where the gravitational sector is made of a function that depends arbitrarily on the Ricci scalar as well as on a scalar field; the most general scalar-tensor theory of gravity, as well as the most general dependence on the Ricci scalar without a scalar field, can be described in full generality in this framework.

We studied the generalized Poisson equations, linking the fluctuating components to the two gauge-invariant generalized gravitational potentials representing the metric fluctuations that cause the weak lensing process itself. This allowed us to fix the contributions from the modified background expansion as well as the fluctuations, both in the Ricci scalar and in the scalar field responsible for the scalar-tensor coupling. We show that both of them are responsible for an anisotropic stress of scalar origin, causing the gravitational potentials to be different already at the linear level. We studied in particular the modifications induced by the time variation of the effective gravitational constant, which are most relevant in nonminimally coupled models in which the gravitational Lagrangian sector is a product of a function depending on a scalar field and on the Ricci scalar; we focus in particular on the EQ scenarios, where the scalar field playing the role of the dark energy and responsible for cosmic acceleration today possesses a quadratic coupling with the Ricci scalar.

Starting from the equation describing the geodesic deviation, we derived the generalized expressions for distortion tensor and projected lensing potential, tracking the effects due to the time variation of the effective gravitational constant and the contribution of the anisotropic stress; in particular, we show how the latter yields a correction proportional to the correlation between the gravitational potentials.

Finally, we specialized our results to the description of two quantities that are most relevant for observations, i.e., the lensing convergence power spectrum as well as the correlation between the lensing potential and the ISW effect, affecting the total intensity and polarization anisotropies in the cosmic microwave background radiation.

By considering again the particular case of EQ cosmologies, we worked out approximate expressions for the corrections induced by the time variation of the effective gravitational constant on the lensing potential, lensing-lensing correlation angular power spectrum, convergence of the angular power spectrum as well as lensing-ISW correlation. We showed that the order of magnitude of these effects is of the order of $8\pi G\xi\phi_0^2$, where ξ is the coupling constant and ϕ_0 is the present value of the dark energy field. It may be noted how such correction may be relevant even if the underlying theory is close to general relativity, i.e., if the Jordan-Brans-

Dicke parameter $\omega_{JBD}=1/32\pi G\xi^2\phi_0^2$ is large, depending on the relative balance between ξ and ϕ_0.

Despite these interesting indications in the particular case of EQ cosmologies, the formulas we developed here have great generality, allowing a direct interpretation of the modern weak lensing observations in the context of cosmologies with generalized theories of gravity.

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