Classical and quantum decay of oscillations: Oscillating self-gravitating real scalar field solitons

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The oscillating gravitational field of an oscillaton of finite mass M causes it to lose energy by emitting classical scalar field waves, but at a rate that is nonperturbatively tiny for small $\mu \equiv GMm/\hbar c$, where m is the scalar field mass: $dM/dt \approx -3\ 797\ 437.776(c^3/G)\mu^{-2}e^{-39.433\ 795\ 197/\mu}[1+O(\mu)]$. Oscillatons also decay by the quantum process of the annihilation of scalarons into gravitons, which is only perturbatively small in μ, giving by itself $dM/dt \approx -0.008\ 513\ 223\ 935(m^2c^2/\hbar)\mu^5[1+O(\mu^2)]$. Thus the quantum decay is faster than the classical one for $\mu \lesssim 39.4338/[\ln(\hbar c/Gm^2)+7\ln(1/\mu)+19.9160]$. The time for an oscillaton to decay away completely into free scalarons and gravitons is $t_{\mathrm{decay}} \sim 2\hbar^6c^3/G^5m^{11} \sim 10^{324}$ yr$(1\ \mathrm{meV}/mc^2)^{11}$. Oscillatons of more than one real scalar field of the same mass generically asymptotically approach a static-geometry $U(1)$ boson star configuration with $\mu=\mu_0$, at the rate $d(GM/c^3)/dt \approx [(C/\mu^4)e^{-\alpha/\mu}+Q(m/m_{\mathrm{Pl}})^2\mu^3](\mu^2-\mu_0^2)$, with μ_0 depending on the magnitudes and relative phases of the oscillating fields, and with the same constants C, α, and Q given numerically above for the single-field case that is equivalent to $\mu_0=0$.

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I. INTRODUCTION

Seidel and Suen [1,2] (see also Tkachev [3] for an even earlier paper) have found numerically that there exist nonsingular oscillating self-gravitating solitonic configurations of a real scalar field, which they called oscillating soliton stars. These have also been studied by several other authors [4–10] and are now generally called *oscillatons*. In the simplest case, which is what I shall consider here, they arise from the Einstein-Klein-Gordon (EKG) equations for gravity plus one or more minimally coupled massive real scalar fields.

The previous numerical evidence suggested that although these oscillatons are oscillating, they appeared to be periodic and stable [1,2,5–10], so that classically, at least, an isolated oscillaton might be expected to last forever. However, here I shall show that oscillatons of finite mass actually decay classically as the oscillating gravitational field leads to the emission of scalar waves. The decay rate is calculated for the case of low-mass classical oscillatons and is found to be nonperturbatively tiny (nonanalytic in the oscillaton mass at zero mass), given by Eq. (122) below, which may be why it has not been clearly seen numerically.

Seidel and Suen [1] did recognize that their numerical results were consistent with quasiperiodic oscillations analogous to the orbit of two black holes that spiral inward while emitting gravitational waves. My results are also similar to this analogy, with the oscillatons classically emitting scalar waves instead of gravitational waves, except that here the classical decay rate goes to zero faster than any power law of the appropriate small parameter (here μ, which is the mass M of the oscillaton, multiplied by the scalar field mass m, and divided by the square of the Planck mass m_{Pl}) as μ is taken to zero.

In addition, oscillatons decay quantum mechanically by the annihilation of scalarons into gravitons, at a rate that is also calculated here for low-mass oscillatons, given by Eq. (139) below. Although this mass-loss rate is also small, it is perturbative and goes as the fifth power of the oscillaton mass, so for sufficiently small oscillaton mass, this quantum decay dominates over the classical mass-loss rate. The time for an oscillaton with an initially large number of scalarons to decay away completely into free scalarons and gravitons then goes as the inverse 11th power of the scalaron mass and hence is very large if the scalaron mass is much less than the Planck mass.

Although my numerical results for the classical and quantum decay rates are for a single real scalar field in a spherical configuration (and for any number of such fields of the same mass oscillating in the same mode except for possible phase shifts [8], which do give a nontrivial effect), I shall start by giving the formalism for the classical decay rates for an arbitrary nearly-Newtonian configuration of an arbitrary number of massive scalar fields, and then do a detailed numerical analysis of the single-field nearly-Newtonian and nearly periodic spherical case for both the classical and quantum decays. Then I shall return to a discussion of the classical and quantum decay rates for multifield oscillatons.

Throughout this paper I shall assume that the mass of each scalar field is much less than the Planck mass, which is a necessary (though not sufficient) requirement for doing a classical analysis and is also necessary for the validity of various equations I shall use for the quantum decay of oscillatons.

II. NOTATION AND UNITS

Consider the case in which there are n real scalar fields Φ_{IJ}, each with mass m_I, minimally coupled to Einstein gravity, and with no other self-interactions. The index I labels the different mass values, and the index J labels the different scalar fields that have the same mass.

In the classical analysis, I shall often use units in which $c=1$ (though sometimes for results I shall insert the appropriate power of c in order to be able to evaluate quantities in conventional units), but I shall not set $\hbar$ or G equal to unity.

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However, to avoid having $\hbar$'s in most of my equations, I shall let the masses m_I have units of inverse time, which is indeed what they would have (at least if $c=1$) in the classical Klein-Gordon equation that each scalar field obeys,

$$(\Box - m_I^2)\Phi_{IJ}=0; \tag{1}$$

i.e., a free zero-spatial-momentum real scalar solution in flat Minkowski spacetime in orthonormal Minkowski coordinates, with the appropriate zero of time, would have the time dependence $\cos(m_I t)$. In terms of the conventional scalaron particle masses, which I shall hereafter denote with the starred subscript, m_{*I}, one has $m_I = m_{*I}c^2/\hbar$. Perhaps it is more perspicuous to write this relation as

$$m_{*I}c^2 = \hbar m_I, \tag{2}$$

so that in terms of the classical quantity m_I (the natural frequency of the scalar field, in radians per second), the energy $m_{*I}c^2$ of a one-particle quantum excitation of the scalar field is indeed $\hbar$ times the frequency of the excitation.

That is, I am taking the view that it is the natural frequencies m_I that are the classical parameters of the scalar fields, and that the masses m_{*I} of the scalaron particles are quantum properties that will not show up in a classical analysis or in the classical decay of the oscillatons (though they will when one considers the quantum annihilation of scalarons into gravitons).

Analogously, to avoid factors of Newton's gravitational constant G in most of my equations, when I consider the gravitational mass M of a scalar field configuration or oscillaton, it is convenient to include G in it (or actually G/c^3 if one uses units in which the speed of light, c, is not unity), so that my M has units of time and is thus half the gravitational (Schwarzschild) radius of the configuration divided by the speed of light. Therefore, if I let M_* be the mass in conventional mass units (e.g., grams or kilograms), what I shall use is

$$M \equiv \frac{GM_*}{c^3}. \tag{3}$$

With these conventions, I can avoid using $\hbar$ and G in most of my intermediate equations, even without using units in which those quantities are set equal to unity.

For example, the simplest classical spherical oscillatons of a single real scalar field (with no nodes) are characterized (up to the overall scale, into which the natural frequency m enters) by the single dimensionless parameter

$$\mu \equiv Mm = \frac{GM_* m_*}{\hbar c} \approx 7.483\,138\,84\times 10^9 \left(\frac{M_*}{M_\odot}\right)\left(\frac{m_* c^2}{\mathrm{eV}}\right), \tag{4}$$

where $M_\odot \approx 1.989\times 10^{33}\ g$ is the mass of the sun in conventional units. (In temporal units, the mass of the sun is $4.925\,490\,95\times 10^{-6}$ s, almost 5 μs, known to much higher accuracy than in conventional units, since the gravitational effect of the sun, proportional to $GM_\odot$, is known much more accurately than G is in conventional units.) By dimensional analysis, one can then easily see that if there is a classical decay of such an oscillaton, one must have (since my M has the dimension of time)

$$\frac{dM}{dt} = -f(\mu), \tag{5}$$

a function purely of the only dimensionless parameter of the oscillaton, μ. [In Eq. (122) below I shall give this function for $\mu \ll 1$, finding that it is nonanalytic at $\mu=0$.]

The price of this simplicity in the units is that one must get used to the mass of the oscillaton having the dimension of time, which is the inverse of the dimension of frequency that is the classical "mass" of the scalar field in its classical Klein-Gordon equation.

We can get a further simplification by using an appropriate redefinition of the scalar fields Φ_{IJ}. Since the square of the time derivative of a scalar field has the dimension of energy density, the square of a scalar field has the dimension of mass divided by length, which is the same as the dimension of c^2/G. Thus (considering also the 8π in Einstein's equations) it is convenient to define the dimensionless scalar field values

$$\phi_{IJ} \equiv \sqrt{8\pi G/c^2}\,\Phi_{IJ}. \tag{6}$$

Then by Einstein's equations, the Ricci tensor generated by the stress-energy tensor of the scalar fields is

$$R_{\alpha\beta} = \sum_{IJ}\left(\phi_{IJ,\alpha}\phi_{IJ,\beta} + \frac{1}{2}g_{\alpha\beta}m_I^2\phi_{IJ}^2\right). \tag{7}$$

This will have the dimension of inverse time squared if the coordinates have the dimension of time and if the metric components $g_{\alpha\beta}$ are dimensionless.

Following the examples of Refs. [10,11], it is also convenient to combine each rescaled dimensionless real scalar field ϕ_{IJ} and its time derivative $\dot\phi_{IJ} \equiv \partial\phi_{IJ}/\partial t$ into a single dimensionless complex quantity,

$$\Psi_{IJ} \equiv \frac{1}{2}e^{im_I t}\left(\phi_{IJ} + \frac{i}{m_I}\dot\phi_{IJ}\right), \tag{8}$$

so

$$\phi_{IJ} = \Psi_{IJ}e^{-im_I t} + \bar\Psi_{IJ}e^{im_I t} \tag{9}$$

and

$$\dot\phi_{IJ} = -im_I\Psi_{IJ}e^{-im_I t} + im_I\bar\Psi_{IJ}e^{im_I t}. \tag{10}$$

In terms of the complex Ψ_{IJ} and its complex conjugate $\bar\Psi_{IJ}$, the time-time, time-space, and space-space components of the Ricci tensor are (using 0 to denote the time coordinate $t=x^0$ and lower-case Latin letters to denote spatial coordinates x^i)

$$R_{00}=\sum_{IJ}\left\{m_I^2\left[(2+g_{00})\Psi_{IJ}\bar{\Psi}_{IJ}-\left(1-\frac{1}{2}g_{00}\right)(\Psi_{IJ}^2e^{-2im_It}+\bar{\Psi}_{IJ}^2e^{-2im_It})\right]\right\}, \tag{11}$$

$$R_{0i}=\sum_{IJ}\left[m_I^2g_{0i}\left(\Psi_{IJ}\bar{\Psi}_{IJ}+\frac{1}{2}(\Psi_{IJ}^2e^{-2im_It}+\bar{\Psi}_{IJ}^2e^{2im_It})\right)+im_I[\bar{\Psi}_{IJ}\Psi_{IJ,i}-\Psi_{IJ}\bar{\Psi}_{IJ,i}-\Psi_{IJ}\Psi_{IJ,i}e^{-2im_It}+\bar{\Psi}_{IJ}\bar{\Psi}_{IJ,i}e^{2im_It}]\right], \tag{12}$$

$$R_{ij}=\sum_{IJ}\left[m_I^2g_{ij}\left(\Psi_{IJ}\bar{\Psi}_{IJ}+\frac{1}{2}(\Psi_{IJ}^2e^{-2im_It}+\bar{\Psi}_{IJ}^2e^{2im_It})\right)+\Psi_{IJ,i}\bar{\Psi}_{IJ,j}+\bar{\Psi}_{IJ,i}\Psi_{IJ,j}+\Psi_{IJ,i}\Psi_{IJ,j}e^{-2im_It}+\bar{\Psi}_{IJ,i}\bar{\Psi}_{IJ,j}e^{2im_It}\right]. \tag{13}$$

III. GAUGE OR COORDINATE CONDITIONS

In finding solutions to the Einstein-Klein-Gordon equations, one must make a choice of coordinates or gauge for the gravitational field. I shall restrict consideration to 3+1 dimensional spacetime. There are four coordinates to be chosen, giving the freedom of four free functions over spacetime for the gauge group of coordinate transformations.

I generally find it convenient to use these four degrees of freedom to set the time-space components of the metric to be zero, $g_{0i}=0$ for the three i, and to set g_{00} to be independent, or nearly independent, of the time coordinate t. This then implies that the hypersurfaces of constant t are orthogonal to the worldlines of constant spatial coordinates x^i, and that along each such worldline, the proper time is nearly proportional to the coordinate time t (with a space-dependent constant of proportionality). For example, if the metric is periodic in time, we can choose g_{00} to be precisely independent of the time coordinate t. However, this still leaves the freedom to make arbitrary spatial coordinate transformations that are independent of t.

If we wish to pin down the spatial coordinates, we could, for example, choose them so that the time average of the spatial metric, $\langle g_{ij}\rangle$, over a time that is long with respect to the reciprocal of the smallest natural frequency difference $|m_I-m_{I'}|$, is as nearly proportional to the 3×3 identity matrix as possible. More explicitly, if we take $\langle g_{ij}\rangle$ to be a spatially dependent 3×3 matrix, we could choose spatial coordinates so that the integral over all space of the square of the traceless part of this matrix is minimized.

This would still not pin down the spatial origin or angular orientation of the resulting quasi-Cartesian spatial coordinate system, but we could choose the spatial origin to be that which gives the center of mass of the asymptotic form of the time-averaged metric. If it is necessary to fix the orientation, we could, for example, fix it so that the asymptotic quadrupole moment has its principle axes lying along the three coordinate axes in some order determined by, say, the ordering of the eigenvalues of the quadrupole moment. Of course, this specification is degenerate in the spherically symmetric case, but then the angular orientation about the center of mass makes no difference.

For our purposes below it is not necessary to be so precise, but I am just illustrating how for a generic nearly periodic metric, it appears to be possible to fix all of the coordinates completely. Of course, there is some arbitrariness in the procedure chosen for this (e.g., whether to take the time average of the spatial metric or of its inverse or of some other matrix function of the spatial metric, and how to define the time average over a finite time if the nonoscillatory part of the metric is slowly varying). But once a sufficiently precise procedure is chosen, the coordinates are in principle rigidly given, and hence so are the metric components for a given spacetime in that coordinate system. That is, the procedure makes the resulting coordinates and metric components become procedure-dependent but gauge-invariant functions over the spacetime.

As a result of all but the last parts of the procedure outlined above, one can write a generic periodic or approximately periodic metric in the form

$$ds^2=-e^{2U(x^k)}dt^2+e^{-2U(x^k)+2V(x^k)}\{[1+2W(t,x^k)]\delta_{ij}+\sigma_{ij}(x^k)+h_{ij}(t,x^k)\}dx^idx^j, \tag{14}$$

where the time averages of the time-dependent quantities, that is the scalar $W(t,x^k)$ and the traceless symmetric tensor $h_{ij}(t,x^k)$, are all zero, and where the spatial coordinates are chosen to minimize the integral over all space of the trace of the square of the time-independent traceless symmetric tensor $\sigma_{ij}(x^k)$.

I can summarize the situation by noting (i) the separation of the metric into the two parts given here, with no cross terms between them, arises from the gauge condition that $g_{0i}=0$; (ii) the form of the first part arises from the gauge condition on the choice of hypersurfaces of constant t that they give g_{00} independent of t; and (iii) the form of the second part is given by first factoring out the dominant Newtonian spatial dependence of the spatial metric and then separating the remaining factor into time-independent and time-dependent isotropic and anisotropic pieces, with the time-dependent terms being chosen each to have zero time average. Generally speaking, the five scalar or tensorial functions appearing in the metric, U, V, W, σ_{ij}, and h_{ij}, tend to be smaller the more non-Newtonian, time dependent, and anisotropic they are.

[If the metric is only approximately periodic, it may be able to be written exactly in the form above for only a limited amount of time, with there then being some ambiguity as to what it means for the time averages of $W(t,x^k)$ and of $h_{ij}(t,x^k)$ to be zero. Alternatively, to be applicable for longer times, either the form above may be only approximate, or else one might need to give U, V, and σ_{ij} some slow time dependence to deal with slow nonperiodic changes in the geometry.]

Of course, there are many other similar forms for the metric from similar procedures, such as having $W(t,x^k)$, $h_{ij}(t,x^k)$, and/or $\sigma_{ij}(x^k)$, or suitable multiples of these quantities, as arguments of an exponential, so I am not claiming that there is a unique preferred form for a periodic or approximately periodic metric, but only that the form above, or its slight generalization to the case when U, V, and σ_{ij} have some slow temporal variation, is sufficient for our purposes.

Note that in the spherically symmetric case, to which I shall turn later, the time-averaged spatial metric is necessarily conformally flat by its spherical symmetry, so $\sigma_{ij}(x^k) = 0$. Both time-dependent quantities, $W(t,x^k)$ and $h_{ij}(t,x^k)$, are generically nonzero. The spatial dependence of any scalar quantity is a function of the one spatial function $r^2 = \delta_{ij}x^ix^j$, so U and V are functions purely of the "radius" r, and W is a function of t and of r. With spherical symmetry, the traceless tensorial quantity has the form $h_{ij}dx^idx^j = h(t,r)(\delta_{ij}dx^idx^j - 3dr^2)$ for some function $h(t,r)$ of both time and radius whose time average is zero.

IV. NEARLY-NEWTONIAN SCALAR FIELD CONFIGURATIONS

In this paper I shall focus on self-gravitating configurations of one or more massive scalar fields in which the gravitational field is very weak (given to an adequate approximation by the linearized Einstein equations), and the scalar fields have a very slow spatial dependence (so the dominant piece of Ψ_{IJ} has a very slow spacetime dependence, though ϕ_{IJ} does have a temporal oscillation of frequency roughly m_I that is not considered slow, since slowness is taken to be relative to these frequencies). See Refs. [11,10] for previous analyses in this limit, which have been a motivation for some of my choices of variables.

In this limiting case, one can work out from the Einstein equations that the metric functions $U(x^k)$ and $W(t,x^k)$ are much smaller in magnitude than unity (but not negligible), and $V(x^k)$, $\sigma_{ij}(x^k)$, and $h_{ij}(t,x^k)$ are negligibly small, at least when there are negligible gravitational waves present. Therefore, the metric (14) takes the form

$$ds^2 \approx -[1+2U(x^k)]dt^2 + [1-2U(x^k)+2W(t,x^k)]\delta_{ij}dx^idx^j. \qquad (15)$$

If $W(t,x^k)=0$, then the approximate metric (15) would be truly Newtonian, but the temporal oscillations of the scalar fields give oscillating components of their stress-energy tensor and hence of the Ricci tensor components (11)–(13) and of the metric (mainly at twice the frequencies of the fields themselves), so $W(t,x^k)$ is nonzero even at the linearized gravity level.

As we shall see below, since the scalar fields are assumed to have slow spatial dependences, a typical magnitude of $W(t,x^k)$ (say its rms value at some spatial location where that is maximized, but of course not its time average, which is zero at all spatial locations by definition) is of the same order of magnitude as a typical magnitude of the square of $U(x^k)$, which we have dropped in expanding the exponential. It is also of the same order of magnitude as a typical value of $V(x^k)$, which we have also dropped. Therefore, it may be thought a bit of a cheat to include the $W(t,x^k)$ term in the metric above but not the $U(x^k)^2$ and $V(x^k)$ terms, which are similar in magnitude.

However, the point is that $W(t,x^k)$ is the largest time-dependent term and is responsible for the dominant contribution to the classical decay of oscillatons and of other self-gravitating real scalar field configurations in the nearly-Newtonian limit. The $U(x^k)^2$ and $V(x^k)$ terms that have been dropped are smaller than the $U(x^k)$ time-independent terms that have been kept, and none of those terms directly contributes to the classical decay. Thus the philosophy is that the metric (15) includes the dominant time-independent corrections to the flat Minkowski metric [the $U(x^k)$ terms] and the dominant time-dependent corrections to the flat metric [the $W(t,x^k)$ term, in the gauge in which g_{00} is independent of t by construction].

Before going to periodic configurations (in the approximation of neglecting the scalar field emission), let us consider the slight generalization to the metric

$$ds^2 \approx -(1+2U)dt^2 + (1-2U+2W)\delta_{ij}dx^idx^j \qquad (16)$$

in which W has a time dependence at frequencies that are roughly twice that of the m_I, but U is now allowed to have some time dependence that is even much slower than its slow spatial dependence.

Now, instead of Eqs. (8)–(10), I shall take

$$\phi_{IJ} \approx \psi_{IJ}e^{-im_It} + \overline{\psi_{IJ}}e^{im_It} \qquad (17)$$

without the restriction $\dot{\psi}_{IJ}e^{-im_It} + \dot{\overline{\psi_{IJ}}}e^{im_It} = 0$ that is true for Ψ_{IJ} from Eqs. (8)–(10). Instead, I shall assume that each ϕ_{IJ} is such that ψ_{IJ} can be chosen to give ϕ_{IJ} approximately and also give

$$|\ddot{\psi}_{IJ}| \ll m_I |\dot{\psi}_{IJ}| \ll m_I^2 |\psi_{IJ}|. \qquad (18)$$

Then the Klein-Gordon equation in the metric (16) with $|W| \ll |U| \ll 1$, which is

$$\ddot{\phi}_{IJ} \approx -m_I^2\phi_{IJ} - 2m_I^2U\phi_{IJ} + c^2\nabla^2\phi_{IJ} \qquad (19)$$

when for the moment we ignore the W term, implies that each ψ_{IJ} approximately obeys the Schrödinger equation

$$\dot{\psi}_{IJ} \approx \frac{ic^2}{2m_I}\nabla^2\psi_{IJ} - im_IU\psi_{IJ}, \qquad (20)$$

where ∇^2 is the flat-space Laplacian, $\nabla^2\psi_{IJ} \equiv \delta^{ij}\psi_{IJ,ij}$.

Note that it is $m_{*I}c^2U = \hbar m_IU$ that is the Newtonian potential energy of the particle of "mass" m_I, and not U itself (which is dimensionless). Also, by having m_I have units of frequency rather than conventional mass units, the explicit appearance of $\hbar$ is avoided in Eq. (20). This is what one would expect, since this Schrödinger equation came from the purely classical Klein-Gordon equation for the real scalar field, rather than from any quantum equation. Note that I have chosen ψ_{IJ} to be dimensionless rather than, say, having the spatial integral of its absolute square be unity (or perhaps some other positive integer), as one would normally normalize the wave function of a truly quantum Schrödinger equation.

It is most straightforward to regard the approximate equivalence between the real second-order Klein-Gordon equation and the complex first-order Schrödinger equation (20) as a procedure that works when one starts with a solution of the Schrödinger equation (20) (with a weak gravitational potential, $|U|\ll 1$) for which $|\nabla^2\psi_{IJ}|\ll m_I^2|\psi_{IJ}|$ (except near possible zeros of ψ_{IJ}) and then uses Eq. (17) to construct from it an approximate solution of the Klein-Gordon equation.

In the reverse direction it is a bit more subtle. If one uses Eq. (8) to define Ψ_{IJ} in terms of ϕ_{IJ} and its time derivative, $\dot{\phi}_{IJ}$, this Ψ_{IJ} itself will be close to the solution ψ_{IJ} of the Schrödinger equation (20). However, since Ψ_{IJ} will generically have a small term roughly proportional to e^{2im_It} as well as its dominant term with a much slower time variation, the time derivative $\dot{\Psi}_{IJ}$ will pick up a relatively significant contribution from the term that is roughly proportional to e^{2im_It} and so will be significantly different from $\dot{\psi}_{IJ}$. Thus Ψ_{IJ} defined by Eq. (8) will not satisfy the Schrödinger equation (20).

However, one can instead define

$$\psi_{IJ}=\Psi_{IJ}+\frac{i}{2m_I}\dot{\Psi}_{IJ}$$
$$\equiv\frac{1}{4m_I^2}e^{im_It}(m_I^2\phi_{IJ}+2im_I\dot{\phi}_{IJ}-\ddot{\phi}_{IJ})$$
$$\approx\frac{1}{2}e^{im_It}\left(\phi_{IJ}+\frac{i}{m_I}\dot{\phi}_{IJ}+U\phi_{IJ}-\frac{c^2}{2m_I^2}\nabla^2\phi_{IJ}\right), \tag{21}$$

where for the last expression I have used the approximate form (19) of the Klein-Gordon equation in the Newtonian part of the metric to evaluate the second time derivative of ϕ_{IJ} in terms of its value and its spatial Laplacian. This ψ_{IJ} then obeys the Schrödinger equation (20) when U is small and slowly varying and when ϕ_{IJ} is oscillating at nearly its natural frequency m_I and has a slow spatial variation in units of m_I.

With this definition of the complex ψ_{IJ} in terms of the real ϕ_{IJ} and its derivatives, Eq. (17) is still a fairly good approximation for ϕ_{IJ} in terms of ψ_{IJ}, but an inversion of Eq. (21) that is accurate to one higher order is

$$\phi_{IJ}\approx\left(\psi_{IJ}-\frac{i}{m_I}\dot{\psi}\right)e^{-im_It}+\left(\overline{\psi}_{IJ}+\frac{i}{m_I}\dot{\overline{\psi}}\right)e^{im_It}$$
$$\approx(1-U)\left[\left(\psi_{IJ}+\frac{c^2}{2m_I^2}\nabla^2\psi_{IJ}\right)e^{-im_It}+\left(\overline{\psi}_{IJ}+\frac{c^2}{2m_I^2}\nabla^2\overline{\psi}_{IJ}\right)e^{im_It}\right]. \tag{22}$$

Nevertheless, although this formula for ϕ_{IJ} is a more accurate inversion of Eq. (21) than is Eq. (17), I do not know that it really gives a more accurate solution of the Klein-Gordon equation than Eq. (17) does from a solution of the Schrödinger equation (20).

After getting an approximate solution of the Klein-Gordon equation in the nearly-Newtonian metric (16) with $|W|\ll|U|\ll 1$ (and temporarily ignoring the small effect of the tiny but rapidly time-varying W term, which shall be discussed later), we need to solve the Einstein equation for the effect of the stress-energy tensor of the scalar fields on the metric. To leading order, the resulting Ricci-tensor components from the stress-energy tensor and the Einstein equation are

$$R_{00}\approx\sum_{IJ}\left(\dot{\phi}_{IJ}^2-\frac{1}{2}m_I^2\phi_{IJ}^2\right)$$
$$\approx\sum_{IJ}\left[m_I^2\left(\psi_{IJ}\overline{\psi}_{IJ}-\frac{3}{2}(\psi_{IJ}^2e^{-2im_It}+\overline{\psi}_{IJ}^2e^{-2im_It})\right)\right], \tag{23}$$

$$R_{0i}\approx 0, \tag{24}$$

$$R_{ij}\approx\sum_{IJ}\frac{1}{2}m_I^2\delta_{ij}\phi_{IJ}^2$$
$$\approx\sum_{IJ}m_I^2\delta_{ij}\left(\psi_{IJ}\overline{\psi}_{IJ}+\frac{1}{2}(\psi_{IJ}^2e^{-2im_It}+\overline{\psi}_{IJ}^2e^{2im_It})\right). \tag{25}$$

The corresponding Einstein tensor components are simpler,

$$G_{00}\approx\sum_{IJ}2m_I^2\psi_{IJ}\overline{\psi}_{IJ}, \tag{26}$$

$$G_{0i}\approx 0, \tag{27}$$

$$G_{ij}\approx-\sum_{IJ}m_I^2\delta_{ij}(\psi_{IJ}^2e^{-2im_It}+\overline{\psi}_{IJ}^2e^{2im_It}). \tag{28}$$

This corresponds to an energy density that has only whatever slow time variation the mass-squared-weighted sum of the squares of the absolute values of the ψ_{IJ}'s may have, and an isotropic pressure that oscillates at the frequencies $2m_I$ and has a time average that is zero to this order of approximation (though at the next order there are small time-independent pressure gradients that hold up the self-gravitating energy density in the approximately periodic or quasiperiodic cases).

Directly from the nearly-Newtonian metric (16) itself, with the spatial derivatives of W being negligibly small, the leading-order linearized Einstein tensor components are

$$G_{00}\approx 2c^2\nabla^2U, \tag{29}$$

$$G_{0i}\approx 0, \tag{30}$$

$$G_{ij}\approx-2\ddot{W}\delta_{ij}. \tag{31}$$

Thus the Einstein equation in the nearly Newtonian case becomes

$$c^2\nabla^2 U \approx \sum_{IJ} m_I^2 \psi_{IJ}\overline{\psi}_{IJ}, \tag{32}$$

$$W \approx -\frac{1}{8}\sum_{IJ}(\psi_{IJ}^2 e^{-2im_I t}+\overline{\psi}_{IJ}^2 e^{2im_I t}). \tag{33}$$

In summary, for nearly Newtonian self-gravitating configurations of real self-gravitating minimally coupled massive scalar fields, the Einstein-Klein-Gordon equations become the coupled approximate partial differential equations (20) and (32), which are the time-dependent Newton-Schrödinger or Schrödinger-Newton equations [12–15], plus the additional algebraic equation (33) for the small rapidly varying term W in the nearly-Newtonian metric (16). The conditions for these nearly-Newtonian equations to be valid are that $\Sigma_{IJ}|\psi_{IJ}|^2 \ll 1$, $|U| \ll 1$, and that the spatial derivatives of the ψ_{IJ}'s are small in comparison with their typical values multiplied by their natural frequencies m_I. [Then the Schrödinger equation (20) implies that the time derivatives of the ψ_{IJ}'s are also small in comparison with their typical values multiplied by their natural frequencies m_I.]

Unless explicitly stated otherwise, in this paper we shall assume that the metric is asymptotically flat with asymptotically Lorentzian coordinates (except for a possible rescaling). That is, we assume that U goes to a time-and direction-independent constant at spatial infinity. Equation (32) then implies that the ψ_{IJ}'s must all asymptotically tend to zero at spatial infinity. Equations (20) and (32) are invariant under shifting U by a constant or a function purely of t, provided the ψ_{IJ}'s are shifted by the appropriate phase factor that is also a function purely of t. (This is simply the gauge transformation of replacing the time coordinate t with a new time coordinate t' that is purely a function of the old time coordinate t, and of rescaling the spatial coordinates appropriately.) One could thus set the asymptotic value of U to be zero, making the coordinates asymptotically Lorentzian without any scaling factors. However, in some cases it is more convenient not to make this restriction, such as when the time dependence of the ψ_{IJ}'s is purely by a time-dependent phase factor, in which case one can cancel the phase factor and make the ψ_{IJ}'s independent of time by an appropriate nonzero but time-independent asymptotic value of U.

For the rest of this section, we shall take the approximate Newton-Schrödinger equations (20) and (32) as exact and so write = signs rather than ≈ equal signs. However, we must bear in mind that these are actually only approximations, valid in the nearly-Newtonian limit, for the actual Einstein-Klein-Gordon equations taken as fundamental in this paper.

The time-dependent Newton-Schrödinger equations (20) and (32), now temporarily re-interpreted as exact equations, may be derived from the classical action (cf. Ref. [16])

$$I_{NS} = \frac{1}{8\pi c^3}\int dt d^3x \sum_{IJ} [im_I(\overline{\psi}_{IJ}\dot{\psi}_{IJ} - \psi_{IJ}\dot{\overline{\psi}}_{IJ}) - c^2(\nabla U)^2 - c^2|\nabla\psi_{IJ}|^2 - 2m_I^2 U|\psi_{IJ}|^2] = \int dt L_{NS} \tag{34}$$

(in units of time squared, to be multiplied by $c^5/\hbar G$ if one wants a dimensionless action), where the classical Lagrangian (with units of time) is

$$\begin{aligned} L_{NS} &= \frac{1}{8\pi c^3}\int d^3x \sum_{IJ} im_I(\overline{\psi}_{IJ}\dot{\psi}_{IJ} - \psi_{IJ}\dot{\overline{\psi}}_{IJ}) - E_U - E_K - E_V \\ &= \frac{1}{8\pi c^3}\int d^3x \sum_{IJ} im_I(\overline{\psi}_{IJ}\dot{\psi}_{IJ} - \psi_{IJ}\dot{\overline{\psi}}_{IJ}) \\ &\quad - MU_\infty - E_U - E_K - 2E_P \\ &= \frac{1}{8\pi c^3}\int d^3x \sum_{IJ} im_I(\overline{\psi}_{IJ}\dot{\psi}_{IJ} - \psi_{IJ}\dot{\overline{\psi}}_{IJ}) - MU_\infty - E. \end{aligned} \tag{35}$$

Here the asymptotic mass of the configuration (essentially the total rest mass, uncorrected for gravitational binding energy), in time units, is

$$M = \frac{1}{4\pi c^3}\int d^3x \sum_{IJ} m_I^2|\psi_{IJ}|^2 = \frac{G}{c^3}\int d^3x\,\rho, \tag{36}$$

with the rest mass density, in conventional units, being

$$\rho = \frac{1}{4\pi G}\sum_{IJ} m_I^2|\psi_{IJ}|^2. \tag{37}$$

When the Newton-Schrödinger action I_{NS} given by Eq. (34) is extremized, so that the Newton-Schrödinger equations (20) and (32) are satisfied, one can readily see that the asymptotic form of the Newtonian potential is

$$U \sim U_\infty - \frac{Mc}{r}, \tag{38}$$

where M is the asymptotic value of the mass, given by Eq. (36) above (in time units, G/c^3 times the mass M_* in conventional mass units, so Mc has units of length and is half the Schwarzschild radius corresponding to the mass), and r is the radial distance from the center of mass, also in units of length.

By Eq. (20) and the asymptotic boundary conditions given, the mass M is an approximately conserved quantity [approximate only to the extent that Eq. (20) is approximate], corresponding under the approximations used to the exactly conserved ADM mass of the spacetime. In some sense it is more nearly the rest mass of the matter (in time units), but since only the zeroth-order approximation is being used for the rest mass density and for the spatial volume element, M is only a zeroth-order approximation for the rest mass as well.

The various other energies appearing in the Newton-Schrödinger Lagrangian (35) (smaller than M by factors that in equilibrium are of the order of a typical value of $|U|$, which must be much less than unity for the nearly-Newtonian approximation to be valid) are the following quantities (in time units, inserting the factor of $1/c$ into the right hand side of each of the equations with squares of spatial gradients, and the factor of $1/c^3$ into those without them, under the assumption that the spatial distances and gradients are being measured in length units rather than in the time units that would avoid the need for all factors of c): the (positive) Newtonian potential gradient energy

$$E_U = \frac{1}{8\pi c}\int d^3x (\nabla U)^2, \tag{39}$$

the (positive) scalar field gradient energy or matter kinetic energy

$$E_K = \frac{1}{8\pi c}\int d^3x \sum_{IJ} |\nabla \psi_{IJ}|^2, \tag{40}$$

the (indefinite in sign, though negative in static equilibrium) matter potential energy

$$E_V = \frac{1}{8\pi c^3}\int d^3x \sum_{IJ} 2m_I^2 U|\psi_{IJ}|^2 = \frac{G}{c^3}\int d^3x \rho U, \tag{41}$$

the (indefinite in sign, though also negative in static equilibrium) rescaled gravitational potential energy

$$E_P = \frac{1}{2}E_V - \frac{1}{2}MU_\infty = \frac{1}{8\pi c^3}\int d^3x \sum_{IJ} m_I^2 (U - U_\infty)|\psi_{IJ}|^2$$
$$= \frac{G}{2c^3}\int d^3x \rho (U - U_\infty), \tag{42}$$

and the total Newtonian energy

$$\begin{aligned} E &= E_U + E_K + E_V - MU_\infty \\ &= E_U + E_K + 2E_P \\ &= \frac{1}{8\pi c^3}\int d^3x \sum_{IJ} [c^2(\nabla U)^2 + c^2|\nabla \psi_{IJ}|^2 \\ &\quad + 2m_I^2 (U - U_\infty)|\psi_{IJ}|^2]. \end{aligned} \tag{43}$$

The extrema of the Newton-Schrödinger action I_{NS} given by Eq. (34), with U fixed to a constant value U_∞ at spatial infinity, and with the ψ_{IJ} functions falling off sufficiently rapidly at spatial infinity, are solutions of the Newton-Schrödinger equations (20) and (32).

When these equations of motion are satisfied, one can readily show (by integration by parts, etc., using the boundary conditions given in the previous sentence) that M is a constant of motion (as mentioned above), that

$$E_V = MU_\infty - 2E_U, \tag{44}$$

that

$$E_P = -E_U, \tag{45}$$

that

$$E = E_K + E_P = E_K - E_U, \tag{46}$$

and that the various energies (all of which have absorbed a factor of G/c^3 to have units of time) have the following time derivatives:

$$\dot{E}_P = \frac{1}{2}\dot{E}_V = -\dot{E}_U = -\dot{E}_K = \frac{G}{c^3}\int d^3x \mathbf{J}\cdot\boldsymbol{\nabla} U, \tag{47}$$

where

$$\mathbf{J} = \sum_{IJ}\left(-\frac{i m_I c^2}{8\pi G}(\overline{\psi_{IJ}}\boldsymbol{\nabla}\psi_{IJ} - \psi_{IJ}\boldsymbol{\nabla}\overline{\psi_{IJ}})\right) \tag{48}$$

is the mass-current flux vector in the conventional units of mass per area per time (when we remember that m_I has units of inverse time and that the ψ_{IJ}'s are dimensionless).

One can then see from Eqs. (46) and (47) that the Newton-Schrödinger equations (20) and (32), along with the asymptotic boundary conditions, imply that the total nonrelativistic Newtonian energy E is conserved, or constant in time. Using the expressions (39) and (40) above, and using Eq. (46), we can see that Eq. (43) may be written, for E in time units (or $E_* = c^5 E/G$ in conventional energy units), as

$$E \equiv \frac{GE_*}{c^5} = \frac{1}{8\pi c}\int d^3x \left(\sum_{IJ} |\nabla \psi_{IJ}|^2 - (\nabla U)^2\right) = \text{const.} \tag{49}$$

This is a first-order correction to the rest mass energy of the configuration in the total ADM mass energy. [But since Eq. (36) for M is only correct for the rest mass to zeroth order, the correct first-order expression for the ADM mass is not simply $M+E$.]

If one has a static solution of the Newton-Schrödinger equations (20) and (32), in which U and each ψ_{IJ}^2 is constant in time, then these are extrema of $E_U + E_K + E_V = E + MU_\infty$. To put it another way, they are extrema of the total Newtonian energy E with the constraint of fixed M. If one finds these extrema by the method of Lagrange multipliers, extremizing $E - \lambda M$, then the Lagrange multiplier is $\lambda = -U_\infty$, which thus is dE/dM for a continuous sequence of such extrema.

If one takes such an extremum static spatial configuration and replaces $U(\mathbf{x})$ by $\tilde{U}(\mathbf{x}) = U(e^\alpha \mathbf{x})$ and each $\psi_{IJ}(\mathbf{x})$ with $\tilde{\psi}_{IJ}(\mathbf{x}) = e^\beta \psi_{IJ}(e^\alpha \mathbf{x})$ for small constants α and β, then for the variation of $E_U + E_K + E_V = E + MU_\infty$ to vanish to first order in both α and β, and with the use also of Eqs. (44)–(46), one can readily see that the static equilibrium configurations have

$$E_U = 2E_K = -2E_V = -E_P = \frac{2}{3}MU_\infty = -2E. \tag{50}$$

The relation that the gravitational potential energy is twice the negative of the kinetic energy, $E_P = -2E_K$, and hence that $E = -E_K$, is just the usual virial relation for an attractive inverse-square force. The relation that $E = -(1/3)MU_\infty$ involves the scaling behavior of the Schrödinger equation and implies that for each ψ_{IJ}^2 to be static (e.g., not to have any time-dependent phase factor), one must have $U_\infty > 0$ (essentially for the system to be bound).

A solution of the approximate time-dependent Newton-Schrödinger equations (20) and (32) is determined by an initial spatial configuration of the ψ_{IJ}'s (and, for the quantities in those equations, but not for the gauge-invariant physical quantities, by the gauge choice of the asymptotic value of U). If the scalar fields all disperse indefinitely, then U will tend to U_∞ everywhere in space asymptotically with time so that the integral of the negative $-(\nabla U)^2$ term in the integral (49) goes to zero, and thus the Newtonian energy E must be nonnegative.

But if Eq. (49) gives a negative Newtonian energy E, then the scalar fields cannot all disperse. To the extent that Eqs. (20) and (32) are accurate, at least some of the scalar field energy must remain gravitationally bound indefinitely. Unless the scalar fields collapse gravitationally into configurations violating the nearly-Newtonian approximation being used (and perhaps leading to continued gravitational collapse into one or more black holes), the scalar fields will continue to oscillate, giving an oscillaton.

Negative Newtonian energy E is thus a sufficient condition for an oscillaton (or for gravitational collapse as a possible alternative) in the nearly-Newtonian limit, though it is not a necessary condition, as one can have an initial configuration which asymptotically tends to a bound part with negative Newtonian energy and a dispersing part with a larger positive Newtonian energy. (See Ref. [2] for an example of this.)

In any case, we see that, at least in the nearly-Newtonian case, gravitationally bound oscillatons are quite generic (unless all but a set of measure zero collapse gravitationally into black holes, which naively seems unlikely), occurring if the single inequality $E < 0$ is true (and also in other cases when just part of the system is bound). That is, any perturbation of a nearly-Newtonian oscillaton within a sufficiently small neighborhood in the space of perturbations gives another oscillaton (unless it collapses). In this sense nearly-Newtonian oscillatons are apparently stable under small perturbations, to the degree that Eqs. (20) and (32) are accurate and continue to remain valid (e.g., when one ignores the scalar field radiation considered below, and when no continued gravitational collapse occurs that takes one outside the validity of these equations).

Of course, if one requires an oscillaton of fixed total mass (which itself is a continuous variable classically, though quantized when one includes the fact that the scalar field particles are quantized) to be periodic with some definite period (say in proper time at spatial infinity, to make the period gauge invariant), then one gets a nonlinear eigenvalue problem with presumably only a discrete set of eigensolutions for each mass (modulo gauge transformations, including spatial translations and rotations and the gauge transformation of shifting U). Thus I would expect there to be discrete periodic oscillatons embedded in an open set of nonperiodic oscillatons for a given total mass.

However, if we include the effect of the oscillating term of the nearly-Newtonian metric (15), the W term given by Eq. (33), we shall see in the next section that it leads to classical emission of each scalar field of mass m_I at frequencies roughly $2m_K \pm m_I$, so that generic oscillatons are *not* classically stable but radiate away their energy and scalar fields. The only exceptions appear to be field configurations that have a $U(1)$ invariance, so that their total stress-energy tensor is independent of time and the W term vanishes.

V. CLASSICAL EMISSION OF SCALAR FIELDS FOR GENERIC SITUATIONS

The oscillating W term of the nearly-Newtonian metric (15) gives, in the Klein-Gordon equation of the scalar field Φ_{IJ}, transitions from its dominant frequency near m_I to frequencies near $2m_K \pm m_I$ that radiate away, carrying off energy and causing a generic oscillaton to decay. Here we assume that the terms given in Eq. (17) describe the nonradiating scalar field oscillations of an oscillaton.

To describe the radiation, extend Eq. (17) to include scalar field oscillations at these emitted frequencies, so the dimensionless rescaled scalar field $\phi_{IJ} \equiv \sqrt{8\pi G/c^2}\Phi_{IJ}$ has the form

$$\phi_{IJ} \approx \psi_{IJ}e^{-im_It} + \overline{\psi}_{IJ}e^{im_It} + \sum_K \big(\chi_{IJK}e^{-i(2m_K+m_I)t} + \overline{\chi}_{IJK}e^{i(2m_K+m_I)t} + \chi'_{IJK}e^{-i(2m_K-m_I)t} + \overline{\chi'}_{IJK}e^{i(2m_K-m_I)t}\big), \tag{51}$$

where not only the ψ_{IJ}'s and their complex conjugates, but also the χ_{IJK}'s and χ'_{IJK}'s and their complex conjugates, are functions varying much more slowly than the frequencies m_I and m_K. Since fields that can radiate away must have frequencies larger than m_I, in the sum above we can omit the χ'_{IJK}'s which have $m_K \leq m_I$.

Of course, the sum over K includes I, so the radiation field occurs even if there is only a single m_I or even just a single real scalar field (one choice for the indices IJ denoting the field). (Then we can omit the χ'_{IJK} and $\overline{\chi'_{IJK}}$ terms, since their frequencies would be just m_I, and they would just give small corrections to the ψ_{IJ} and $\overline{\psi}_{IJ}$ terms that are not radiating away by assumption.)

In principle one should include a whole infinite series of frequencies by adding to m_I all possible positive and negative integer multiples of all m_K's, but the additional terms will generally be smaller yet, and so to a good approximation it is sufficient to consider only the ψ_{IJ}, χ_{IJK}, and χ'_{IJK} terms and their complex conjugates.

Now the term with the $e^{-i(2m_K+m_I)t}$ time dependence in the Klein-Gordon equation (1) for the field ϕ_{IJ} in the metric (16) with the oscillating W term included, given by Eq. (33), yields

$$[\nabla^2+4m_K(m_K+m_I)]\chi_{IJK}\approx-\frac{3}{4}m_Im_K\sum_L \psi_{KL}^2\psi_{IJ}, \tag{52}$$

and the term with the $e^{-i(2m_K-m_I)t}$ time dependence gives

$$[\nabla^2+4m_K(m_K-m_I)]\chi'_{IJK}\approx\frac{3}{4}m_Im_K\sum_L \psi_{KL}^2\overline{\psi}_{IJ}. \tag{53}$$

We shall assume, except where explicitly discussed below, that there are no scalar waves incoming from infinity, so we shall impose outgoing wave boundary conditions on the χ_{IJK}'s and χ'_{IJK}'s. Then they have the form

$$\chi_{IJK}(\mathbf{x})\approx\frac{3m_Im_K}{16\pi}\int d^3\mathbf{x}'\frac{e^{2i\sqrt{m_K(m_K+m_I)}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\times\sum_L \psi_{KL}^2(\mathbf{x}')\psi_{IJ}(\mathbf{x}'), \tag{54}$$

$$\chi'_{IJK}(\mathbf{x})\approx-\frac{3m_Im_K}{16\pi}\int d^3\mathbf{x}'\frac{e^{2i\sqrt{m_K(m_K-m_I)}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\times\sum_L \psi_{KL}^2(\mathbf{x}')\overline{\psi}_{IJ}(\mathbf{x}'). \tag{55}$$

These represent scalar waves that are propagating outward at asymptotic speeds

$$v_{IJK}=2\,\frac{\sqrt{m_K(m_K+m_I)}}{2m_K+m_I} \tag{56}$$

and

$$v'_{IJK}=2\,\frac{\sqrt{m_K(m_K-m_I)}}{2m_K-m_I}, \tag{57}$$

respectively, which generically are within a factor of the order of unity of the speed of light (taken to be unity in these equations).

In contrast, the ψ_{IJ}'s are assumed to be localized almost entirely within some region that we shall call the system. Since the ψ_{IJ}'s are assumed to be slowly varying with respect to the natural frequencies m_I, we assume that the ψ_{IJ}'s within the system do not change much during the light travel time across the system, or during the time it takes for the waves represented by the χ_{IJK}'s and χ'_{IJK}'s to traverse the system. That is why we can use an instantaneous approximation for the propagators in the formulas (54) and (55) for the χ_{IJK}'s and χ'_{IJK}'s.

If we surround the system by a sphere much larger than the dominant region over which the ψ_{IJ}'s are significant, then we can calculate the flux of mass out through that sphere in the scalar waves represented by the χ_{IJK}'s and χ'_{IJK}'s given above. When we average this over the oscillation periods and do a bit of algebra that is not repeated here, we get a classical mass loss rate of the oscillaton that is

$$-\frac{dM}{dt}\approx\sum_{IJKLM}\frac{9m_I^2m_K^2}{256\pi^2}\int\frac{d^3\mathbf{x}d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\times[(2m_K+m_I)\sin(2\sqrt{m_K(m_K+m_I)}|\mathbf{x}-\mathbf{x}'|)\times\psi_{KL}^2(\mathbf{x})\psi_{IJ}(\mathbf{x})\overline{\psi}_{KM}^2(\mathbf{x}')\overline{\psi}_{IJ}(\mathbf{x}')+(2m_K-m_I)\sin(2\sqrt{m_K(m_K-m_I)}|\mathbf{x}-\mathbf{x}'|)\times\psi_{KL}^2(\mathbf{x})\overline{\psi}_{IJ}(\mathbf{x})\overline{\psi}_{KM}^2(\mathbf{x}')\psi_{IJ}(\mathbf{x}')]. \tag{58}$$

In the second of the two terms inside the integrand, whenever m_K-m_I is negative, that term is to be omitted, since the corresponding χ'_{IJK} is spatially exponentially damped rather than having the oscillatory behavior representing an outgoing wave. (Of course, if $m_K-m_I=0$, this second term is just zero, so we need not consider it in that case either.)

Remembering that I am using units and conventions in which M has the units of time, in which the m_I's have the units of temporal frequency (inverse time), in which the ψ_{IJ}'s and their complex conjugates are dimensionless, and in which either the spatial coordinates x^i (represented above by the 3-vector $\mathbf{x}$) have the units of time or else the speed of light is set equal to unity, it is easy to see that both sides of Eq. (58) are dimensionless.

Since the classical mass loss rate formula (58) is rather complicated when there are several real massive scalar fields of different masses, it may help to give it explicitly when there is only one real scalar field of mass (or actually natural frequency) m:

$$-\frac{dM}{dt}\approx\frac{27m^5}{256\pi^2}\int\frac{d^3\mathbf{x}d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\sin(\sqrt{8}m|\mathbf{x}-\mathbf{x}'|)\psi^3(\mathbf{x})\overline{\psi}^3(\mathbf{x}'). \tag{59}$$

When there are two scalar fields of the same mass m, then one gets

$$-\frac{dM}{dt}\approx\frac{27m^5}{256\pi^2}\int\frac{d^3\mathbf{x}d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}\sin(\sqrt{8}m|\mathbf{x}-\mathbf{x}'|)[\psi_1^2(\mathbf{x})+\psi_2^2(\mathbf{x})][\overline{\psi}_1^2(\mathbf{x}')+\overline{\psi}_2^2(\mathbf{x}')][\psi_1(\mathbf{x})\overline{\psi}_1(\mathbf{x}')+\psi_2(\mathbf{x})\overline{\psi}_2(\mathbf{x}')]. \tag{60}$$

In particular, when ψ_1 and ψ_2 each have the same magnitude everywhere and are everywhere 90° out of phase, or $\psi_1=\pm i\psi_2$ [so the two real fields can be interpreted as forming a single complex field with a global $U(1)$ symmetry], then the mass loss rate is zero. This is a case in which the stress-energy tensor of the scalar fields does not have an oscillating component, and so there is no oscillating W term in the metric (16).

There is also the analogous case in which one has an arbitrary number (greater than one) of fields of each mass m_I, when one has zero everywhere for the sum of the squares of the ψ_{IJ}'s (not the squares of the absolute values of these quantities) for each fixed I (for each different mass m_I, the sum being over J that labels the different fields with fixed I and hence with fixed mass m_I). Then again the stress-energy tensor and the metric (16) has no oscillating term, and so there is no generation of outgoing χ_{IJK} or χ'_{IJK} waves.

In these cases, if the ψ_{IJ}'s are not stationary (or stationary up to a time-dependent phase factor) but are slowly changing their form, then although there may be no oscillations at the frequencies m_I or their sums or differences, the metric would still have a slow time dependence, and this would presumably lead to some scalar field radiation, though at an amount presumably considerably reduced from what it would be the case if the sum of the squares of the ψ_{IJ}'s for at least one I were different from zero so that there would be the more rapid oscillations in the metric.

The only case in which I would expect absolutely no scalar radiation would be the case in which the metric is absolutely stationary. Otherwise it would seem extremely unlikely that the outgoing radiation at all possible multiples of the metric oscillation frequency, plus or minus the natural frequency of the fields that can potentially be emitted, would be zero. However, I have not tried to find a rigorous proof that there are not exceptional counter-examples to this conjecture.

VI. CLASSICAL EMISSION OF SCALAR FIELDS WITH SPHERICAL SYMMETRY

A simple subset of the set of all oscillatons is the set in which the metric and all of the scalar fields have spherical symmetry. In the nearly-Newtonian limit (which I am taking to exclude gravitational waves), the spherical symmetry of the metric follows from the spherical symmetry of the scalar fields, and the spherical symmetry of the scalar fields (including the outgoing waves) follows from the spherical symmetry of the ψ_{IJ}'s. Therefore, we basically just need to assume that each $\psi_{IJ}=\psi_{IJ}(t,r)$.

The Klein-Gordon equation (1) for the scalar field implies that ψ_{IJ} obeys the Schrödinger equation (20), which for the spherical symmetric case becomes

$$\dot{\psi}_{IJ}\approx\frac{ic^2}{2m_I r}(r\psi_{IJ})''-im_I U\psi_{IJ}, \tag{61}$$

where, except for the prime on χ'_{IJK} and on dummy variables inside integrals, a prime henceforth denotes a partial derivative with respect to r (or later, with respect to a rescaled radial variable $x=kr/c$). One can avoid the explicit r's in this equation by defining

$$f_{IJ}(t,r)\equiv r\psi_{IJ}(t,r), \tag{62}$$

which makes the Schrödinger equation take the form

$$\dot{f}_{IJ}\approx\frac{ic^2}{2m_I}f''_{IJ}-im_I U f_{IJ}. \tag{63}$$

The Newtonian part of the Einstein equations, Eq. (32), can be integrated in the spherically symmetric case to give at each time

$$U\approx U_\infty-\frac{1}{c^2 r}\int_0^r dr'\int_{r'}^\infty dr'' r''\sum_{IJ} m_I^2|\psi_{IJ}(r'')|^2. \tag{64}$$

Here of course the primes and double primes on the r's inside the integrals just denote dummy variables to be integrated over, and not derivatives with respect to r.

The spherically symmetric analogues of Eqs. (54) and (55) for the χ_{IJK}'s and χ'_{IJK}'s obeying Eqs. (52) and (53) (to solve the Klein-Gordon equation) are (with $c=1$ for simplicity here and in many formulas following)

$$\begin{aligned}\chi_{IJK}(r)\approx\frac{3m_I m_K}{8\sqrt{m_K(m_K+m_I)}r}\Bigg(&e^{2i\sqrt{m_K(m_K+m_I)}r}\\&\times\int_0^\infty dr' r'\sin[\sqrt{m_K(m_K+m_I)}r]\sum_L\psi_{KL}^2(r')\\&\times\psi_{IJ}(r')-\int_r^\infty dr' r'\sin[\sqrt{m_K(m_K+m_I)}(r'-r)]\\&\times\sum_L\psi_{KL}^2(r')\psi_{IJ}(r')\Bigg),\end{aligned} \tag{65}$$

$$\begin{aligned}\chi'_{IJK}(r)\approx\frac{3m_I m_K}{8\sqrt{m_K(m_K-m_I)}r}\Bigg(&e^{2i\sqrt{m_K(m_K-m_I)}r}\\&\times\int_0^\infty dr' r'\sin[\sqrt{m_K(m_K-m_I)}r]\\&\times\sum_L\psi_{KL}^2(r')\bar{\psi}_{IJ}(r')\\&-\int_r^\infty dr' r'\sin[\sqrt{m_K(m_K-m_I)}(r'-r)]\\&\times\sum_L\psi_{KL}^2(r')\bar{\psi}_{IJ}(r')\Bigg).\end{aligned} \tag{66}$$

The classical mass loss rate becomes

$$\begin{aligned}-\frac{dM}{dt}\approx\sum_{IJK}\frac{9m_I^2m_K^2}{16}\Bigg(&\frac{2m_K+m_I}{2\sqrt{m_K(m_K+m_I)}}\\&\times\left|\int_0^\infty dr r\sin[2\sqrt{m_K(m_K+m_I)}r]\sum_L\psi_{KL}^2(r)\right.\\&\left.\times\psi_{IJ}(r)\right|^2+\frac{2m_K-m_I}{2\sqrt{m_K(m_K-m_I)}}\left|\int_0^\infty dr r\right.\\&\left.\times\sin[2\sqrt{m_K(m_K-m_I)}r]\sum_L\psi_{KL}^2(r)\bar{\psi}_{IJ}(r)\right|^2\Bigg).\end{aligned} \tag{67}$$

As in Eq. (58), in the second term that has the factors of $\sqrt{m_K(m_K-m_I)}$, if m_K-m_I is negative, the corresponding term in the sum over modes is to be omitted, since it corresponds to a χ'_{IJK}-mode frequency $2m_K-m_I$ that is below the natural frequency m_I and hence to a mode that is not freely propagating at large radial distance to carry off mass.

When only one real scalar field of mass m is present, the classical mass loss rate becomes

$$-\frac{dM}{dt}\approx\frac{27m^4}{2^{11/2}}\left|\int_0^\infty drr\,\sin(\sqrt{8}mr)\psi^3(r)\right|^2. \qquad (68)$$

With two scalar fields of identical masses present, one gets

$$-\frac{dM}{dt}\approx\frac{27m^4}{2^{11/2}}\left(\left|\int_0^\infty drr\,\sin(\sqrt{8}mr)(\psi_1^2+\psi_2^2)\psi_1\right|^2+\left|\int_0^\infty drr\,\sin(\sqrt{8}mr)(\psi_1^2+\psi_2^2)\psi_2\right|^2\right). \qquad (69)$$

Again one can readily see that if the sum of the squares of the complex ψ fields with identical masses are zero, then there is no classical mass loss, at least at this level of approximation. With just two fields of identical masses, this condition is that $\psi_1=\pm i\psi_2$.

VII. SIMPLEST SPHERICAL OSCILLATONS

Now let us focus on the simplest case, in which there is a single massive scalar field in a finite-mass spherically symmetric nearly-Newtonian configuration that is very nearly periodic in time and has no nodes. In particular, require that ψ be spherically symmetric, independent of time (except for a possible slowly varying phase factor), and nowhere zero (no nodes, though asymptotically zero at spatial infinity).

By a suitable choice of the hypersurfaces of constant time, one can cancel the phase factor to make ψ real and positive everywhere, which is what I shall assume, at the cost of having U approach a nonzero constant at spatial infinity. Then the time-dependent Newton-Schrödinger equations (20) and (32) for a single scalar field with a single real ψ become the time-independent Newton-Schrödinger equations [12–15] (where for simplicity I am using units in which $c=1$ so that I can drop many occurrences of factors of c that one can easily put back in by dimensional analysis if needed with other choices of units),

$$\nabla^2\psi\approx 2m^2U\psi, \qquad (70)$$

and

$$\nabla^2U\approx m^2\psi^2. \qquad (71)$$

In the case of spherical symmetry which I am now also assuming, the equations take the form (with ψ real) of two coupled second-order ordinary differential equations,

$$\psi''+\frac{2}{r}\psi'\approx 2m^2U\psi, \qquad (72)$$

and

$$U''+\frac{2}{r}U'\approx m^2\psi^2. \qquad (73)$$

As noted previously [12,14], this system of equations has the scale invariance

$$(\psi,U,r)\mapsto(\lambda^2\psi,\lambda^2U,\lambda^{-1}r). \qquad (74)$$

Inserting explicitly the speed of light c so that the radius r can have units of length instead of time as it implicitly does above, I shall set

$$r=\frac{cx}{k}, \qquad (75)$$

$$\psi=\frac{k^2S}{\sqrt{2}m^2}, \qquad (76)$$

$$U=\frac{-k^2V}{2m^2}, \qquad (77)$$

so with k having the units of frequency or inverse time, x, S, and V are the dimensionless variables used by Ref. [14] (except that what I now call x, they call r). Note that this V, which is just a rescaling of the Newtonian potential U with its sign reversed, is not to be confused with the $V(t,x^k)$ in the metric (14), which is negligible in the nearly-Newtonian metric.

Although so far I have used a prime generally to denote a derivative with respect to the radius r, when I am using the dimensionless rescaled radial variable x instead as the independent variable, I shall use a prime to denote a derivative with respect to x. Then Eqs. (72) and (73) (where the prime denoted d/dr) become the dimensionless time-independent Newton-Schrödinger equations

$$(xS)''=-xSV \qquad (78)$$

and

$$(xV)''=-xS^2. \qquad (79)$$

Note that I have now replaced the $\approx$ signs with $=$ signs, even though these equations are only an approximation to the actual Einstein-Klein-Gordon equations, though an approximation that becomes arbitrarily good in the nearly-Newtonian limit.

These time-independent Newton-Schrödinger equations are the same as Eqs. (6a) and (6b) of Ref. [14], except for the replacement of their radius r by my dimensionless rescaled x. The fact that the arbitrary constant k does not appear in these equations illustrates their scale invariance.

Another form of the equations that is helpful for some of the analysis below is to use

$$p \equiv \frac{1}{2}(S-V) \tag{80}$$

instead of V in Eqs. (78) and (79), which then become

$$(xS)'' = -xS(S-2p) \tag{81}$$

and

$$(xp)'' = xSp. \tag{82}$$

Yet another set of variables to use that were particularly convenient for numerical integration of the equations and for the interpretation of the results are

$$X \equiv xS, \tag{83}$$

$$\mathcal{M} \equiv -x^2 V', \tag{84}$$

$$w \equiv -(xV)' = \frac{\mathcal{M}}{x} - V. \tag{85}$$

The variable X is the same as that used in Ref. [15]. The asymptotic value of $\mathcal{M}(x)$, say $\mathcal{M}_\infty$, is what is called B in Ref. [14] and what is called I in Eq. (2.19) of Ref. [15]. The asymptotic value of $w(x)$, say w_∞, which is positive, is by Eq. (85) the same as the asymptotic value of $-V(x)$, say $-V_\infty$, which is the same as $-A$ in Ref. [14].

$\mathcal{M}(x)$ has the interpretation of the rescaled mass interior to the sphere at x. From the fact that asymptotically one has $U \sim U_\infty - M/r$ and $V \sim V_\infty + \mathcal{M}/x$ (assuming a finite-mass oscillaton in which the mass-energy density, proportional to S^2, asymptotically rapidly approaches zero), one can see that the mass (in units of time) interior to a sphere of radius $r = cx/k$ is

$$M(r) = \frac{k}{2m^2}\mathcal{M}(x). \tag{86}$$

The variable w can be written as

$$w = \frac{2m^2}{k^2}(U + rg), \tag{87}$$

where g is the acceleration of gravity at the same radius r where the gravitational potential U is evaluated,

$$g = \frac{dU}{dr} = \frac{M(r)}{r^2}. \tag{88}$$

Thus w can be interpreted as a rescaled value that the gravitational potential U would have at twice the radius r if U had a uniform gradient from r to $2r$, with this uniform gradient having the same value of the actual gradient at radius r. If the mass-energy density dropped precisely to zero outside some radius, w would be constant outside this radius, at the value $w = -V_\infty = (2m^2/k^2)U_\infty$. In the actual case in which the mass-energy density, proportional to S^2, drops exponentially toward zero, w is exponentially close to $-V_\infty$ and is therefore a good integration variable to use to evaluate the asymptotic value of V, that is V_∞.

The differential equations in terms of these variables are

$$X'' = \left(w - \frac{\mathcal{M}}{x}\right)X \equiv -VX, \tag{89}$$

$$\mathcal{M}' = X^2, \tag{90}$$

$$w' = \frac{X^2}{x}. \tag{91}$$

The initial conditions for these variables (initial in x, of course, not in time, since all of the quantities being considered presently are independent of time in the approximation that we initially ignore the mass decay rate) are that at $x = 0$, we have $X = 0$, $X' = S_0$, $\mathcal{M} = 0$, and $w = -V_0$.

As discussed in Refs. [14,15], for an everywhere regular static spherically symmetric solution to the Newton-Schrödinger equations, S and V must be smooth everywhere, and S must be decreasing exponentially at spatial infinity ($x \to \infty$). At the origin ($x = 0$), S and V must have finite values, S_0 and V_0 respectively, and must have zero dimensionless radial derivatives, $S' = 0$ and $V' = 0$.

Because of the scale invariance, the only independent nontrivial parameter for a solution regular at the origin is the ratio S_0/V_0. Integrating out from the origin gives a solution that diverges at finite radius (with S going to $+\infty$ and with V going to $-\infty$ there) if $V_0 \leqslant 0$, so we shall choose $V_0 > 0$. Using the scale invariance, without loss of generality we can and shall set $V_0 = 1$, leaving the nontrivial parameter to be S_0. If $S_0 = 0$, then we just get the trivial solution $S = 0$, $V = V_0$, which is flat spacetime with no matter, a solution we shall discard as previously studied by other people. By the symmetry of the equations under $S \mapsto -S$, we can thus choose $S_0 > 0$.

If S_0 is too large, Eq. (79) implies that xV (initially growing as x) bends down rapidly, so that V goes negative while $X = xS$ is still growing. [Initially $X = xS$ also grows linearly with x, as $S_0 x$, but, like xV, it also bends down. However, Eq. (78) implies that it does not bend down so fast as xV bends down, for $S > V > 0$.] Then when V becomes negative, X' grows with x, and so X grows faster and faster, and eventually so does S, by Eq. (78) or the equivalent Eq. (89). Equation (79) implies that then xV and eventually also V gets more and more negative. In fact, then S goes to $+\infty$ and V goes to $-\infty$ at a singularity of infinite mass at finite x. Of course, we shall discard these solutions.

On the other hand, if S_0 is positive but too small (e.g., less than $V_0 = 1$ [15]), Eq. (79) implies that V will stay positive long enough for Eq. (78) to imply that xS will oscillate (with characteristic period in the x variable of $2\pi/\sqrt{V}$ if V were constant). However, we want the regular solution with no nodes, the solution with the largest value of S_0 that does not lead to a singularity. This value of S_0 is an eigenvalue for the system.

At this eigenvalue, $X = xS$ will bend over from increasing at $x = 0$ ($X' = S_0$ there) to decreasing again toward zero value asymptotically, but never crossing zero. xV will also bend over from increasing (at unit rate) at $x=0$ (since we have chosen $V_0=1$) and will cross zero to become negative and will eventually keep decreasing at the asymptotically rate given by the asymptotic negative value of V, say V_∞. That is, S will start out at S_0 with zero derivative with respect to x ($S'=0$ initially) but will then bend down to approach zero asymptotically at large x (while then bending back upward just enough to keep from crossing zero, but never quite leveling out, except asymptotically). Similarly, as a function of x from $x=0$ to $x=\infty$, V will start out at $V_0=1$ also with zero derivative with respect to x ($V'=0$ initially) but will then bend down and cross zero before bending back up to level off toward some negative constant asymptotic value V_∞.

For this eigensolution, since X starts at zero and initially increases linearly with x, and since $\mathcal{M}'=X^2$ with $\mathcal{M}(0)=0$, and since w starts at $-V_0=-1$ with $w'=X^2/x$, we have initially (near the origin, $x\ll 1$),

$$\mathcal{M}\sim\frac{1}{3}S_0^2x^3-\frac{1}{15}S_0^2x^5, \tag{92}$$

$$w\sim-1+\frac{1}{2}S_0^2x^2, \tag{93}$$

$$X\sim S_0x-\frac{1}{6}S_0x^3, \tag{94}$$

$$S\sim S_0-\frac{1}{6}S_0x^2, \tag{95}$$

$$V\sim 1-\frac{1}{6}S_0^2x^2, \tag{96}$$

$$p\sim\frac{1}{2}(S_0-1)\left(1+\frac{1}{6}S_0x^2\right). \tag{97}$$

Then as x is increased to ∞, X will at first increase, while bending downward and eventually passing a maximum and then decreasing. While decreasing, X will pass an inflection point (at the point at which V crosses below zero) and will then bend upward to level out asymptotically as it also approaches zero asymptotically. At the same time (here meaning during the same evolution in x from 0 to ∞), $\mathcal{M}$ will start from 0 with zero slope and curvature and initially grow as the cube of x (i.e., as the volume interior to the sphere of radius $r=cx/k$) but eventually will reach an inflection point (at the point at which X reaches its maximum) and then gradually level off to approach its asymptotic value $\mathcal{M}_\infty$. The variable w will start at -1 at $x=0$ with zero slope and will bend up to cross zero, before bending downward to level off asymptotically and approach its asymptotic value $w_\infty=-V_\infty>0$.

For use below in calculating the quantum decay rate of this type of oscillaton, it is also of interest to integrate the variable $D(x)$ given by the differential equation

$$D'=x^2S^4=\frac{X^4}{x^2} \tag{98}$$

with the boundary condition $D(0)=0$. Since the mass density in conventional units is (with m and k in frequency units)

$$\rho=\frac{m^2|\psi|^2}{4\pi G}=\frac{k^4S^2}{8\pi Gm^2}, \tag{99}$$

the asymptotic value of D, namely D_∞, is proportional to the integrated square of the mass-energy density (and hence to the total annihilation rate of two scalarons into two gravitons):

$$D_\infty=\frac{16\pi m^4}{k^5}\int (G\rho)^2d^3x. \tag{100}$$

I have used the differential equation routine of MAPLE 8 to evaluate to high accuracy the eigenvalue S_0 and the asymptotic values $\mathcal{M}_\infty$ and w_∞. The eprint version of this paper [17] explains in more detail the numerical procedure and gives all of the 30–35 significant digits of the results, but here I shall give only a smaller number of digits.

The values I obtained, rounded to 19 digits, were

$$S_0\approx 1.088\ 637\ 079\ 429\ 044\ 996, \tag{101}$$

$$\mathcal{M}_\infty\approx 3.618\ 701\ 237\ 823\ 656\ 810, \tag{102}$$

$$w_\infty\approx 1.065\ 731\ 278\ 365\ 451\ 059, \tag{103}$$

$$D_\infty\approx 1.320\ 680\ 334\ 028\ 957\ 064. \tag{104}$$

One can see that my value for S_0 confirms all but the last of the 15 digits given for this quantity by Ref. [14].

From these numbers, one can of course construct various combinations of them, such as the scale-invariant quantity A/B^2 that Ref. [14] discusses:

$$\frac{A}{B^2}=-\frac{w_\infty}{\mathcal{M}_\infty^2}\approx-0.081\ 384\ 603\ 921\ 072\ 995. \tag{105}$$

The first two nonzero digits of this quantity seem to agree with the value [14] plotted in their Fig. 4, but they do not list its numerical value.

From the scaling relation given by Eq. (86), we can get the small dimensionless mass parameter of the nearly-Newtonian oscillaton,

$$\mu\equiv Mm=\frac{k}{2m}\mathcal{M}_\infty. \tag{106}$$

Since in the end we want to express other properties of the oscillaton in terms of μ, we shall actually invert this to get the scaling parameter k (which has units of frequency, as does m) as

$$k=\frac{2m\mu}{\mathcal{M}_\infty}. \tag{107}$$

For example, the value of the Newtonian potential at infinity is

$$U_\infty = -\frac{k^2}{2m^2}V_\infty = \frac{2w_\infty}{\mathcal{M}_\infty^2}\mu^2 \approx 0.162\ 769\ 207\ 842\ 145\ 990\ \mu^2. \tag{108}$$

We can then express the fractional binding energy of a nearly-Newtonian oscillaton as

$$\frac{E}{M} = -\frac{1}{3}U_\infty = -\frac{k^2}{6m^2}V_\infty = -\frac{2w_\infty}{3\mathcal{M}_\infty^2}\mu^2 = -\epsilon\mu^2 \tag{109}$$

with

$$\epsilon = \frac{2w_\infty}{3\mathcal{M}_\infty^2} \approx 0.054\ 256\ 402\ 614\ 048\ 663\ 473. \tag{110}$$

If we take the number of scalar particles to be

$$N = \frac{M_*}{m_*} = \frac{c^5 M}{\hbar G m}, \tag{111}$$

so the conventional rest mass of the oscillaton is N times the conventional rest mass of the scalar field quantum, $M_* = N m_*$, then the conventional total energy of the oscillaton is, to first order in μ^2,

$$E_{\mathrm{tot}} = N m_* c^2 (1 + E/M) = N m_* c^2 - N^3 m_*^5 \epsilon G^2/\hbar^2. \tag{112}$$

This would be the same value for a boson star with a complex scalar field having a $U(1)$ symmetry. For that problem the value was calculated by Ruffini and Bonazzola [18] over 34 years ago, getting $\epsilon = 0.1626$. Except for the last digit, this corresponds to three times the value above.

One can also compare my numerical results with the 5-place results for a boson star by Friedberg, Lee, and Pang [12] (for $n=0$ nodes). In terms of my calculated parameters and my numerical results, their calculated parameters would be

$$\hat{\gamma}_0 = -V_0/S_0 = 1/S_0 \approx 0.918\ 579\ 771\ 804\ 638\ 252, \tag{113}$$

$$\hat{\gamma}_\infty = w_\infty/S_0 \approx 0.978\ 959\ 194\ 486\ 001\ 441, \tag{114}$$

$$\hat{\gamma}_1 = \mathcal{M}_\infty/S_0^{1/2} \approx 3.468\ 256\ 171\ 397\ 572\ 160. \tag{115}$$

They got $\hat{\gamma}_0 = -0.91858$, $\hat{\gamma}_\infty = 0.97896$, and $\hat{\gamma}_1 = 3.46826$, in perfect agreement with my results rounded to five digits after the decimal point.

VIII. CLASSICAL EMISSION FROM THE SIMPLEST SPHERICAL OSCILLATONS

Now that the nodeless spherically symmetric nearly-Newtonian configurations have been found (determined by the single scaling parameter $\mu = Mm$), we need to use Eq. (68) to evaluate the classical mass loss rate. By using Eqs. (75), (76), and (107), we can write the classical mass loss rate, or power emitted in classical scalar radiation (dimensionless when the mass M is in time units, i.e., $M = GM_*/c^2$ in terms of the mass M_* in conventional mass units), as

$$P_c \equiv -\dot{M}_{\mathrm{classical}} \approx \frac{27\mu^8}{\sqrt{2}\mathcal{M}_\infty^8}F^2, \tag{116}$$

$$F = \int_0^\infty dx\, x S^3(x)\sin(ax), \tag{117}$$

$$a = \frac{\sqrt{2}\mathcal{M}_\infty}{\mu}. \tag{118}$$

Because the nearly-Newtonian configurations have $\mu \ll 1$, the μ-dependent constant a is very large, $a \gg 1$. Therefore, the sin(ax) factor in the integral (117) for F oscillates very rapidly and nearly washes out the integral for large a, causing F to be very small.

We can estimate the value of F by the following method of contour integration: Since $S(x)$ is an even function of x, the integrand is an even function of x, and so the integral along the real axis from 0 to ∞ may be replaced by half of the integral along the real axis from $-\infty$ to $+\infty$. Then the real variable x may be extended to the complex variable z, as a function of which the integrand is analytic except at poles of $S(z)$. One may split $\sin(az) = 0.5ie^{-iaz} - 0.5ie^{iaz}$ into the first exponential, which drops exponentially along the negative imaginary axis for the complex z, and the second exponential, which drops exponentially along the positive imaginary axis for the complex z. By splitting up the integral into the corresponding two pieces, the first piece may be replaced by a contour integral making a clockwise loop around the lower half plane ($\mathrm{Im}\, z < 0$), and the second piece may be replaced by a contour integral making a counterclockwise loop around the upper half plane ($\mathrm{Im}\, z > 0$), which gives an equal contribution.

Now one of these contour integrals, say the second one (since they each give equal contributions), may be evaluated by finding the residues at each of the poles of the integrand. There are a series of poles of $S(z)$ running up the imaginary z axis. Because of the e^{iaz} factor from sin(az), the dominant residue will come from the pole closest to the real axis, say at $z = iy_0$.

From an analysis [14,15] of the time-independent Newton-Schrödinger equations in their dimensionless form (78) and (79), one can see that the solutions are analytic over the complex z plane, except for movable (moving if S_0 were changed) double poles with coefficients -6. In particular, near the pole at $z = iy_0$, S has the asymptotic form

$$S(z) \sim \frac{-6}{(z - iy_0)^2}. \tag{119}$$

Because it is $S^3(z)$ that appears in the integral (117) for F, which thus has a 6th-order pole at $z = iy_0$, one must integrate this factor by parts five times, giving the 5th derivative of the e^{iaz} factor, and hence 5 powers of a, in the dominant term of the result. (I shall drop terms with lower powers of a that arise from differentiating the z factor rather than the e^{iaz} factor, since they involve higher powers of the small quantity μ.) As a result, one finds that

$$F \approx -1.8\pi a^5 y_0 e^{-ay_0}. \tag{120}$$

Thus there is one more parameter that must be determined numerically before we can evaluate explicitly the dominant term (for $\mu \ll 1$) in the classical mass loss rate, namely the value y_0 that locates the pole in $S(z)$ at $z = iy_0$ that is above the real axis but nearest to it.

To find y_0, one can integrate the time-independent Newton-Schrödinger equations (78) and (79) up the imaginary z axis (after replacing the real radial variable x with the complex radial variable z). For more details of the numerical procedure and for the result to more than 30 significant digits, see Ref. [17]. Rounded to 19 digits, the pole location is at

$$y_0 \approx 3.852\ 750\ 221\ 596\ 529\ 692. \tag{121}$$

Now that we have the last parameter, y_0, that we need to evaluate F by Eq. (120), we can go back and plug the result into Eqs. (116) and (118) to give the classical mass loss rate as

$$P_c \equiv -\dot{M}_{\mathrm{classical}} \approx \frac{C}{\mu^2} e^{-\alpha/\mu}, \tag{122}$$

where

$$\alpha = \sqrt{8}\mathcal{M}_\infty y_0 \approx 39.433\ 795\ 197\ 160\ 163\ 094, \tag{123}$$

and where

$$C = \frac{2^{3/2}3^7}{5^2}\pi^2\alpha^2 \approx 3\ 797\ 437.776\ 333\ 014\ 909. \tag{124}$$

I should emphasize that this is what I believe to be just the dominant term in the classical emission of scalar waves from a nearly-Newtonian oscillaton when $\mu \ll 1$. I would expect this expression to have a relative error of the order of μ.

To put it another way, one can presumably write

$$\ln\frac{1}{P_c} = \frac{\alpha}{\mu} - 2\ln\frac{1}{\mu} - \ln C + O(\mu)$$
$$\approx \frac{39.433\,795\,197\,160\,163\,094}{\mu}$$
$$-2\ln\frac{1}{\mu} - 15.149\ 837\ 127\ 888\ 728\ 199 + O(\mu), \tag{125}$$

so that the calculations performed here have given the three leading terms in an expansion for $\ln(1/P_c)$.

It is beyond the scope of this paper to do the nonlinear gravitational calculations to find the classical mass loss rate when μ is not small, but if one neglects the $O(\mu)$ corrections, one can make a very crude estimate for the mass loss rate even up to the maximum value of μ, say μ_{max}.

For what are generally called boson stars (stationary spherical configurations of a complex massive scalar field whose phase rotates in a circle in the complex plane, equivalent to two real scalar fields oscillating 90° out of phase), the maximum value of μ is 0.633 [12]. For oscillatons of a real massive scalar field, the initial calculations [1] gave $\mu_{\mathrm{max}} \approx 0.6$. Alcubierre *et al.* [9] have given $\mu_{\mathrm{max}} = 0.607$.

For example, if the coefficient of the $O(\mu)$ term of Eq. (125) were the same magnitude (but of uncertain sign) as the coefficient of the $1/\mu$ term, namely $\alpha \approx 39.4338$, then just taking this single term with, say $\mu = 0.633$, would change $\ln(1/P_c)$ by roughly ± 25.0, or a total range of roughly 50.0 for this quantity, giving an uncertainty in the mass decay rate by a factor of about $e^{50.0} \sim 5\times 10^{21}$. One might hope that the uncertainty is a lot less, but without calculating the $O(\mu)$ and higher terms, I do not see how one can be sure.

Despite this proviso, if we did naively insert the boson star maximum mass parameter $\mu_{\mathrm{max}} \approx 0.633$ into Eq. (122), we would get a mass loss rate (dimensionless, since our masses denoted by M have the factor of G/c^3 inserted to give them the dimension of time) of about 8×10^{-21}. However, if it could be larger or smaller by a factor of roughly $e^{25.0} \sim 7\times 10^{10}$, the dimensionless mass loss rate could be as large as roughly 6×10^{-10} or as small as roughly 10^{-31}.

In any case, unless the coefficient of μ in the $O(\mu)$ term of Eq. (125) (or actually this entire correction term divided by μ) were negative and had a larger magnitude than the coefficient of the $1/\mu$ term, it seems that the dimensionless mass loss rate is always less than about $10^{-9}\mu$. This means that during a one-radian change in the phase of the scalar field oscillation (a time $t = 1/m$), the oscillaton would have lost less than one-billionth of its mass. If instead the correction to applying Eq. (122) to $\mu = \mu_{\mathrm{max}}$ were negligible, then during the 2π-longer period of a full scalar field oscillation, even a maximum-mass oscillaton would have lost less than a billionth of a billionth of its mass.

This result shows why the numerical analyses to date have not shown any instability of the oscillaton, since its mass loss rate is so low.

Figure 6 of Ref. [9] for $\mu = 0.5726$ shows an apparent numerical mass loss rate of about $3\times 10^{-9} \sim 5\times 10^{-9}\mu$ in dimensionless units, but since this is several times higher than the crude guess above for the upper limit for the mass loss rate, and is $\sim 2\times 10^{14} \sim e^{33} \sim e^{58\mu}$ times larger than what my rashly interpolated formula would predict for that μ, namely $P_c \sim 1.4\times 10^{-23}$, I suspect that the authors are indeed correct in attributing this to "a small amount of numerical dissipation still present in our numerical method." However, I cannot completely rule out the possibility that the $O(\mu)$ term in Eq. (125) is roughly -58μ at $\mu = 0.5726$, so that there conceivably might be mass loss comparable to that given by Fig. 6 of Ref. [9].

Later we shall find that for sufficiently small μ, the classical mass loss rate (122) is dominated by a quantum mass loss rate. However, when the classical mass loss rate dominates, and when $\mu \ll 1$, the time $t_2 - t_1$ to evolve from μ_1 to μ_2 is approximately

$$t_2 - t_1 \approx \frac{\mu_2^4}{\alpha C m} e^{\alpha/\mu_2} - \frac{\mu_1^4}{\alpha C m} e^{\alpha/\mu_1}. \tag{126}$$

For example, we could define $t(\mu)$ to be the time to decay from $\mu = \mu_{\max}$ to some smaller value of μ. Then if $\mu_{\max} - \mu \gg \mu_{\max}^2/\alpha \sim 0.00934$, then the magnitude of the first term on the right hand side of Eq. (126) (with $\mu_2 = \mu$) is much greater than the magnitude of the second term (say with $\mu_1 = \mu_{\max} \approx 0.607$). Since this is necessarily the case for $\mu \ll 1$ where Eq. (126) is applicable, we may then drop the second term and say that the time to decay down to $\mu \ll 1$ from $\mu_{\max}$ by the classical emission of scalar radiation is

$$t(\mu) \approx \frac{\mu^4}{\alpha C m} e^{\alpha/\mu}. \tag{127}$$

IX. PRECISELY PERIODIC BUT INFINITE-MASS OSCILLATONS

Although it is beyond the scope of the present paper, which just gives numerical results for $\mu \ll 1$, it would be of interest to be able to calculate the function $t(\mu)$ for all $\mu < \mu_{\max}$. To calculate "exact" results (i.e., exact up to numerical errors in solving the differential equations), one needs a precise definition of $\mu_{\max}$ and of $\mu(t)$ for the time thereafter.

It is rather hard to define $\mu_{\max}$ precisely (and the initial oscillaton configuration that gives this maximum mass), since any initial configuration is losing mass (assuming boundary conditions of no incoming scalar waves), so one could start with a wide variety of initial configurations. But essentially one would like to start with one out of a set of initial configurations that lose mass as slowly as possible for each initial mass, and then choose the maximum-mass element from the set members that do not have rapid mass loss (e.g., at a rate roughly given by the dynamical timescale $1/m$). If the initial decay time scale is of the order of $[\mu_{\max}^4/(\alpha C m)]e^{\alpha/\mu_{\max}}$, as the weak-field formulas would suggest, then to the degree that this is much larger than $1/m$, one can define the maximum-mass initial configuration to that accuracy, i.e., with a relative error that would be expected to be of the order of $(\alpha C/\mu_{\max}^4)e^{-\alpha/\mu_{\max}}$. However, to define it to more accuracy (e.g., "precisely") would require more care.

Although I do not see any highly preferred way to make the definition precise, I can propose the following *ad hoc* method, which leads to a consideration of precisely periodic but infinite-mass oscillatons:

Temporarily relax the condition of no incoming waves that has been fundamental to the discussion so far. Then it appears that one can have precisely periodic oscillatons, though now with infinite mass, so that the spacetime is not quite asymptotically flat (although having curvatures falling off fairly rapidly, so, for example, the spatial integral of the Kretschman invariant, the total four-fold contraction of the square of the Riemann tensor, is finite at any time).

The idea of these precisely periodic (though infinite mass) oscillatons is that not only does the oscillation of the scalar field at frequency m (with respect to a suitably scaled coordinate time t in a gauge or choice of coordinates in which $g_{0i} = 0$ and g_{00} is independent of t) have precisely the right phase to drop exponentially to zero at large spatial distance by the gravitational binding of that mode, but also the oscillations at all odd multiples of m also have precisely the right phases to drop exponentially to zero at large spatial distance by the gravitational binding of those modes as well, when one takes into account the gravitational field not only of the mode at frequency m but also the gravitational field of all the higher-frequency modes.

That is, one has a sequence of relations for the modes at each frequency that start as follows for the lowest mode: The mode with frequency m is chosen to have the right phase (e.g., the right relation of its initial-in-r value at $r = 0$, determined by the quantity S_0 above, to the value of g_{00} there, determined by the quantity V_0 above in the weak-field or nearly-Newtonian limit), so that once the stress-energy tensor of the scalar field (predominantly from this mode at small radii) causes $-g_{00}$ to rise above unity at sufficiently large radius r (so that this mode there has a proper-time frequency that is less than its natural frequency m and hence has a concave or exponential radial dependence for larger values of r), one has only the asymptotically exponentially decaying behavior of the mode at larger values of r.

However, the oscillation of the g_{ij} components of the metric (at frequencies that are even multiples of m) couples, via the Klein-Gordon equation for the minimally coupled massive scalar field, the modes of the scalar field that have frequencies that are different odd multiples of m. Thus, for example, the mode with frequency $3m$ is excited and propagates out from the region of the oscillaton where the mode with frequency m dominates the stress-energy tensor.

So far, this is just like the decaying oscillaton described above. If there are no incoming waves at frequency $3m$, the outgoing waves carry off energy from the oscillaton, which decays (and hence is not precisely periodic in time).

On the other hand, for the precisely periodic but infinite-mass oscillaton, there are both incoming and outgoing waves at frequency $3m$, and their stress-energy tensor (which at sufficiently large r dominates over that of the mode of frequency m, since that m-frequency mode is asymptotically exponentially decaying at sufficiently large r) eventually causes $-g_{00}$ to rise above 9, so that then the mode with frequency $3m$ also has an approximately exponential radial dependence at larger r. With the right choice of phase of this mode [i.e., of how big is the part that goes roughly as $\sin(\sqrt{8}mr)/r$ at small r], only the decaying exponential is present, and the mode at frequency $3m$ also goes to zero exponentially rapidly at sufficiently large r. (The mode at frequency $3m$ starts to fall off exponentially in radius at a much larger radius than the mode at frequency m does.) That is, the phase is chosen so that at each radius where this mode is oscillating radially, there are equal parts of outgoing and incoming waves, and then the rising gravitational potential from the stress-energy tensor of mostly that mode makes the mode gravitationally bound so that the outgoing waves reflect off the potential barrier and become the incoming modes at smaller radii.

Of course, there is also coupling to modes with frequency $5m$, and these modes must also be given the right phase to have equal parts of outgoing and incoming waves for radii at which $-g_{00}<25$, so that when one gets so far out that the energy density of that mode, mostly, causes $-g_{00}$ to rise above 25 and that mode also to develop an exponential behavior, one has only the approximately exponentially decaying piece.

And so it goes for all higher modes with frequencies that are odd multiples of m, say $(2n+1)m$. They must each be chosen so that have equal amounts of outgoing and incoming waves for $-g_{00}<(2n+1)^2$, where that mode oscillates in the radial direction, and then that mode is totally reflected by the gravitational potential for $-g_{00}>(2n+1)^2$.

Since this process must continue indefinitely in order that the oscillaton be precisely periodic in time and not be losing energy to outgoing waves of any mode, one must have $-g_{00}$ rising indefinitely, so the metric is not asymptotically flat and indeed has $M(r)$, the mass interior to radius r, rising indefinitely with r: these are infinite-mass oscillatons. (They are also almost certainly unstable to small perturbations, but that is another story.)

These precisely periodic but infinite-mass oscillatons are of course not physically realistic, but they do make interesting theoretical solutions of the coupled Einstein-Klein-Gordon equations that one may use for such purposes as defining a precise canonical (though rather *ad hoc*) maximum mass for a finite-mass oscillaton without nodes.

For example, one could take a precisely periodic oscillaton and then evaluate $M(t,r)$ at a time t when the scalar field at that r is passing through zero, and at a value of r where the time average of the energy density of the mode with frequency m is equal to the time average of the energy density of the mode with frequency $3m$. (If the scalar field passes though zero more than twice in each coordinate time period $2\pi/m$, then choose one of the times at which the field at that r is zero but the magnitude of the time derivative of the field is a minimum among all of these times when the field is zero there. If that still does not uniquely specify the time within one period of the metric, which is a half-period π/m of the scalar field, then go to the minimum of the magnitude of the second time derivative of the field among those times, etc, until the degeneracy is broken. If the degeneracy never is broken, then presumably the mass does not depend on which of those times it is evaluated.)

The value of $M(t,r)$ at that value of t and of r could then be said to be a canonical (though *ad hoc*) finite precise value for the mass of a decaying finite-mass oscillaton without nodes that is in some sense represented by the precisely periodic infinite-mass (at $r=\infty$) oscillaton.

The idea of the representation is that in the precisely periodic oscillaton, the value of r determined above would be where the mode with frequency m has decayed exponentially to a very small value, where its time-averaged energy density has dropped to that of the tiny amount of outgoing and incoming waves at frequency $3m$. For the finite-mass decaying oscillaton, if one does not include the energy density of the modes with frequency $3m$ or higher outside this radius (e.g., if one assumes that they are just starting to be emitted and so at that time are negligible outside this radius), the energy density of the mode with frequency m is so small at that radius (and dropping roughly exponentially with radius) that there is a negligible addition to the mass from that energy density in the finite-mass oscillaton.

That is, it is negligible unless we are wanting to get some absolutely precise number to assign as the mass of the oscillaton, in which case we must go from the slightly-poorly-defined initial configuration of a finite-mass decaying oscillaton to the precisely defined (but unphysical) infinite-mass oscillaton.

To get a representation of the maximum mass of a decaying finite-mass oscillaton by this sort of precise mathematical definition, one must choose the right precisely periodic oscillaton. There can be an arbitrary number of nodes in the mode with frequency m before it decays approximately exponentially in the radial direction, and to model the decaying finite-mass oscillaton which has no nodes in its mode with frequency m, we want the precisely periodic oscillaton also to have no nodes in that mode. On the other hand, we want the modes at frequencies that are higher odd multiples of m, say $(2n+1)m$ for each positive integer n, to have as low a value of energy density possible, which would mean they should each have the largest number of nodes possible before $-g_{00}$ rises above $(2n+1)^2$ [at a rate with respect to radius that is given mainly by the energy density in the outgoing and incoming waves at that frequency, once $-g_{00}$ rises above $(2n-1)^2$ and the mode at the next lower frequency, $(2n-1)m$, is reflected back inward].

Smaller numbers of nodes are possible for each of the higher-frequency modes if there is an extra magnitude of outgoing and incoming waves at that frequency, so there appears to be a whole infinite sequence of integers to be specified for the generic precisely periodic oscillaton, even after specifying, say, the time-averaged energy density at the center to give the one continuous parameter of the classical oscillaton that is the nonlinear-gravity analogue of the scale of the nearly-Newtonian oscillaton. This means that there should be uncountably many periodic oscillatons at the same value of the central time-averaged density.

It is plausible that they form a fractal set of perpetually oscillating spatially inhomogeneous but spherically symmetric periodic-in-time solutions of the Einstein-Klein-Gordon equation, somewhat analogous to the apparently fractal set of homogeneous but nonperiodic cosmological solutions of the same equations [19].

However, here we want just the simplest example, with the fastest possible falloff of the energy density, to use to define a precise value for the maximum mass of a finite-mass oscillaton with no nodes for the mode with frequency m and only outgoing components for the modes with higher frequencies—these frequencies being only approximate in the slightly nonperiodic case in which the finite-mass oscillaton is actually slowly decaying with time.

Once we have chosen the simplest precisely periodic oscillaton for each possible value of the central time-averaged energy density, we still need to choose the one that gives the maximum value of the mass defined by the procedure above. That will be the unique precisely periodic oscillaton that we will use to represent in some sense the initial state of a finite-mass decaying oscillaton, and it will give us a precise (though rather *ad hoc*) mathematical definition of μ_{max}.

Now to get a definite initial state [say for giving a precise definition of $\mu(t)$ and of its inverse, $t(\mu)$] for a finite-mass decaying oscillaton that is represented by the unique precisely periodic oscillaton above, one could take the initial metric and scalar field values and time derivatives of the precisely periodic infinite-mass oscillaton at the time t defined above for determining the value of $M(t,r)$ that was used to represent the maximum mass of a decaying oscillaton, out to the value of r that was also determined by the procedure above (where the time average of the energy density of the mode with frequency m had dropped to the same time average of the mode with frequency $3m$). One could then truncate the precisely periodic oscillaton at that radius, replacing its initial data on that hypersurface with initial data that agreed for smaller values of r but which for larger values of r has zero for both the scalar field and its time derivative, and the Schwarzschild values for the initial data of the metric. (That is, the solution can be taken to be vacuum Schwarzschild on the initial time surface for larger values of r.)

Next, simply let this truncated oscillaton initial data evolve by solving the Einstein-Klein-Gordon equations, to represent the decay of an initially maximal-mass oscillaton.

As energy flows out from the decaying oscillaton in the form of scalar waves moving slower than the velocity of light, the ADM mass and also the mass at future null infinity stay constant, so we need a different definition of the mass of the decaying oscillaton to define a nontrivial time dependence for it. For that, we can simply choose $M(t,r)$, with the coordinate value of r kept fixed at the value where, in the periodic oscillaton that provided the initial data out to that radius, the time-averaged energy density of the mode with frequency m equaled that in the mode with frequency $3m$.

Here the coordinate value of r, once determined for the precisely periodic oscillaton, is assumed to be kept rigid, with no gauge freedom, say by continued use in the decaying oscillaton of the gauge choice that $g_{0i}=0$ and that g_{00} has no periodic component. However, since g_{00} has a slow secular evolution for the decaying oscillaton, I am not quite sure how to fix the gauge absolutely precisely in this case, since I do not see how uniquely to disentangle this slow secular evolution from a periodic component. Perhaps if one wanted an absolutely precise mathematical definition, which has been the aim of this long discussion, one should use instead as the sphere where $M(t,r)$ is to be evaluated as a function of r, a sphere that has a constant circumference (the Schwarzschild r coordinate, after taking r to be the circumference divided by 2π, though not the r coordinate that makes g_{00} independent of time for the precisely periodic oscillaton).

There is also a slight ambiguity in defining t absolutely precisely if g_{00} has a slow secular evolution rather than being precisely independent of t as it can be in the precisely periodic oscillaton with a suitable choice for the time and spatial coordinates. One commonly used precise way to define it would be to define the constant-t hypersurfaces to be orthogonal to the worldlines at constant angular coordinates on the spheres of constant circumference, which leads to so-called Schwarzschild coordinates.

One disadvantage of the Schwarzshild (t,r) coordinates is that inside the oscillaton, the worldtubes of constant Schwarzschild r (circumference divided by 2π) oscillate in and out, relative to coordinates in which not only is $g_{0i}=0$ (as is true also for the Schwarzschild coordinates by construction) but also g_{00} is constant (for the precisely periodic oscillaton). Then since the hypersurfaces of constant t in Schwarzschild coordinates are by construction orthogonal to the worldlines of constant Schwarzschild r (and of constant angular position on the 2-spheres), these hypersurfaces also bend forward and back by a rather considerable amount in the interior of the oscillaton, giving periodic effects that depend on this global choice of Schwarzschild coordinates and are actually considerably larger than any effect in the local geometry.

However, at the value of r defined above, where the time-average of the energy density of the mode with frequency m has dropped down to equal the corresponding value for the tiny outgoing and incoming wave modes with frequency $3m$, one is where the energy density of the oscillaton is so low that the radial oscillations of the world tubes of constant Schwarzschild r are very small and should not have a significant effect.

One still has to define the t labeling of this foliation into constant-t hypersurfaces. When the metric is asymptotically flat, as it would be for the decaying oscillaton, then one can define it so that t is the proper time along the world tube of infinite circumference. However, with this normalization, the coordinate-time period of the approximately periodic decaying oscillaton would be shifted from the value $2\pi/m$ that it is given by construction (normalization of t) in the precisely periodic oscillaton.

Another simple but inequivalent choice would be to choose t to be proportional to proper time of the central worldline at $r=0$. Then one can have the approximate period of the approximately periodic decaying oscillaton be very near $2\pi/m$ by choosing the constant of proportionality between proper time and coordinate time to be the same as it is in the precisely periodic oscillaton at $r=0$ in the gauge for that solution in which g_{00} is independent of time. However, one should note that as the oscillaton decays and the gravitational potential at the center changes relative to that at infinity, the nearly periodic oscillations of the scalar field will have its approximate period of oscillation secularly shifted from the value $2\pi/m$ that it has initially by this choice of t.

A third choice for the labeling of the constant-Schwarzschild-t hypersurfaces would be to choose at $r=0$ the time coordinate t so that the quantity ψ defined by Eq. (21) would be real and positive for all t. This essentially forces the coordinate t to be chosen so that, at the center at least, the coordinate period of oscillation of the scalar field Φ (or of the rescaled dimensionless field ϕ) is fixed to be $2\pi/m$. This is perhaps the best choice if one wants to count the number of oscillations of the scalar field, and to make a precise count even when that number is not an integer. Henceforth we shall assume that we have made this choice for the Schwarzschild time coordinate t (which will *not* be the same choice that would make t equal to proper time at radial infinity).

Once a suitably time-evolving sphere is defined to represent the outer surface of the decaying oscillaton (with the waves that go outside that radius being considered outgoing waves rather than part of the oscillaton), and once one has a precise time coordinate t, one can in principle solve numerically for the mass as a function of this t. Multiplying the mass $M(t)$ (in time units) by the constant m (the natural frequency of the scalar field, in inverse time units) gives $\mu(t)$, the dimensionless measure of the evolving mass of the oscillaton.

Assuming that $\mu(t)$ is monotonically decreasing with t (which it conceivably need not be within each period, since the waves need not be purely outgoing over the entirety of a period, though they should be when averaged over an integral number of periods), one can invert this relation to get $t(\mu)$ to see the time needed for an oscillaton to decay from the maximum-mass configuration to a smaller value of μ. In the nearly-Newtonian limit in which $\mu \ll 1$, one would expect that this time should asymptotically approach that given by Eq. (127), not in the sense that the difference between the actual $t(\mu)$ and that given by this formula would go to zero, but in the sense that the ratio of the actual $t(\mu)$ to that given by Eq. (127) would tend to unity as μ tends to zero in this purely classical calculation.

Of course, it would be interesting to calculate numerically the value of μ_{max} and of the ratio of the actual $t(\mu)$ to that given by Eq. (127), the latter being the function

$$R(\mu) \equiv \frac{\alpha C m}{\mu^4} e^{-\alpha/\mu} t(\mu). \tag{128}$$

In particular, it would be interesting to calculate the large dimensionless number

$$q \equiv m t(e^{-1}\mu_{max}), \tag{129}$$

which is the angle by which the phase of the oscillating scalar field advances (2π times the number of scalar field oscillations) as the oscillaton decays from its maximum mass to a mass that is smaller by a factor of e. The numerical calculations done above for the nearly-Newtonian limit imply that

$$q \approx R(e^{-1}\mu_{max}) \frac{\mu_{max}^4}{e^4 \alpha C} e^{e\alpha/\mu_{max}}. \tag{130}$$

This dimensionless number q, which is somewhat of an analogue of an average Q value of the oscillaton system for masses within a factor of e of the maximum mass, is a pure mathematical number determined purely by the Einstein-Klein-Gordon equations (with no dependence upon the scale set by the scalar field natural frequency m) and by the mathematically-precise (but admittedly rather *ad hoc*) procedure given above.

It is interesting that q is apparently quite large (at least if the correction factor $R(e^{-1}\mu_{max})$ is not too many orders of magnitude smaller than unity), a counterexample to the folklore that numbers defined purely mathematically tend to be within a few orders of magnitude of unity. For example, if $\mu_{max}=0.607$ and if $R(e^{-1}\mu_{max})$ were unity, then one would get $q \approx 8.20 \times 10^{65}$. [However, there are other counterexamples that are even more extreme, such as the unknown prime n that is the first positive integer greater than one such that the number of primes less than or equal to n, namely $\pi(n)$, is greater than $\mathrm{Li}(n)$, the principal value of the integral of $1/\ln x$ from $x=0$ to $x=n$, which forms a good asymptotic estimate for $\pi(n)$.]

It would certainly be worthwhile to calculate numerically the values of μ_{max} and of q, as well as the function $R(\mu)$ for values of μ up to μ_{max}, but since that cannot be done within the nearly-Newtonian calculation reported here, it is beyond the scope of this paper and will have to wait for future research.

It would also be amusing mathematically to get a quantitative description of the particular precisely periodic oscillaton described qualitatively above that I have used to represent (by its part within the radius r defined above) the maximum-mass decaying oscillaton. In particular, what is the quantitative asymptotic behavior of the metric and of the scalar field [each mode of which, of frequency $(2n+1)m$, eventually decays approximately exponentially in the radial direction as the integrated gravitational effect of this and other modes causes $-g_{00}$ to rise above $(2n+1)^2$ so that the mode becomes gravitationally bound]? However, this will also be left to future work.

X. QUANTUM DECAY OF SINGLE-FIELD OSCILLATONS

Besides the classical decay of finite-mass oscillatons to outgoing scalar radiation, there are also quantum decay processes that appear to be dominated by the annihilation of two scalar particles into two gravitons. This rate also goes to zero as μ is taken to zero, but only as a power-law in μ, so for sufficiently small μ (depending on the ratio of m to the Planck value), it actually dominates over the classical decay into scalar radiation.

The annihilation cross section for two nonrelativistic scalar particles to annihilate into two gravitons has been given by DeWitt [20] to be (where I have inserted the factors of the speed of light, c, that he and I usually set equal to unity)

$$\sigma_{\rm NR}=\frac{2\pi G^2 m_*^2}{c^3 v}=\frac{2\pi\hbar^2 G^2 m^2}{c^7 v}, \tag{131}$$

where the first form is in terms of the scalar particle mass in conventional mass units, which I have been calling m_*, and the second form is in terms of the scalar field natural frequency in inverse time units, which I have been calling m, which is given by Eq. (2) as $m=m_* c^2/\hbar$.

Therefore, if we have a number density n of scalar field particles of one species (more than one species will be considered in the next section), and hence a conventional mass density

$$\rho=m_* n=(\hbar m/c^2)n, \tag{132}$$

the annihilation rate (per time) for one particle passing through is

$$R=n\sigma_{\rm NR}v=(2\pi\hbar^2 G^2 m^2/c^7)n. \tag{133}$$

Two scalar particles annihilate in each such process, but when one takes the square of the number density, there is also a factor of 2 overcounting the number of pairs of identical particles, so these two factors of 2 cancel each other and give the number rate per time per volume by which scalar particles annihilate as

$$-\frac{dN}{dt\,d({\rm vol})}=Rn=(2\pi G^2 m_*^2/c^3)n^2=(2\pi\hbar^2 G^2 m^2/c^7)n^2 =2\pi G^2\rho^2/c^3. \tag{134}$$

In the nearly-Newtonian limit (which is where the nonrelativistic annihilation cross section formula would apply), the mass density with one scalar field present, and represented by the dimensionless complex field ψ given by Eq. (21), is

$$\rho=\frac{m^2|\psi|^2}{4\pi G}. \tag{135}$$

If one integrates over an oscillaton with one real scalar field (one complex ψ), the total annihilation rate per time is

$$-\frac{dN}{dt}=\frac{1}{8\pi c^3}\int m^4|\psi|^4 d^3x. \tag{136}$$

In particular, for a spherical oscillaton with real ψ, one gets

$$-\frac{dN}{dt}=\frac{1}{2c^3}\int m^4\psi^4 r^2 dr. \tag{137}$$

Now if we use the fact that by Eq. (3), the total mass in time units of the oscillaton is $M\equiv GM_*/c^3$, and if we use the fact that the mass in conventional units is the conventional mass m_* of each scalar particle times the number N of such particles (neglecting the small correction due to the kinetic energy of the scalar particles and the gravitational binding energy), $M_*=m_*N=(\hbar m/c^2)N$, then we get the rate per time at which the total mass M in time units decays away, the dimensionless rate

$$P_q\equiv-\dot{M}_{\rm quantum}=-\frac{d\mu}{d(mt)}=-\frac{\hbar G}{c^5}m\frac{dN}{dt} \approx\frac{\hbar G}{8\pi c^8}\int m^5\psi^4 d^3x=\frac{\hbar G}{2c^8}\int m^5\psi^4 r^2 dr. \tag{138}$$

For the nearly-Newtonian spherical oscillaton analyzed above, by using Eqs. (75), (76), (83), (98), (99), (100), and (106), one can show that the quantum decay rate (per time) in the mass (in time units) is

$$P_q\equiv-\dot{M}_{\rm quantum}\approx 4\frac{\hbar G}{c^5}m^2\frac{D_\infty}{\mathcal{M}_\infty^5}\mu^5=Q\frac{m^2}{m_{\rm Pl}^2}\mu^5=Qt_{\rm Pl}^2 m^2\mu^5, \tag{139}$$

where

$$m_{\rm Pl}=\sqrt{\frac{c^5}{\hbar G}}\approx 1.855\times 10^{43}\ \ {\rm s}^{-1} \tag{140}$$

is the Planck frequency,

$$t_{\rm Pl}=\sqrt{\frac{\hbar G}{c^5}}\approx 5.391\times 10^{-44}\ \ {\rm s}, \tag{141}$$

is the Planck time, the reciprocal of the Planck frequency, and where

$$Q=\frac{4D_\infty}{\mathcal{M}_\infty^5}\approx 0.008\,513\,223\,934\,732\,691\,876, \tag{142}$$

using the numerical results given in Eqs. (102) and (104).

When both the classical decay rate given by Eq. (122) and the quantum decay rate given by Eq. (139) are both significant, one has the total decay rate being given by

$$-\frac{dM}{dt}=-\frac{d\mu}{d(mt)}=P_c+P_q\approx\frac{C}{\mu^2}e^{-\frac{\alpha}{\mu}}+Qt_{\rm Pl}^2 m^2\mu^5. \tag{143}$$

It may also be of interest to calculate the expected number of scalarons and gravitons emitted during one period of oscillation of the oscillaton scalar field, which is a time $2\pi/m$. Since by far the most dominant scalaron emission is at frequency $3m$, the energy of almost every scalaron emitted is $3\hbar m$, so the expected number of scalarons emitted in one period is

$$N_s=-\frac{2\pi/m}{3\hbar m}\left(\frac{dM_* c^2}{dt}\right)_{\rm classical}\approx\frac{2\pi}{3}\left(\frac{m_{\rm Pl}}{m}\right)^2\frac{C}{\mu^2}e^{-\alpha/\mu}. \tag{144}$$

Similarly, the most dominant graviton emission is at frequency m (as two scalarons of this frequency annihilate into two gravitons of the same frequency), so the expected number of gravitons emitted in one period is

$$N_g = -\frac{2\pi/m}{\hbar m}\left(\frac{dM_* c^2}{dt}\right)_{\rm quantum} \approx 2\pi Q\mu^5 \approx 0.053\,490\,163\,543\,442\,036\,464\mu^5. \tag{145}$$

It is interesting that although it is the classical emission power (into scalar waves or scalarons) P_c that depends only on μ, with the quantum emission power (into gravitons) depending also on $m/m_{\rm Pl}$, for the expected number of particles emitted in one oscillaton period, it is the quantum emission into gravitons that depends only on μ (with an expected number of gravitons per period never larger than unity, and in fact never larger than roughly 0.0044 if the maximum value for μ is 0.607 and if the formula above indeed applies to this large a value of μ), whereas the emission into scalarons (classical) gives a number depending also on $m/m_{\rm Pl}$ (and which can be larger than unity for sufficiently small mass m).

The expected number of scalarons emitted per oscillaton period is unity, $N_s = 1$, for

$$\frac{m}{m_{\rm Pl}} \approx \frac{\sqrt{2\pi C/3}}{\mu} e^{-\alpha/2\mu} \approx \frac{2820.165\,789\,522\,802\,747}{\mu} e^{-19.716\,897\,598\,580\,081\,547/\mu}. \tag{146}$$

If this formula were to apply for $\mu = 0.607$, which is the most recent numerical result for the maximum mass parameter of an oscillaton [9], then this would occur for

$$m_* \approx 3.63\times 10^{-11} m_{*\rm Pl} \approx 4.43\times 10^8 \ {\rm GeV}/c^2. \tag{147}$$

Then there would be some allowed oscillaton mass value at which the oscillaton would emit an expected number of one scalaron per oscillation period, for any scalaron mass less than some scalaron mass value that is given crudely by Eq. (147) that uses Eq. (122) outside its $\mu \ll 1$ domain of validity.

If we define for use here (not the same x as used previously for the rescaled radial variable)

$$x \equiv \frac{\alpha}{\mu} \approx \frac{39.433\,795\,197\,160\,163\,094}{\mu} \tag{148}$$

and

$$\gamma = \frac{C}{\alpha^7 Q} \approx 0.003\,008\,268\,339\,955\,585\,529, \tag{149}$$

then the ratio of the classical mass-loss rate to the quantum mass-loss rate is, for $\mu \ll 1$,

$$\frac{P_c}{P_q} = \gamma \frac{m_{\rm Pl}^2}{m^2} x^7 e^{-x}, \tag{150}$$

which is unity for

$$x - 7\ln x = \ln\left(\gamma \frac{m_{\rm Pl}^2}{m^2}\right), \tag{151}$$

or

$$\begin{aligned} x &= \ln\left(\gamma\frac{m_{\rm Pl}^2}{m^2}\right) + 7\ln x \\ &= \ln\left(\gamma\frac{m_{\rm Pl}^2}{m^2}\right) + 7\ln\left[\ln\left(\gamma\frac{m_{\rm Pl}^2}{m^2}\right) + 7\ln x\right] \\ &= \ln\left(\gamma\frac{m_{\rm Pl}^2}{m^2}\right) \\ &\quad + 7\ln\left\{\ln\left(\gamma\frac{m_{\rm Pl}^2}{m^2}\right) + 7\ln\left[\ln\left(\gamma\frac{m_{\rm Pl}^2}{m^2}\right) + 7\ln x\right]\right\}, \end{aligned} \tag{152}$$

to give the first three steps in an iterative procedure for solving for the value of $x = \alpha/\mu$ that gives $P_c = P_q$.

The decay time for an oscillaton to have its dimensionless decay parameter decay from μ_1 to a smaller value of μ is given by

$$m(t-t_1) = \int_\mu^{\mu_1} \frac{d\mu}{P_c + P_q}. \tag{153}$$

Define the m-dependent constant r [for the use in this immediate section; not the same r as used elsewhere for the radial variable] as the largest real solution to the equation

$$\frac{r}{(\ln r)^3} = s \equiv \frac{1}{\alpha^4 Q t_{\rm Pl}^2 m^2}. \tag{154}$$

For brevity also define here q [now not the large dimensionless constant angle defined in Eq. (129)] to be the function of the classically dimensionless mass μ that is

$$q \equiv \frac{\mu^4}{\alpha C} e^{\alpha/\mu}. \tag{155}$$

Then when $\ln s = -\ln(\alpha^4 Q t_{\rm Pl}^2 m^2) \gg 1$ (which implies that $\ln r \gg 1$), and when Eq. (143) also holds, one can show that the integral on the right hand side of Eq. (153) can be approximated by the following explicit function of r, q, and q_1 (the value of q when μ has its initial value μ_1):

$$m(t-t_1) \approx \frac{r}{4(\ln r)^3}\{[\ln(r+q)]^4 - [\ln(r+q_1)]^4\}. \tag{156}$$

We can use Eq. (154) to rewrite this equation in the form

$$4\alpha^4 Q t_{\rm Pl}^2 m^3 (t-t_1) \approx [\ln(r+q)]^4 - [\ln(r+q_1)]^4. \tag{157}$$

This gives an approximate formula for the decay time, $t-t_1$, for an oscillaton in terms of its mass M and the scalar field mass m (which then determine the dimensionless mass parameter μ), assuming that one knows the initial value μ_1 of the dimensionless mass parameter.

For other purposes, one might know the decay time and from it want to get a relationship between the oscillaton mass M and the dimensionless mass parameter μ. For this purpose, it is helpful to define from the decay time the large dimensionless parameter [not the same a as the μ-dependent constant defined in Eq. (118)]

$$a\equiv\left(\frac{16}{\alpha^4 Q}\right)^{1/3}\left(\frac{t-t_1}{t_{\rm Pl}}\right)^{2/3}\gg 1. \tag{158}$$

It is also helpful to define a new dimensionless mass parameter

$$\nu\equiv\frac{M}{[4\alpha^4 Q t_{\rm Pl}^2(t-t_1)]^{1/3}}, \tag{159}$$

which depends on the oscillaton mass M and on the decay time $t-t_1$ but not on the mass m of the scalar field as the other dimensionless mass parameter $\mu=Mm$ does.

Then, for example, Eq. (157) becomes

$$\mu\approx\nu\{[\ln(r+q)]^4-[\ln(r+q_1)]^4\}^{1/3}, \tag{160}$$

and Eq. (154) becomes

$$\frac{r}{(\ln r)^3}=s=\frac{a\nu^2}{\mu^2}. \tag{161}$$

For $s\gg 1$, this solution to this equation is very roughly

$$r\sim s(\ln s)^3=\frac{a\nu^2}{\mu^2}\left(\ln\frac{a\nu^2}{\mu^2}\right)^3. \tag{162}$$

Now we can insert r from Eq. (162) and the definition of q from Eq. (155) into Eq. (160) to get the following rough explicit algebraic relation between μ, μ_1, and ν for fixed $t-t_1$, and hence for fixed a defined by Eq. (158):

$$\left(\frac{\mu}{\nu}\right)^3\sim\left\{\ln\left[\frac{a\nu^2}{\mu^2}\left(\ln\frac{a\nu^2}{\mu^2}\right)^3+\frac{\mu^4}{\alpha C}e^{\alpha/\mu}\right]\right\}^4 -\left\{\ln\left[\frac{a\nu^2}{\mu^2}\left(\ln\frac{a\nu^2}{\mu^2}\right)^3+\frac{\mu_1^4}{\alpha C}e^{\alpha/\mu_1}\right]\right\}^4. \tag{163}$$

However, I should note that although this relation is a good approximation to Eq. (160) when μ is very small, it can be off by about 20–30 % when μ is large.

For $\alpha/\mu_1<\alpha/\mu\ll\ln a$, so that the classical decay dominates, this relationship can be solved explicitly for ν to give

$$\nu\approx\frac{\alpha C a\mu}{4(\mu^4 e^{\alpha/\mu}-\mu_1^4 e^{\alpha/\mu_1})}. \tag{164}$$

In this limit, this Eq. (164) is actually a better approximation to Eq. (160) than is Eq. (163). At the other extreme, for $\alpha/\mu\gg\ln a$, so that the quantum annihilation dominates during most of the decay, this relationship can also be solved explicitly for ν to give a fairly accurate approximation to Eq. (160), namely

$$\nu\approx\alpha^{-4/3}\mu^{7/3}\left(1-\frac{\mu^4}{\mu_1^4}-4\frac{\mu}{\alpha}\ln\frac{\mu^4}{\alpha C}\right)^{-1/3}. \tag{165}$$

In between these two extremes, i.e., for $\alpha/\mu\sim\ln a$, I do not see how to give any simple expression that would solve explicitly for either μ or ν in terms of the other (say for fixed a and μ_1), though of course one could solve Eqs. (160) and (161) numerically for either μ or ν in terms of a specific value of the other, for fixed decay time $t-t_1$ and hence for fixed a given by Eq. (158).

Let us put in some numbers for these quantities. If we take μ_1 to be the maximum mass of an oscillaton given by [9], namely $\mu_1=0.607$, then we get

$$q_1\equiv\frac{\mu_1^4}{\alpha C}e^{\alpha/\mu_1}\approx 1.48\times 10^{19}. \tag{166}$$

If we take the oscillaton decay time $t-t_1$ to be the present age of the universe, about 13.7 billion years or $8.02\times 10^{60}t_{\rm Pl}$ in Planck units, then

$$a\equiv\left(\frac{16}{\alpha^4 Q}\right)^{1/3}\left(\frac{t-t_1}{t_{\rm Pl}}\right)^{2/3}\approx 3.68\times 10^{39}, \tag{167}$$

or $\ln a\approx 91.10$.

In the previous paragraph it was noted that Eq. (164) applies for $\alpha/\mu\ll\ln a$ and that Eq. (165) applies for $\alpha/\mu\gg\ln a$, so these inequalities are saturated and one has $\alpha/\mu=\ln a$ at $\mu\approx 0.433$. Therefore, since μ cannot be much larger than this value, one never really has the validity of the inequality $\alpha/\mu\ll\ln a$, but in actuality Eq. (164) is a good approximation to Eq. (160) for $e^{\alpha/\mu}\ll a$, and indeed at $\mu=\mu_1$, one has $e^{\alpha/\mu}/a\approx 4.44\times 10^{-12}\ll 1$.

However, one must still remember that Eq. (160), or equivalently Eq. (156) or Eq. (157), are valid approximations only to the extent that Eqs. (122) and (139) are valid for the classical and quantum decays rates P_c and P_q respectively. In this paper these formulas were derived under the assumption that $\mu\ll 1$, so they are not likely to be accurate for the small μ values where $e^{\alpha/\mu}\ll a$, particularly for P_c with its very strong dependence on μ. Hence the formulas used here for when the classical decay appears to dominate should be taken with a big grain of salt, as merely provisional formulas that might give some rough qualitative indication of the true quantitative behavior.

Nevertheless, to get some idea of this rough qualitative behavior, let us for now assume that Eqs. (122) and (139) are valid for all μ less than its maximum value, at, say, $\mu=\mu_1=0.607$, and put in some various possible values for m, μ, and/or M.

For example, taking the example used by Seidel and Suen [1] in which the scalar field mass is typical of that of an axion, $m_* = 10^{-5}$ eV/c^2 or $m \approx 1.519\times 10^{10}$ s$^{-1} \approx 8.19 \times 10^{-34} m_{\rm Pl}$, then $\ln(\gamma m_{\rm Pl}^2/m^2) \approx 146.564$, so one finds that $P_c = P_q$ at $x \approx 183.031$, or $\mu \approx 0.2154$ (if the formulas above really apply to this large a value of μ), which corresponds to an oscillaton mass in time units of $M \approx 1.418\times 10^{-11}$ s or an oscillaton mass in conventional units of $M_* \approx 5.726\times 10^{27} g = 0.9585 M_\oplus$ (about 96% of the mass of the earth).

For this oscillaton, assuming that the equations above did apply for $\mu \approx 0.2154$ even though this is not much smaller than unity, one gets that the total power emitted would be $P = P_c + P_q \approx 5.3\times 10^{-72}$, and the logarithmic rate of decrease of the mass would be $-d\ln M/dt \approx 3.7\times 10^{-61}$ s^{-1}. The oscillaton would contain about $N = M_*/m_* = (m_{\rm Pl}/m)^2 \mu \approx 3.2\times 10^{65}$ scalarons, and so in each period of oscillation of the oscillaton, there would be about $(2\pi/3)(m_{\rm Pl}/m)^2 P_c \approx 8.3\times 10^{-6}$ scalarons emitted (each of energy roughly $3m_*c^2$) and about $(2\pi)(m_{\rm Pl}/m)^2 P_q \approx 2.5 \times 10^{-5}$ gravitons emitted (each of energy roughly m_*c^2). That is, one would need to wait on average about 120 000 periods of oscillation between the emission of successive scalarons, and about 80 000 periods of oscillation between the emission of successive pairs of gravitons (since they come out predominantly in pairs, with the pair having two-thirds the energy of a typical scalaron that is emitted). Therefore, although this oscillaton is not absolutely stable, for astronomical purposes it is very nearly stable.

If this oscillaton with $\mu \approx 0.2154$ had actually decayed from $\mu = \mu_1 = 0.607$, that would have taken a time $t - t_1 \sim 10^{51}$ yr, again using Eqs. (122) and (139) outside their true range of validity just to give a qualitative answer.

On the other hand, if an oscillaton with this value of the scalar field mass, $m_* = 10^{-5}$ eV/c^2, were to have started at $\mu_1 = 0.607$ at the beginning of the universe and had decayed up until its present age, it would now have $\mu \approx 0.459$ (again taking the classical decay rate formula outside its range of validity, $\mu \ll 1$). But even though this calculated value for μ today is not likely to be actually correct, it is interesting that it is significantly below the initial value. Thus if this result is at least qualitatively correct, an initially maximum-mass oscillaton with this value of the scalar field mass would have decayed by a significant amount during a time comparable to the age of the universe.

To take a more extreme example, if we imagine that there is a scalar field (quintessence?) of natural frequency m that has the value of the current Hubble expansion rate, $H_0 \approx 2.3\times 10^{-18}$ s$^{-1} \approx 1.24\times 10^{-61} m_{\rm Pl}$, which corresponds to $m_*c^2 \approx 1.51\times 10^{-33}$ eV, then $\ln(\gamma m_{\rm Pl}^2/m^2) \approx 274.678$, so then $P_c = P_q \approx 4\times 10^{-129}$ at $x \approx 314.945$, or $\mu \approx 0.1252$, giving an oscillaton mass in time units of $M \approx 5.4\times 10^{16}$ s ≈ 1.7 billion years, or an oscillaton mass in conventional units of $M_* \approx 2.2\times 10^{55} g = 1.1\times 10^{22} M_\odot$ (of the same order of magnitude as that of all the observable galaxies in the universe, by the same coincidence that this mass is roughly within a factor of 10 or so of what is needed to close the universe). For this example, one would need to wait on average about 1800 000 periods of oscillation between the emission of successive scalarons, and about 1200 000 periods of oscillation between the emission of successive pairs of gravitons.

If we took an oscillaton with a scalar field of this $m = H_0$, started it with $\mu_1 = 0.607$, and applied the formulas above, we would find that during the age of the universe, it would have had μ decay only by about 6.3×10^{-22}, an insignificant reduction in its value. Besides the usual caveat about the inapplicability of Eq. (126) to this large μ value—note that Eq. (127) does not apply here, since in this case the two terms in Eq. (126) are very nearly equal—there is also the error from the fact that the age of the universe corresponds to only about 0.99 of a radian of the phase of the oscillation of an oscillaton with a scalar field mass equaling the current value of the Hubble constant, whereas the formulas above apply only for time periods containing many oscillations.

For values of x smaller than the solution of Eq. (151) or (152), so that μ is larger than the corresponding critical μ value for that $m/m_{\rm Pl}$, then the classical decay rate dominates (i.e., for large oscillaton masses). On the other hand, for values of x larger than the solution of Eq. (151) or (152), so that μ is smaller than the corresponding critical μ value for that $m/m_{\rm Pl}$, then the quantum decay rate dominates (i.e., for small oscillaton masses).

It is interesting that even with perhaps about the smallest conceivable value of the scalar field mass in the present universe, that of the Hubble constant, the quantum emission dominates over the classical emission when μ is only as small as about 1/8 (and of course for all smaller values of μ). That is, for almost any conceivable oscillaton in the present universe, if the dimensionless mass parameter μ is smaller than roughly 1/8, the classical emission of scalar waves would be even less than the tiny quantum emission of gravitons from the annihilation of pairs of scalar particles in the oscillaton. This illustrates how rapidly the classical emission drops as μ is made small.

When the mass of the scalar field is much smaller than the Planck mass, as it must be for one to have nearly-Newtonian oscillatons containing a large number of scalar particles, as is implicitly assumed in the analysis above, then by Eq. (151) or (152) one finds that x is large in comparison with unity for $P_c = P_q$. However, because α is itself rather large, it is not necessarily the case that this value of x corresponds to a small value of μ or even a value of μ less than $\mu_{\rm max}$.

For example, if Eq. (150) were valid not just for very small μ but also for values of μ up to $\mu_{\rm max}$, and if we take $\mu_{\rm max} = 0.607$, then we would find that $P_c = P_q$ at $\mu = \mu_{\rm max} = 0.607$ for $m \approx 9.47\times 10^{-10} m_{\rm Pl}$ or $m_* \approx 9.47\times 10^{-10} m_{*\rm Pl} \approx 1.16\times 10^{10}$ GeV/c^2. Assuming that this is correct, then for larger values of the scalar field mass, the quantum decay would dominate ($P_q > P_c$) for all values of μ up to and including $\mu_{\rm max}$. On the other hand, for smaller values of the scalar field mass, which is more realistic if the scalar field is, say, an axion, then there is always a mass range [say $\mu_c < \mu < \mu_{\rm max}$ with μ_c being given roughly by α/x with x the solution of Eq. (151) or (152) when it gives a small

μ_c], where the classical emission dominates for the mass loss rate, though for $\mu<\mu_c<\mu_{\max}$ the quantum mass-loss decay rate would dominate.

It may be of interest to estimate the present upper bound on μ for oscillatons with scalar field masses other than the two examples above, assuming that the oscillatons formed in the early universe and have been decaying for a time comparable to the age of the universe. The maximum present value of μ that they would have would be what they would have if they started with the maximum allowed initial value of μ, which here I shall take to be $\mu_1=0.607$, as above. I shall also assume that the scalar field mass energy is $m_*c^2 \ll 10^{10}$ GeV, so that the decay from $\mu=\mu_1$ within the present lifetime of the universe would be in the regime where the classical decay dominates (see below for more details on this) and where I shall assume that Eq. (126) holds.

Then if I use $x\equiv\alpha/\mu$ defined by Eq. (148), one gets (with a new use for y)

$$\frac{e^x}{x^4}\approx y \tag{168}$$

where for this section I shall define

$$y\equiv\frac{C}{\alpha^3}[m(t-t_1)+q_1]$$
$$\approx 4.067\,595\times10^{34}\left(\frac{m_*c^2}{\text{eV}}\right)\left(\frac{t-t_1}{1.37\times10^{10}\ \text{yr}}\right)+9.188\,619\times10^{20} \tag{169}$$

(not the same y that denoted the rescaled imaginary radial coordinate in Sec. VIII).

An explicit approximation that solves Eq. (168) for x in terms of y to at least 8-digit accuracy is

$$x\approx\ln y+4\ln(\ln y+4\ln\{\ln y+4\ln[\ln y+4\ln(\ln y+4\ln x_1)]\}), \tag{170}$$

where

$$x_1\equiv\frac{\alpha}{\mu_1}\approx 64.965, \tag{171}$$

$$4\ln x_1=4\ln(\alpha/\mu_1)\approx 16.695. \tag{172}$$

Then we get

$$\mu\approx\frac{39.433\ \ 795}{\ln y+4\ln(\ln y+4\ln\{\ln y+4\ln[\ln y+4\ln(\ln y+16.695)]\})}. \tag{173}$$

However, Eq. (168) itself is not that accurate, since it was derived on the assumption of the accuracy of Eq. (126), which is in doubt, since that equation was derived for $\mu\ll1$, whereas here for $m_*c^2\ll10^{10}$ GeV, one gets μ in the range roughly between 0.3 and 0.6, which is not much less than unity.

Once we have an estimate for μ (no doubt rather crude, since it does not give $\mu\ll1$ where it would be valid), or for the maximum value of μ, as a function of the scalar field mass, we can easily get the oscillaton mass $M=\mu/m$ in time units or $M_*=\hbar c\mu/(Gm_*)$ in conventional mass units. In terms of the solar mass $M_\odot\approx1.989\times10^{33}g\approx0.9137\times10^{38}m_{*\mathrm{Pl}}$, one can use Eq. (4) to write

$$M_*\approx 1.336\,337\,63\times10^{-10}M_\odot\left(\frac{1\ \text{eV}}{m_*c^2}\right)\mu. \tag{174}$$

Combining this with Eq. (173) then gives

$$M_*\approx\frac{5.269\ 686\ 43\times10^{-9}M_\odot[(1\ \text{eV})/(m_*c^2)]}{\ln y+4\ln(\ln y+4\ln\{\ln y+4\ln[\ln y+4\ln(\ln y+16.695)]\})}, \tag{175}$$

where y is given by Eq. (169). This would be the estimated value for the oscillaton mass if it started at $\mu_1=0.607$ and would be an upper limit for the mass if 0.607 were the maximum value of μ at which it could have started. Again, this formula is applicable for $m_*c^2\ll10^{19}$ eV (where the classical decay dominates for an oscillaton starting with $\mu_1=0.607$ and decaying for up to 13.7 billion years), and it also assumes the dubious correctness of Eq. (122) for the resulting fairly large values of μ as given by Eq. (173).

For example, if we let $\mu(m_*c^2)$ be the value of μ that an oscillaton of scalaron mass-energy m_*c^2 would decay to, from $\mu_1=0.607$, in a time of 13.7 billion years, then we had shown above that $\mu(10^{-5}\ \mathrm{eV})\approx 0.459$. We can also readily calculate the following values of $\mu(m_* c^2)$ for other values of $m_* c^2$:

$$\begin{aligned}
\mu(10^{-35}\ \mathrm{eV}) &\approx \mu_1 - 4.41\times 10^{-24},\\
\mu(10^{-30}\ \mathrm{eV}) &\approx \mu_1 - 4.41\times 10^{-19},\\
\mu(10^{-25}\ \mathrm{eV}) &\approx \mu_1 - 4.41\times 10^{-14},\\
\mu(10^{-20}\ \mathrm{eV}) &\approx \mu_1 - 4.41\times 10^{-9},\\
\mu(10^{-15}\ \mathrm{eV}) &\approx \mu_1 - 4.41\times 10^{-4},\\
\mu(10^{-10}\ \mathrm{eV}) &\approx 0.534,\\
\mu(10^{-5}\ \mathrm{eV}) &\approx 0.459,\\
\mu(1\ \mathrm{eV}) &\approx 0.402,\\
\mu(10^{5}\ \mathrm{eV}) &\approx 0.358,\\
\mu(10^{10}\ \mathrm{eV}) &\approx 0.323,\\
\mu(10^{15}\ \mathrm{eV}) &\approx 0.295.
\end{aligned} \tag{176}$$

Similarly, we can calculate $M_*(m_* c^2)$, the value of M_* that an oscillaton of scalaron mass-energy $m_* c^2$ would decay to, from $\mu_1 = 0.607$ and

$$M_{*1} = \frac{\hbar c \mu_1}{G m_*} \approx 8.111\times 10^{-11} M_\odot \frac{1\ \mathrm{eV}}{m_* c^2}, \tag{177}$$

in a time of 13.7 billion years, as having the following values:

$$\begin{aligned}
M_*(10^{-35}\ \mathrm{eV}) &\approx 8.111\times 10^{24} M_\odot,\\
M_*(10^{-30}\ \mathrm{eV}) &\approx 8.111\times 10^{19} M_\odot,\\
M_*(10^{-25}\ \mathrm{eV}) &\approx 8.111\times 10^{14} M_\odot,\\
M_*(10^{-20}\ \mathrm{eV}) &\approx 8.111\times 10^{9} M_\odot,\\
M_*(10^{-15}\ \mathrm{eV}) &\approx 81055 M_\odot,\\
M_*(10^{-10}\ \mathrm{eV}) &\approx 0.7133 M_\odot,\\
M_*(10^{-5}\ \mathrm{eV}) &\approx 6.128\times 10^{-6} M_\odot,\\
M_*(1\ \mathrm{eV}) &\approx 5.375\times 10^{-11} M_\odot,\\
M_*(10^{5}\ \mathrm{eV}) &\approx 4.790\times 10^{-16} M_\odot,\\
M_*(10^{10}\ \mathrm{eV}) &\approx 4.322\times 10^{-21} M_\odot,\\
M_*(10^{15}\ \mathrm{eV}) &\approx 3.938\times 10^{-26} M_\odot.
\end{aligned} \tag{178}$$

These are either the estimates for the masses, if the oscillatons started 13.7 billion years ago with $\mu = 0.607$, or are estimates for the upper bounds of the oscillaton masses, if 0.607 is merely an upper bound on the initial value of μ.

XI. QUANTUM DECAY WHEN THE PARTICLE NUMBER GETS SMALL

When $\mu \ll 1$ and when Eq. (150) gives $P_c/P_q \ll 1$, then the quantum mass-loss rate dominates and is given to good accuracy by Eq. (139), but only so long as the number of scalar particles in the oscillaton,

$$N \approx \frac{M_*}{m_*} = \frac{c^5 M}{\hbar G m} = \frac{c^5 \mu}{\hbar G m^2} = \frac{m_{\mathrm{Pl}}^2}{m^2}\mu, \tag{179}$$

is large in comparison with unity.

That is, the quantum mass loss rate dominates and is given to good accuracy by Eq. (139) if

$$\frac{m^2}{m_{\mathrm{Pl}}^2} \ll \mu < \mu_c \approx \frac{\alpha}{x} \sim \frac{\alpha}{\ln(\gamma m_{\mathrm{Pl}}^2/m^2)} \approx \frac{19.716\,897\,598\,580\,082}{61.769 + \ln(\mathrm{eV}/m_* c^2)} \ll 1. \tag{180}$$

The right-hand side of this requirement is actually a bit stronger than what is needed, which is that both $\mu < \mu_c$ and that $\mu \ll 1$, but it is not really necessary that $\mu_c \ll 1$. We may note that for this inequality to have any range of validity for μ, we need $m \ll m_{\mathrm{Pl}}$, which we have been assuming throughout this paper and shall continue to assume.

When an oscillaton is decaying, it will eventually get down to having a small number N of scalar particles, and Eq. (139) will cease to be accurate. In principle one could solve the N-body Schrödinger equation with Newtonian attractive potentials between the N scalar particles for the ground-state wave function (ignoring for the moment the annihilations into gravitons) and then calculate the overlap between two particles to get the two-particle annihilation rate into two gravitons. However, I did not do this calculation for $N > 2$ and am not familiar with the literature where it might have been done.

Just as the annihilation rate for large N goes as the fifth power of μ and of N, one would also expect that the annihilation rate for a small number N would also decrease rapidly as N is reduced, reaching a minimum for $N = 2$ (if one can reach this number, though if one has an odd number when N is somewhat larger, and if the two-body annihilations dominate so that the scalar particles predominantly annihilate in pairs, then one would most likely end up with a three-particle state before the final decay to two gravitons and one free scalar particle).

So if the oscillaton decays down to two scalar particles before annihilating completely, the decay of the final two particles is likely to take more time than the entire decay down to that point.

The final annihilation rate is easily calculable from using the ground-state solution of the two-particle Schrödinger equation. One readily gets that the probability density for one of the particles to be at the location of the other, the quantity that takes the role of the number density n in Eq. (133), is

$$n=\frac{1}{4\pi}\frac{m^9}{m_{\rm Pl}^6c^3}, \tag{181}$$

continuing to use m with units of inverse time. Then by Eq. (133), the annihilation rate per time (for the two particles to annihilate) is

$$R=\frac{m^{11}}{2m_{\rm Pl}^{10}}. \tag{182}$$

If one uses the first Eq. (138) to convert this to the quantum expectation value of a mass-loss rate and uses the fact that for this 2-particle state, $\mu=2(m/m_{\rm Pl})^2$, one gets

$$\begin{aligned}P_q&\equiv-\dot{M}_{\rm quantum}=-\frac{dM}{dt}=-\frac{d\mu}{d(mt)}\\&=\left(\frac{m}{m_{\rm Pl}}\right)^{12}=\frac{\hbar^6G^6}{c^{30}}m^{12}=\frac{1}{32}\frac{m^2}{m_{\rm Pl}^2}\mu^5\\&\approx3.670\ 759\ 777\ 914\ 995\ 479Qt_{\rm Pl}^2m^2\mu^5.\end{aligned} \tag{183}$$

That is, the actual rate at which the 2-particle state annihilates is a factor of about 3.67 times what one would get by blindly extrapolating down to $N=2$ particles the rate given by Eq. (139), which actually applies only for very many particles, $N\gg1$ (as well as $\mu\ll1$). Equation (183) contains the largest positive power of $\hbar$ (6), the largest positive power of G (6), and the largest negative power of the speed of c (30), that I can ever recall seeing in a formula, though I am not used to using formulas in which I have not just set $\hbar=G=c=1$.

If instead we take the reciprocal of the annihilation rate R given by Eq. (182) as the expectation value of the decay time from the 2-scalar-particle state to the 2-graviton state, and also as an estimate for the total decay time for an oscillaton (since it presumably dominates over the time to get down to 2 particles, assuming that the number of particles is even when one gets close enough to the 2-particle state that one can ignore the probability that an odd number of scalar particles will annihilate), then we get a total decay time of

$$t_{\rm decay}\approx1/R=\frac{2m_{\rm Pl}^{10}}{m^{11}}=\frac{2c^{25}}{\hbar^5G^5m^{11}}=\frac{2\hbar^6c^3}{G^5m_*^{11}}=2t_{\rm Pl}\left(\frac{m_{*\rm Pl}}{m_*}\right)^{11}. \tag{184}$$

For example, if we take a typical axion mass, $m_*=10^{-5}\ {\rm eV}/c^2$, then Eq. (183) gives $P_q\approx9.11\times10^{-398}$ and Eq. (184) gives $t_{\rm decay}=1.88\times10^{346}$ yr. To take the more extreme example in which m has the value of the current Hubble expansion rate, $H_0\approx2.3\times10^{-18}\ {\rm s}^{-1}\approx1.24\times10^{-61}m_{\rm Pl}$, then Eq. (183) gives $P_q\approx1.3\times10^{-731}$ and Eq. (184) gives $t_{\rm decay}\approx1.3\times10^{680}$ yr.

Thus the complete quantum decay of an oscillaton can take a very long time and probably would not be a suitable subject for an experimental Ph.D. thesis. On the other hand, the slowness of both the classical and quantum decay of oscillatons of light scalar fields shows that if they form in the universe, they can last for astronomically long times.

XII. QUANTUM DECAY OF MULTIPLE-FIELD OSCILLATONS

The analysis above was for oscillatons having just one real massive scalar field, minimally coupled to gravity. Although in this paper we shall not go beyond minimal coupling or consider other scalar-field self couplings (other than mass terms), we started with a general discussion of an arbitrary number of minimally coupled massive scalar fields and their classical decay rates, so it would be of interest to say also how the quantum decay goes when there are more than one scalar field.

When the scalar fields all have different masses, then the separate decay processes are incoherent, so the rates for each just add, with the rate for each (in the nearly-Newtonian limit) going as the spatial integral of the square of the mass density for that scalar field, with the coefficient as given above.

This is at least so if we average over times long in comparison with the reciprocals of the differences of the scalar field masses in frequency units, which will hereby be assumed—if any mass differences are short in comparison with the reciprocal of the decay time of interest, then for that time we may consider these scalar fields as having the same mass. Intermediate cases in which the decay times of interest are comparable to the reciprocal of any mass differences will not be discussed here.

Therefore, we may consider separately all of the fields at one mass (or one range of masses if the range is much less than the reciprocal of the decay time being considered).

The simplest case is that in which there are two equal-mass scalar fields that are oscillating at 90° out of phase. This is equivalent to one complex scalar field that has a global $U(1)$ symmetry and hence a conserved particle number that presumably cannot decay away, at least by perturbative quantum effects such as what DeWitt [20] used to calculate the annihilation of scalar particles into gravitons.

Presumably there are nonperturbative gravitational effects in which a nonzero particle number, though conserved by the global $U(1)$ symmetry perturbatively, forms or tunnels into a real or virtual black hole that then decays into a different particle number (e.g., zero). Thus at some level the global $U(1)$ invariance is surely broken by gravity. In Ref. [21] we used a model of gravitational foam to estimate that this rate would be disastrously high for point scalar particles, suggesting that perhaps no such particles could exist in our universe. If so, this would of course rule out the whole idea of oscillatons (unless they were made of composite scalars that are not pointlike down to near the Planck scale). But since our ideas were admittedly rather speculative, here I shall assume that the nonperturbative effects violating global $U(1)$ invariance are suppressed to give rates much smaller than the particle-antiparticle annihilation into gravitons that is allowed by the perturbative analysis that preserves the global $U(1)$ invariance.

If this is indeed so, when there are two equal-mass scalar fields that are oscillating at 90° out of phase, effectively the decay of each individual scalar field must destructively interfere so that the total decay rate is zero.

At first this sounds impossible, since if one has a state with N_1 scalar particles of the first field and N_2 of the second, then the final state in which two particles of the first field annihilate into gravitons would have N_1-2 particles of the first field and N_2 of the second, which would be orthogonal to the final state in which instead two particles of the second field annihilate into gravitons, leaving N_1 particles of the first field and N_2-2 particles of the second field. Therefore, how could there possibly be any destructive interference to prevent the particles from annihilating?

However, this objection can be circumvented if the quantum state of the oscillaton does not have a definite number of particles of both kinds. (Indeed, that would have to be the case if they are oscillating 90° out of phase, since phase is in some sense a conjugate variable to particle number. Note that the total particle number could be precise, so that the total phase is undefined, so long as the individual particle numbers are sufficiently indefinite that the relative phase between the two fields is well defined.) For example, the quantum state for the particles could be a coherent state that is an eigenstate of the annihilation operators for the two kinds of particles.

Then in the case that two particles of the first real scalar field decay, although the expectation value of the number of particles of that field would have been reduced by two, the final state would not need to be orthogonal to the state that would result if instead two particles of the second real scalar field were to annihilate. Therefore, the two decay processes can interfere. When the two scalar fields oscillate 90° out of phase, their combination is equivalent to a single complex scalar field with a $U(1)$ symmetry that prevents perturbative quantum decay into gravitons.

The gravitational signal of this $U(1)$ symmetry would be that the stress-energy tensor would have no oscillations, so one would be back to the case of a boson star that seems to be completely stable (except presumably to nonperturbative tunneling processes in which some or all of the particles tunnel into a black hole, or a virtual black hole, that would either transcend or violate what would otherwise be the conservation of the global $U(1)$ charge [21]).

Now consider the case in which there are an arbitrary number, say n, scalar fields at some mass m_I. We shall continue to assume the nearly-Newtonian limit, in which the dimensionless rescaled real massive scalar fields ϕ_{IJ}, defined by Eq. (6), have the form given by Eq. (17) in terms of the complex dimensionless scalar fields ψ_{IJ} that are very slowly varying spatially, and even more slowly varying temporally, if at all, on the length scale c/m_I and on the time scale $1/m_I$. Then the real scalar fields ϕ_{IJ} are essentially oscillating nearly periodically with frequency m_I.

When two scalar particles annihilate into (predominantly) two gravitons, the graviton wavelengths are roughly c/m_I, which is a much shorter length scale than the length scale of the variation of the fields ϕ_{IJ}, so each region of size of the order of c/m_I annihilates essentially independently. (That is, the quantum states of the outgoing gravitons are essentially orthogonal for the annihilation in the separate regions, if one uses graviton wavepackets that have sizes more nearly comparable to their wavelengths than to the much bigger size of the oscillaton.) Thus we can just add up the annihilation rates in each region, effectively getting an integral over the oscillaton of the annihilation rate in each region.

Since we shall only be interested in the annihilation over many oscillations of the oscillaton, we shall only consider the annihilation rate averaged over many such periods.

In each region of size somewhat bigger than c/m_I where we are calculating the average annihilation rate, each of the real scalar fields of mass m_I is oscillating essentially with constant amplitude and period, staying in phase with each other scalar field over a time long compared with the oscillation period that is very nearly $2\pi/m_I$. In this region, we can perform an $O(n)$ transformation of the n scalar fields so that all but two of the fields are transformed to zero for the time of interest (to the accuracy of the nearly-Newtonian approximation), and the two that remain nonzero are oscillating 90° out of phase, say

$$\phi_1=\sum_J O_{1J}\phi_{IJ}\approx 2c_1\cos[m_I(t-t_0)], \tag{185}$$

$$\phi_2=\sum_J O_{2J}\phi_{IJ}\approx -2c_2\sin[m_I(t-t_0)], \tag{186}$$

and

$$\phi_i=\sum_J O_{iJ}\phi_{IJ}\approx 0 \tag{187}$$

for $i>2$, with real positive amplitudes c_1 and c_2 that are nearly constant over the spacetime region where the time-averaged annihilation rate is being calculated. Without loss of generality we can choose the $O(n)$ transformation to give $c_1\geqslant c_2$.

I am not bothering to include the subscript I, which tells what the mass m_I is, on what I am calling ϕ_i. Indeed, the fact that I am giving ϕ_i only a single subscript is used here to distinguish it from the $O(n)$-related scalar fields ϕ_{IJ} without having to put primes on ϕ_i as I would have if it had the same number of indices as ϕ_{IJ}.

We can also define the $O(n)$-transformed complex scalar fields

$$\psi_1=\sum_J O_{1J}\psi_{IJ}\approx c_1 e^{im_I t_0}, \tag{188}$$

$$\psi_2=\sum_J O_{2J}\psi_{IJ}\approx -ic_2 e^{im_I t_0}, \tag{189}$$

and

$$\psi_i = \sum_J O_{iJ}\psi_{IJ} \approx 0 \tag{190}$$

for $i > 2$.

Now for the quantum analysis, we can replace the two real scalar fields with the one complex scalar field

$$\begin{aligned}\Phi &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\\ &\approx \frac{1}{\sqrt{2}}[(\psi_1 + i\psi_2)e^{-im_I t} + (\overline{\psi_1} + i\overline{\psi_2})e^{im_I t}]\\ &\approx \frac{1}{\sqrt{2}}[(c_1 + c_2)e^{-im_I(t-t_0)} + (c_1 - c_2)e^{im_I(t-t_0)}]\\ &= \Phi_+ e^{-im_I t} + \Phi_- e^{im_I t}\end{aligned} \tag{191}$$

(with Φ not to be confused with the original real scalar fields Φ_{IJ} or the slowly varying complex dimensionless scalar fields ψ_{IJ} that were used to represent each dimensionless real scalar field ϕ_{IJ}; those all had subscripts that will not appear on the single rapidly varying complex Φ that combines the two rapidly varying real scalar fields ϕ_1 and ϕ_2).

Because of the $U(1)$ invariance of the field equations and stress-energy tensor of the two nonzero real scalar fields, ϕ_1 and ϕ_2, which are represented by this complex scalar field Φ, there is a conserved global $U(1)$ charge. The part of the classical field Φ that has the phase factor $e^{-im_I(t-t_0)}$ and the coefficient Φ_+ can be said to represent particles with positive global $U(1)$ charge and with particle number density

$$n_{I+} \approx \frac{m_I c^2}{4\pi G\hbar}|\Phi_+|^2 \approx \frac{m_I c^2}{4\pi G\hbar}\left|\frac{c_1 + c_2}{2\sqrt{2}}\right|^2 \approx \frac{m_I c^2}{4\pi G\hbar}\left|\frac{\psi_1 + i\psi_2}{\sqrt{2}}\right|^2 \tag{192}$$

and the part of the classical field Φ that has the phase factor $e^{+im_I(t-t_0)}$ and the coefficient Φ_- can be said to represent antiparticles with negative global $U(1)$ charge and with antiparticle number density

$$n_{I-} \approx \frac{m_I c^2}{4\pi G\hbar}|\Phi_-|^2 \approx \frac{m_I c^2}{4\pi G\hbar}\left|\frac{c_1 - c_2}{2\sqrt{2}}\right|^2 \approx \frac{m_I c^2}{4\pi G\hbar}\left|\frac{\overline{\psi_1} + i\overline{\psi_2}}{\sqrt{2}}\right|^2. \tag{193}$$

In the classical limit that we are assuming, we can express these number densities in terms of the mass density and mean-squared pressure of the fields of mass m_I in the following way:

By using Eq. (26) for $G_{00} \approx 8\pi G\rho$ or Eq. (37) directly for the mass density ρ (which is nearly constant in time), and splitting it up into the contributions from the fields of the different masses m_I, one gets

$$\begin{aligned}\rho_I &\approx \frac{m_I^2}{4\pi G}\sum_J |\psi_{IJ}|^2 = \frac{m_I^2}{4\pi G}(|\psi_1|^2 + |\psi_2|^2)\\ &\approx \frac{m_I^2}{4\pi G}(c_1^2 + c_2^2) \approx \frac{m_I\hbar}{c^2}n_I = m_{I*}n_I,\end{aligned} \tag{194}$$

where

$$n_I = n_{I+} + n_{I-} = \frac{m_I c^2}{4\pi G\hbar}(c_1^2 + c_2^2) = \frac{m_I c^2}{4\pi G\hbar}\sum_J |\psi_{IJ}|^2 \tag{195}$$

is the total number density of all the scalar fields of mass m_I, or, equivalently, of both the particles and the antiparticles of the single complex scalar field Φ that classically represents all of the real fields (i.e., we are ignoring vacuum fluctuations in the transformed scalar fields ϕ_i with $i > 2$ that are classically zero).

Similarly, by using Eq. (28) for $G_{ij} \approx (8\pi G/c^2)P\delta_{ij}$ for the oscillating nearly-isotropic pressure P, and also splitting it up into the contributions from the fields of the different masses m_I, one gets

$$P_I \approx -\frac{m_I^2 c^2}{8\pi G}\sum_J (\psi_{IJ}^2 e^{-2im_I t} + \overline{\psi}_{IJ}^2 e^{2im_I t}). \tag{196}$$

In this case, if one takes the time-average of the square of the total pressure of all the n scalar fields of mass m_I (or equivalently of the single complex scalar field Φ), one gets

$$\begin{aligned}\langle P_I^2\rangle &\approx \frac{c^4}{2}\left(\frac{m_I^2}{4\pi G}\right)^2\left|\sum_J \psi_{IJ}^2\right|^2 \approx \frac{c^4}{2}\left(\frac{m_I^2}{4\pi G}\right)^2|\psi_1^2 + \psi_2^2|^2\\ &\approx \frac{c^4}{2}\left(\frac{m_I^2}{4\pi G}\right)^2(c_1^2 - c_2^2)^2 \approx 2m_I^2\hbar^2 n_{I+}n_{I-}.\end{aligned} \tag{197}$$

Because P_I is oscillating sinusoidally, its maximum value, say $P_{I\max}$ (as a function of time at each spatial location) is $\langle 2P_I^2\rangle^{1/2}$.

Then from Eqs. (194), (195), and (197), one can solve for the number densities of both the particles and the antiparticles of the complex scalar field Φ of mass m_I:

$$n_{I+} \approx \frac{1}{2m_I\hbar}(\rho_I c^2 + \sqrt{\rho_I^2 c^4 - \langle 2P_I^2\rangle}), \tag{198}$$

$$n_{I-} \approx \frac{1}{2m_I\hbar}(\rho_I c^2 - \sqrt{\rho_I^2 c^4 - \langle 2P_I^2\rangle}), \tag{199}$$

where $\langle 2P_I^2\rangle$ is given in terms of the ψ_{IJ}'s by Eq. (197).

Another way to express this relationship, using $P_{I\max} = \langle 2P_I^2\rangle^{1/2}$, is to note that

$$\rho_I c^2 + P_{I\max} = m_I\hbar(\sqrt{n_{I+}} + \sqrt{n_{I-}})^2, \tag{200}$$

$$\rho_I c^2 - P_{I\max} = m_I\hbar(\sqrt{n_{I+}} - \sqrt{n_{I-}})^2. \tag{201}$$

Thus when $P_{I\max}=0$, there are just particles but no antiparticles of the complex scalar field, and when $P_{I\max}=\rho_I c^2$, there are equal numbers of particles and antiparticles. This latter possibility is the case when there is only one real scalar field, in which case it is a fiction to say that there is the complex scalar field Φ at all, but the classical real scalar field does act as if it were composed of equal numbers of particles and antiparticles of the fictitious complex scalar field, $n_{I+}=n_{I-}=(1/2)n_I$. (Of course, then there are no vacuum fluctuations of the nonexistent imaginary component of the complex Φ, but here we are taking the classical limit and are only considering the effects of real particles and not of any vacuum fluctuations.)

By making the arbitrary requirement that the $O(n)$ transformation lead to real $c_1 \geqslant c_2 \geqslant 0$ for the coefficients of the two nonzero scalar fields after the transformation, we have made an arbitrary choice of what to call particles and what to call antiparticles (with the number density of particles never less than the number of antiparticles by this choice) and hence of which expression in Eqs. (198) and (199) has the minus sign in front of the square root.

Now when we have the possibility of both particles and antiparticles of the complex scalar field Φ, the $U(1)$ invariance prevents the annihilation of a pair of particles or of a pair of antiparticles (at least at the perturbative level) but allows the annihilation of a particle-antiparticle pair. On the other hand, the particle-antiparticle annihilation cross-section is twice that given by Eq. (131) [20] for two real scalar field particles, i.e.,

$$\sigma_{+-}=\frac{4\pi G^2 m_*^2}{c^3 v}=\frac{4\pi\hbar^2 G^2 m^2}{c^7 v}. \tag{202}$$

This means that when we have N_{I+} particles and N_{I-} antiparticles, both of these numbers decrease at the rate

$$\begin{aligned}-\frac{dN_{I+}}{dt}&=-\frac{dN_{I-}}{dt}=\int d^3x n_{I+}n_{I-}\sigma_{+-}v\\ &=\frac{4\pi\hbar^2G^2m_I^2}{c^7}\int d^3x n_{I+}n_{I-}=\frac{\pi G^2}{c^7}\int d^3x\langle 2P_I^2\rangle\\ &=\frac{\pi G^2}{c^7}\int d^3x P_{I\max}^2=\frac{m_I^4}{16\pi c^3}\int d^3x\left|\sum_J \psi_{IJ}^2\right|^2.\end{aligned} \tag{203}$$

The total number decay rate is of course twice this. When $n_{I+}=n_{I-}$, so that $P_{I\max}=\rho_I c^2$, then the total rate agrees with the integral of Eq. (134).

When we multiply the total number decay rate for each mass m_I (which is in frequency units) by the mass $m_{*I}=\hbar m_I/c^2$ in conventional units and sum over all I, we get the total mass loss rate by scalar particle annihilation in conventional mass units:

$$-\frac{dM_*}{dt}=\frac{\hbar}{8\pi c^5}\sum_I m_I^5\int d^3x\left|\sum_J \psi_{IJ}^2\right|^2. \tag{204}$$

We can then multiply this by G/c^3 to get the dimensionless rate of decrease of $M=GM_*/c^3$, the mass in units of time:

$$-\frac{dM}{dt}=\frac{\hbar G}{8\pi c^8}\sum_I m_I^5\int d^3x\left|\sum_J \psi_{IJ}^2\right|^2. \tag{205}$$

XIII. QUANTUM AND CLASSICAL EMISSION FROM THE SIMPLEST MULTIFIELD SPHERICAL OSCILLATONS

Now consider the particular case in which all of the real scalar fields have the same mass, so we can drop the subscript I that labels the mass. Furthermore, restrict attention to the case in which all of the real scalar fields are oscillating with the same quasistationary nodeless spherically symmetric mode (that of the simplest spherical oscillaton, except that now there are more than one real scalar field that may be oscillating with different phases).

By the argument above, we can perform an $O(n)$ transformation so that only two real scalar fields are then oscillating with nonzero amplitude and are 90° out of phase. By the assumption that all of the scalar fields are oscillating in the same mode (up to phase), this $O(n)$ transformation is constant over space, and the ratio of the amplitudes of the two resulting nonzero modes are also constant. By the procedure above, it can be replaced by a single complex field.

Let N_+ be the number of particles of the complex scalar field, N_- be the number of antiparticles, and $N=N_++N_-$ be the total number of particles and antiparticles. Then if the ψ given by Eq. (76) (real in this case) represents the simplest spherical oscillaton with one real scalar field described above, the complex oscillaton with two real scalar fields and the same total mass is represented (after a shift in the origin of time) by

$$\psi_1=\frac{\sqrt{N_+}+\sqrt{N_-}}{\sqrt{2N}}\psi, \tag{206}$$

$$\psi_2=-i\frac{\sqrt{N_+}-\sqrt{N_-}}{\sqrt{2N}}\psi. \tag{207}$$

Then the complex scalar field is given by

$$\begin{aligned}\Phi&=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2)\\ &=\frac{1}{\sqrt{2}}[(\psi_1e^{-imt}+\bar{\psi}_1e^{imt})+i(\psi_2e^{-imt}+\bar{\psi}_2e^{imt})]\\ &=\left(\sqrt{\frac{N_+}{N}}e^{-imt}+\sqrt{\frac{N_-}{N}}e^{imt}\right)\psi.\end{aligned} \tag{208}$$

Now, by using Eq. (203) and doing an analysis analogous to that which led to Eq. (139), one can show that the quantum annihilation of scalaron particle-antiparticle pairs for the spherically-symmetric nodeless complex oscillaton gives

$$-\frac{dN_+}{dt}=-\frac{dN_-}{dt}=2Qt_{\rm Pl}^{10}m^{11}N^3N_+N_-, \tag{209}$$

where the numerical constant Q is given in Eq. (142).

These equations have the obvious constant of motion being the number of particles minus the number of antiparticles, say,

$$N_0\equiv N_+-N_-. \tag{210}$$

Then, since the total number of particles is $N=N_++N_-$, one can write

$$N_+=\frac{1}{2}(N+N_0), \tag{211}$$

$$N_-=\frac{1}{2}(N-N_0). \tag{212}$$

Similarly, one can write the dimensionless mass parameter of the classical configuration as

$$\mu\equiv Mm=t_{\rm Pl}^2m^2N \tag{213}$$

and also define a constant dimensionless mass parameter as

$$\mu_0\equiv t_{\rm Pl}^2m^2N_0. \tag{214}$$

As both N_+ and N_- decay away at equal rates, the oscillaton asymptotically approaches the configuration with $N=N_+=N_0$ and $N_-=0$, which is a static boson star with $\mu=\mu_0$. Thus μ_0 is the asymptotic (minimum) value of μ.

In terms of N and N_0, Eq. (209) gives

$$-\frac{dN}{dt}=Qt_{\rm Pl}^{10}m^{11}N^3(N^2-N_0^2) \tag{215}$$

from the annihilation of scalaron particle-antiparticle pairs into pairs of gravitons (i.e., ignoring the classical emission into scalar radiation).

Alternatively, we can write the evolution of the dimensionless mass parameter as

$$-\frac{d\mu}{dt}=Qt_{\rm Pl}^2m^3\mu^3(\mu^2-\mu_0^2) \tag{216}$$

from the quantum annihilation. This differential equation has the algebraic solution

$$\frac{\mu^2-\mu_0^2}{\mu^2}e^{\mu_0^2/\mu^2}=e^{-2Qt_{\rm Pl}^2m^3\mu_0^4(t-t_0)} \tag{217}$$

where t_0 is an arbitrary constant of integration. Thus at late times, μ approaches μ_0 exponentially rapidly, and the configuration approaches that of a static boson star with the conserved number of particles.

It may also be of interest to give the decay rates from the classical emission of scalar radiation in this multifield case with the simplest spherical configuration. By using Eq. (69), one can deduce that the classical scalar field emission leads to

$$-\frac{dN_+}{dt}=-\frac{dN_-}{dt}\approx 2C\frac{m_{\rm Pl}^6}{m^5}\frac{N_+N_-}{N^4}e^{-\alpha m_{\rm Pl}^2/m^2N}, \tag{218}$$

$$-\frac{dN}{dt}\approx C\frac{m_{\rm Pl}^6}{m^5}\frac{N^2-N_0^2}{N^4}e^{-\alpha m_{\rm Pl}^2/m^2N}, \tag{219}$$

and

$$-\frac{d\mu}{dt}\approx Cm\frac{\mu^2-\mu_0^2}{\mu^4}e^{-\alpha/\mu}. \tag{220}$$

To get the totals for these rates, one must add the corresponding expressions for the quantum rates from Eqs. (209), (215), and (216) respectively. For example, the total rate at which the mass (in time units) decreases is

$$\begin{aligned}-\frac{dM}{dt}&=-\frac{d\mu}{m\,dt}\approx C\frac{\mu^2-\mu_0^2}{\mu^4}e^{-\alpha/\mu}+Qt_{\rm Pl}^2m^2\mu^3(\mu^2-\mu_0^2)\\&=\left(\frac{C}{m^2M^4}e^{-\alpha/mM}+Qt_{\rm Pl}^2m^7M^3\right)(M^2-M_0^2),\end{aligned} \tag{221}$$

where $M_0=\mu_0/m=t_{\rm Pl}^2mN_0$ is the asymptotic mass of the final boson star in time units, and where the numerical constants $C\approx 3797\,438$, $\alpha\approx 39.4338$, and $Q\approx 0.008\,513\,224$ were given to 19 decimal places in Eqs. (124), (123), and (142) respectively.

One can see that at late times, for $\mu_0>0$, as μ approaches very near to μ_0, $\mu-\mu_0\ll\mu_0^2/\alpha$, μ approaches μ_0 exponentially rapidly:

$$\mu\sim\mu_0+\exp[-(C\mu_0^{-3}e^{-\alpha/\mu_0}+Qt_{\rm Pl}^2m^2\mu_0^4)2mt]. \tag{222}$$

Finally, if we use the scalar field mass in conventional mass units, $m_*=\hbar m/c^2$, and the oscillaton mass also in conventional mass units, $M_*=c^3M/G$, then the total mass loss rate from the simplest spherical multiple-field oscillaton with minimum conventional mass $M_{*0}=c^3M_0/G=N_0m_*$, for $m_*\ll M_*\ll 10^{-10}M_\odot[(1\text{ eV})/(m_*c^2)]$, is

$$\begin{aligned}-\frac{dM_*}{dt}\approx&\left(C\frac{\hbar^2c^5}{G^3}\frac{1}{m_*^2M_*^4}e^{-\alpha\hbar c/(Gm_*M_*)}\right.\\&\left.+Q\frac{G^5}{\hbar^6c^3}m_*^7M_*^3\right)(M_*^2-M_{*0}^2).\end{aligned} \tag{223}$$

Again may I remind the reader that the conventional mass m_* of the scalar field is a quantum quantity with $\hbar$ in it, which is why there is an explicit $\hbar^2$ in the numerator of the first (classical) term for the mass decay rate (from scalar field radiation), to cancel the implicit $\hbar^2$ in the m_*^2 term in the denominator. Similarly, in the second (quantum) term, once the implicit factor of $\hbar^7$ in the m_*^7 term is taken into account, there is one positive power of $\hbar$ appearing, as one would expect for this first-order quantum perturbative contribution (the annihilation of pairs of scalarons into pairs of gravitons).

XIV. CONCLUSIONS

Oscillatons without a $U(1)$ invariance (those without a static geometry) are unstable both classically (to emitting scalar waves) and quantum mechanically (to having scalarons annihilate into gravitons). The classical rate dominates for large $\mu = Mm$ but drops very fast with decreasing μ and is nonanalytic at $\mu=0$: $P_c \approx (C/\mu^2)e^{-\alpha/\mu}$, Eq. (122), with the numerical constants C given by Eq. (124) and α given by Eq. (123). The quantum rate also drops as μ drops, but only as a power law in μ: $P_q \approx Q(m/m_{\rm Pl})^2\mu^5$, Eq. (139), with the numerical constant Q given by Eq. (142). The quantum rate dominates for $\mu \lesssim 1/8$ for $m \gtrsim H_0$ (the current Hubble expansion rate, a lower bound on m for any oscillaton existing in our universe today).

An oscillaton that starts at $\mu_1 \approx \mu_{\rm max} \approx 0.607$ [9] has a significant drop in μ (more than 10%) over a lifetime comparable with the age of the universe if $m_* c^2 \gtrsim 2 \times 10^{-11}$ eV or $M_* \lesssim 3.6 M_\odot$. However, unless $m_* c^2 \gtrsim 2.3 \times 10^{13}$ eV $=2300$ GeV or $M_* \lesssim 1.8 \times 10^{-24} M_\odot \approx 3.5 \times 10^9 g$, in this decay time μ does not decrease by more than a factor of 2.

These numerical approximations are using formulas derived in this paper for nearly-Newtonian configurations, which have $\mu \ll 1$. For accurate results for the not-too-small values of μ that would arise from the decay, within astronomical times, of oscillatons that started with the maximum mass possible for any reasonable value of the scalaron mass, one would need to extend the results derived here to the strong-gravity regime. This is research that shall be left to the future.

Multifield oscillatons [8], in which different real scalar fields of the same mass oscillate out of phase, do not decay away completely but instead asymptotically approach a stable $U(1)$-invariant configuration with a static metric (a boson star), at a rate given by Eq. (221) or (223).

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