

Selection rules for J^{PC} exotic hybrid meson decay in large N_c

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The coupling of a neutral hybrid $\{1,3,5 \dots\}^{-+}$ exotic particle (or current) to two neutral (hybrid) meson particles with the same J^{PC} and $J=0$ is proved to be sub-leading to the usual large- N_c QCD counting. The coupling of the same exotic particle to certain two (hybrid) meson currents with the same J^{PC} and $J=0$ is also sub-leading. The decay of a $\{1,3,5 \dots\}^{-+}$ hybrid particle to $\eta\pi^0, \eta'\pi^0, \eta'\eta, \eta(1295)\pi^0, \pi(1300)^0\pi^0, \eta(1440)\pi^0, a_0(980)^0\sigma$ or $f_0(980)\sigma$ is sub-leading, assuming that these final state particles are (hybrid) mesons in the limit of large N_c .

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I. INTRODUCTION

States of quantum chromodynamics (QCD) can definitively be said not to be conventional mesons when these states have exotic J^{PC} , which cannot be constructed for conventional mesons in the quark model, or equivalently, cannot be built from local currents with only a quark and an anti-quark field. Here J denotes the internal angular momentum, P (parity) the reflection through the origin and C (charge conjugation) particle-antiparticle exchange. These are conserved quantum numbers of QCD.

With the experimental discovery of isovector J^{PC} exotics, the question of their interpretation has come into focus. QCD with a large number of colors N_c offers a systematic expansion in $1/N_c$ with considerable phenomenological success [1,2], which can address this question. This is because a glueball (built from only gluons) and a (hybrid) meson (quark-antiquark with additional gluons) do not mix in large N_c [1]. Furthermore, four-quark states (two quark-antiquark pairs) are absent [2]. In large N_c the isovector J^{PC} exotics *must* therefore be hybrid mesons, as glueballs are isoscalar. Here it is proved for the first time that certain decays of hybrid mesons that are allowed by the conserved quantum numbers of QCD are sub-leading to their usual large- N_c counting, providing a consistency check for the hybrid nature of the state.

Selection rules for J^{PC} exotic hybrid decay *amplitudes*, e.g., the amplitude for $J^{PC}=1^{-+}$ hybrid particle $\rightarrow \eta\pi, \eta'\pi$, were noticed in non-field theoretic analyses [3]. In QCD it was found that these selection rules are really properties of certain three-point *Green's functions* [4,5]. The first attempt to obtain hadronic properties from the Green's functions [4] contained some errors [5]. These properties were subsequently extracted in finite- N_c QCD, e.g. the physical $N_c=3$ [5]. The properties were of limited physical relevance since they pertained to the coupling of currents to particles, e.g., a 1^{-+} hybrid current to $\eta\pi$. Also, for technical reasons, the scope of the deductions was limited. As will be seen below, these reasons disappear in large N_c , because three and more particles do not contribute. The large- N_c

treatment of the results of Ref. [5] is the subject of this paper. The results of this paper can be comprehended by only reading Sec. IV, which also explicates the experimental consequences of this paper. Section II proves two results for the coupling of particles to currents by first proving two preliminary results. Section III uses the results of the previous section to prove a result for the coupling of particles to particles.

II. COUPLING OF CURRENTS TO PARTICLES

The decay of a $J^{PC}=\{1,3,5 \dots\}^{-+}$ particle to two identical $J=0$ particles vanishes by Bose symmetry, because the final state particles are in an odd partial wave. The analogous statement for a Green's function built from a $J^{PC}=\{1,3,5 \dots\}^{-+}$ current and two $J=0$ currents is that the "identical current" part, or symmetric part, of the Green's function vanishes [5]. The OZI rule allowed contributions to the Green's function only has a symmetric part [5], so that they do not contribute to the Green's function. The expression that does not contain an OZI rule allowed contribution is

$$\begin{aligned} & \int_{-\infty}^{\infty} dt e^{iEt} \hat{O}_{\mathbf{p}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{p}\cdot\mathbf{y})} \langle 0 | B(\mathbf{x}, t) C(\mathbf{y}, t) A_{\mu}(0) | 0 \rangle \\ &= \sum_n (2\pi)^4 \delta^3(\mathbf{p}_n) \delta(E_n - E) \hat{O}_{\mathbf{p}} \\ & \times \langle 0 | \left(\int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} B(\mathbf{x}, 0) \right) C(0) | n \rangle \langle n | A_{\mu}(0) | 0 \rangle. \quad (1) \end{aligned}$$

This is proved in Eqs. (2), (3) and (14) of Ref. [5] with no approximations. The left-hand side (LHS) of the equation contains the time integral and spatial Fourier transform of a three-point Green's function which describes the "decay" of A into B and C . The expression is in Minkowski (physical) space with E and \mathbf{p} real numbers. The LHS is expanded on the right-hand side (RHS) by inserting an infinite set of asymptotic stable states n with energy E_n and momentum \mathbf{p}_n in order to extract physical predictions. The delta functions indicate that the asymptotic states are at rest and have energy E . The gauge-invariant local currents B , C and A_{μ} have the flavor structure of a neutral (hybrid) meson (linear combinations of $\bar{u}u, \bar{d}d, \dots$ quark fields), and can contain gluon

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fields [6]. The currents B and C both have the same color-Dirac-derivative-gluon structure for a given flavor, a finite number of derivatives (when expanded as a power series) and $J=0$. Also, the currents $B(0)$ and $C(0)$ have equal P and C . The current $A_\mu(0)$ is assumed to have $P=-$ and odd J (with Lorentz indices denoted by μ). Conservation of charge conjugation then implies that this current is $J^{PC} = \{1,3,5 \dots\}^{-+}$ exotic, so that it should contain at least one gluon field: a hybrid meson current.

Equation (1) would be of limited interest were it not for the fact that the action of the operator \hat{O}_p (containing a finite number of derivatives in powers of \mathbf{p}) allowed the demonstration that the LHS contains only OZI rule forbidden contributions, and hence is $\mathcal{O}(1)$ to leading order in the large- N_c power counting, as opposed to the usual $\mathcal{O}(N_c)$. The remainder of this paper exploits this behavior of the LHS and deduces the consequences for the RHS. The strategy is to keep only the leading contributions to the RHS in large N_c , and then to equate to the LHS. It is shown in the remainder of this section that the leading contribution to the RHS comes from either $J^{PC} = \{1,3,5 \dots\}^{-+}$ one-hybrid-meson states, or from two- (hybrid) meson states with $J=0$ and the same J^{PC} [Eq. (2)]. The RHS contains a product of two matrix elements for each of the leading contributions. One of the matrix elements will be shown to have the usual large- N_c count-

ing. The other matrix element will be shown to have one order in N_c lower counting than usual [Eqs. (3),(4)], in order that the LHS has an order lower counting than usual, as required.

On the RHS the usual large- N_c counting for $\langle 0|B(\mathbf{x},0)C(0)|n\rangle$ is order $\sqrt{N_c}$, N_c , $\sqrt{N_c}$ or 1 for n respectively a one-, two-, three- or four- (hybrid) meson asymptotic state [1,7]. The counting for $\langle n|A_\mu(0)|0\rangle$ is respectively $\sqrt{N_c}$, 1, $1/\sqrt{N_c}$ or $1/N_c$ [1,7]. The product of the countings of the two matrix elements is N_c , N_c , 1 and $1/N_c$ respectively. If the asymptotic states contained glueballs the counting of the product will be lower than N_c . Hence only one- and two-particle (hybrid) meson states contribute in large N_c , and they contribute at $\mathcal{O}(N_c)$, as they should to equal the usual counting of the LHS. Also, the one-particle states that contribute to $\langle n|A_\mu(0)|0\rangle$ at $\mathcal{O}(\sqrt{N_c})$ are *only* neutral hybrid mesons with the same J^{PC} as the current $A_\mu(0)$ [7]. These states cannot be mesons because they are J^{PC} exotic. The two-particle states that contribute to $\langle 0|B(\mathbf{x},0)C(0)|n\rangle$ at $\mathcal{O}(N_c)$ are *only* two neutral (hybrid) mesons [2] with the same J^{PC} as the currents $B(0)$ and $C(0)$. It follows that only one-hybrid-meson and two- (hybrid) meson states contribute on the RHS to leading order in large N_c . Using this the RHS of Eq. (1) can be simplified to read (Appendix A 1)

$$2\pi \sum_{\sigma} \delta(m_{\sigma}-E) \hat{O}_p \langle 0 | \left(\int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} B(\mathbf{x},0) \right) C(0) | \sigma \mathbf{0} \rangle \langle \sigma \mathbf{0} | A_{\mu}(0) | 0 \rangle + \frac{1}{(2\pi)^2} \sum_{\sigma_1 \sigma_2} \left(1 - \frac{\delta_{\sigma_1 \sigma_2}}{2} \right) K(E) \times \int d\Omega_{\mathbf{k}_1} \hat{O}_p \langle 0 | \left(\int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} B(\mathbf{x},0) \right) C(0) | \sigma_1 k_1 \sigma_2 k_2 \rangle \langle \sigma_1 k_1 \sigma_2 k_2 | A_{\mu}(0) | 0 \rangle \Big|_{k_1+k_2=(\mathbf{0},E)}, \quad (2)$$

where the first sum is over the one-hybrid-meson states σ (implicitly including the different polarizations); and the second over the two- (hybrid) meson states σ_1 and σ_2 , in such a way that a particular two-particle state is summed over only in the permutation $\sigma_1 \sigma_2$ (and not $\sigma_2 \sigma_1$) in order to avoid double counting. Throughout the text the convention is that non-boldfaced variables starting with k (and p, q, x, y, z) indicate four-vectors.

The kinematical variable dependence of the one-particle terms in Eq. (2) is only on E and \mathbf{p} . If the hybrid particles have a discrete spectrum, there would only be contributions for discrete values of E when $E = m_{\sigma}$. Because of the constraint $k_1 + k_2 = (\mathbf{0}, E)$, the two-particle terms also only depend on the kinematical variables E and \mathbf{p} . Because $K(E)$ only has support above the two-particle threshold, the two-particle terms all vanish below the lowest threshold, and contains a (continuous in E) contribution from each two-particle state above its threshold.

In each of the terms in Eq. (2), one of the matrix elements has the usual large- N_c counting. For the one-particle terms this is $\langle \sigma \mathbf{0} | A_{\mu}(0) | 0 \rangle$. The reason is that, barring accidental

cancellations, this matrix element *has* to have exactly its usual large- N_c counting ($\mathcal{O}(\sqrt{N_c})$), in order not to violate the counting for two-point Green's functions (the *first preliminary* proved in Appendix A 2). For the two-particle terms in Eq. (2), $\langle 0 | B(\mathbf{x},0) C(0) | \sigma_1 k_1 \sigma_2 k_2 \rangle$ has the usual large- N_c counting ($\mathcal{O}(N_c)$). The reason is that, barring accidental cancellations, it *has* to be exactly $\mathcal{O}(N_c)$, in order not to violate the counting for four-point Green's functions (the *second preliminary* proved in Appendix A 3).

Equations (1),(2) can schematically be written as LHS = RHS = $\sum_m a_m b_m + \sum_m c_m d_m$, where the sums are over σ for the one-particle terms $\sum_m a_m b_m$ and over $\sigma_1 \sigma_2 \Omega_{\mathbf{k}_1}$ for the two-particle terms $\sum_m c_m d_m$. (As shown in Appendix B, the integral over $\Omega_{\mathbf{k}_1}$ can be written as a finite sum due to a partial wave expansion.) The sum over σ is finite, since the one-particle states must have mass equal to E . The sum over $\sigma_1 \sigma_2$ is finite because there is a finite number of two-particle thresholds below E . If the LHS is subleading in the large- N_c counting, the RHS is, but this does *not* imply that $a_m b_m$ is subleading, nor that $c_n d_n$ is, because it is possible that both

$a_m b_m$ and $c_m d_m$ have the usual large- N_c counting, and there is a cancellation between the terms yielding a subleading RHS. The arguments in Appendix B have the consequence that $a_m b_m$ and $c_m d_m$ are *each* subleading, i.e., $\mathcal{O}(1)$. This follows by evaluating the equation LHS = $\sum_m a_m b_m + \sum_m c_m d_m$ multiple times for different currents, so that a matrix equation is obtained which is then inverted. This can only be done if the number of terms on the RHS is finite, as it is here. If three- or more-particle terms contributed on the RHS, as would generally be the case for the physical $N_c = 3$, it is not clear that the number of terms on the RHS is finite, and that the inversion can be done. For this reason, the fact that three or more particles are subleading in the large- N_c expansion, is important to make progress in the derivation. The remainder of this section is devoted to two results, which although exhaustively derived in Appendix B, are mostly evident at this point.

The first preliminary [$\langle \sigma \mathbf{0} | A_\mu(0) | 0 \rangle$ has to be $\mathcal{O}(\sqrt{N_c})$] means that b_m is $\mathcal{O}(\sqrt{N_c})$. Together with $a_m b_m$ is $\mathcal{O}(1)$, this implies that a_m is $\mathcal{O}(1/\sqrt{N_c})$, which yields most of the *first result* of the paper that the coupling of currents to a particle

$$\langle 0 | B(\mathbf{x}, t) C(\mathbf{y}, t) | \sigma \mathbf{0} \rangle = \mathcal{O}\left(\frac{1}{\sqrt{N_c}}\right), \quad (3)$$

where its usual counting is $\mathcal{O}(\sqrt{N_c})$. This holds for a neutral on-shell hybrid meson particle σ at rest with $J^{PC} = \{1, 3, 5 \dots\}^{-+}$. Also, B and C are neutral gauge-invariant local (hybrid) meson currents at space-time positions x and y at equal time with flavor structure a linear combination of $\bar{u}u, \bar{d}d, \dots$, the same color-Dirac-derivative-gluon structure for a given flavor, a finite number of derivatives and $J = 0$. The currents $B(0)$ and $C(0)$ should have equal P and C .

The second preliminary [$\langle 0 | B(\mathbf{x}, 0) C(0) | \sigma_1 k_1 \sigma_2 k_2 \rangle$ has to be $\mathcal{O}(N_c)$] yields most of the fact that c_m is $\mathcal{O}(N_c)$. Since $c_m d_m$ is $\mathcal{O}(1)$, this implies that d_m is $\mathcal{O}(1/N_c)$, which yields most of the *second result* that the coupling of particles to a current

$$\langle \sigma_1 k_1 \sigma_2 k_2 | A_\mu(z) | 0 \rangle = \mathcal{O}\left(\frac{1}{N_c}\right), \quad (4)$$

where its usual counting is $\mathcal{O}(1)$. This holds for neutral on-shell (hybrid) meson particles σ_1 and σ_2 with identical J^{PC} and $J = 0$, and with arbitrary four-momenta k_1 and k_2 . Also, $A_\mu(z)$ is a neutral gauge-invariant local hybrid meson $J^{PC} = \{1, 3, 5 \dots\}^{-+}$ current with Lorentz indices μ at space-time position z with flavor structure a linear combination of $\bar{u}u, \bar{d}d, \dots$

III. COUPLING OF PARTICLES TO PARTICLES

In this section the dependence on the current $A_\mu(z)$ is removed from the matrix element in Eq. (4) to obtain a result [Eq. (6)] that does not depend on the current, but on the physically relevant T-matrix. Because the matrix element in Eq. (4) has one order in N_c lower counting than the usual,

the T-matrix element [Eq. (6)] will have an order lower counting than usual. It is shown in Appendix A 4 that

$$\begin{aligned} & \langle \sigma_1 k_1 \sigma_2 k_2 \text{ out} | A_\mu(z) | 0 \rangle - \langle \sigma_1 k_1 \sigma_2 k_2 \text{ in} | A_\mu(z) | 0 \rangle \\ &= \sum_\sigma \left[\frac{i2\pi}{\sqrt{N_c}} \langle \sigma(k_1 + k_2) \text{ in} | A_\mu(z) | 0 \rangle \delta(\sqrt{m_1^2 + \mathbf{k}_1^2} \right. \\ & \quad \left. + \sqrt{m_2^2 + \mathbf{k}_2^2} - m_\sigma) \right] \left[\sqrt{N_c} \langle \sigma_1 k_1 \sigma_2 k_2 | T | \sigma(k_1 + k_2) \rangle \right] \end{aligned} \quad (5)$$

when the restriction to the rest frame, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}$, applies. This equation states that when the difference of the coupling of remote future ‘‘out’’ states and remote past ‘‘in’’ states to the current is considered, valuable information about the physically relevant T-matrix is obtained: the *third result* that the coupling of a particle to particles

$$\langle \sigma_1 k_1 \sigma_2 k_2 | T | \sigma \mathbf{0} \rangle = \mathcal{O}\left(\frac{1}{N_c^{3/2}}\right), \quad (6)$$

where its usual counting is $\mathcal{O}(1/\sqrt{N_c})$. This holds for neutral on-shell (hybrid) meson particles σ_1 and σ_2 with $J = 0$, identical J^{PC} , and four-momenta k_1 and k_2 in the rest frame $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}$; and for a neutral on-shell hybrid meson particle σ at rest with $J^{PC} = \{1, 3, 5 \dots\}^{-+}$.

Even though the third result is proved in Appendix B using techniques analogous to those used to derive the first two results, its plausibility can be verified by using Eq. (6) in conjunction with the first preliminary [$\langle \sigma \mathbf{0} | A_\mu(z) | 0 \rangle$ has to be $\mathcal{O}(\sqrt{N_c})$], to obtain that the RHS of Eq. (5) is $\mathcal{O}(1/N_c)$, consistent with the LHS given by the second result [Eq. (4)] as $\mathcal{O}(1/N_c)$.

IV. REMARKS

The three results of the paper are Eqs. (3), (4) and (6), including the discussion under each equation. These results are theorems of large- N_c QCD field theory with no approximations, and are valid within the generic large- N_c framework [1,2,7].

The first result [Eq. (3)] implies that certain four-quark currents are not good interpolators for hybrid meson particles. This may have implications for Euclidean space lattice QCD, even though the result was derived only in Minkowski space. A special case of the second result [Eq. (4)] was previously derived [5] for an $\eta\pi^0$ asymptotic state for certain quark masses within a certain kinematical range.

The third result [Eq. (6)] is of direct experimental relevance. For *example*, the decay amplitudes (couplings of a particle to particles) of a $\{1, 3, 5 \dots\}^{-+}$ hybrid to $\eta\pi^0, \eta'\pi^0, \eta'\eta, \eta(1295)\pi^0, \pi(1300)^0\pi^0, \eta(1440)\pi^0, a_0(980)^0\sigma$ or $f_0(980)\sigma$ are $\mathcal{O}(1/N_c^{3/2})$, while the usual counting is $\mathcal{O}(1/\sqrt{N_c})$, assuming that these final state particles are (hybrid) mesons in the limit of large N_c . Hence the widths of these decays are $1/N_c^2$ suppressed with respect to their usual counting. This is the same suppression that large

N_c predicts for decays forbidden by the OZI rule [1], implying that the suppressions predicted here should be similar phenomenologically. The selection rule is most useful when OZI allowed decay is expected to be important in the absence of the selection rule. In the example above, this is true for a hybrid composed dominantly of $u\bar{u}$ and $d\bar{d}$. On the other hand, it can be deduced from Eq. (6) that the coupling of a 1^{-+} hybrid to $\eta_c\eta$ is $\mathcal{O}(1/N_c^{3/2})$, but this is less useful as the OZI allowed coupling [of the $c\bar{c}$ component of the hybrid to $\eta_c(c\bar{c})$ and the $c\bar{c}$ component of the η] is not expected to be important. Interestingly, even in the unlikely case where the η' or σ is a pure glueball in the limit of large N_c , the decay amplitude of $\{1,3,5\dots\}^{-+}$ hybrids to $\eta'\pi^0, \eta'\eta, a_0(980)^0\sigma$ or $f_0(980)\sigma$ would be $\mathcal{O}(1/N_c)$ [1], which is still subdominant to the usual counting. In case σ is a meson-meson state in the limit of large N_c , the predictions mentioned do not apply. Beside the 0^{-+} and 0^{++} particles mentioned, examples can also be given of 0^{+-} and 0^{--} exotic particles in the final states. The large- N_c selection rules, in contrast to the selection rules discussed in Sec. I, also apply when both final state mesons do not have the same radial excitation, e.g., the $\eta(1295)\pi^0, \pi(1300)^0\pi^0$ and $\eta(1440)\pi^0$ final states. Assuming isospin symmetry the results can also be extended to charged states by use of the Wigner-Eckart theorem, as will now be done.

Consider the decay of a 1^{-+} isovector hybrid with isospin symmetry. Decay to $\eta\pi, \eta'\pi, \eta(1295)\pi, \eta(1440)\pi$ and $a_0(980)\sigma$, which is ordinarily important, is suppressed. The experimental $\pi_1(1600)$ [8] is a 1^{-+} exotic isovector resonance. It has not been seen in $\eta\pi$. A 1^{-+} enhancement at 1.6 GeV has prominently been seen in $\eta'\pi$ [8], although the branching ratio is not dominant if the enhancement is resonant ($B[\pi_1(1600)\rightarrow f_1\pi]/B[\pi_1(1600)\rightarrow \eta'\pi]=3.80\pm 0.78$ [9]). If the enhancement is dominantly nonresonant, as has been advocated [10], the branching ratio is very small. The decay $\pi_1(1600)\rightarrow \eta(1295)\pi$ is found to be small relative to $f_1\pi$ in an analysis of the $\eta\pi^+\pi^-\pi^0$ final state [9], although an earlier report stated that $\pi_1(1600)$ was seen in $f_1\pi$ and $\eta(1295)\pi$ at a similar magnitude in $K^+\bar{K}^0\pi^-\pi^0$ [11]. If $\pi_1(1600)$ is found to have a large branching ratio to $\eta'\pi$, that would be inconsistent with large- N_c expectations which are otherwise consistent with its being a hybrid meson [12]. As discussed above, decay to $\eta'\pi$ is large- N_c suppressed when η' is either a meson or a glueball in the limit of large N_c , although the suppression is less when η' is a glueball. Hence a sizable $\eta'\pi$ branching ratio can arise through a large glueball component of an η' meson [3], which violates the large- N_c prediction that meson-glueball mixing is suppressed. The recently discovered $\pi_1(2000)$ has not been seen in $\eta\pi, \eta'\pi$ and $\eta(1295)\pi$ [9], consistent with its being a hybrid meson.

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APPENDIX A: DIVERSE RESULTS

1. Derivation of Eq. (2)

For the one-particle states, $\Sigma_n = \Sigma_\sigma \int d^3p_\sigma / (2\pi)^3$, and the momentum $\mathbf{p}_\sigma = \mathbf{0}$ due to the momentum δ function, so that

$E_\sigma = \sqrt{m_\sigma^2 + \mathbf{p}_\sigma^2} = m_\sigma$ because the particles in an asymptotic state are on-shell. For the two-particle states $\Sigma_n = \Sigma_{\sigma_1\sigma_2} (1 - \delta_{\sigma_1\sigma_2}/2) \int d^3k_1 / (2\pi)^3 \int d^3k_2 / (2\pi)^3$, where the factor $(1 - \delta_{\sigma_1\sigma_2}/2)$ is 1/2 for the phase space of identical particles. Substituting $\mathbf{p}_n = \mathbf{k}_1 + \mathbf{k}_2$ and $E_n = \sqrt{\mathbf{k}_1^2 + m_1^2} + \sqrt{\mathbf{k}_2^2 + m_2^2}$ in the phase space integration

$$\int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} (2\pi)^4 \delta^3(\mathbf{p}_n) \delta(E_n - E) f(k_1, k_2) = \frac{K(E)}{(2\pi)^2} \int d\Omega_{\mathbf{k}_1} f(k_1, k_2) \Big|_{k_1+k_2=(\mathbf{0}, E)}, \quad (\text{A1})$$

where

$$K(E) \equiv \int_0^\infty k_1^2 dk_1 |\delta(\sqrt{\mathbf{k}_1^2 + m_1^2} + \sqrt{\mathbf{k}_1^2 + m_2^2} - E)| = \frac{1}{8E^4} (E^4 - (m_1 + m_2)^2 (m_1 - m_2)^2) \times \sqrt{(E^2 - (m_1 + m_2)^2)(E^2 - (m_1 - m_2)^2)} \quad (\text{A2})$$

if $E \geq m_1 + m_2$; and $K(E)$ vanishes if $E < m_1 + m_2$.

2. First preliminary

The following two-point function is $\mathcal{O}(N_c)$ in the Feynman-diagrammatic large- N_c counting [7]

$$\langle 0 | A_\mu(x_1) A_\nu(x_2) | 0 \rangle = \sum_n \langle 0 | A_\mu(x_1) | n \rangle \langle n | A_\nu(x_2) | 0 \rangle, \quad (\text{A3})$$

and *only* one-(hybrid)-mesons n contribute at leading order [7]. Hence there must be a non-empty set of states n for which $\langle 0 | A_\mu(x_1) | n \rangle \langle n | A_\nu(x_2) | 0 \rangle$ of $\mathcal{O}(N_c)$. If A, μ, ν, x_1 and x_2 are changed the set of states for which this is true may change. As these variables are changed, a specific state n should regularly be part of the set, since there is nothing special about it. Hence for a specific particle σ with four-momentum p_σ , it must be possible to choose $A, \mu = \nu$ and $x_1 = x_2 = z$ such that $\langle 0 | A_\mu(x_1) | n \rangle \langle n | A_\nu(x_2) | 0 \rangle = |\langle \sigma p_\sigma | A_\mu(z) | 0 \rangle|^2$ is $\mathcal{O}(N_c)$. This implies that $\langle \sigma p_\sigma | A_\mu(z) | 0 \rangle$ is $\mathcal{O}(\sqrt{N_c})$, as promised.

3. Second preliminary

The following four-point function is $\mathcal{O}(N_c^2)$ [2]

$$\langle 0 | B(x_1) C(x_2) B(x_3) C(x_4) | 0 \rangle = \sum_n \langle 0 | B(x_1) C(x_2) | n \rangle \langle n | B(x_3) C(x_4) | 0 \rangle, \quad (\text{A4})$$

and *only* two-(hybrid) meson states n contribute at leading order [2]. Similar to the argument for the first preliminary, it

must be possible to choose B , C , $x_1=x_3=(\mathbf{x},0)$, $x_2=x_4=0$ and a specific state $|\sigma_1 k_1 \sigma_2 k_2\rangle$, such that

$$\begin{aligned} & \langle 0|B(x_1)C(x_2)|n\rangle\langle n|B(x_3)C(x_4)|0\rangle \\ & = |\langle 0|B(\mathbf{x},0)C(0)|\sigma_1 k_1 \sigma_2 k_2\rangle|^2 \end{aligned}$$

is $\mathcal{O}(N_c^2)$. Whence the promised result.

4. Derivation of Eq. (5)

All derivations so far left unspecified whether the asymptotic states were “in” or “out” states. Consider the specific case of “out” states, and insert a complete set of “in” states:

$$\begin{aligned} & \langle \sigma_1 k_1 \sigma_2 k_2 \text{ out} | A_\mu(z) | 0 \rangle \\ & = \sum_\sigma \int \frac{d^3 q}{(2\pi)^3} \langle \sigma_1 k_1 \sigma_2 k_2 \text{ out} | \sigma q \text{ in} \rangle \\ & \quad \times \langle \sigma q \text{ in} | A_\mu(z) | 0 \rangle + \sum_{\sigma'_1 \sigma'_2} \left(1 - \frac{\delta_{\sigma'_1 \sigma'_2}}{2} \right) \\ & \quad \times \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3} \langle \sigma_1 k_1 \sigma_2 k_2 \text{ out} | \sigma'_1 q_1 \sigma'_2 q_2 \text{ in} \rangle \\ & \quad \times \langle \sigma'_1 q_1 \sigma'_2 q_2 \text{ in} | A_\mu(z) | 0 \rangle. \end{aligned} \quad (\text{A5})$$

Restrict to $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}$ in this subsection. The usual large- N_c counting for $\langle \sigma_1 k_1 \sigma_2 k_2 | n \rangle$ is order $1/\sqrt{N_c}$, 1 (no scattering), or $1/\sqrt{N_c}$ for n respectively a one-, two- or three- (hybrid) meson state [1,7]. The counting for $\langle n | A_\mu(z) | 0 \rangle$ is respectively $\sqrt{N_c}$, 1 or $1/\sqrt{N_c}$ [1,7]. The product of the countings of the two matrix elements is 1, 1 and $1/N_c$ respectively. If the asymptotic states contained glueballs the counting of the product will be lower than 1. Hence only one- and two- (hybrid) meson states contribute in large N_c , as indicated in Eq. (A5). As before, the one-particle states that contribute are *only* neutral hybrid mesons with the same J^{PC} as the current $A_\mu(0)$.

Connection with the S matrix is now made by using $\langle m \text{ out} | n \text{ in} \rangle = \langle m \text{ in} | S | n \text{ in} \rangle$ [13]. In the second part of Eq. (A5) write the two-body scattering

$$\begin{aligned} & \langle \sigma_1 k_1 \sigma_2 k_2 \text{ out} | \sigma'_1 q_1 \sigma'_2 q_2 \text{ in} \rangle \\ & = \langle \sigma_1 k_1 \sigma_2 k_2 \text{ in} | \sigma'_1 q_1 \sigma'_2 q_2 \text{ in} \rangle \end{aligned}$$

where it was used that only no-scattering occurs at $\mathcal{O}(1)$ [1]. The latter overlap is simply an overlap between free bosonic states in the same basis. It can be evaluated [13] and equals

$$\begin{aligned} & (2\pi)^6 (\delta^3(\mathbf{k}_1 - \mathbf{q}_1) \delta_{\sigma_1 \sigma'_1} \delta^3(\mathbf{k}_2 - \mathbf{q}_2) \delta_{\sigma_2 \sigma'_2} \\ & + \delta^3(\mathbf{k}_1 - \mathbf{q}_2) \delta_{\sigma_1 \sigma'_2} \delta^3(\mathbf{k}_2 - \mathbf{q}_1) \delta_{\sigma_2 \sigma'_1}). \end{aligned} \quad (\text{A6})$$

In the first part in Eq. (A5) introduce the T matrix, defined as the transition from an “in” to an “out” state

$$\begin{aligned} & \langle \sigma_1 k_1 \sigma_2 k_2 \text{ out} | \sigma q \text{ in} \rangle = i(2\pi)^4 \delta^4(k_1 + k_2 - q) \\ & \quad \times \langle \sigma_1 k_1 \sigma_2 k_2 \text{ in} | T | \sigma q \text{ in} \rangle, \end{aligned} \quad (\text{A7})$$

using that no-scattering does not contribute, given that this is the overlap of a two- with a one-particle state; and employing the definition of the S matrix in terms of the “reduced” T matrix [13]. Dropping the “in” label on the RHS of Eq. (A7), because $\langle m \text{ in} | T | n \text{ in} \rangle = \langle m \text{ out} | T | n \text{ out} \rangle$ [13], and substituting Eqs. (A6),(A7) in Eq. (A5), yield Eq. (5).

APPENDIX B: WHY EACH TERM IN EQ. (2) IS SUBLEADING

This appendix starts by proving that the angular integration in Eq. (2) can be written as a finite sum. The first two results of the paper are then derived. [This is done directly, without first showing that *each* of the terms in Eq. (2) is subleading as discussed in the main text. However, once the first two results are established, it follows that each of the terms is subleading.] The third result is subsequently derived.

It will be convenient for the derivation to write the integral over the solid angle $\Omega_{\mathbf{k}_1}$ in Eq. (2) as a finite sum. This can be done by performing a partial wave expansion of the overlap $\langle \sigma_1 k_1 \sigma_2 k_2 | A_\mu(0) | 0 \rangle$ by explicitly considering its Lorentz structure. It is a function of k_1 and k_2 , or equivalently of $k_1 - k_2$ and $k_1 + k_2$. First consider the Lorentz scalars that can be built from these two variables: $(k_1 - k_2)^2$, $(k_1 + k_2)^2$ and $(k_1 - k_2) \cdot (k_1 + k_2)$. It is easily shown that the on-shell conditions $k_1^2 = m_1^2$ and $k_2^2 = m_2^2$ imply that the latter two variables can be expressed in terms of the first variable. Hence the only independent Lorentz scalar is $(k_1 - k_2)^2$. Second consider the case where A_μ has $J=1$, i.e. is a vector. The overlap can be written as a linear combination of $k_{1\mu}$ times a Lorentz scalar, and $k_{2\mu}$ times a Lorentz scalar, since this is the most general structure transforming like a vector. Denote the two Lorentz scalars by $\langle \sigma_1 \sigma_2 | A(0) \rangle_i ((k_1 - k_2)^2)$, with $i=1,2$. Define $\mathcal{L}_\mu^i(k_1, k_2) = k_{i\mu}$. Then the overlap

$$\begin{aligned} & \langle \sigma_1 k_1 \sigma_2 k_2 | A_\mu(0) | 0 \rangle \\ & = \sum_i \mathcal{L}_\mu^i(k_1, k_2) \langle \sigma_1 \sigma_2 | A(0) \rangle_i ((k_1 - k_2)^2). \end{aligned} \quad (\text{B1})$$

It is evident that the procedure can be performed for arbitrary J , and that appropriate \mathcal{L}_μ^i can always be constructed, with the partial wave i ranging over a finite number of integers. Equation (B1) is the promised partial wave expansion of the overlap for general J . The functions \mathcal{L}_μ^i depend purely on kinematical variables and all dynamical information is contained in the scalar functions $\langle \sigma_1 \sigma_2 | A(0) \rangle_i ((k_1 - k_2)^2)$, which depend kinematically only on $(k_1 - k_2)^2$. When Eq. (B1) is substituted in the two-particle terms of Eq. (2) the integral over the solid angle $\Omega_{\mathbf{k}_1}$ can be written as a finite sum over i , as promised, since $(k_1 - k_2)^2$ does not depend on the solid angle.

Rewrite Eqs. (1),(2) as

$$W_{BC} = \sum_{\sigma=1}^{N_{\Sigma}} M_{BC\sigma} V_{\sigma} + \sum_{\sigma_1\sigma_2i=1}^{N_{\Pi}} \tilde{M}_{BC\sigma_1\sigma_2i} \tilde{V}_{\sigma_1\sigma_2i}, \quad (\text{B2})$$

where the explicit dependence on the currents B and C are indicated. The LHS of Eq. (1) (divided by N_c) is

$$W_{BC} \equiv \frac{1}{N_c} \int_{-\infty}^{\infty} dt e^{iEt} \hat{O}_{\mathbf{p}} \int d^3x d^3y e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{p}\cdot\mathbf{y})} \langle 0|B(\mathbf{x},t)C(\mathbf{y},t)A_{\mu}(0)|0\rangle. \quad (\text{B3})$$

The one-particle matrix elements of Eq. (2) (divided by N_c) are given by

$$M_{BC\sigma} \equiv N_c^{\alpha-1/2} 2\pi \delta(m_{\sigma}-E) \hat{O}_{\mathbf{p}} \langle 0| \left(\int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} B(\mathbf{x},0) \right) C(0) | \sigma \mathbf{0} \rangle, \quad (\text{B4})$$

$$V_{\sigma} \equiv N_c^{-\alpha-1/2} \langle \sigma \mathbf{0} | A_{\mu}(0) | 0 \rangle, \quad (\text{B5})$$

where α is a real number specified below. The two-particle matrix elements of Eq. (2) (divided by N_c) are

$$\tilde{M}_{BC\sigma_1\sigma_2i} \equiv \frac{1}{N_c} \frac{1}{(2\pi)^2} \left(1 - \frac{\delta_{\sigma_1\sigma_2}}{2} \right) K(E) \int d\Omega_{\mathbf{k}_1} \hat{O}_{\mathbf{p}} \langle 0| \left(\int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} B(\mathbf{x},0) \right) C(0) | \sigma_1 k_1 \sigma_2 k_2 \rangle \mathcal{L}_{\mu}^i(k_1, k_2) \Big|_{k_1+k_2=(\mathbf{0},E)}, \quad (\text{B6})$$

$$\tilde{V}_{\sigma_1\sigma_2i} \equiv \langle \sigma_1 \sigma_2 | A(0) \rangle_i ((k_1 - k_2)^2) \Big|_{k_1+k_2=(\mathbf{0},E)}, \quad (\text{B7})$$

using Eq. (B1). As discussed in the main text, each side of Eq. (B2) depends on the kinematical variables \mathbf{p} and E . For a specific choice of these variables, there are $N_{\Sigma} \geq 0$ one-particle states σ contributing, and $N_{\Pi} \geq 0$ two-particle states and partial waves $\sigma_1\sigma_2i$ contributing. It will be useful to think of the group of labels $\sigma_1\sigma_2i$ as a single label. The remainder of the discussion is only of interest if it is not the case that $N_{\Sigma} = N_{\Pi} = 0$.

From the second preliminary [$\langle 0|B(\mathbf{x},0)C(0)|\sigma_1k_1\sigma_2k_2\rangle$ has to be $\mathcal{O}(N_c)$] it follows that \tilde{M} in Eq. (B6) is $\leq \mathcal{O}(1)$. This obtains by noting that $\hat{O}_{\mathbf{p}}$ is independent of color, as is shown at the end of this appendix, and that \mathcal{L}_{μ}^i and K are purely kinematical functions with no color dependence. The possibility that there are accidental cancellations in the various integrations, which could make $\tilde{M} < \mathcal{O}(1)$, is incorporated by indicating that $\tilde{M} \leq \mathcal{O}(1)$. The possibility of cancellations will be taken into account in the derivations below. By choosing α appropriately, M in Eq. (B4) is *defined* to be exactly $\mathcal{O}(1)$. Hence both M and \tilde{M} are $\leq \mathcal{O}(1)$. Evaluate Eq. (B2) for $N_{\Sigma} + N_{\Pi}$ different currents B, C . Since V and \tilde{V} in Eqs. (B5) and (B7) are independent of the choice of currents B and C , this amounts to constructing a matrix equation $\mathbf{W} = \mathbf{M}\mathbf{V}$. The $N_{\Sigma} + N_{\Pi}$ dimensional column vector \mathbf{W} is built from the evaluations of W_{BC} for different values of B, C ; the $(N_{\Sigma} + N_{\Pi}) \times (N_{\Sigma} + N_{\Pi})$ dimensional matrix \mathbf{M} contains $M_{BC\sigma}$ and $\tilde{M}_{BC\sigma_1\sigma_2i}$; and the $N_{\Sigma} + N_{\Pi}$ dimensional column vector \mathbf{V} is built from V_{σ} and $\tilde{V}_{\sigma_1\sigma_2i}$. Because both M and \tilde{M} are $\leq \mathcal{O}(1)$ it follows that each entry of the matrix \mathbf{M} is also $\leq \mathcal{O}(1)$. This means that, barring accidental cancellations, the determinant of \mathbf{M} , $\det \mathbf{M}$, which is a sum of

products of the entries of \mathbf{M} , is exactly $\mathcal{O}(1)$, i.e., is non-zero and finite in the large- N_c limit. Note that even if some of the entries of the matrix \mathbf{M} are $< \mathcal{O}(1)$, it is still possible for $\det \mathbf{M}$ to be exactly $\mathcal{O}(1)$. If $\det \mathbf{M} < \mathcal{O}(1)$ the derivations below are invalid. This possibility can be excluded by an appropriate choice of currents. Since $\det \mathbf{M} = \mathcal{O}(1)$ the inverse of \mathbf{M} exists, and $\mathbf{V} = \mathbf{M}^{-1}\mathbf{W}$. Since $\mathbf{M}^{-1} = \text{adj } \mathbf{M} / \det \mathbf{M}$, where $\text{adj } \mathbf{M}$ is the adjoint matrix of \mathbf{M} , which is a sum of products of entries of \mathbf{M} , it follows that the entries of $\mathbf{M}^{-1} \leq \mathcal{O}(1)$. Noting from the main text that the LHS of Eq. (1) is $\mathcal{O}(1)$ [strictly speaking, it is $\leq \mathcal{O}(1)$, since it is only known that OZI allowed $\mathcal{O}(N_c)$ contributions are not present [5], and that the highest order OZI forbidden contribution is $\mathcal{O}(1)$ [1] if there are no accidental cancellations], so that each entry of the vector $\mathbf{W} \leq \mathcal{O}(1/N_c)$, it follows from $\mathbf{V} = \mathbf{M}^{-1}\mathbf{W}$ that each entry of the vector \mathbf{V} is $\leq \mathcal{O}(1/N_c)$. This implies that V and \tilde{V} are both $\leq \mathcal{O}(1/N_c)$.

It is instructive to study the kinematical variable dependence of \tilde{V} in Eq. (B7), which can only be relevant to the discussion if E is above the two-particle threshold $m_1 + m_2$, since the two-particle term in Eq. (2) only has support above this threshold. This, together with the constraint $k_1 + k_2 = (\mathbf{0}, E)$ in Eq. (B7), and the on-shell character of the particles, can be shown to imply that $(k_1 - k_2)^2 = 2(m_1^2 + m_2^2) - E^2 \in (-\infty, (m_1 - m_2)^2]$. Using $\tilde{V} \leq \mathcal{O}(1/N_c)$, and Eqs. (B1) and (B7), it follows that $\langle \sigma_1 k_1 \sigma_2 k_2 | A_{\mu}(0) | 0 \rangle \leq \mathcal{O}(1/N_c)$ with the constraint that $(k_1 - k_2)^2 \in (-\infty, (m_1 - m_2)^2]$. However, it is possible to show by only considering the on-shell nature of the particles, that the same constraint holds. Hence the constraint adds no new information, and is dropped henceforth. Thus

$$\begin{aligned} \langle \sigma_1 k_1 \sigma_2 k_2 | A_\mu(z) | 0 \rangle &= e^{i(k_1+k_2) \cdot z} \langle \sigma_1 k_1 \sigma_2 k_2 | A_\mu(0) | 0 \rangle \\ &\leq \mathcal{O}(1/N_c), \end{aligned} \quad (\text{B8})$$

using space-time translational invariance $A_\mu(z) = e^{iP \cdot z} A_\mu(0) e^{-iP \cdot z}$, with P^ν the QCD four-momentum operator. It is evident that equality in Eq. (B8) would have been attained were it not for the possibility of accidental cancellations. These cancellations can be eliminated by an appropriate choice of the currents. Whence the result in Eq. (4).

The observation that $V \leq \mathcal{O}(1/N_c)$, together with the first preliminary [$\langle \sigma \mathbf{0} | A_\mu(0) | 0 \rangle$ has to be $\mathcal{O}(\sqrt{N_c})$], implies from Eq. (B5) that $\alpha \geq 1$. Since M was defined to be $\mathcal{O}(1)$ this implies from Eq. (B4) that $\langle 0 | (\int d^3x e^{i\mathbf{p} \cdot \mathbf{x}} B(\mathbf{x}, 0)) C(0) | \sigma \mathbf{0} \rangle$ is $\leq \mathcal{O}(1/\sqrt{N_c})$, noting that m_σ is $\mathcal{O}(1)$ [7]. Inverting the Fourier transform, it follows that $\langle 0 | B(\mathbf{x}, 0) C(0) | \sigma \mathbf{0} \rangle \leq \mathcal{O}(1/\sqrt{N_c})$. Using space-time translational invariance analogous to Eq. (B8)

$$\begin{aligned} \langle 0 | B(\mathbf{x}, t) C(\mathbf{y}, t) | \sigma \mathbf{0} \rangle &= e^{-im_\sigma t} \langle 0 | B(\mathbf{x} - \mathbf{y}, 0) C(0) | \sigma \mathbf{0} \rangle \\ &\leq \mathcal{O}(1/\sqrt{N_c}). \end{aligned} \quad (\text{B9})$$

From the same arguments as those below Eq. (B8), the result in Eq. (3) is deduced.

It remains to prove that Eq. (6) can be deduced from Eq. (5). The proof is analogous to the proof already given in this appendix, and the various steps are outlined. The notation of the vectors and matrices will be the same except that the

label BC will be replaced by the label A . Call the LHS of Eq. (5) W_A . The first and second terms in long brackets on the RHS of Eq. (5) are called $M_{A\sigma}$ and V_σ respectively. Equation (5) is then of the form of Eq. (B2) with $N_\Pi = 0$. From the first preliminary [$\langle \sigma \mathbf{0} | A_\mu(z) | 0 \rangle$ has to be $\mathcal{O}(\sqrt{N_c})$], it follows that $M_{A\sigma}$ is $\mathcal{O}(1)$. Evaluate Eq. (B2) for N_Σ different currents A . This again gives a matrix equation $\mathbf{W} = \mathbf{M}\mathbf{V}$. Barring accidental cancellations, $\det \mathbf{M} = \mathcal{O}(1)$, so that its inverse exists. From Eq. (4) $\mathbf{W} \leq \mathcal{O}(1/N_c)$, and together with $\mathbf{M}^{-1} \leq \mathcal{O}(1)$, the equation $\mathbf{V} = \mathbf{M}^{-1}\mathbf{W}$ implies that $\mathbf{V} \leq \mathcal{O}(1/N_c)$. Along the same lines as before this establishes the result in Eq. (6).

It is lastly outlined why $\hat{O}_\mathbf{p}$ does not depend on color. This is done by following the derivation of $\hat{O}_\mathbf{p}$ in the Appendix of Ref. [5], employing the notations of that reference. Color appears when the (anti)commutators are evaluated, e.g., as δ^{ab} in $\{\psi_\xi^a(\mathbf{x}, t), \bar{\psi}_\xi^b(\mathbf{y}, t)\} = -\delta^{ab} \vec{\gamma}_{\xi\xi} \cdot \vec{\partial}_\mathbf{x} \delta^3(\mathbf{x} - \mathbf{y})$. The color and Dirac indices in the commutators are then contracted with the remaining quark and gluon fields and subsumed in $f_\mu(x, y, z)$. The construction of $\hat{O}_\mathbf{p}$ only depends on the number of derivatives acting on $\delta^3(\mathbf{x} - \mathbf{y})$ and not on $f_\mu(x, y, z)$, making it independent of color, as promised. Let us give an example of how this observation is used above. Suppose $\hat{O}_\mathbf{p} = \partial/\partial \mathbf{p}$, then in Eqs. (B4) and (B6) it occurs as $\hat{O}_\mathbf{p} \int d^3x \exp(i\mathbf{p} \cdot \mathbf{x}) g(\mathbf{x}) = i \mathbf{x} \int d^3x \exp(i\mathbf{p} \cdot \mathbf{x}) g(\mathbf{x})$. It is evident that $\hat{O}_\mathbf{p}$ does not affect the large- N_c counting of the function g .

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