All-orders infrared freezing of observables in perturbative QCD

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(Received 17 October 2003; published 13 July 2004)

We consider a Borel sum definition of all-orders perturbation theory for Minkowskian QCD observables such as the $R_{e^+e^-}$ ratio, and show that both this perturbative component and the additional nonperturbative all-orders operator product expansion (OPE) component can remain *separately* well-defined for all values of energy $\sqrt{s}$, with the perturbative component dominating as $s\to\infty$, and with both components contributing as $s\to 0$. In the infrared $s\to 0$ limit the perturbative correction to the parton model result for $R_{e^+e^-}$ has an all-orders perturbation theory component which smoothly freezes to the value $\mathcal{R}(0)=2/b$, where $b=(33-2N_f)/6$ is the first QCD beta-function coefficient, with N_f flavors of massless quark. For freezing one requires $N_f<9$. The freezing behavior is manifested by the ''contour-improved'' or ''analytic perturbation theory'' (APT), in which an infinite subset of analytical continuation terms are resummed to all-orders. We show that for the Euclidean Adler-*D* function, $D(Q^2)$, the perturbative component remains defined into the infrared if *all* the renormalon singularities are taken into account, but no analogue of the APT reorganization of perturbation theory is possible. We perform phenomenological comparisons of suitably smeared low-energy data for the $R_{e^+e^-}$ ratio, with the perturbative freezing predictions, and find good agreement.

DOI: 10.1103/PhysRevD.70.014002 PACS number(s): 12.38.Bx, 12.38.Aw, 12.38.Qk, 13.60.Hb

I. INTRODUCTION

In this paper we wish to address the question of whether QCD perturbation theory can be used to make predictions in the low-energy infrared regime where one expects nonperturbative effects to dominate. Such an extension of the applicability of perturbation theory, beyond the ultraviolet regime of asymptotic freedom, would obviously enable one to test QCD in new ways. Reorganizations of fixed-order perturbation theory exhibiting a stable infrared freezing behavior have previously been formulated and studied; these include the so-called ''analytic perturbation theory'' (APT) approach initiated by Shirkov and Solovtsov in Refs. [1] (for a review see Ref. [2]), and the variational perturbation theory (VPT) approach [3]. Our discussion will address the more fundamental question of how all-orders QCD perturbation theory, and the nonperturbative operator product expansion (OPE) contribution, can remain defined in the low-energy regime. We will discover that this is possible for Minkowskian observables, and that the APT approach should be asymptotic to the all-orders perturbative result which also exhibits the same infrared freezing behavior found with APT. For Euclidean quantities, however, we will find that all-orders perturbation theory only exhibits stable infrared behavior if one has complete information on the perturbative corrections to all-orders, and that this behavior is not in general related to the infrared behavior found using APT.

We will focus our discussion on the $R_{e^+e^-}(s)$ ratio, at c.m. energy $\sqrt{s}$. This is a Minkowskian quantity derived by analytical continuation from the Euclidean QCD vacuum polarization function. The corrections to the parton model result for $R_{e^+e^-}(s)$ will consist of a perturbative part, which can be developed as a power series in the renormalized QCD coupling $a(s)\equiv\alpha_s(s)/\pi$, and a nonperturbative part which can be developed as an OPE in powers of Λ^2/s, the first term corresponding to the lowest dimension relevant operator, the gluon condensate, being proportional to $(\Lambda^2/s)^2$. The key point is that the combination of the all-orders perturbation series and OPE must be well-defined at all values of *s*, since $R_{e^+e^-}(s)$ is a physical quantity. Each part by itself, however, exhibits pathologies. Specifically, the perturbation series exhibits $n!$ growth in the perturbative coefficients, at large-orders *n*. Attempts to define the all-orders sum of the perturbation series using a Borel integral run into the difficulty that there are singularities on the integration contour termed infrared renormalons [4]. It turns out, however, that the resulting ambiguity in defining the Borel integral is of the same form as ambiguities in the coefficient functions involved in the OPE, and so choosing a particular regulation of the Borel integral (such as principal value) induces a corresponding definition of the coefficient functions, and the sum of the two components is well-defined [4,5]. There is a further crucial pathology of the Borel integral, which we shall refer to as the ''Landau divergence.'' This means that at a critical energy $s=s_L$, the Borel integral diverges. It should be stressed that the value of s_L should not be confused with the ''Landau pole'' or ''Landau ghost'' in the QCD coupling $a(s)$. The ''Landau ghost'' is completely unphysical and scheme-dependent, whereas the divergence of the Borel integral is completely scheme-independent [4]. For Minkowskian quantities such as $R_{e^+e^-}$ there is an oscillatory factor in the Borel transform in the integrand, arising from the analytical continuation from Euclidean to Minkowskian, which means that the Borel integral is finite at $s=s_L$, and diverges for $s<s_L$. To go to lower energies than s_L we shall show that one needs to modify the form of the Borel integral, the modified form now having singularities on the integration contour corresponding to *ultraviolet* renormalons, correspondingly to go below $s=s_L$ one needs to resum the OPE to all-orders and recast it as a modified expansion in powers of s/Λ^2. One

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then finds that the ambiguities in regulating this modified Borel integral, are of the same form as ones in the modified OPE, and for $s<s_L$ the sum of the two components is again well-defined. In the infrared $s\to 0$ limit the modified OPE resulting from resummation can contain a constant term independent of s even though such a term is not invisible in perturbation theory, so both the perturbative and nonperturbative components will contribute to the infrared freezing limit. The oscillatory factor in the Borel integral means that it freezes smoothly to $2/b$ in the infrared, where $b=(33-2N_f)/6$ is the first SU(3) QCD beta-function coefficient, with N_f quark flavors.

The arguments sketched above suggest that the all-orders perturbative and nonperturbative components for Minkowskian quantities such as $R_{e^+e^-}(s)$ can *separately* remain defined at all energies, with the perturbative part being dominant in the ultraviolet and both components contributing in the infrared limit. One can then compare all-orders perturbative predictions with data, having suitably smeared and averaged over resonances [6] to suppress the nonperturbative mass threshold effects. In practice, of course, we do not have exact all-orders perturbative information. We know exactly the perturbative coefficients of the corrections to the parton model result for $R_{e^+e^-}$ to next-next-leading order (NNLO), i.e. including terms of order α_s^3 [7]. Clearly, conventional fixed-order perturbation theory for $R_{e^+e^-}$ will not exhibit the freezing behavior in the infrared to be expected for the all-orders perturbation theory. What is required is a rearrangement of fixed-order perturbation theory which has freezing behavior in the infrared. As we have discussed in a recent paper [8] the resummation to all-orders of the convergent subset of analytical continuation terms (''large-π^2'' terms), arising when the perturbative corrections to the Euclidean Adler-$D(-s)$ function at a given order are continued to the Minkowskian $R_{e^+e^-}(s)$, recasts the perturbation series as an expansion in a set of functions $A_n(s)$ which are well-defined for all values of s, vanishing as $s\to\infty$ in accord with asymptotic freedom, and with all but $A_1(s)$ vanishing in the infrared limit, with $A_1(s)$ approaching $2/b$ to provide infrared freezing behavior to all-orders in perturbation theory. This ''contour-improved'' perturbation theory (CIPT) approach is equivalent to the analytic perturbation theory (APT) mentioned above [1] in the case of $R_{e^+e^-}(s)$. We gave explicit expressions for the functions $A_n(s)$. At the two-loop level these can be expressed in terms of the Lambert W-function [9,10]. To make contact with the all-orders perturbative result represented as a Borel integral, we note that the CIPT-APT reorganization of perturbation theory corresponds to leaving the oscillatory factor in the Borel transform intact whilst expanding the remaining factor as a power series. Integrating term-by-term then yields the functions $A_n(s)$. The presence of the oscillatory factor in these integrals guarantees that the $A_n(s)$ are well-defined at all energies. The CIPT-APT series should thus be asymptotic to the Borel integral at both ultraviolet and infrared energies. Whilst a reorganized fixed-order perturbation series exhibiting stable infrared freezing behavior is possible for Minkowskian quantities, we shall show that it is not possible for Euclidean observables such as the Adler D-function, $D(Q^2)$. For Euclidean observables the Borel integral does not contain the oscillatory factor and so is potentially divergent at $s=s_L$, although as we shall show in the so-called leading-b approximation [4,5], the divergence is cancelled if *all* the infrared and ultraviolet renormalon singularities are included, and once again perturbative and nonperturbative components which are separately well-defined at all energies can be obtained. This is only possible in the leading-b approximation, however.

The plan of the paper is as follows. In Sec. II we shall describe the CIPT-APT reorganisation of fixed-order perturbation theory for $R_{e^+e^-}$, reviewing the results of Ref. [8]. In Sec. III we consider how for Minkowskian observables one can define all-orders perturbation theory, and the all-orders nonperturbative OPE in such a way that each component remains well-defined at all energies. The link between the all-orders perturbative result and the reorganized CIPT-APT fixed-order perturbation theory is emphasized. We then briefly consider the corresponding problem for Euclidean observables. In Sec. IV we perform some phenomenological studies in which we compare low energy experimental data for $R_{e^+e^-}(s)$ with the CIPT-APT perturbative predictions. Section V contains a discussion and conclusions.

II. INFRARED FREEZING OF $R_{e^+e^-}$-CIPT-APT

We begin by defining the $R_{e^+e^-}$ ratio at c.m. energy $\sqrt{s}$,

$$R_{e^+e^-}(s)\equiv\frac{\sigma_{tot}(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}=3\sum_f Q_f^2(1+\mathcal{R}(s)). \tag{1}$$

Here the Q_f denote the electric charges of the different flavors of quarks, and $\mathcal{R}(s)$ denotes the perturbative corrections to the parton model result, and has a perturbation series of the form,

$$\mathcal{R}(s)=a+\sum_{n>0} r_n a^{n+1}. \tag{2}$$

Here $a\equiv\alpha_s(\mu^2)/\pi$ is the renormalized coupling, and the coefficients r_1 and r_2 have been computed in the $\overline{\text{MS}}$ scheme with renormalization scale $\mu^2=s$ [7]. We can consider the s-dependence of $\mathcal{R}(s)$ at NNLO,

$$s\frac{d\mathcal{R}(s)}{ds}=-\frac{b}{2}\rho(\mathcal{R})\equiv-\frac{b}{2}\mathcal{R}^2(1+c\mathcal{R}+\rho_2\mathcal{R}^2). \tag{3}$$

Here $c=(153-19N_f)/12b$ is the second universal QCD beta-function coefficient, and ρ_2 is the NNLO effective charge beta-function coefficient [11], an RS-invariant combination of r_1, r_2 and beta-function coefficients. The condition for $\mathcal{R}(s)$ to approach the infrared limit $\mathcal{R}^*$ as $s\to 0$ is for the effective charge beta-function to have a nontrivial zero, $\rho(\mathcal{R}^*)=0$. At NNLO the condition for such a zero is $\rho_2<0$. Putting $N_f=2$ active flavors we find for the NNLO RS invariant, $\rho_2=-9.72$, so that $\mathcal{R}(s)$ apparently freezes in the infrared to $\mathcal{R}^*=0.43$. The freezing behavior was first investigated in a pioneering paper by Mattingly and Stevenson

[12] in the context of the principle of minimal sensitivity (PMS) approach. However, it is not obvious that we should believe this apparent NNLO freezing result [13]. In fact ρ_2 is dominated by a large $b^2\pi^2$ term arising from analytical continuation (AC) of the Euclidean Adler $D(-s)$ function to the Minkowskian $R(s)$, with $\rho_2=9.40-\pi^2b^2/12$. Similarly the N^3LO invariant ρ_3 will contain the large AC term $-5c\pi^2b^2/12$. This suggests that in order to check freezing we need to resum the AC terms to *all-orders*.

$R_{e^+e^-}$ is directly related to the transverse part of the correlator of two vector currents in the Euclidean region,

$$(q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(s) = 4\pi^2 i\int d^4x e^{iq\cdot x}\langle 0|T[j_\mu(x)j_\nu(0)]|0\rangle, \quad (4)$$

where $s=-q^2>0$. To avoid an unspecified constant it is convenient to take a logarithmic derivative with respect to s and define the Adler D-function,

$$D(s)=-s\frac{d}{ds}\Pi(s). \quad (5)$$

This can be represented by Eq. (1) with the perturbative corrections $\mathcal{R}(s)$ replaced by

$$\mathcal{D}(s)=a+\sum_{n>0} d_n a^{n+1}. \quad (6)$$

The Minkowskian observable $\mathcal{R}(s)$ is related to $\mathcal{D}(-s)$ by analytical continuation from Euclidean to Minkowskian. One may write the dispersion relation,

$$\mathcal{R}(s)=\frac{1}{2\pi i}\int_{-s-i\epsilon}^{-s+i\epsilon} dt\frac{\mathcal{D}(t)}{t}. \quad (7)$$

Written in this form it is clear that the "Landau pole" in the coupling $a(s)$, which lies on the positive real s-axis, is not a problem, and $\mathcal{R}(s)$ will be defined for all s. The dispersion relation can be reformulated as an integration around a circular contour in the complex energy-squared s-plane [14,15],

$$\mathcal{R}(s)=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\mathcal{D}(se^{i\theta}). \quad (8)$$

One should note, however, that this is only equivalent to the dispersion relation of Eq. (7) for values of s above the "Landau pole." Expanding $\mathcal{D}(se^{i\theta})$ as a power series in $\overline{a}\equiv a(se^{i\theta})$, and performing the θ integration term-by-term, leads to a "contour-improved" perturbation series, in which at each order an infinite subset of analytical continuation terms present in the conventional perturbation series of Eq. (2) are resummed. It is this complete analytical continuation that builds the freezing of $\mathcal{R}(s)$. We shall begin by considering the "contour-improved" series for the simplified case of a one-loop coupling. The one-loop coupling will be given by

$$a(s)=\frac{2}{b\ln(s/\tilde{\Lambda}^2_{\overline{MS}})}. \quad (9)$$

As described above one can then obtain the "contour-improved" perturbation series for $\mathcal{R}(s)$,

$$\mathcal{R}(s)=A_1(s)+\sum_{n=1}^{\infty} d_n A_{n+1}(s), \quad (10)$$

where the functions $A_n(s)$ are defined by

$$A_n(s)\equiv\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\overline{a^n}=\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\frac{a^n(s)}{[1+ib\theta a(s)/2]^n}. \quad (11)$$

This is an elementary integral which can be evaluated in closed-form as [8]

$$A_1(s)=\frac{2}{\pi b}\arctan\left(\frac{\pi b a(s)}{2}\right)$$
$$A_n(s)=\frac{2a^{n-1}(s)}{b\pi(1-n)}\mathrm{Im}\left[\left(1+\frac{ib\pi a(s)}{2}\right)^{1-n}\right](n>1). \quad (12)$$

We then obtain the one-loop "contour-improved" series for $\mathcal{R}(s)$,

$$\mathcal{R}(s)=\frac{2}{\pi b}\arctan\left(\frac{\pi b a(s)}{2}\right)+d_1\left[\frac{a^2(s)}{(1+b^2\pi^2a^2(s)/4)}\right]+d_2\left[\frac{a^3(s)}{(1+b^2\pi^2a^2(s)/4)^2}\right]+\cdots. \quad (13)$$

The first arctan term is well-known, and corresponds to resumming the infinite subset of analytical continuation terms in the standard perturbation series of Eq. (2) which are independent of the d_n coefficients. Subsequent terms correspond to resumming to all-orders the infinite subset of terms in Eq. (2) proportional to $d_1, d_2, \ldots$, etc. In each case the resummation is *convergent*, provided that $|a(s)|<2/\pi b$. In the ultraviolet $s\to\infty$ limit the $A_n(s)$ vanish as required by asymptotic freedom. In the infrared $s\to 0$ limit, the one-loop coupling $a(s)$ has a "Landau" singularity at $s=\tilde{\Lambda}^2_{\overline{MS}}$. However, the functions $A_n(s)$ resulting from resummation, if analytically continued, are well-defined for all real values of s. $A_1(s)$ smoothly approaches from below the asymptotic infrared value $2/b$, whilst for $n>1$ the $A_n(s)$ vanish. Thus, as claimed, $\mathcal{R}(s)$ is asymptotic to $2/b$ to all-orders in perturbation theory. We postpone the crucial question of how to define all-orders perturbation theory in the infrared region until the next section. We should also note that the functions $A_n(s)$ in Eq. (12) can also be obtained by simple manipulation of the dispersion relation in Eq. (7), which is defined for all real s. This avoids the possible objection that the contour integral in Eq. (8) is only defined for s above the "Landau pole."

Beyond the simple one-loop approximation the freezing is most easily analyzed by choosing a renormalization scheme in which the beta-function equation has its two-loop form,

$$\frac{\partial a(\mu^2)}{\partial \ln \mu^2} = -\frac{b}{2}a^2(\mu^2)(1+ca(\mu^2)). \tag{14}$$

This corresponds to a so-called 't Hooft scheme [16] in which the nonuniversal beta-function coefficients are all zero. Here $c=(153-19N_f)/12b$ is the second universal beta-function coefficient. For our purposes the key feature of these schemes is that the coupling can be expressed analytically in closed-form in terms of the Lambert W function, defined implicitly by $W(z)\exp(W(z))=z$ [17,18]. One has

$$a(\mu^2) = -\frac{1}{c[1+W_{-1}(z(\mu))]}$$

$$z(\mu) \equiv -\frac{1}{e}\left(\frac{\mu}{\tilde{\Lambda}_{\overline{MS}}}\right)^{-b/c}, \tag{15}$$

where $\tilde{\Lambda}_{\overline{MS}}$ is defined according to the convention of [19], and is related to the standard definition [20] by $\tilde{\Lambda}_{\overline{MS}} = (2c/b)^{-c/b}\Lambda_{\overline{MS}}$. The "$-1$" subscript on W denotes the branch of the Lambert W function required for asymptotic freedom, the nomenclature being that of Ref. [18]. Assuming a choice of renormalization scale $\mu^2=xs$, where x is a dimensionless constant, for the perturbation series of $\mathcal{D}(s)$ in Eq. (5), one can then expand the integrand in Eq. (6) for $\mathcal{R}(s)$ in powers of $\overline{a}\equiv a(xse^{i\theta})$, which can be expressed in terms of the Lambert W function using Eq. (15),

$$\overline{a} = \frac{-1}{c[1+W(A(s)e^{iK\theta})]} \tag{16}$$

where

$$A(s) = \frac{-1}{e}\left(\frac{\sqrt{xs}}{\tilde{\Lambda}_{\overline{MS}}}\right)^{-b/c}, \quad K=\frac{-b}{2c}. \tag{17}$$

The functions $A_n(s)$ in the "contour-improved" series are then given, using Eqs. (15), (16), by

$$A_n(s) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \overline{a^n} = \frac{1}{2\pi}\int_{-\pi}^{0} d\theta \frac{(-1)^n}{c^n} \times [1+W_1(A(s)e^{iK\theta})]^{-n} + \frac{1}{2\pi}\int_{0}^{\pi} d\theta \frac{(-1)^n}{c^n} \times [1+W_{-1}(A(s)e^{iK\theta})]^{-n}. \tag{18}$$

Here the appropriate branches of the W function are used in the two regions of integration. As discussed in Refs. [9,10], by making the change of variable $w=W(A(s)e^{iK\theta})$ we can then obtain

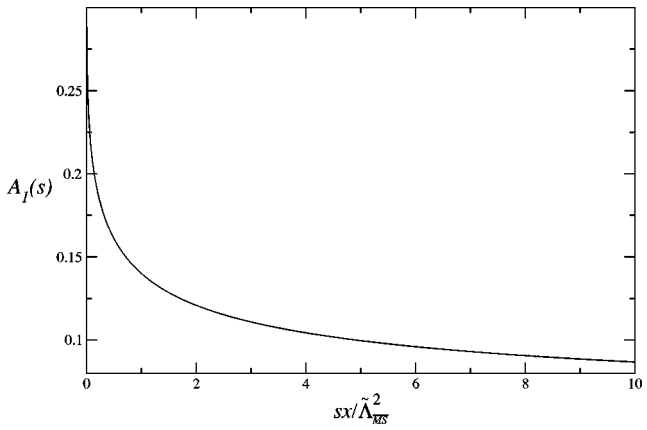

FIG. 1. The function $A_1(s)$ of Eq. (20) versus $sx/\tilde{\Lambda}^2_{\overline{MS}}$. We assume $N_f=2$ flavors of quark.

$$A_n(s) = \frac{(-1)^n}{2iKc^n\pi}\int_{W_1(A(s)e^{-iK\pi})}^{W_{-1}(A(s)e^{iK\pi})} \frac{dw}{w(1+w)^{n-1}}. \tag{19}$$

Noting that $W_1(A(s)e^{-iK\pi})=[W_{-1}(A(s)e^{iK\pi})]^*$, we can evaluate the elementary integral to obtain for $n=1$,

$$A_1(s) = \frac{2}{b} - \frac{1}{\pi Kc}\,\mathrm{Im}[\ln(W_{-1}(A(s)e^{iK\pi}))], \tag{20}$$

where the $2/b$ term is the residue of the pole at $w=0$. For $n>1$ we obtain

$$A_n(s) = \frac{(-1)^n}{c^nK\pi}\,\mathrm{Im}\left[\ln\left(\frac{W_{-1}(A(s)e^{iK\pi})}{1+W_{-1}(A(s)e^{iK\pi})}\right) + \sum_{k=1}^{n-2}\frac{1}{k(1+W_{-1}(A(s)e^{iK\pi}))^k}\right]. \tag{21}$$

Crucially the contribution from the poles at $w=0$ and $w=-1$ cancel exactly. Equivalent expressions have been obtained in the APT approach [10]. Provided that $b/c>0$, which will be true for $N_f<9$, we find the same behavior as in the one-loop case with the $A_n(s)$ vanishing in the ultraviolet limit consistent with asymptotic freedom, and with $A_n(s)$ vanishing in the infrared limit for $n>1$, and $A_1(s)$ freezing to $2/b$. To the extent that the freezing holds to all-orders in perturbation theory it should hold irrespective of the choice of renormalization scheme (RS). The use of the 't Hooft scheme simply serves to make the freezing manifest. In Figs. 1–3 we plot the functions $A_1(s)$,$A_2(s)$ and $A_3(s)$, respectively, as functions of $(sx/\tilde{\Lambda}^2_{\overline{MS}})$.

Having shown how fixed-order perturbation theory can be reorganized so that it exhibits well-behaved freezing behavior in the infrared, we turn in the next section to a discussion of how all-orders perturbation theory and the all-orders nonperturbative OPE, can be defined in such a way that they remain well-defined at all energies.

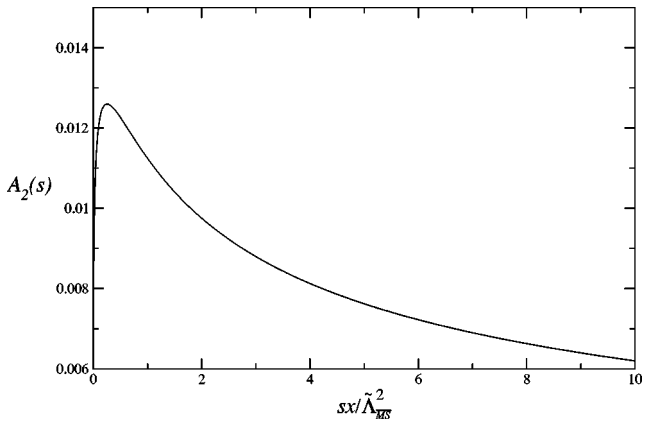

FIG. 2. As Fig. 1 but for $A_2(s)$ of Eq. (21).

III. ALL-ORDERS PERTURBATION THEORY AND OPE

The corrections to the Adler D function, $\mathcal{D}(Q^2)$, can be split into a perturbative part, $\mathcal{D}_{PT}(Q^2)$, and a nonperturbative operator product expansion (OPE) part, $\mathcal{D}_{NP}(Q^2)$,

$$\mathcal{D}(Q^2)=\mathcal{D}_{PT}(Q^2)+\mathcal{D}_{NP}(Q^2). \tag{22}$$

The PT component is formally just the resummed perturbation series of Eq. (6),

$$\mathcal{D}_{PT}(Q^2)=a(Q^2)+\sum_{n>0} d_n a^{n+1}(Q^2). \tag{23}$$

In addition one has the nonperturbative OPE contribution,

$$\mathcal{D}_{NP}(Q^2)=\sum_n \frac{C_n(Q^2,\mu^2)\langle\mathcal{O}_n(\mu^2)\rangle}{Q^{2n}}, \tag{24}$$

where the sum is over the relevant operators $\mathcal{O}_n$ of dimension $2n$. μ denotes the factorization scale, and C_n is the coefficient function. For the Adler D function the lowest dimension relevant operator is the dimension four gluon condensate,

$$\langle 0|G^a_{\mu\nu}G^{\mu\nu}_a|0\rangle. \tag{25}$$

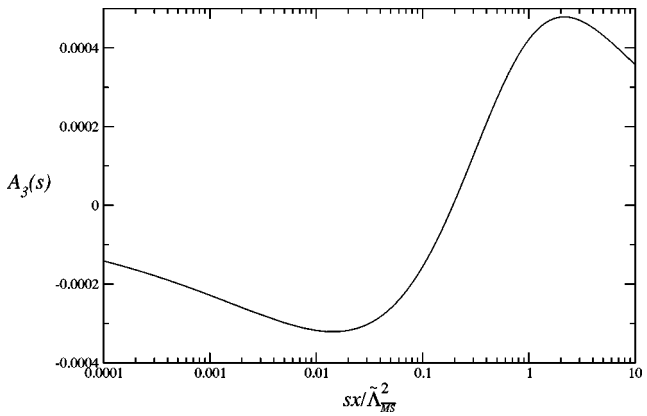

FIG. 3. As Fig. 1 but for $A_3(s)$ of Eq. (21).

It will be convenient to scale out the dimensionful factor $\tilde{\Lambda}^{2n}$ from the operator expectation value, and combine it with the coefficient function to obtain the overall coefficient $\mathcal{C}_n(Q^2,\mu^2)$. We can then write the $\mathcal{D}_{NP}(Q^2)$ component in the form

$$\mathcal{D}_{NP}(Q^2)=\sum_n \mathcal{C}_n\left(\frac{\tilde{\Lambda}^2}{Q^2}\right)^n. \tag{26}$$

We have suppressed the μ^2 and Q^2 dependence of the coefficient $\mathcal{C}_n$. The coefficients are themselves series expansions in a:

$$\mathcal{C}_n=Ka^{\delta_n}(\mu^2)[1+O(a)]. \tag{27}$$

Here K is an undetermined nonperturbative normalization, and δ_n is related to the anomalous dimension of the operator concerned.

The definition of the all-orders perturbative component in Eq. (23) needs care. The series has zero radius of convergence in the coupling a. A direct way of seeing this is to consider the large-N_f expansion of the perturbative coefficient d_n,

$$d_n=d_n^{[n]}N_f^n+d_n^{[n-1]}N_f^{n-1}+\cdots+d_n^{[0]}. \tag{28}$$

The leading large-N_f coefficient, $d_n^{[n]}$, can be computed exactly to all-orders since it derives from a restricted set of diagrams in which a chain of n fermion bubbles (renormalon chain) is inserted in the initiating quark loop. Working in the so-called V-scheme, which corresponds to $\overline{\text{MS}}$ subtraction with scale $\mu^2=e^{-5/3}Q^2$, one finds the exact large-N_f result [21–23],

$$d_n^{[n]}(V)=\frac{-2}{3}(n+1)\left(\frac{-1}{6}\right)^n\Bigg[-2n-\frac{n+6}{2^{n+2}}+\frac{16}{n+1}\sum_{n/2+1>m>0} m(1-2^{-2m})\times(1-2^{2m-n-2})\zeta_{2m+1}\Bigg]n!. \tag{29}$$

The $n!$ growth of coefficients means that the perturbation series is at best an asymptotic one. To arrive at a function to which it is asymptotic one can use a Borel integral representation, writing

$$\mathcal{D}_{PT}(Q^2)=\int_0^\infty dz e^{-z/a(Q^2)}B[\mathcal{D}](z). \tag{30}$$

Here $B[\mathcal{D}](z)$ is the Borel transform, defined by

$$B[\mathcal{D}(z)]=\sum_{n=0}^{\infty}\frac{z^n d_n}{n!}. \tag{31}$$

On performing the Borel integral term-by-term one reconstructs the divergent formal perturbation series for $\mathcal{D}_{PT}$. If the series for the Borel transform has finite radius of conver-

gence, by analytical continuation to the whole region of integration one can then define the Borel sum, provided that the Borel integral exists. On general grounds [24,25] one expects that in renormalizable field theories the Borel transform will contain branch point singularities on the real axis in the complex z plane, at positions $z=z_k\equiv 2k/b$ corresponding to infrared renormalons, IR_k, $k=1,2,3,\ldots$, and at $z=-z_k$ corresponding to so-called ultraviolet renormalons, UV_k. Here b is the first beta-function coefficient, so that for QED with N_f fermion species $b=-\frac{2}{3}N_f$, whilst for SU(3) QCD with N_f active quark flavors, $b=(33-2N_f)/6$. Thus in QED there are ultraviolet renormalon singularities on the positive real axis, and hence the Borel integral will be ambiguous. In QCD with $N_f<33/2$ flavors, so that the theory is asymptotically free, and $b>0$, there are infrared renormalon singularities on the positive real axis making the Borel integral again ambiguous. For both field theories all-orders perturbation theory by itself is not sufficient. The presence of singularities on the integration contour means that there is an ambiguity depending on whether the contour is taken above or below each singularity. It is easy to check that, taking $\mathcal{D}$ in the Borel integral of Eq. (30) to be a generic QED or QCD observable with branch point singularities $(1-z/|z_k|)^{-\gamma_k}$ in the Borel plane, the resulting ambiguity for the singularity at $z=|z_k|$ is of the form

$$\Delta\mathcal{D}_{PT}\sim Ke^{-|z_k|/a(Q^2)}a^{1-\gamma_k}, \tag{32}$$

where K is complex. Using the one-loop form for the coupling, $a(Q^2)=2/b\,\ln(Q^2/\tilde{\Lambda}^2)$, one finds that in the QCD case,

$$\Delta\mathcal{D}_{PT}\approx Ka^{1-\gamma_k}\left(\frac{\tilde{\Lambda}^2}{Q^2}\right)^k. \tag{33}$$

This has exactly the same structure as a term in the OPE expansion, Eq. (26), and one sees that the branch point exponent γ of the IR renormalon is related to the anomalous dimension of the operator, with $\delta_k=1-\gamma_k$. The idea is that the coefficient, $\mathcal{C}_k$, in particular the constant K, is ambiguous in the OPE because of nonlogarithmic UV divergences [26,27]. This ambiguity can be compensated by the IR renormalon ambiguity in the PT Borel integral, and so regulating the Borel integral, using for instance a principal value (PV) prescription, induces a particular definition of the coefficient functions in the OPE, and the PT and OPE components are then separately well-defined. That this scenario works in detail can be confirmed in toy models such as the nonlinear $O(N)$ σ model [26,28]. For the QED case the ambiguity corresponds to a $Q^2/\tilde{\Lambda}^2$ effect. So that all-orders QED perturbation theory is only defined if there are in addition power corrections in Q^2. Such effects are only important if $Q^2\sim\tilde{\Lambda}^2$, here $\tilde{\Lambda}$ corresponds to the Landau ghost in QED, $\tilde{\Lambda}^2\sim 10^{560}m^2$, with m the fermion mass. Thus in QED such power corrections can have no phenomenological consequences and can be completely ignored.

Our exact information about the Borel transform, $B[\mathcal{D}](z)$, for the QCD Adler D function is restricted to the large-N_f result of Eq. (29). In QCD we expect large-order behavior in perturbation theory of the form $d_n\approx Kn^\gamma(b/2)^n n!$, involving the QCD beta-function coefficient $b=(33-2N_f)/6$. Motivated by the structure of renormalon singularities in QCD one can then convert the N_f expansion into the so-called b-expansion [29–32], by substituting $N_f=(33/2-3b)$ to obtain

$$d_n=d_n^{(n)}b^n+d_n^{(n-1)}b^{n-1}+\cdots+d_n^{(0)}. \tag{34}$$

The leading-b term $d_n^{(L)}\equiv d_n^{(n)}b^n$ is then used to approximate d_n. Since $d_n^{(L)}=(-3)^n d_n^{[n]}b^n$, it is known to all-orders from the large-N_f result. This approach is sometimes also referred to as ''naive non-Abelianization'' [29]. It can be motivated by considering a QCD skeleton expansion [33], and corresponds to simply taking the first ''one-chain'' term in the expansion. It does not include the multiple exchanges of renormalon chains needed to build the full asymptotic behavior of the perturbative coefficients, and there are no firm guarantees as to its accuracy. The leading-b result for the Borel transform of the Adler-D function in the V-scheme can then be obtained from Eq. (29):

$$B[\mathcal{D}^{(L)}](z)=\sum_{j=1}^{\infty}\frac{A_0(j)+zA_1(j)}{\left(1+\frac{z}{z_j}\right)^2}+\frac{B_0(2)}{\left(1-\frac{z}{z_2}\right)}+\sum_{j=3}^{\infty}\frac{B_0(j)+zB_1(j)}{\left(1-\frac{z}{z_j}\right)^2}, \tag{35}$$

so that one sees in the leading-b limit a set of single and double pole renormalon singularities at the expected positions. The residues of the UV_j poles, $A_0(j)$ and $A_1(j)$, are given by [30]

$$A_0(j)=\frac{8}{3}\frac{(-1)^{j+1}(3j^2+6j+2)}{j^2(j+1)^2(j+2)^2},$$

$$A_1(j)=\frac{4}{3}\frac{b(-1)^{j+1}(2j+3)}{j^2(j+1)^2(j+2)^2}. \tag{36}$$

Because of the conformal symmetry of the vector correlator [34] the IR_j residues, $B_0(j)$ and $B_1(j)$, are directly related to the UV_j ones, with $B_0(j)=-A_0(-j)$ and $B_1(j)=-A_1(-j)$ for $j>2$. $B_0(1)=B_1(1)=B_1(2)=0$, and $B_0(2)=1$. Notice the absence of an IR_1 renormalon singularity. This is consistent with the correspondence between OPE terms and IR renormalon ambiguities noted above, since there is no relevant operator of dimension 2 in the OPE. The singularity nearest the origin is then the UV_1 singularity at $z=-2/b$, which generates the leading asymptotic behavior,

$$d_n^{(L)}\approx\frac{(12n+22)}{27}n!\left(\frac{-b}{2}\right)^n. \tag{37}$$

We shall now consider the correction, $\mathcal{R}(s)$, to the parton model result for $R_{e^+e^-}$. This may be split into a perturbative

component $\mathcal{R}_{PT}(s)$, and an OPE component $\mathcal{R}_{NP}(s)$, analogous to Eqs. (23), (24). Inserting the Borel representation for $\mathcal{D}_{PT}$ of Eq. (30) into the dispersion relation of Eq. (7) one finds the representation

$$\mathcal{R}_{PT}(s)=\frac{1}{2\pi i}\int_{-s-i\epsilon}^{-s+i\epsilon}\frac{dt}{t}\int_0^\infty dz e^{-z/a(t)}B[\mathcal{D}](z). \quad (38)$$

It will be convenient to consider the all-orders perturbative result in leading-b approximation to start with, in which case the coupling $a(t)$ will have its one-loop form, $a(t)=2/(b\ln(t/\tilde{\Lambda}_V^2))$, where we assume the V-scheme. In this case the t integration is trivial and one finds

$$\mathcal{R}^{(L)}_{PT}(s)=\int_0^\infty dz e^{-z/a(s)}\frac{\sin(\pi bz/2)}{\pi bz/2}B[\mathcal{D}^{(L)}](z), \quad (39)$$

where $B[\mathcal{D}^{(L)}](z)$ (in the V-scheme) is given by Eq. (35). It is now possible to explicitly evaluate $\mathcal{R}^{(L)}_{PT}(s)$ in terms of generalized exponential integral functions $\mathrm{Ei}(n,w)$, defined for $\mathrm{Re}\,w>0$ by

$$\mathrm{Ei}(n,w)=\int_1^\infty dt\frac{e^{-wt}}{t^n}. \quad (40)$$

One also needs the integral

$$\int_0^\infty dz e^{-z/a}\frac{\sin(\pi bz/2)}{z}=\arctan\left(\frac{\pi ba}{2}\right). \quad (41)$$

Writing the ''sin'' as a sum of complex exponentials and using partial fractions one can then evaluate the contribution to $\mathcal{R}^{(L)}_{PT}(s)$ coming from the UV renormalon singularities, i.e. from the terms involving $A_0(j)$ and $A_1(j)$ in Eq. (35) [30]

$$\mathcal{R}^{(L)}_{PT}(s)|_{UV}=\frac{2}{\pi b}\left(\frac{8\zeta_2}{3}-\frac{11}{3}\right)\arctan\left(\frac{\pi ba(s)}{2}\right)+\frac{2}{\pi b}\sum_{j=1}^{j=\infty}(A_0(j)\phi_+(1,j)+(A_0(j)-A_1(j)z_j)\phi_+(2,j)), \quad (42)$$

where $\zeta_2=\pi^2/6$ is the Riemann zeta-function, and we have defined

$$\phi_+(p,q)\equiv e^{z_q/a(s)}(-1)^q\mathrm{Im}[\mathrm{Ei}(p,(1/a(s))+i\pi bz_q/2)]. \quad (43)$$

To evaluate the remaining contribution involving the IR renormalon singularities we need to regulate the integral to deal with the singularities on the integration contour. For simplicity we could choose to take a principal value prescription. We need to continue the $\mathrm{Ei}(n,w)$ defined for $\mathrm{Re}\,w>0$ by Eq. (40), to $\mathrm{Re}\,w<0$. With the standard continuation one arrives at a function analytic everywhere in the cut complex w-plane, except at $w=0$; with a branch cut running along the negative real axis. Explicitly [35]

$$\mathrm{Ei}(n,w)=\frac{(-w)^{n-1}}{(n-1)!}\left[-\ln w-\gamma_E+\sum_{m=1}^{n-1}\frac{1}{m}\right]-\sum_{\substack{m=0\\ m\neq n-1}}\frac{(-w)^m}{(m-n+1)m!}, \quad (44)$$

with $\gamma_E=0.5722\ldots$ Euler's constant. The $\ln w$ contributes the branch cut along the negative real w-axis. To obtain the principal value of the Borel integral one needs to compensate for the discontinuity across the branch cut, and make the replacement $\mathrm{Ei}(n,w)\to\mathrm{Ei}(n,w)+i\pi\,\mathrm{sgn}(\mathrm{Im}\,w)$. This leads one to introduce, analogous to Eq. (43),

$$\phi_-(p,q)\equiv e^{-z_q/a(s)}(-1)^q\mathrm{Im}[\mathrm{Ei}(p,(-1/a(s))-i\pi bz_q/2)]-\frac{e^{-z_q/a(s)}(-1)^q z_q^{p-1}}{(p-1)!}\pi\,\mathrm{Re}[((1/a(s))+i\pi b/2)^{p-1}]. \quad (45)$$

The principal value of the IR renormalon contribution is then given by [30]

$$\mathcal{R}^{(L)}_{PT}(s)|_{IR}=\frac{2}{\pi b}\left(\frac{14}{3}-\frac{8\zeta_2}{3}\right)\arctan\left(\frac{\pi ba(s)}{2}\right)+\frac{2B_0(2)}{\pi b}\phi_-(1,2)+\frac{2}{\pi b}\sum_{j=3}^{\infty}(B_0(j)\phi_-(1,j)+(B_0(j)+B_1(j)z_j)\phi_-(2,j)). \quad (46)$$

The perturbative component is then the sum of the UV and (regulated) IR contributions,

$$\mathcal{R}^{(L)}_{PT}(s)=\mathcal{R}^{(L)}_{PT}(s)|_{UV}+\mathcal{R}^{(L)}_{PT}(s)|_{IR}=\frac{2}{\pi b}\arctan\left(\frac{\pi ba(s)}{2}\right)+\frac{2}{\pi b}\sum_{j=1}^{\infty}(A_0(j)\phi_+(1,j)+(A_0(j)-A_1(j)z_j)\phi_+(2,j))+\frac{2B_0(2)}{\pi b}\phi_-(1,2)+\frac{2}{\pi b}\sum_{j=3}^{\infty}(B_0(j)\phi_-(1,j)+(B_0(j)+B_1(j)z_j)\phi_-(2,j)). \quad (47)$$

Note that the ζ_2 contributions cancel, and one obtains the arctan term, which is the leading contribution, $A_1(s)$, in the CIPT-APT reformulation of fixed-order perturbation theory. The connection between the Borel representation and the $A_n(s)$ will be further clarified later.

We now turn to the infrared behavior of the regulated Borel integral. In the one-loop (leading-b) case the V-scheme coupling $a(s)$ becomes infinite at $s=s_L\equiv\tilde{\Lambda}_V^2$. The $e^{-z/a(s)}$ term in the Borel integrand approaches unity at $s=s_L$, but the trigonometric factor $\sin(\pi bz/2)/(\pi bz/2)$ ensures that the integral is defined at $s=s_L$. For $s<s_L$, how-

ever, $a(s)$ becomes negative, and the $e^{-z/a(s)}$ factor diverges at $z=\infty$, the Borel transform in the V-scheme does not contain any exponential z-dependence to compensate, so the Borel integral is not defined. We shall refer to this pathology of the Borel integral at $s=s_L$ as the "Landau divergence." It is important to stress that the Landau divergence is to be carefully distinguished from the Landau pole in the coupling. The Landau pole in the coupling depends on the chosen renormalization scale. At one-loop choosing an $\overline{\rm MS}$ scale $\mu^2=xs$, the coupling $a(xs)$ has a Landau pole at $s=\tilde{\Lambda}^2_{\overline{MS}}/x$, the Borel integral of Eq. (39) can then be written in terms of this coupling as

$$\mathcal{R}^{(L)}_{PT}(s)=\int_0^\infty dz e^{-z/a(xs)}\frac{\sin(\pi bz/2)}{\pi bz/2}[xe^{5/3}]^{bz/2}B[\mathcal{D}^{(L)}](z). \tag{48}$$

In a general scheme the Borel transform picks up the extra factor $[xe^{5/3}]^{bz/2}$ multiplying the V-scheme result. The Borel integrand is scheme (x) invariant. The extra factor has to be taken into account when identifying where the integral breaks down, and one of course finds the Landau divergence to be at the same x-independent energy, $s=s_L=e^{5/3}\tilde{\Lambda}^2_{\overline{MS}}=\tilde{\Lambda}^2_V$. Thus the Borel representation of Eq. (38) for $\mathcal{R}^{(L)}_{PT}(s)$ only applies for $s\geqslant s_L$. For $s<s_L$ the one-loop (V-scheme) coupling $a(s)$ becomes negative. We can rewrite the perturbative expansion of $\mathcal{R}_{PT}(s)$ as an expansion in $(-a(s))$,

$$\begin{aligned}\mathcal{R}_{PT}(s)&=a(s)+r_1a^2(s)+r_2a^3(s)+\cdots+r_na^{n+1}(s)+\cdots\\&=-[(-a(s))-r_1(-a(s))^2+r_2(-a(s))^3+\cdots\\&\quad+(-1)^nr_n(-a(s))^{n+1}+\cdots].\end{aligned} \tag{49}$$

The expansion in $(-a(s))$ follows from the modified Borel representation

$$\begin{aligned}\mathcal{R}_{PT}(s)&=-\int_0^\infty dz e^{-z/(-a(s))}B[\mathcal{R}](-z)\\&=\int_0^{-\infty} dz e^{z/(-a(s))}B[\mathcal{R}](z).\end{aligned} \tag{50}$$

This modified form of Borel representation will be valid when $\mathrm{Re}(a(s))<0$, and involves an integration contour along the negative real axis. Thus, it is now the *ultraviolet* renormalons UV_k which render the Borel integral ambiguous. The ambiguity in taking the contour around these singularities [analogous to Eq. (33)] now involves $(s/\tilde{\Lambda}^2)^k$. Of course, it is now unclear how these ambiguities can cancel against the corresponding OPE ambiguities. The key point is that since only the sum of the PT and OPE components is well-defined, the Landau divergence of the Borel integral at $s=s_L$, must be accompanied by a corresponding breakdown in the validity of the OPE as an expansion in powers of $(\tilde{\Lambda}^2/s)$, at the same energy. The idea is illustrated by the following toy example, where the OPE is an alternating geometric progression,

$$\begin{aligned}\mathcal{R}_{NP}(s)&=\left(\frac{\tilde{\Lambda}^2}{s}\right)-\left(\frac{\tilde{\Lambda}^2}{s}\right)^2+\left(\frac{\tilde{\Lambda}^2}{s}\right)^3-\cdots\\&=\frac{\dfrac{\tilde{\Lambda}^2}{s}}{1+\dfrac{\tilde{\Lambda}^2}{s}}=\frac{1}{1+\dfrac{s}{\tilde{\Lambda}^2}}\\&=1-\left(\frac{s}{\tilde{\Lambda}^2}\right)+\left(\frac{s}{\tilde{\Lambda}^2}\right)^2-\left(\frac{s}{\tilde{\Lambda}^2}\right)^3-\cdots.\end{aligned} \tag{51}$$

At any value of s, $\mathcal{R}_{NP}(s)$ is given by the equivalent functions in the middle line. For $s>\tilde{\Lambda}^2$ these have a valid expansion in powers of $\tilde{\Lambda}^2/s$, the standard OPE, given in the top line. For $s<\tilde{\Lambda}^2$ the standard OPE breaks down, but there is a valid expansion in powers of $s/\tilde{\Lambda}^2$ given in the bottom line. Thus for $s<s_L$ the OPE should be resummed and recast in the form

$$\mathcal{R}_{NP}(s)=\sum_n \tilde{\mathcal{C}}_n\left(\frac{s}{\tilde{\Lambda}^2}\right)^n. \tag{52}$$

It is crucial to note that this reorganised OPE can contain a $\tilde{\mathcal{C}}_0$ term which is independent of s, as indeed is the case in the toy example of Eq. (51). Of course, an analogous $\mathcal{C}_0$ term in the standard OPE in Eq. (26) is clearly excluded since it would violate asymptotic freedom, and all the terms in the regular OPE are perturbatively invisible. As a result $\mathcal{R}_{NP}(s)$ can have a nonvanishing infrared limit, and both components can contribute to the infrared freezing behavior. It should be no surprise that perturbation theory by itself cannot determine the infrared behavior of observables, but the existence of a well-defined perturbative component which, as we shall claim, can be computed at all values of the energy using a reorganized APT version of fixed-order perturbation theory, is a noteworthy feature. The remaining terms present in this modified OPE should then be in one-to-one correspondence with the UV_n renormalon singularities in the Borel transform of the PT component, and the PT renormalon ambiguities can cancel against corresponding OPE ones, and again each component separately be well-defined. The modified coefficients $\tilde{\mathcal{C}}_n$ will have a form analogous to Eq. (27),

$$\tilde{\mathcal{C}}_n=Ka^{\tilde{\delta}_n}(\mu^2)[1+O(a)]. \tag{53}$$

The anomalous dimension is that of an operator which can be identified using the technique of Parisi [24]. The anomalous dimension corresponding to $\tilde{\mathcal{C}}_1$ for the Adler D function has been computed [36]. The ambiguity for the modified Borel representation of Eq. (50), taking UV_k to be a branch point singularity $(1-z/z_k)^{\tilde{\gamma}_k}$, is

$$\Delta R_{PT}\approx Ka^{1-\tilde{\gamma}_k}\left(\frac{s}{\tilde{\Lambda}^2}\right)^k. \tag{54}$$

Comparing with Eq. (53) one finds $\tilde{\delta}_k = 1 - \tilde{\gamma}_k$. The modified Borel representation for $\mathcal{R}^{(L)}_{PT}$ valid for $s < s_L$ will be

$$\mathcal{R}^{(L)}_{PT}(s) = -\int_0^\infty dz e^{-z/(-a(s))} B[\mathcal{R}^{(L)}](-z). \qquad (55)$$

This may again be written explicitly in terms of Ei(n,w) functions. One simply needs to change $a(s) \to -a(s)$, $z_j \to -z_j$, and $A_1(j) \to -A_1(j)$, $B_1(j) \to -B_1(j)$ in Eq. (47). One finds that the result of Eq. (47) is invariant under these changes, apart from the additional terms which we added to the Ei(n,w) in continuing from Re $w > 0$ to Re $w < 0$, in order to obtain the principal value. In fact the PV Borel integral is not continuous at $s = s_L$. Continuity is obtained if rather than the principal value we use the standard continuation of the Ei(n,w) defined by Eq. (44). That is we redefine

$$\phi_-(p,q) \equiv e^{-z_q/a(s)}(-1)^q \mathrm{Im}[\mathrm{Ei}(p,(-1/a(s)) - i\pi b z_q/2)]. \qquad (56)$$

This simply corresponds to a different regulation of singularities. We then see that Eq. (47) for $\mathcal{R}^{(L)}_{PT}(s)$ is a function of $a(s)$ which is well-defined at all energies, and freezes to $2/b$ in the infrared. We note that the branch of the arctan changes at $s = s_L$, so that its value smoothly changes from zero at $s = \infty$ to π at $s = 0$. The reformulated OPE of Eq. (52) together with the perturbative component determines the infrared freezing behavior, and in the ultraviolet the perturbative component dominates. The key point is that both components can be described by functions of s which are well-defined at all energies. The apparent Landau divergence simply reflects the fact that the Borel integral and OPE series, which are used to describe the PT and NP components, each have a limited range of validity in s. The connection with the CIPT-APT rearrangement of fixed-order perturbation theory is now clear. It is obtained by keeping the $\sin(\pi bz/2)/(\pi bz/2)$ term in the Borel transform intact, and expanding the remainder in powers of z. Ordinary fixed-order perturbation theory, of course, corresponds to expanding the whole Borel transform in powers of z. The retention of the oscillatory sin factor in the Borel transform ensures that the reformulated perturbation theory remains defined at all energies. One then finds that for $s \geqslant s_L$,

$$A_n(s) = \int_0^\infty dz e^{-z/a(s)} \frac{\sin(\pi bz/2)}{\pi bz/2} \frac{z^{n-1}}{(n-1)!}, \qquad (57)$$

where the one-loop $A_n(s)$ are given by Eqs. (12). Similarly for $s \leqslant s_L$ one finds

$$A_n(s) = \int_0^{-\infty} dz e^{z/(-a(s))} \frac{\sin(\pi bz/2)}{\pi bz/2} \frac{z^{n-1}}{(n-1)!}. \qquad (58)$$

Thus the CIPT-APT fixed-order result should be an asymptotic approximation to the Borel integral at both large and small values of s. In Fig. 4 we compare the all-orders leading-b result for $\mathcal{R}^{(L)}_{PT}(s)$ given by Eq. (47), with the NNLO CIPT-APT prediction,

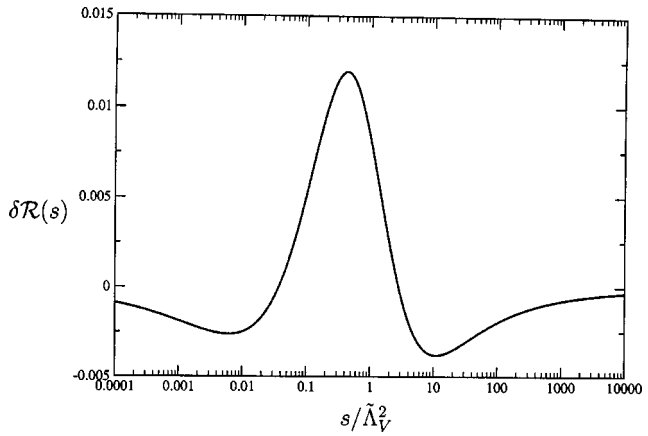

FIG. 4. $\delta\mathcal{R}(s) = \mathcal{R}^{(L)}_{PT}(s) - \mathcal{R}^{(L)}_{APT}(s)$, at the one loop level for 2 flavors of quark.

$$\mathcal{R}^{(L)}_{APT}(s) = A_1(s) + d_1^{(L)} A_2(s) + d_2^{(L)} A_3(s). \qquad (59)$$

The one-loop $A_n(s)$ are given by Eqs. (12) and as in Eq. (47) the V-scheme is assumed. We assume $N_f = 2$ quark flavors. One sees that there is good agreement at all values of $s/\tilde{\Lambda}_V^2$.

We now turn to the full QCD result beyond the one-loop approximation, and as in Sec. II it will be sufficient to consider the two-loop result since one can always use an 't Hooft scheme. Consider the Borel representation for $\mathcal{R}_{PT}(s)$ of Eq. (38). We shall assume that, as in the leading-b approximation, the Borel transform $B[\mathcal{D}](z)$ in the V-scheme does not contain any exponential dependence on z, but is simply a combination of branch point singularities. It is then clear that the Landau divergence occurs when the factor $e^{-z/a(-s)}$ becomes a diverging exponential, that is when $\mathrm{Re}(1/a(-s)) < 0$. Thus the critical energy s_L is given by the condition $\mathrm{Re}(1/a(-s)) = 0$. At one-loop level one has

$$\frac{1}{a(-s)} = \frac{b}{2} \ln\left(\frac{s}{\tilde{\Lambda}_V^2}\right) + \frac{i\pi b}{2}, \qquad (60)$$

and so the condition yields $s = s_L = \tilde{\Lambda}_V^2$, as we found before. At the two-loop level the situation is slightly different. Integrating the two-loop beta-function in Eq. (14) now gives

$$\frac{1}{a(-s)} + c \ln\left[\frac{ca(-s)}{1+ca(-s)}\right] = \frac{b}{2} \ln\left(\frac{s}{\tilde{\Lambda}_V^2}\right) + \frac{i\pi b}{2}. \qquad (61)$$

The vanishing of $\mathrm{Re}(1/a(-s))$ then corresponds to the solution of the transcendental equation

$$\mathrm{Re}\left\{c \ln\left[\frac{ca(-s)}{1+ca(-s)}\right]\right\} = \frac{b}{2} \ln\left(\frac{s}{\tilde{\Lambda}_V^2}\right). \qquad (62)$$

Assuming $N_f = 2$ flavors one finds $s = s_L = 0.4574\tilde{\Lambda}_V^2$. Since the Borel integral is scheme-invariant so must the value of s_L be, in particular the breakdown of the Borel representation would occur in any scheme, not just a 't Hooft one. We can perform the t-integration in Eq. (38) in closed form, and arrive at the two-loop Borel representation

$$\mathcal{R}_{PT}(s)=\frac{-2}{\pi b}\int_0^\infty dz\,\mathrm{Im}\Bigg[\frac{e^{-z/a(-s+i\epsilon)}}{z} - ce^{zc}\mathrm{Ei}\Bigg(1,zc+\frac{z}{a(-s+i\epsilon)}\Bigg)\Bigg]B[\mathcal{D}](z). \tag{63}$$

The factor in the square bracket plays the role of the $e^{-z/a(s)}\sin(\pi bz/2)/(\pi bz/2)$ factor in the one-loop case. It provides an oscillatory factor so that at $s=s_L$ the Borel representation remains defined. For $s<s_L$ one must switch to a modified Borel representation as in Eq. (50), writing

$$\mathcal{R}_{PT}(s)=-\frac{1}{2\pi i}\int_{-s-i\epsilon}^{-s+i\epsilon}\frac{dt}{t}\int_0^\infty dz e^{-z/(-a(t))}B[\mathcal{D}](-z). \tag{64}$$

Which, performing the t-integration gives

$$\mathcal{R}_{PT}(s)=\frac{2}{\pi b}\int_0^\infty dz\,\mathrm{Im}\Bigg[-\frac{e^{-z/(-a(-s+i\epsilon))}}{z} - ce^{-zc} \times\mathrm{Ei}\Bigg(1,-zc+\frac{z}{(-a(-s+i\epsilon))}\Bigg)\Bigg]B[\mathcal{D}](-z). \tag{65}$$

Unfortunately we cannot write down a function analogous to Eq. (47) which gives $\mathcal{R}_{PT}(s)$ at all energies, because we do not know $B[\mathcal{D}](z)$ exactly. The two-loop situation, however, is the same as that at one-loop. The regulated representation of Eq. (63) applies for $s\geqslant s_L$, with the corresponding standard OPE. Below $s=s_L$ one needs the modified representation of Eq. (65) together with the resummed OPE recast in the form of Eq. (52). The perturbative component $\mathcal{R}_{PT}(s)$ then freezes to $2/b$ in the infrared, we can see this if we split $B[\mathcal{D}](-z)$ into $(1+(B[\mathcal{D}](-z)-1))$. The part of the integrand proportional to $B[\mathcal{D}](-z)-1$ vanishes for all z from $0\to\infty$ in the infrared limit. The remaining term integrates to give us $A_1(s)$ which freezes to $2/b$ as $s\to 0$. The nonperturbative component $\mathcal{R}_{NP}(s)$ given by the reformulated OPE together with the perturbative component determine the infrared freezing behavior. There is again a direct connection with the CIPT-APT reformulation of fixed-order perturbation theory. Using integration by parts one can show that for $s\geqslant s_L$

$$A_n(s)=\frac{-2}{\pi b}\int_0^\infty dz\,\mathrm{Im}\Bigg[\frac{e^{-z/a(-s+i\epsilon)}}{z}-ce^{zc} \times\mathrm{Ei}\Bigg(1,zc+\frac{z}{a(-s+i\epsilon)}\Bigg)\Bigg]\frac{z^{n-1}}{(n-1)!}, \tag{66}$$

where the $A_n(s)$ correspond to the two-loop results in Eqs. (20), (21). Once again CIPT-APT corresponds to keeping the oscillatory function in the Borel transform intact, and expanding the remainder in powers of z. Similarly for $s\leqslant s_L$ one has

$$A_n(s)=\frac{2}{\pi b}\int_0^\infty dz\,\mathrm{Im}\Bigg[-\frac{e^{-z/(-a(-s+i\epsilon))}}{z}-ce^{-zc} \times\mathrm{Ei}\Bigg(1,-zc+\frac{z}{(-a(-s+i\epsilon))}\Bigg)\Bigg]\frac{(-z)^{n-1}}{(n-1)!}. \tag{67}$$

Thus, as in the one-loop case, the CIPT-APT reformulation of fixed-order perturbation theory will be asymptotic to the Borel representations at small and large energies. We would like, as in Fig. 4 for the one-loop case, to compare how well the fixed-order CIPT-APT perturbation theory corresponds with the all-orders Borel representation. We are necessarily restricted to using the leading-b approximation since this is the extent of the exact all-orders information at our disposal. One possibility is to simply use the leading-b result for the Borel transform, $B[\mathcal{D}^{(L)}](z)$, in the two-loop Borel representation of Eq. (63). The difficulty though is that with $a(-s)$ the two-loop coupling, the Borel integral is now scheme-dependent, since $B[\mathcal{D}^{(L)}](z)$ has a scale dependence which exactly compensates that of the *one*-loop coupling. Using a renormalization scale $\mu^2=xs$ our result for $\mathcal{R}_{PT}(s)$ has an unphysical x-dependence. This difficulty is exacerbated if we attempt to match the result to the exactly known perturbative coefficients d_1 and d_2, which we could do by adding an additional contribution $(d_1-d_1^{(L)})z+(d_2-d_2^{(L)})(z^2/2)$ to the Borel transform. Thus, as has been argued elsewhere, such matching of leading-b results to exact higher-order results yields completely *ad hoc* predictions, which may be varied at will by changing the renormalization scale [37,38]. The resolution of this difficulty follows if one accepts that the standard RG-improvement of fixed-order perturbation theory is incomplete, in that only a subset of RG-predictable UV logarithms involving the energy scale s are resummed. Performing a complete resummation of these logs together with the accompanying logs involving the renormalization scale, yields a scale-independent result. This complete renormalization group improvement (CORGI) approach [39] applied to $\mathcal{D}(s)$ corresponds to use of a renormalisation scale $\mu^2=e^{-2d/b}s$, where d denotes the NLO perturbative correction d_1 in Eq. (23), in the $\overline{\mathrm{MS}}$ scheme with $\mu^2=s$. In the CORGI scheme we have the perturbation series,

$$\mathcal{D}(t)=a_0(t)+X_2a_0^3(t)+X_3a_0^4(t)+\cdots+X_na_0^{n+1}+\cdots, \tag{68}$$

where $a_0(t)$ is given by Eq. (15) with $z=(-1/e)(\sqrt{t}/\Lambda_D)^{-b/c}$, where $\Lambda_D\equiv e^{d/b}\tilde{\Lambda}_{\overline{MS}}$, and X_n are the CORGI invariants, and only X_2 is known. We can then attempt to perform the leading-b CORGI resummation,

$$\mathcal{D}^{(L)}_{CORGI}(t)=a_0(t)+X_2a_0^3(t)+\sum_{n>2}X_n^{(L)}a_0^{n+1}(t)+\cdots, \tag{69}$$

so that the exactly known NNLO X_2 coefficient is included, with the remaining unknown coefficients approximated by

$X_3^{(L)}$, $X_4^{(L)}$, . . . , the leading-b approximations. We stress that $a_0(t)$ denotes the full CORGI coupling defined in Eq. (15). One can define this formal sum using the Borel representation of $\mathcal{D}$ in Eq. (30), with the result for $B[\mathcal{D}^{(L)}]$ in Eq. (35). The integral can be expressed in closed form in terms of the exponential integral functions $\mathrm{Ei}(n,w)$ of Eq. (40), with the result [9]

$$\begin{aligned}\mathcal{D}^{(L)}(1/a(t)) = &\sum_{j=1}^{\infty} z_j\{-e^{z_j/a(t)}\mathrm{Ei}(1,z_j/a(t))[(z_j/a(t))\\ &\times(A_0(j)-z_jA_1(j))-z_jA_1(j)]+(A_0(j)\\ &-z_jA_1(j))\}-e^{-z_j/a(t)}B_0(2)\mathrm{Ei}(1,-z_j/a(t))\\ &+\sum_{j=3}^{\infty}\{-e^{-z_j/a(t)}\mathrm{Ei}(1,-z_j/a(t))[(z_j/a(t))\\ &\times(B_0(j)+z_jB_1(j))]-(B_0(j)+z_jB_1(j))\}.\end{aligned} \tag{70}$$

To define the infrared renormalon contribution we have assumed the standard continuation of $\mathrm{Ei}(n,w)$ from $\mathrm{Re}\,w>0$ to $\mathrm{Re}\,w<0$, defined by Eq. (44). In [9] a principal value was assumed, which corresponds to adding $-i\pi\,\mathrm{sgn}(\mathrm{Im}(z_j/a(t)))$ to the $\mathrm{Ei}(1,-z_j/a(t))$ term. As we found for $\mathcal{R}_{PT}^{(L)}(s)$ the principal value is not continuous at $s=s_L$, whereas the standard continuation is. The formal resummation in Eq. (69) then corresponds to [9]

$$\mathcal{D}_{CORGI}^{(L)}(t)=\mathcal{D}^{(L)}\left(\frac{1}{a_0(t)}+d_1^{(L)}(V)\right)+(X_2-X_2^{(L)})a_0^3(t), \tag{71}$$

once again $a_0(t)$ is the full CORGI coupling, and $d_1^{(L)}(V)$ denotes the NLO leading-b correction in the V-scheme. Inserting $\mathcal{D}_{CORGI}(t)$ inside the dispersion relation of Eq. (7) one can then define

$$\mathcal{R}_{CORGI}^{(L)}(s)=\frac{1}{2\pi i}\int_{-s-i\epsilon}^{-s+i\epsilon}dt\,\frac{\mathcal{D}_{CORGI}^{(L)}(t)}{t}. \tag{72}$$

This can be evaluated numerically, if we have $\mathcal{R}_{CORGI}^{(L)}(s_1)$ then we can obtain

$$\begin{aligned}\mathcal{R}_{CORGI}^{(L)}(s_2)=\mathcal{R}_{CORGI}^{(L)}(s_1)+\frac{1}{2\pi i}\Bigg(&\int_{-s_2-i\epsilon}^{-s_1-i\epsilon}dt\frac{\mathcal{D}_{CORGI}^{(L)}(t)}{t}\\ &+\int_{-s_1+i\epsilon}^{-s_2+i\epsilon}dt\frac{\mathcal{D}_{CORGI}^{(L)}(t)}{t}\Bigg).\end{aligned} \tag{73}$$

If we set s_1 to be large enough we can evaluate $\mathcal{R}_{CORGI}^{(L)}(s_1)$ using the circular contour in the s-plane, as in Eq. (8). Combining this circular integral with the integrals above and below the real negative axis we arrive at $\mathcal{R}_{CORGI}^{(L)}(s_2)$ where s_2 can be as far into the infrared as we want. The all-orders CORGI result can be compared with the NNLO CIPT-APT CORGI result,

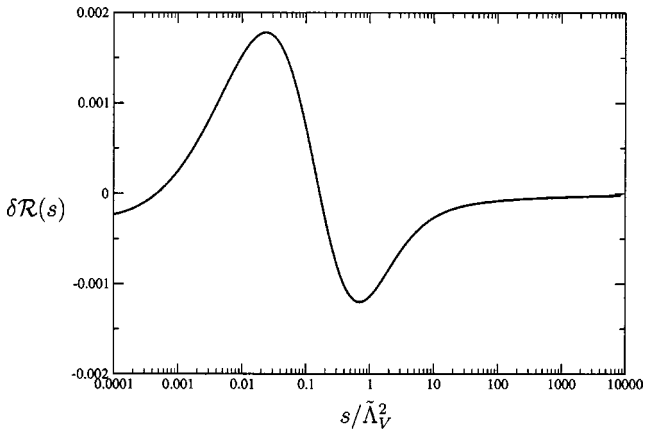

FIG. 5. $\delta\mathcal{R}(s)=\mathcal{R}_{CORGI}^{(L)}(s)-\mathcal{R}_{APT}(s)$, at the two loop level for 2 flavors of quark.

$$\mathcal{R}_{APT}(s)=A_1(s)+X_2A_3(s). \tag{74}$$

Here the $A_n(s)$ are the two-loop results of Eqs. (20), (21), with $A(s)=(-1/e)(\sqrt{s}/\Lambda_D)^{-b/c}$ in the CORGI scheme. Analogous to Fig. 4 we plot in Fig. 5 the comparison of the all-orders and NNLO APT CORGI results; $N_f=2$ quark flavors are assumed. As in the one-loop case there is extremely close agreement at all values of s. For the fits to low-energy $R_{e^+e^-}(s)$ data to be presented in the next section, therefore, we shall use the NNLO CORGI APT result.

Before turning to phenomenological analysis in Sec. IV, we conclude this section with a brief discussion of the situation for Euclidean observables. We can define the Adler D function in the Euclidean region by inverting the integral transform corresponding to the dispersion relation of Eq. (7). That is we can write

$$\mathcal{D}(Q^2)=Q^2\int_0^{\infty}\frac{ds}{(s+Q^2)^2}\mathcal{R}(s). \tag{75}$$

One can certainly define a Euclidean version of APT by inserting the Minkowskian $A_n(s)$ in the right-hand side of Eq. (75), and defining

$$A_n^{(E)}(Q^2)=Q^2\int_0^{\infty}\frac{ds}{(s+Q^2)^2}A_n(s). \tag{76}$$

The one-loop result would be [1]

$$A_1^{(E)}(Q^2)=\frac{2}{b}\left[\frac{1}{\ln(Q^2/\tilde{\Lambda}^2)}+\frac{\tilde{\Lambda}^2}{\tilde{\Lambda}^2-Q^2}\right]. \tag{77}$$

This Euclidean APT coupling freezes in the infrared to $2/b$, but this behavior is induced by the second nonperturbative contribution, which cancels the forbidden tachyonic Landau pole singularity present in the first perturbative term. There is now no direct connection, however, between this Euclidean APT coupling and the Borel representation for $D_{PT}(Q^2)$ of Eq. (30). Since there is now no oscillatory factor present in the Borel integral it is potentially divergent at $s=s_L$. We can explicitly exhibit this divergent behavior working in leading-

b approximation. The Borel integral can then be explicitly evaluated in terms of Ei functions as we have seen in Eq. (70). Using Eq. (44) for the Ei function one then finds a divergent behavior as $s \to s_L$ proportional to $\ln a$,

$$\mathcal{D}_{PT}(s) \to \left[\sum_{j=1}^{\infty} (z_j^2 A_1(j) + z_j^2 B_1(j)) - z_2 B_0(2) \right] \ln a + \cdots, \tag{78}$$

where the ellipsis denotes terms finite as $s \to s_L$. However, remarkably, the factor in the square bracket vanishes, and the result is finite at $s = s_L$, provided that *all* the renormalon singularities are included. The contribution of any individual renormalon is divergent. The cancellation follows because of an exact relation between the residues of IR and UV renormalons [Eq. (36)],

$$z_j^2 A_1(j) = -z_{j+3}^2 B_1(j+3). \tag{79}$$

This results in cancellations in the sum, leaving a residual term $z_3^2 B_1(3)$ which then cancels with the $z_2 B_0(2)$ term. An analogous relation $A_0(j) = -B_0(j+2)$ has been noted in [30]. It seems that these relations are underwritten by the conformal symmetry of the vacuum polarization function [34], but further investigation is warranted. The above finiteness at $s = s_L$ means that one can obtain a $\mathcal{D}_{PT}(Q^2)$ component well-defined in the infrared by changing to the modified form of Borel representation for $s < s_L$. One finds that this component becomes negative before approaching the freezing limit $\mathcal{D}_{PT}(0) = 0$. Similar behavior is found for the Gross-Llewellyn Smith and polarized and unpolarized Bjorken structure function sum rules, whose complete renormalon structure is also known in leading-*b* approximation [30]. Phenomenological investigations are planned [40]. Comparable investigations in the standard APT approach have been reported in [41]. Unfortunately, nothing is known about the full renormalon structure beyond leading-*b* approximation. Such knowledge would be tantamount to a full solution of the Schwinger-Dyson equations. Correspondingly no analogue of the APT reorganization of fixed-order perturbation theory asymptotic to $\mathcal{D}_{PT}$ is possible in the Euclidean case.

We finally note that in the case of $R_{e^+e^-}$ and D it is possible to say something about the separate infrared freezing behaviors of the PT and NP components. Arguments of spontaneous chiral symmetry breaking in the limit of a large number of colors [34] imply that $D(0) = 0$, or equivalently $\mathcal{D}(0) = -1$. Furthermore according to Ref. [42] $\mathcal{R}$ and $\mathcal{D}$ should have the same infrared freezing limit. This argument follows directly from Eq. (8) if the circular contour is shrunk to zero. These exact results then suggest that $\mathcal{D}_{NP}(0) = -1$ to be consistent with the leading-*b* result $\mathcal{D}_{PT}(0) = 0$ obtained above. For $\mathcal{R}$ one infers that $\mathcal{R}_{NP}(0) = -1 - (2/b)$ to be consistent with the $\mathcal{R}_{PT}(0) = 2/b$ result.

IV. COMPARISON OF NNLO APT WITH LOW ENERGY $R_{e^+e^-}$ DATA

In this section we wish to compare the NNLO CORGI APT perturbative predictions with low energy experimental data for $R_{e^+e^-}$. The discussion so far has assumed massless quarks. To include quark masses we use the approximate result [6,43]

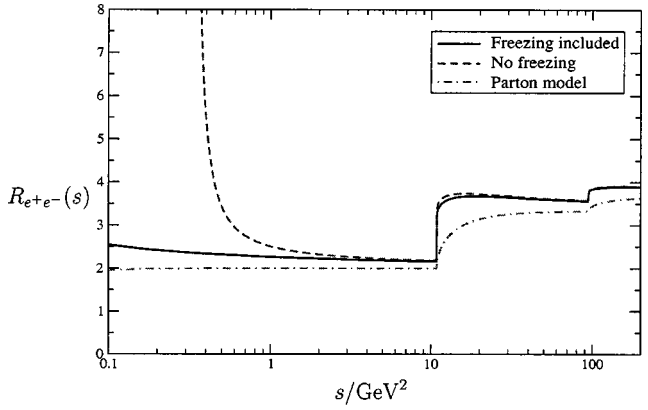

FIG. 6. Comparison of CORGI APT and the standard NNLO CORGI calculations of $R_{e^+e^-}(s)$ at low energies.

$$R_{e^+e^-}(s) = 3 \sum_f Q_f^2 T(v_f)[1 + g(v_f)\mathcal{R}(s)], \tag{80}$$

with the sum over all active quark flavors, i.e. those with masses $m_f < \sqrt{s}/2$, and where

$$v_f = (1 - 4m_f^2/s)^{1/2},$$

$$T(v) = v(3 - v^2)/2,$$

$$g(v) = \frac{4\pi}{3}\left[\frac{\pi}{2v} - \frac{3+v}{4}\left(\frac{\pi}{2} - \frac{3}{4\pi} \right) \right]. \tag{81}$$

For the theoretical predictions we shall take $\mathcal{R}(s)$ to be the NNLO CIPT-APT CORGI result of Eq. (74). Starting with $\tilde{\Lambda}^{(5)}_{\overline{MS}} = 216$ MeV for $N_f = 5$, corresponding to the world average value $\alpha_s(M_Z) = 0.1172$ [44], we demand that $\mathcal{R}(s)$ remains continuous as we cross quark mass thresholds. This then determines $\tilde{\Lambda}^{(N_f)}_{\overline{MS}}$ for $N_f = 4,3,2$. We take standard values for current quark masses for the light quarks [44]: $m_u = 3.0$ MeV, $m_d = 6.75$ MeV, $m_s = 117.5$ MeV, and also from [44] we take the values for pole masses of the heavy quarks $m_c = 1.65$ GeV, and $m_b = 4.85$ GeV. The approximate result [6] uses pole masses in Eq. (81), so we use pole masses where we can. Using these values for the quark masses and $\alpha_s(M_Z)$, we plot the resulting $R_{e^+e^-}(s)$ in Fig. 6. The solid line corresponds to the CORGI APT result for $\mathcal{R}(s)$ in Eq. (74). The dashed curve corresponds to the standard NLO fixed-order CORGI result,

$$\mathcal{R}_{CORGI}(s) = a_0(s) + \left(X_2 - \frac{\pi^2 b^2}{12} \right) a_0^3(s). \tag{82}$$

The standard fixed-order result breaks down at $s = \Lambda_D^2 = 0.4114\ \mathrm{GeV}^2$, where there is a Landau pole. The APT result smoothly freezes in the infrared. The dashed-dot curve shows the parton model result [i.e. assuming $\mathcal{R}(s) = 0$].

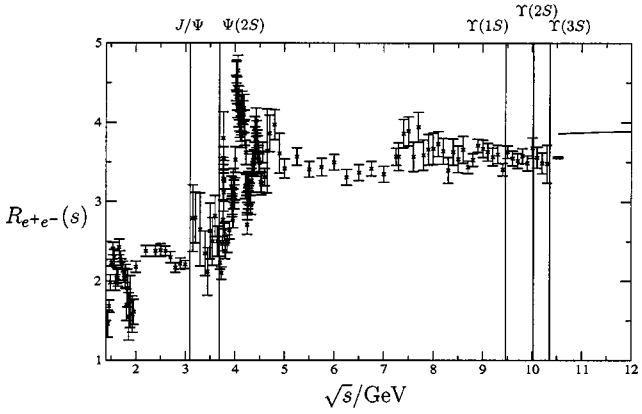

FIG. 7. Data used to compare with model, statistical errors shown only.

For a recent comprehensive review of the experimental data for $R_{e^+e^-}(s)$ at low energies see Ref. [45]. The experimental data we have used comes from a variety of sources. From the two pion threshold up to $\sqrt{s}=1.43$ GeV we use references [46], the data from these references is given as individual exclusive channels which must be combined to obtain the full hadronic cross section. In the region between 1.43 GeV and 2.0 GeV we use data from [47,48]; Refs. [49,50] are used in the region between 2.0 GeV and 5.0 GeV. From 5.0 GeV to 7.25 GeV we use [51], and from 7.25 GeV to 10.52 GeV we use [52,53]. These sets of data all give the inclusive total hadronic cross section. Above 10.52 GeV we insert the NNLO CORGI APT prediction for $R_{e^+e^-}$, this is represented by the continuous line in Fig. 7.

In order to simplify the analysis of the data we did not use overlapping data sets, instead where one data set overlapped another we simply took the better, smaller error, data set in the region of the overlap in $\sqrt{s}$. Errors were dealt with by taking each data point and calculating the effect of its statistical and its systematic error. The effect of its statistical error was added in quadrature with the other statistical errors. The contribution from the systematic error was added to the other systematic errors from the same data set, then the contribution from the systematic errors of each data set were added in quadrature with each other and the contribution from the statistical errors.

We also need to consider the effect of narrow resonances not included in the data; we employ the same approach as used in [12]. We assume that the narrow resonances have a relativistic Breit-Wigner form

$$R_{res}(s)=\frac{9}{\alpha^2}B_{ll}B_h\frac{M^2\Gamma^2}{(s-M^2)^2+M^2\Gamma^2}, \tag{83}$$

where α is the QED coupling, and M,Γ,B_{ll},B_h are the mass, width, lepton branching ratio, and hadron branching ratio respectively. We are assuming a narrow resonance i.e. Γ is small, so we approximate the resonance with a delta function

$$R_{res}(s)=\frac{9}{\alpha^2}B_{ll}B_hM\Gamma\pi\frac{M\Gamma/\pi}{(s-M^2)^2+M^2\Gamma^2}$$

$$\approx\frac{9}{\alpha^2}B_{ll}B_hM\Gamma\pi\delta(s-M^2). \tag{84}$$

The compilation of data for $R_{e^+e^-}$ is shown in Fig. 7. Narrow resonances are indicated by the vertical lines. Unfortunately it is not possible to directly compare the experimental data with the theoretical predictions. This is because there is not a direct correspondence between the quark mass thresholds in perturbation theory and the hadronic resonances. This difficulty can be overcome if one employs a ''smearing procedure.'' We shall employ the method proposed by Poggio, Quinn and Weinberg [6], defining the smeared quantity

$$\overline{R}_{e^+e^-}(s;\Delta)=\frac{\Delta}{\pi}\int_0^\infty dt\frac{R_{e^+e^-}(t)}{(t-s)^2+\Delta^2}. \tag{85}$$

$R_{e^+e^-}(s)$ itself is related to the vacuum-polarization function $\Pi(s)$ of Eq. (4) by

$$2iR_{e^+e^-}(s)=\Pi(s+i\epsilon)-\Pi(s-i\epsilon), \tag{86}$$

that is it is the discontinuity across the cut. The smeared $\overline{R}_{e^+e^-}(s;\Delta)$ can be written as

$$2i\overline{R}_{e^+e^-}(s;\Delta)=\Pi(s+i\Delta)-\Pi(s-i\Delta). \tag{87}$$

If Δ is sufficiently large one is kept away from the cut, and is insensitive to the infrared singularities which occur there. If both data and theory are smeared they can then be compared. In this way one hopes to minimize the contribution of the $\mathcal{R}_{NP}$ component. One needs to choose Δ sufficiently large that resonances are averaged out. For the charm region it turns out that $\Delta=3$ GeV^2 is a good choice, whilst for lower energies $\Delta=1$ GeV^2 is adequate. In Fig. 8(a) we choose $\Delta=1$ GeV^2. $\overline{R}_{e^+e^-}(s;\Delta)$ obtained from the data is represented by the solid line. The dashed-dot line is the smeared NNLO CORGI APT prediction, assuming the quark mass thresholds as above with the exception of the charm quark whose mass is taken to be $m_c=1.35$ GeV for reasons which we shall shortly discuss. The dashed line is the parton model prediction. The shaded region denotes the error in the data. It is clear that in the charm region the averaging is insufficient, although for lower energies the agreement is extremely good. In Fig. 8(b) we show the corresponding plot with $\Delta=3$ GeV^2. There is now good agreement between smeared theory and experiment over the whole s range, for $m_c=1.35$ GeV. Whilst we have indicated an error band associated with the data, we have not indicated an error band for the theory prediction. There are several potential sources of error to consider. The first is the choice of renormalization scale. Our viewpoint would be that the use of the CORGI scale corresponds to a complete resummation of ultraviolet logarithms, which in the process results in a cancellation of μ-dependent logarithms contained in the coupling $a(\mu^2)$

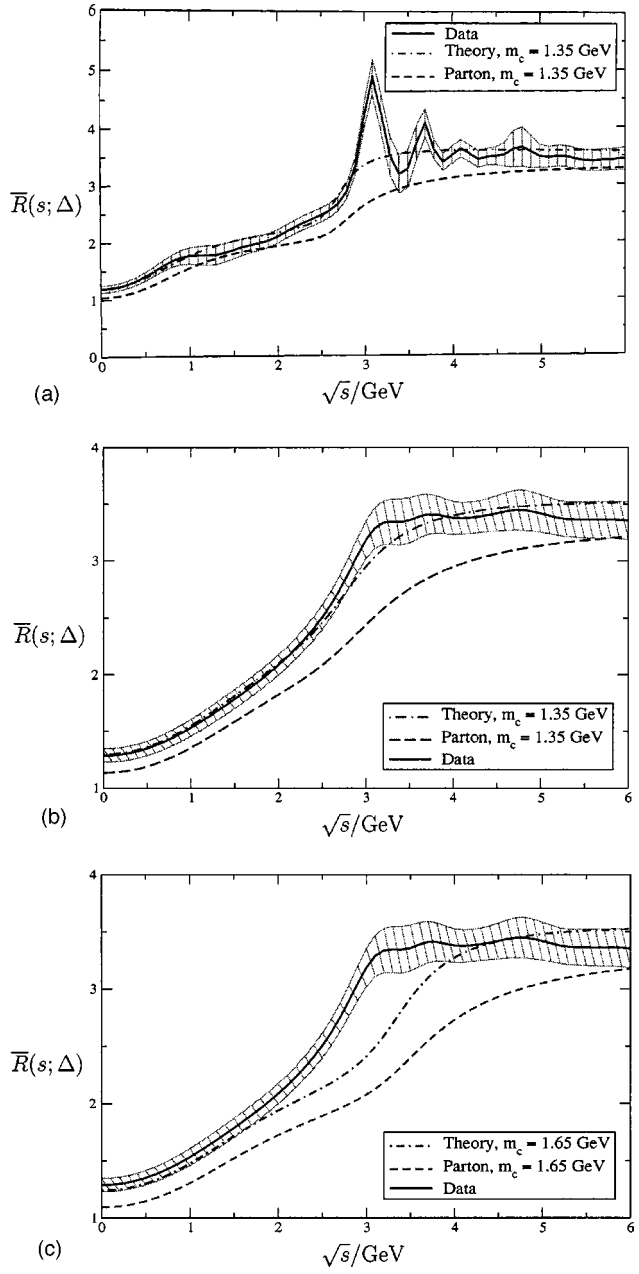

FIG. 8. (a) $\overline{R}(s;\Delta)$ in the charm region, with $\Delta=1$ GeV^2. (b) $\overline{R}(s;\Delta)$ in the charm region, with $\Delta=3$ GeV^2. (c) $\overline{R}(s;\Delta)$ in the charm region, with $\Delta=3$ GeV^2 here $m_c=1.65$ GeV.

and in the perturbative coefficients. As we have argued elsewhere [39] attempts to estimate a theoretical error on the perturbative predictions by making *ad hoc* changes in the renormalization scale are simply misleading, and give no information on the importance of uncalculated higher-order corrections. A common approach, for instance, is to use scales $\mu^2=xs$ where x is varied between $x=\frac{1}{2}$ and $x=2$, with $x=1$ providing a central value. We should note, however, that were we to have used such a procedure it would not have led to a noticeable difference in the theory curves, since the APT has greatly reduced scale-dependence, as has been noted elsewhere [54]. A more important uncertainty is the precise value of the quark masses assumed, and in particular the choice of the charm quark mass m_c. To illustrate how this affects the results we show in Fig. 8(c) the curves obtained if we assume $m_c=1.65$ GeV. As can be seen the theory curve is now inconsistent with the data in the charm region, although for lower energies where the charm quark has decoupled, the agreement is again good.

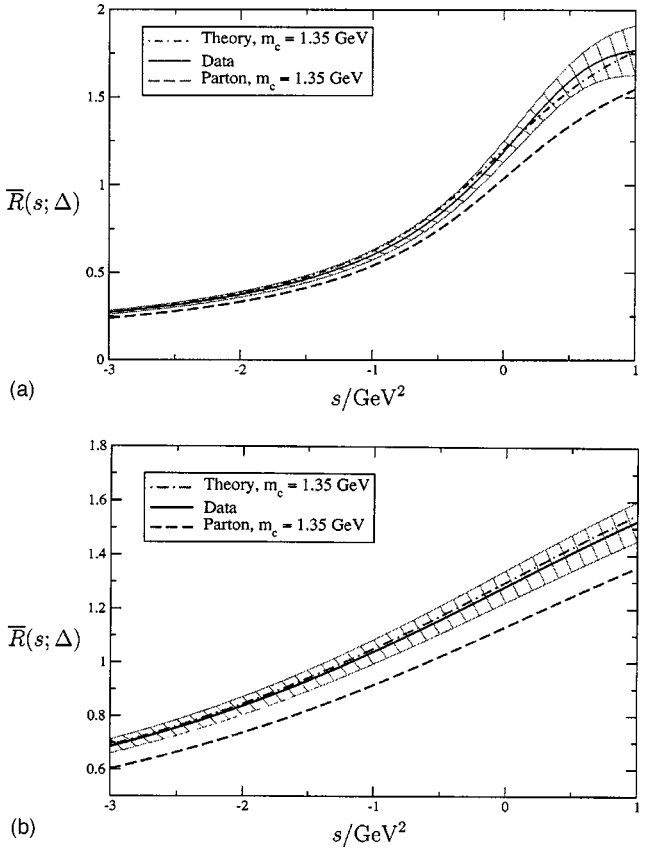

FIG. 9. (a) $\overline{R}(s;\Delta)$ in the spacelike region, with $\Delta=1$ GeV^2. (b) $\overline{R}(s;\Delta)$ in the spacelike region, with $\Delta=3$ GeV^2.

The uncertainty in the mass of the charm quark is exceptionally large. Looking at the different references used in [44] a value $m_c=1.35$ GeV for the pole mass is reasonable, and agrees well with [55] which is referenced in [44]. Part of the problem is the relationship between the pole mass and the $\overline{\text{MS}}$ mass for the charm quark, where the α_s^3 contribution is larger than the α_s^2 contribution. Obtaining the pole mass through $\overline{\text{MS}}$ mass calculations, which is done in [44], is not very satisfactory. Reference [55], which also fits low-energy $R_{e^+e^-}$ data, gives a pole mass of $m_c=1.33$–1.4 GeV, and so the choice of 1.35 GeV is reasonable.

It is possible to extend the smearing to spacelike values of s. We give the corresponding curves for $\overline{R}_{e^+e^-}(s;\Delta)$, with $m_c=1.35$ GeV, over the range $-3<s<1$ GeV^2 in Figs. 9(a), 9(b), for $\Delta=1$ GeV^2, and $\Delta=3$ GeV^2, respectively. The agreement between theory and experiment is extremely good in both cases.

In Fig. 10 we show $\overline{R}_{e^+e^-}(s;\Delta)$ in the upsilon region. The choice $\Delta=10$ GeV^2 works quite well; we show the theory predictions for different m_b values. A direct comparison between theory and data which does not involve smearing is

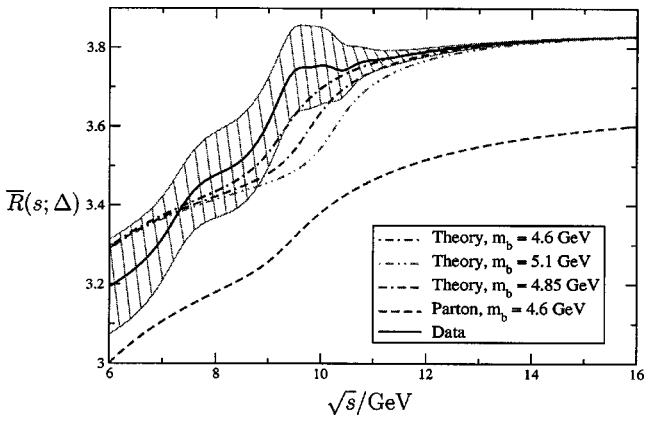

FIG. 10. $\overline{R}(s;\Delta)$ in the upsilon region, with $\Delta = 10$ GeV^2.

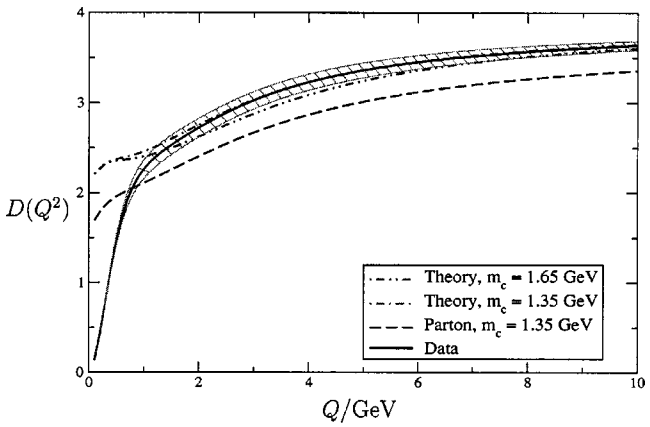

FIG. 12. $D(Q^2)$ calculated using APT.

possible if one evaluates the area under the $R_{e^+e^-}(s)$ data, that is evaluates the integral,

$$I(s) \equiv \int_{4m_\pi^2}^{s} R_{e^+e^-}(t)dt, \tag{88}$$

where s lies well above the low-energy resonances in the continuum. We show the theory and experimental $I(s)$ over the range $5 < \sqrt{s} < 9$ GeV in Fig. 11. There is extremely good agreement. Finally we can avoid smearing by transforming $R_{e^+e^-}(s)$ to obtain $D(Q^2)$ in the Euclidean region, using the dispersion relation of Eq. (75)

$$D(Q^2) = Q^2 \int_{4m_\pi^2}^{\infty} \frac{ds}{(s+Q^2)^2} R_{e^+e^-}(s). \tag{89}$$

In practice we cannot integrate up to infinity so we just take the sufficiently large upper limit of 10^6 GeV^2. As noted earlier above $\sqrt{s} = 10.52$ GeV the NNLO CORGI APT prediction is used for the data. The theory and data results are shown in Figs. 12, 13. There is good agreement. Our results are comparable to the fit obtained in [56], and to the results in [57]. We should also note that very similar plots and fits to those we have presented in this section are included in Ref. [58], which uses instead the so-called variational perturbation theory (VPT) approach [3].

V. DISCUSSION AND CONCLUSIONS

The analytic perturbation theory (APT) approach advocates the ''analytization'' of the terms in standard perturbation theory so that the perturbative expansion is recast as an expansion in a basis of functions that have desirable analytic properties, in particular the absence of unphysical ''Landau poles'' in Q^2 [1]. The functions in the Euclidean and Minkowski regions are interrelated by the integral transforms of Eq. (7) ($\mathcal{D} \to \mathcal{R}$) and Eq. (75) ($\mathcal{R} \to \mathcal{D}$). In a previous paper we pointed out the Minkowskian formulation of APT for the quantity $R_{e^+e^-}$ was equivalent to the all-orders resummation of a convergent subset of analytical continuation terms [8]. This reorganization of fixed-order perturbation theory gives apparent infrared freezing to the limit $2/b$ to all-orders in perturbation theory, and the functions $A_n(s)$ at two-loop level could be written in closed form in terms of the Lambert W function. However, one might question whether this all-orders perturbative freezing has any physical relevance. It is well-known that all-orders perturbation theory by itself is insufficient, and that it must be complemented by the nonperturbative operator product expansion (OPE) [4,5]. It is clear that the OPE breaks down as $s \to 0$,

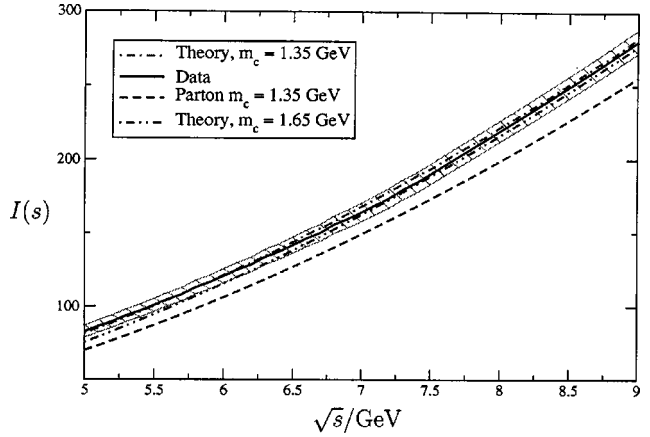

FIG. 11. Area under $R_{e^+e^-}(s)$.

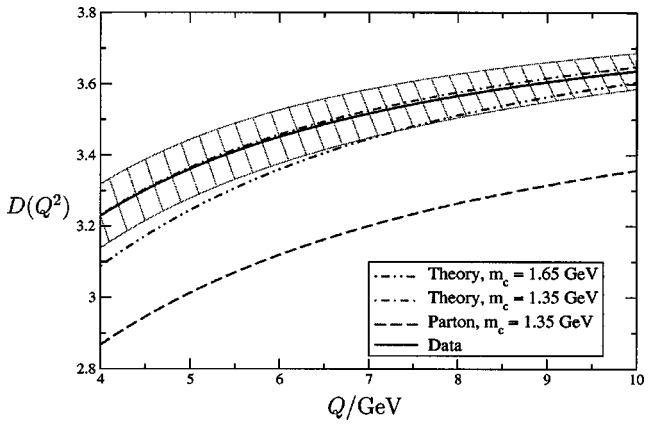

FIG. 13. Same as Fig. 12 but viewed over a smaller range.

since it is an expansion in powers of $\tilde{\Lambda}^2/s$. In this paper we have shown how both the PT and the OPE components can remain defined in the infrared limit. Writing a Borel representation for the PT component one finds that it is ambiguous because of the presence of singularities on the integration contour, termed infrared renormalons [4]. These ambiguities, however, are of precisely the same form as OPE terms, and a regulation of the singularities in the Borel integrand induces a definition of the OPE coefficients, allowing the two components to be defined. We showed that the Borel integral representation inevitably breaks down at a critical energy s_L which we referred to as the ''Landau divergence.'' For Minkowskian quantities the Borel transform contains an oscillatory factor which means that the Borel integral remains defined at $s=s_L$. For $s<s_L$ one needs to switch to an alternative Borel representation, which has ambiguities due to ultraviolet renormalon singularities on the integration contour. Correspondingly the OPE should be resummed and recast in the form of an expansion in powers of $s/\tilde{\Lambda}^2$. The UV renormalon ambiguities in the Borel integral are then of the same form as the terms in the modified OPE, and regulating the modified Borel integral induces a definition of the coefficients in the modified OPE, allowing both components to be defined. The modified Borel integral freezes to $2/b$ in the infrared thanks to the presence of the oscillatory factor, whilst the modified OPE component will also contribute to the infrared freezing behavior since resummation of the standard OPE can result in s-independent terms which can give a nonzero freezing limit, as in the toy example of Eq. (51). As we noted we did not expect to be able to determine the infrared behavior from perturbation theory alone, but the existence of a perturbative component which can be defined using a reorganized version of fixed-order perturbation theory at all energies is important. In particular the perturbative component dominates in the ultraviolet and may possibly provide a good approximation into the low-energy region. We explicitly constructed the all-orders Borel representations using the all-orders leading-b approximation for $\mathcal{R}(s)$ [30], and a one-loop coupling. We could express the Borel integral in closed form in terms of exponential integral functions [Eq. (47)]. With the standard continuation of the $\mathrm{Ei}(n,w)$ functions defined by Eq. (44) the result for $\mathcal{R}_{PT}^{(L)}(s)$ of Eq. (47) is a function of s which is well-defined at all energies, freezing to $2/b$ in the infrared, and continuous at $s=s_L$. The two-loop Borel representation was also discussed. The details are similar to the one-loop case, with a modified oscillatory factor and a shifted value of s_L, the modified Borel representation again freezes to $2/b$ in the infrared. At both one-loop and two-loops the APT modification of fixed-order perturbation theory corresponds to keeping the oscillatory factor in the Borel integrand intact, and expanding the remainder. As a result the APT results should be asymptotic to the Borel representations at all energies, underwriting the validity of the all-orders perturbative freezing behavior. It should be noted that we have somewhat oversimplified our discussion of the OPE contribution. The OPE coefficients are not constant, as in the toy example of Eq. (51), but are functions of a, $\mathcal{C}_n(a)$. Each coefficient will involve a perturbation series in a which is divergent with $n!$ growth of coefficients, and can be defined using a Borel representation. As defined by analytic continuation from the OPE for $\mathcal{D}_{NP}$ to that for $\mathcal{R}_{NP}$, the corresponding Borel integrands will contain the same oscillatory factors, enabling $\mathcal{C}_n(a)$ to remain defined at $s=s_L$, and for $s<s_L$ one switches to the modified Borel representation. We should note that the difficulty of uniquely extending the Borel representation for Minkowskian quantities into the infrared has also been discussed in Ref. [59], but with differing conclusions to us. A more closely related discussion concerning the significance and interpretation of the Landau pole is given in Ref. [32]. The modified Borel representation of Eq. (50) and the promotion of UV renormalon singularities to the positive axis in the Borel z-plane has also been discussed in Ref. [34].

Whilst the Minkowskian version of APT is underwritten by a Borel representation valid at all energies, this is not the case for the Euclidean version. There is no oscillatory factor in the integrand in the Euclidean case, and the Borel integral will potentially diverge as one approaches s_L. However, we showed that working in leading-b approximation $\mathcal{D}_{PT}$ was finite at s_L thanks to a cancellation between the infinite set of IR and UV renormalon residues. For individual renormalon singularities the Borel integral is divergent. By switching to the modified Borel representation one can then define a $\mathcal{D}_{PT}$ component which in fact freezes to zero in the infrared. This is interesting and similar perturbative freezing is also found for structure function sum rules [40]. The key point, however, is that no analogue of the Minkowskian APT reorganization of fixed-order perturbation theory is possible in the Euclidean case, and one is restricted to the leading-b approximation in exhibiting the perturbative freezing.

In the final section we performed fits of NNLO APT results to low energy $R_{e^+e^-}$ data. We needed to introduce quark masses approximately, and in order to avoid ambiguities due to the precise location of quark mass thresholds, and to minimize the contribution of the $\mathcal{R}_{NP}$ component, we used a smearing procedure. Extremely good agreement between theory and data was found.

An obvious further application would be to use the APT approach in the analysis of the tau decay ratio and in particular the estimation of the uncertainty in $\alpha_s(M_Z)$ which such measurements imply [9,54]. In Ref. [9] this was estimated by comparing NNLO CIPT in the CORGI approach, with an all-orders resummation based on the leading-b result. However, in fact CIPT for the tau decay ratio is *not* equivalent to the APT approach and corresponds to an expansion in a different basis of functions. In particular the resulting functions are *discontinuous* at $s=s_L$. We hope to study this further in a future publication.

ACKNOWLEDGMENTS

We thank Andrei Kataev and Paul Stevenson for entertaining discussions on infrared freezing in perturbative QCD. D.M.H. gratefully acknowledges support from PPARC UK.

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