edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964), p. 565.

[10]One can proceed nonrigorously by means of perturbation theory. However, even when the mass is not zero, there exist unsolved problems in the $\lambda\phi^4$ theory. See, for example, J. Glimm and A. Jaffe, in *Statistical Mechanics and Quantum Field Theory,* edited by C. DeWitt and R. Stora (Gordon and Breach, New York, 1971), p. 1.

[11]L. Parker, Ph.D. thesis, Harvard University, 1966 (unpublished); L. Parker, Phys. Rev. 183, 1057 (1969).

[12]Ya. B. Zel'dovich and A. A. Starobinsky, Zh. Eksp. Teor. Fiz. 61, 2161 (1971) [Sov. Phys. JETP 34, 1159 (1972)].

[13]Dimensional considerations for the Friedmann universes, as well as the earlier results, indicate that the Planck mass is the mass with which μ_0 must be compared in order to determine when the gravitationally induced particle creation will yield an energy density which is small with respect to the energy density acting as the source of the isotropic expansion.

[14]The direct demonstration is given in S. A. Fulling, Ph.D. thesis, Princeton University, 1972 (unpublished).

[15]Canonical quantization in curved space-time is discussed by the following: R. Utiyama, Phys. Rev. 125, 1727 (1962); R. Utiyama and B. S. DeWitt, J. Math. Phys. 3, 608 (1962); and C. Møller, *Royaumont Conference Proceedings* (National Center for Scientific Research, Paris, 1962), who noted the consistency and general covariance of the method. A recent discussion of the method is given in Ref. 14, where further references are given.

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Presymmetry of Classical Relativistic Fields*

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The physical act of accelerating observation instruments has an obvious counterpart in Newtonian physics: It is the transformation induced by $(\vec{x}, t) \mapsto (\vec{x} + \frac{1}{2}\vec{a}t^2, t)$. In special relativity, a theoretical counterpart for physical acceleration has been introduced only for a small subset of measurement procedures, such as clocks and yardsticks, but not for such instruments as accelerometers. We propose a general theoretical counterpart for physical accelerations in Minkowski space. In the algebra $\mathcal{O}$ of observation procedures, it induces automorphisms, not of $\mathcal{O}$ but of a point subalgebra $\mathcal{O}_x \subset \mathcal{O}$ that is associated to a single point x. From the postulates of presymmetry, we deduce a generalized version of Newton's second law; an acceleration-invariant subset $\bigcup_{(x|x_0=0)} \mathcal{O}_{xc}$ of instant observation procedures provides complete predictive power. The main technical contribution of the paper is the introduction of a topological algebra $\mathcal{O}$ in which local subsets associated to a single point are proper (although unbounded) subalgebras, whose automorphisms are discussed.

I. INTRODUCTION

The motivation for the present study is the desire to obtain clues in classical field theory for a more complete and physically plausible axiom system of quantum field theory. The canonical commutation relations of the early field theories had to be rejected because they produced inconsistencies, but the new mathematically consistent axiomatic field theories of Wightman and Araki-Haag lack an important element: kinematics.[1] This deficiency is exemplified by the lack of limitation on the order of the equations of motion. In a nonrelativistic n-particle problem, the equations of motion are $3n$ second-order differential equations for purely kinematic reasons. The Lagrangian field theories have analogous properties, but no such limitation is contained in axiomatic field theories as they stand. Yet a limitation of this kind is indispensable for a physical theory with effective predictive power. Is it possible to add simple and physically cogent assumptions to the axioms of quantum field theory such that a generalized kind of Newton's second law follows as a theorem? For the simpler case of nonrelativistic quantum mechanics, this question was answered positively in a previous paper.[2]

Classical relativistic field theory may provide another clue for answering the main question at hand. The present paper establishes what we hope to be physically reasonable basic assumptions from which a generalized form of Newton's second law follows as a theorem. Couched in abbreviated terms, it states that an acceleration-independent subset of instant observation procedures provides full predictive power for the future.

For the systematic comparison of two theories, it is at least necessary that they purport to answer the same questions. More precisely, there must be a common conceptual framework and the two theories should appear as special cases. Since quantum mechanics predicts the mean values of observables with respect to states, the classical theory must be reformulated in terms of an algebra of observables and of states. Furthermore, it has been found recently that problems related to presymmetry (as contrasted to the usual space-time symmetry, either Poincaré or Galilei) require an additional structure, the algebra of observation procedures. It is only in this algebra $\mathcal{O}$ that physical accelerations of measuring instruments induce automorphisms, and not in the algebra $\mathfrak{A}$ of observables. Hence, the present paper starts by constructing an Abelian algebra of observation procedures. The great technical advantage of this classical algebra is that observation procedures and observables at one space-time point are proper but unbounded elements of a topological algebra.

This strategy of attacking easier objectives as a preparation for the main operation has the apparent drawback that an elaborate formal apparatus is used to obtain old results in a new form and from new assumptions. Yet we believe that these results are interesting by themselves, and not only as by-products of a search for insights on quantum field theory. Maxwell's theory, the prototype of relativistic field theories, was expressed in the form of second-order equations because Maxwell believed in mechanical ether models and in Newton's second law. All other relativistic field theories follow this model, although the original mechanical motivation is forgotten. The challenge is to derive the result from considerations of space-time invariance. To understand how the order of the equations is related to space-time transformations, it is necessary to define acceleration transformations of the algebra of observation procedures. The usual mathematical framework of partial differential equations is insufficient for this purpose.

The elementary nonmathematical objects with which a class of field theories deals are observation procedures designed to measure conditions at space-time points $P=(\vec{x}, t)$. Experience shows that a small subset of such conditions or properties measured at all points at an instant t is sufficient for full predictive power; for instance, it is sufficient to measure temperature for heat conduction and the six components of electric and magnetic intensities for the Maxwell field. In the algebra $\mathcal{O}$ of observation procedures (contrary to the algebra $\mathfrak{A}$ of observables) the images $F_{i,x}$ of such nonmathematical procedures are distinct and algebraically independent for all space-time points x. By abuse of language, we shall identify these procedures with their images in $\mathcal{O}$ and refer to them as generators of canonical point algebras $\mathcal{O}_{cx}$ associated to space-time points x.

We shall assume a specific structure of the algebra $\mathcal{O}$ and then show that it has properties in agreement with reasonable idealizations of empirical facts. In this construction, we are guided both by the algebra $\mathcal{O}$ of observation procedures for nonrelativistic quantum mechanics[2] and by the algebra $\mathfrak{A}$ of observables for classical field theory.[3]

II. THE ALGEBRA $\mathcal{O}$ OF OBSERVATION PROCEDURES AND THE CONVEX SET $\mathcal{S}$ OF STATE-PREPARING PROCEDURES

Traditionally, the theory of classical fields has the mathematical form of partial differential equations. The algebraic theory has the more elaborate mathematical structure of algebras of functionals on a carrier space of functions, and the partial differential equations of the traditional theory appear in the subordinate role of inducing homeomorphisms of the carrier space, which in turn induce algebraic automorphisms of the space of functionals.

In a previous paper,[3] a specific Abelian algebra $\mathfrak{A}$ of observables was associated to a classical field. One knows from the study of nonrelativistic quantum mechanics[2] that a systematic theory of transformations induced by accelerations requires the consideration of a larger structure, namely the algebra $\mathcal{O}$ of observation procedures. In this algebra, observation procedures at different instants are algebraically independent. The relationship between $\mathcal{O}$ and $\mathfrak{A}$ is a many-one morphism Φ.

The algebra $\mathfrak{A}$ was defined as the algebra $C(\omega)$ of all real continuous functionals on a linear space ω of bounded, real, vector-valued functions ψ on the Euclidean space E^3. The prime physical interpretation was the association between the nonmathematical observation procedures $\varphi_i(\vec{x})$ at a point x and time $t=0$ (the field) and the functional

$$f_{i,\vec{x}}(\psi)=\psi_i(\vec{x}) \ . \tag{2.1}$$

By extension, it is natural to define the algebra $\mathcal{O}$ of observation procedures as the algebra $C(\Omega)$ of all real-valued continuous functionals $F(\varphi)$ on a space Ω of real, bounded, vector-valued functions on the Minkowski space M, and to identify the point-observation procedure $\varphi_i(x)$ $(x \in M)$ with the functional

$$F_{i,x}(\varphi)=\varphi_i(x) \ . \tag{2.2}$$

By analogy with the topological assumptions on the algebra $\mathfrak{A}$, we assume that the algebra $\mathcal{O}=C(\Omega)$ is equipped with the c topology,[3] and that the carrier space Ω is a linear space of real vector-valued functions φ on Minkowski space M, equipped with the norm

$$\|\varphi\| = \sup_x \left(\sum_{i=1}^{N} |\varphi_i(x)|^2 \right)^{1/2} \tag{2.3}$$

and piecewise differentiable. We assume that Ω contains *all* functions φ of the class defined.

The great technical advantage of this algebra over its quantum-theoretical analog mentioned in the Introduction is the existence of subalgebras of observation procedures and of observables associated to a single point x. These elements are, of course, unbounded, but they are proper members of a topological algebra. No "smearing" is necessary, and this circumstance opens the door to the definition of those automorphisms of point algebras of observation procedures that are induced by local accelerations.

For any set $\{F_i\}$ of elements of $\mathcal{O}$, the topological closure of the set of all inhomogeneous polynomials in F_i will be called the algebra topologically generated by $\{F_i\}$ and denoted by $(\{F_i\})_{\text{cl}}$. By definition, every topologically closed subalgebra contains the multiples of the unit functional $I(\varphi) = 1$. The following theorem shows that the unbounded functionals (2.2) topologically generate the algebra $\mathcal{O}$.

Theorem 1:

$$\mathcal{O} = \left(\bigcup_{\substack{x\in M\\ i=1,\ldots,N}} F_{i,x} \right)_{\text{cl}} .$$

Proof: Let

$$B = \bigcup_{\substack{x\in M\\ i=1,\ldots,N}} F_{i,x} .$$

Then clearly $B \subset \mathcal{O}$.

Further, B separates Ω in the following sense. For each pair $\varphi_1, \varphi_2 \in \Omega$, $\varphi_1 \neq \varphi_2$, there exists an $F\in B$ such that

$$F(\varphi_1) \neq F(\varphi_2) .$$

The fact that $(B)_{\text{cl}} = \mathcal{O}$ follows now from the Stone-Weierstrass theorem.[4]

Theorem 1 allows a more general physical interpretation. If expectations of the functional $F_{i,x}$ are (approximately) equal to dial readings or printouts obtained by the nonmathematical procedure $\varphi_i(x)$ (the field at x), then all real-valued functions of such printouts are now similarly associated to elements of $\mathcal{O}$, and all elements of $\mathcal{O}$ are approximated by such functions.

Let $\mathcal{O}_{cx}$ be the algebra generated by the point observation procedures $F_{i,x}$ at a fixed point x, i.e.,

$$\left(\bigcup_i F_{i,x} \right)_{\text{cl}} = \mathcal{O}_{cx} . \tag{2.4}$$

It will be called the canonical algebra of point observation procedures at x.

Those observation procedures that "look only" at events in the space-time region R and are "blind" to what happens outside R are characterized by the equation

$$\mathcal{O}(R) = \{F \mid F(\varphi) = F(\chi_R \varphi)\} , \tag{2.5}$$

where χ_R is the characteristic function of R. The subalgebra $\mathcal{O}(R)$ is called the algebra of local observation procedures (or the local subalgebra) in R. It will be shown that for any open set $R \subset M$ (with respect to the Euclidean topology), the topological closure of the union of canonical point algebras associated to $x\in R$ is equal to $\mathcal{O}(R)$.

Theorem 2:

$$\mathcal{O}(R) = \left(\bigcup_{x\in R} \mathcal{O}_{cx} \right)_{\text{cl}} \tag{2.6}$$

if R is open.

Proof: Before proceeding as in Theorem 1, one must take into account the fact that, because of the constraint (2.5), $\mathcal{O}(R)$ now is not the set of all continuous functionals on Ω. However, it is shown in the appendix of Ref. 2 that this constraint can be eliminated by working in the space of equivalence classes of functions. The proof is then as in Theorem 1.

For a spacelike hyperplane $h \subset M$, the canonical algebra $\mathcal{O}_{hc}$ will be defined by

$$\mathcal{O}_{hc} = \left(\bigcup_{x\in h} \mathcal{O}_{cx} \right)_{\text{cl}} , \tag{2.7}$$

but since h is not an open set, this is not a local subalgebra of $\mathcal{O}$. As in Ref. 1, this canonical subalgebra of $\mathcal{O}$ is isomorphic to the algebra $\mathfrak{A}$ of observables, as shown by the following theorem.

Theorem 3:

$$\mathcal{O}_{hc} \cong \mathfrak{A} .$$

Proof: Without loss of generality, we assume that h is the hyperplane $t=0$. Then

$$\mathcal{O}_{hc} = \left(\bigcup_{\substack{\vec{x}\in E^3\\ i=1,\ldots,N}} F_{i,\vec{x}} \right)_{\text{cl}} .$$

Let

$$g \in \omega ,$$

$$\Phi_g = \{\varphi \in \Omega,\ \varphi(\vec{x}, 0) = g(\vec{x})\} .$$

The map $\Phi_g \to g$ is one-one from the set $\bigcup_{g\in\omega} \Phi_g$ onto ω. Now, if $F\in \mathcal{O}_{hc}$ and $\varphi_1, \varphi_2 \in \Phi_g$, we have

$$F(\varphi_1) = F(\varphi_2) \equiv F(\Phi_g) .$$

We now define a map π from $\mathcal{O}_{hc}$ onto $\mathfrak{A}$ by

$$\pi F = f$$

such that

$$f(g)=F(\Phi_g)\ ,\quad \text{for all } g\in\omega\ ,$$

and it is easy to check that this map is in fact an isomorphism.

After defining the algebra $\mathcal{O}$, we turn to the set $\mathcal{S}$ of state-preparing procedures. As pointed out in Refs. 5 and 2, one can operationally define a convex linear combination of two state-preparing procedures. Hence, $\mathcal{S}$ is a convex linear set. A subset $\mathcal{S}_h$ is characterized by a spacelike hyperplane h such that the emission of the prepared sample occurs on h. At this instant, all interaction between the sample and the preparing equipment ceases. It is assumed that there exists a (canonical) subset $\mathcal{O}_{hc}$ of observation procedures whose expectation values always remain fixed for a given procedure $s\in\mathcal{S}_h$, regardless of the external field or interaction within the prepared sample. Hence, the procedure $s\in\mathcal{S}_h$ is associated to a state σ (a linear, continuous, positive, and normalized form) on the algebra $\mathcal{O}_{hc}$, and $\mathcal{S}_h$ itself is isomorphic to the set of these states, i.e., it is the intersection $\mathcal{O}_{hc}^{*+n}$ of the positive cone of the dual $\mathcal{O}_{hc}^{*+}$ with the surface $\sigma(I)=1$.

Let a family of parallel hyperplanes be parametrized by τ. Then the convex linear closure of the union $\cup_\tau \mathcal{S}_\tau$ is assumed to be the set $\mathcal{S}$ given by

$$\left(\bigcup_\tau \mathcal{S}_\tau\right)_{\text{cl}}=\mathcal{S}\ , \tag{2.8}$$

with the inclusion of all finite convex sums. While each subset $\mathcal{S}_\tau$ is isomorphic to the set of states on $\mathcal{O}_{\tau c}$, the set $\mathcal{S}$ is not the set of states on $\mathcal{O}$.

III. AUTOMORPHISMS INDUCED BY SPACE-TIME MOTIONS

In accordance with Ref. 2, an observation procedure $\alpha\in\mathcal{O}$, which we will identify with the instructions for its performance, can be considered as a pair $(b,\{P_n\})$ composed of a blueprint b for the building of the hardware and a set $\{P_n\}$ of the space-time points in V_4 at which various marks on the apparatus are to be positioned, switches are to be thrown, or buttons are to be pressed. A modified instruction is obtained by a permutation g: $P\rightarrowtail gP$ of the events P_n which is induced by a transformation g of space-time. In general, such altered instructions cannot be implemented physically, or they lead to meaningless acts: For instance, a scale transformation might call for the crushing of an ammeter, but a crushed ammeter is not a measuring instrument. What characterizes those transformations g that create new observation procedures, i.e., those that induce transformations within the set of all observation procedures? In Ref. 2, it was assumed that these are the automorphisms of space-time, and that they induce automorphisms of the algebra $\mathcal{O}$ of observation procedures through

$$\alpha=(b,\{P_n\})\rightarrowtail V_g\alpha=(b,\{gP_n\})\ . \tag{3.1}$$

A distinguished generating subset of $\mathcal{O}$, consisting of the linear canonical point functionals $F_{i,x}$ defined by Eq. (2.2), has a direct and simple physical interpretation. It is natural to require that automorphisms induced by space-time motions leave stable the linear manifold generated by $\{F_{i,x}\}$. Consider the action of the Poincaré group on the canonical point algebras $\mathcal{O}_{cx}$. An element Λ of that Lorentz group L that leaves x fixed acts on a generating element $F_{\mu,x}$ of $\mathcal{O}_{cx}$ by

$$V_\Lambda F_{\mu,x}=S_{\mu\nu}(\Lambda)F_{\nu,x}\ , \tag{3.2}$$

where the matrices $S_{\mu\nu}(\Lambda)$ form a finite-dimensional linear representation of L. More generally, an element (Λ, a) of the Poincaré group induces the transformation

$$V(\Lambda, a)F_{\mu,x}=S_{\mu\nu}(\Lambda)F_{\nu,\Lambda x+a}\ . \tag{3.3}$$

The finite-dimensional representations $S_{\mu\nu}$ of the Lorentz group are completely reducible.[6] Each irreducible subspace may be identified with a field, and hence may be associated to a particular collection of blueprints $b^{(\gamma)}$ through $F_{i,x}^{(\gamma)}=(b^{(\gamma)},\{P_n\})$.

Since the canonical procedures $F_{\mu,x}$ generate the algebra $\mathcal{O}$, the automorphisms induced by the Poincaré group are completely defined. It is convenient to induce this automorphisms by a linear transformation $Q(\Lambda, a)$: $\Omega\to\Omega$ of the carrier space Ω. Thus

$$(Q(\Lambda, a)\varphi)_i(x)=S_{ik}\varphi_k(\Lambda x+a) \tag{3.4}$$

so that, by Eq. (2.2),

$$(V(\Lambda, a)F)(\varphi)=F[Q(\Lambda, a)\varphi]\ . \tag{3.5}$$

Are these all the space-time automorphisms? It is possible to transfer some observation instruments to an accelerated vehicle without "ruining" them. The operational test for proper functioning of observation instruments is the preservation of algebraic relations. Two ammeters which, when at rest, had dial readings in the proportion 2 : 1 for all states are expected to maintain this algebraic relation. Three observation procedures q_x, q_y, q_θ, whose results were invariably related by

$$q_\theta=q_x\cos\theta+q_y\sin\theta$$

when at rest, are expected to maintain this relationship after being set in motion if they function properly. More generally, automorphism is an operationally verifiable condition for the existence of a presymmetry, and experience suggests that there exist enough instruments with sufficiently sturdy design to justify such an idealization, at least for a class of accelerations. The difficulty in preserving true functioning of measuring instruments despite space-time motions is not peculiar to accelerations. A magnetometer, properly calibrated in an empty room, should read "the same" when transported to a region of high fields other than magnetic. Clearly, there are conditions that will melt or otherwise spoil any magnetometer, but the experimenter has at least two remedies: He can shield his instrument or he can correct the "false" reading by using theory. The first device is impossible for gravitational or inertial forces, but the second is reasonable and is in fact used constantly for satellite observations.

According to a widely accepted view, one has to use the concept of curved space to describe accelerated laboratories. Yet, the argument is not convincing, since one can easily associate a classical nonrelativistic transformation $(\vec{x}, t) \rightarrowtail (\vec{x} + \frac{1}{2}\vec{g}t^2, t)$ to a physical act of accelerating observation instruments. This possibility is quite compatible with the absence of what is loosely called physical equivalence between the static and the accelerated laboratories.

Some authors[7] have given a partial answer to the question of accelerated motion of measuring devices in flat space. The article by Heintzmann and Mittelstaedt[8] is perhaps the most explicit statement of this view. According to these authors, a restricted set of standard measuring instruments is not affected by their acceleration, but only by their instantaneous velocity. Our interest, on the other hand, centers on the change of observation results in those instruments (e.g., accelerometers) that are influenced by instantaneous acceleration. The acceleration invariance of a special subset of procedures is the content of a theorem to be derived. Thus, the present work may be viewed as a more general theory of acceleration transformations; it may contain the situation viewed by Møller and others as a special case.

If accelerated motions can be assumed to induce automorphisms, there remains the choice between automorphisms of the whole algebra $\mathcal{O}$ and those of subalgebras of $\mathcal{O}$.

IV. LOCAL ACCELERATIONS

For Newtonian space-time, the automorphisms of the Galilean space-time preserve two bilinear forms: both the space distances $|\vec{x}_1 - \vec{x}_2|$ for pairs of simultaneous points $P_i = (\vec{x}_i, t)$ with $i = 1, 2$ and also the time distances $t_1 - t_2$ for any two points P_i. These automorphisms are not exhausted by the Galilei group; they are the rigid-body motions; automorphisms of the algebra $\mathcal{O}$ of observation procedures induced by them were discussed in Ref. 2. In special relativity, the automorphisms of M preserve the Minkowski pseudodistance; they form the extended Poincaré group. No extension to a larger group with accelerations is possible. For a theory of acceleration transformations, there are only two possibilities: Abandon the flat affine space-time as did Einstein, or modify the assumed relationship between the allowed transformations g of M and transformations of the algebra $\mathcal{O}$ of observation procedures induced by g. In this paper, the second possibility is explored.

A subset of observation procedures that measure events at or close to a point P_0 in space-time is modified in a manner that can be described intuitively as the act of imparting an accelerated motion to the apparatus. The transformation g that induces this change by $(b, \{P_n\}) \rightarrowtail (b, \{gP_n\})$ will be assumed to preserve only a part of the structure of Minkowski space. Since the laboratory is small, it is not required that the structure of distant regions be preserved. Hence, it is assumed only that the Minkowski distance between any point P and the reference point P_0 is preserved. Consider a transformation in the (x, t) plane, namely,

$$\begin{aligned} g_a x &= x\cosh f(x, t) + t\sinh f(x, t) , \\ g_a t &= x\sinh f(x, t) + t\cosh f(x, t) . \end{aligned} \tag{4.1}$$

It preserves the invariant $x^2 - t^2$ with any function $f(x, t)$. The natural implementation of such an acceleration is accomplished by an impact on a solid piece of apparatus at the space-time point P_0 which we assume to be the origin of the coordinate system. Then, no material point can move before the arrival of a signal from the origin. Hence, the function $f(x, t)$ must vanish outside the forward light cone. It stands to reason that the effect of the impact must be independent of the state of uniform motion of the apparatus; i.e., the transformation g_a commutes with the Lorentz transformations

$$\begin{aligned} lx &= x\cosh\alpha + t\sinh\alpha , \\ lt &= x\sinh\alpha + t\cosh\alpha \end{aligned} \tag{4.2}$$

for all values of α. It follows that f depends only on the invariant $x^2 - t^2$.

Similarly, a rotatory motion in the (x, y) plane, namely,

$$\begin{aligned} g_r x &= x\cos f(x, y, t) + y\sin f(x, y, t) , \\ g_r y &= -x\sin f(x, y, t) + y\cos f(x, y, t) , \end{aligned} \tag{4.3}$$

preserves the invariant $x^2 + y^2$. It is naturally implemented by letting a rotating axle parallel to the

z direction impact the point O of a solid apparatus at the instant $t=0$. To maintain causality, the function f must vanish outside the forward light cone defined by $t^2-x^2-y^2 \geq 0$. It stands to reason that the effect of the mechanical impact is not intrinsically altered by a previously accomplished rotation of the apparatus, i.e., that the transformation g_r commutes with the rotation

$$rx = x\cos\varphi + y\sin\varphi \,, \quad ry = -x\sin\varphi + y\cos\varphi \tag{4.4}$$

for all angles φ. It follows that f can only be a function of the invariant $x^2+y^2-t^2$ and vanishes outside the forward light cone.

To describe the acceleration transformations compactly, consider the Lorentz group as generated by the usual six canonical one-parameter subgroups, the parameters being three angles and three pseudoangles, namely,

$$L^i(\alpha)x_\mu = \Lambda^i_{\mu\nu}(\alpha)x_\nu, \quad i=1,\ldots,6 \,, \tag{4.5}$$

where $\Lambda_{\mu\nu}$ is considered as a function, trigonometric or hyperbolic. Then

$$L^i(\alpha)L^i(\beta) = L^i(\alpha+\beta) \,. \tag{4.6}$$

A subgroup of acceleration transformations has the form

$$g^i(f)x_\mu = \Lambda^i_{\mu\nu}[f(x_\rho x^\rho)]x_\nu \,, \tag{4.7}$$

where f vanishes outside the forward light cone. Then

$$g^i(f)g^i(h) = g^i(f+h) \,. \tag{4.8}$$

With substitution as a multiplication, these transformations generate a group. They preserve a part of the structure of Minkowski space: the Minkowski pseudodistance $d(P,P_0)$ for all points $P\in M$, and the pseudodistances $d(P_1,P_2)$ between any two points P_1, P_2 that are on the same hyperboloid; i.e., $d(P_1,P_0)=d(P_2,P_0)$. This set will be referred to as the group M_x of meromorphisms of Minkowski space with x as a fixed point.

Some smoothness properties for the function f are physically indispensable. We choose the simplest. Let n be a fixed non-negative integer and consider the one-parameter group

$$g_{i,n}(\beta) = \Lambda^i_{\mu\nu}(\beta|x|^n)x_\nu \,,$$

where $|x| = (x_0^2 - x_1^2 - x_2^2 - x_3^3)^{1/2}$ in the forward light cone and zero otherwise. The set

$$S \equiv \bigcup_{i=1,\ldots,6} \; \bigcup_{n=0,1,\ldots} g_{i,n}(\beta_{i,n})$$

is composed of $6\times\infty$ one-parameter groups, and we will be interested in the group generated from this set by multiplying a finite number of elements, i.e., in the group

$$g\in M_x \Leftrightarrow g = \prod_{l=1}^{q} g_l \,, \quad g_l \in S \,, \quad q<\infty \,. \tag{4.9}$$

Thus, our group is not closed, since the product of an infinite number of elements may not belong to the group. M_x is not a Lie group, since it contains elements with an arbitrary, though finite, number of parameters. Too little is known about the theory of infinite-dimensional Lie groups[9] to allow the use of a Lie algebra. However, we can associate a generator $a_{i,n}$ to each one-parameter subgroup $g_{i,n}(\beta)$, and thereby obtain an infinite-dimensional Lie algebra.

Proceeding formally, one obtains the commutation relations of this Lie algebra in the usual way. Let c_{ijk} be the standard set of structure constants of the Lie algebra of the Lorentz group. Then, the infinite set of generators $a_{i,n}$ has the commutation relations

$$[a_{i,n}, a_{j,m}] = c_{ijk}\, a_{n+m,k} \,. \tag{4.10}$$

The elements $a_{i,0}$ can be associated to the Lorentz group itself. This Lie algebra is used only as a mnemonic aid.

V. THE ALGEBRA $\mathfrak{A}$ OF OBSERVABLES

As in Ref. 3, the algebra $\mathfrak{A}$ of observables is assumed to be the algebra $C(\omega)$ of all continuous functionals f on the linear space ω of all piecewise differentiable real vector-valued functions $\psi_i(x_1,x_2,x_3)$ on E^3. As mentioned before, this algebra is isomorphic to each canonical subalgebra $\mathcal{O}_{hc}\subset\mathcal{O}$ associated to a spacelike hyperplane h. It follows from Sec. II that a subset $\mathcal{S}_h\subset\mathcal{S}$ of instant state-preparing procedures is isomorphic to the set $S(\mathfrak{A})$ of states on $\mathfrak{A}$.

The results of observations in a given external field are summarized by the map (expectation)

$$\mathcal{E}: \mathcal{S}\times\mathcal{O} \to R \tag{5.1}$$

of the Cartesian product of $\mathcal{S}$ and $\mathcal{O}$ into the reals such that the mean value of repeated observations by the procedure $\alpha\in\mathcal{O}$ on a state produced by procedure $s\subset\mathcal{S}$ is approximately

$$\mathcal{E}(s,\alpha) \approx \frac{1}{N}\sum_1^N s_n(\alpha) \,. \tag{5.2}$$

As in Ref. 2, each expectation defines an equivalence class $E(\mathcal{E})$ and a kernel $K(\mathcal{E})$ in $\mathcal{S}$ and $\mathcal{O}$, respectively, by

$$K(\mathcal{E}) = \{\alpha \mid \mathcal{E}(s,\alpha)=0 \text{ for all } s\} \tag{5.3}$$

and

$$[s_1, s_2 \in E(\mathcal{E})] \Rightarrow [\mathcal{E}(s_1,\alpha) = \mathcal{E}(s_2,\alpha) \text{ for all } \alpha] \,. \tag{5.4}$$

The two quotient sets $\mathcal{O}/K$ and $\mathcal{S}/E$ define surjective morphisms

$$\Phi: \mathcal{O} \to \mathfrak{A} \cong \mathcal{O}/K \tag{5.5}$$

and

$$\Psi: \mathcal{S} \to S(\mathfrak{A}) \cong \mathcal{S}/E \tag{5.6}$$

up to isomorphisms of $\mathfrak{A}$ and S.

Physical evidence is consistent with the postulate of *strong causality:* Knowledge of the expectation values for the subalgebra of observation procedures $\mathcal{O}_{hc}$ associated to a spacelike hyperplane (instant) h is sufficient for complete prediction. In mathematical language, the image $\Phi\mathcal{O}_{hc}$ of the instant canonical algebra $\mathcal{O}_{hc}$ is the whole algebra $\mathfrak{A}$ of observables procedures, so that the expectation value of every element $A \in \mathfrak{A}$ is known. Consider now those automorphisms V_g of $\mathcal{O}$ that are induced by elements $g \in P$ that carry spacelike hyperplanes h into other spacelike hyperplanes h'. These are the time-translation and velocity-boost transformations. Since V_g is an automorphism of $\mathcal{O}$, its restriction to $\mathcal{O}_{hc}$ is an isomorphism onto $\mathcal{O}_{h'c}$.

According to Theorem 3, $\mathcal{O}_{hc}$ is isomorphic to $\mathfrak{A}$. Hence, the morphism Φ together with the automorphism V_g of $\mathcal{O}$ induces an automorphism Q_g of $\mathfrak{A}$ by

$$\Phi V_g \alpha = Q_g \Phi \alpha, \quad \alpha \in \mathcal{O}_{hc} .$$

However, a one-parameter subgroup $\{V_\tau\}$ that carries a subalgebra $\mathcal{O}_{hc}$ onto a family of isomorphic subalgebras $\mathcal{O}_{h+\tau,c}$ associated to hyperplanes $h+\tau$ does not necessarily induce a group $\{Q_\tau\}$ on $\mathfrak{A}$.

A particular subalgebra $\mathcal{O}_{0c}$, generated by the point subalgebras $\mathcal{O}_{xc}$ with $x_0 = t = 0$ and $\vec{x}$ ranging over E^3, is isomorphic to the algebra $C(\omega)$, and we make the convention that the morphisms Φ are completely defined by the requirement that the map

$$\Phi\mathcal{O}_{0c} = \mathfrak{A}$$

be the natural isomorphism

$$\Phi F_{i,\vec{x},0} = f_{i,\vec{x}}, \quad f_{\vec{x},i}(\psi) = \psi_i(\vec{x}) . \tag{5.7}$$

The local structure in $\mathcal{O}$ induces a corresponding local structure in $\mathfrak{A}$ by the definition

$$\Phi\mathcal{O}_x = \mathfrak{A}_x \tag{5.8}$$

and, for open sets R,

$$\Phi\mathcal{O}(R) = \mathfrak{A}(R) . \tag{5.9}$$

Since every subalgebra $\mathcal{O}_{xc}$ belongs to a canonical subalgebra $\mathcal{O}_{hc}$ associated to a spacelike hyperplane, and the latter are isomorphic to $\mathfrak{A}$, there is an isomorphism

$$\mathcal{O}_{xc} \cong \mathfrak{A}_{xc} = \Phi\mathcal{O}_{xc} . \tag{5.10}$$

The algebraic structure of $\mathcal{O}_{hc}$ is carried over onto $\mathfrak{A}$. In particular, if x_1 and x_2 are on a spacelike hyperplane h, i.e., if

$$\mathcal{O}_{x_1c} \cap \mathcal{O}_{x_2c} = \{\alpha I\} , \tag{5.11}$$

then also

$$\mathfrak{A}_{x_1c} \cap \mathfrak{A}_{x_2c} = \{\alpha I\} , \tag{5.12}$$

i.e., canonical point algebras associated to spacelike points are algebraically independent. It was shown in Ref. 2 that this fact implies causal independence for any two subalgebras $\mathfrak{A}(V_1)$ and $\mathfrak{A}(V_2)$, where $V_1 \cap V_2 = \phi$ and both space volumes belong to $t=0$.

We turn to the derivation of the causal shadow structure of $\mathfrak{A}$.

Theorem 4: If x is in the causal shadow of $V \subset h$, then $\mathfrak{A}_{xc} \subset \mathfrak{A}_c(V)$.

Proof: From strong causality,

$$\mathfrak{A}_{xc} \subset \mathfrak{A}_{hc} .$$

By Theorem 3, $\mathfrak{A}_{xc} \cap \mathfrak{A}_{x'c} = \{\alpha I\}$ if x' is in h but not in V. That is, if $x' \in h - V$, then

$$\mathfrak{A}_{xc} \cap \mathfrak{A}_{(h-V),c} = \{\alpha I\} .$$

Therefore $\mathfrak{A}_{xc} \subset \mathfrak{A}_c(V)$.

To show that our construction satisfies the theory of relativity, we must prove local independence. Theorem 4 states that if x is in the causal shadow of the three-dimensional volume V, then $\mathfrak{A}_{xc} \subset \mathfrak{A}_c(V)$.

Let D_V be the double cone with V as basis. According to Theorem 2, we have

$$\mathfrak{A}(D_V) = \left(\bigcup_{x \in D_V} \mathfrak{A}_{xc} \right)_{\text{cl}} ,$$

and hence

$$\mathfrak{A}(D_V) \subset \mathfrak{A}_c(V) .$$

But since $D_V \supset V$, we have also

$$\mathfrak{A}(D_V) \supset \mathfrak{A}_c(V)$$

and therefore

$$\mathfrak{A}(D_V) = \mathfrak{A}_c(V) .$$

For convenience, we will write

$$\mathfrak{A}_1 \asymp \mathfrak{A}_2$$

to express causal independence of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. In Ref. 3 it was proved that for instant spacelike regions V_1, V_2,

$$\mathfrak{A}_c(V_1) \asymp \mathfrak{A}_c(V_2) .$$

From the equality $\mathfrak{A}(D_V) = \mathfrak{A}_c(V)$, and hence, for spacelike double cones D_{V_1} and D_{V_2} with instant bases V_1 and V_2, one has

$$\mathfrak{A}(D_{V_1}) \asymp \mathfrak{A}(D_{V_2}) .$$

For any two open regions R_1 and R_2 that are spacelike, there exist two double cones $D_i \supset R_i$, $i=1, 2$, such that D_1 is spacelike to D_2 and both have bases on the same hyperplane $t=$const. Thus

$$\mathfrak{A}(D_1) \asymp \mathfrak{A}(D_2)$$

and obviously

$$\mathfrak{A}(R_1) \asymp \mathfrak{A}(R_2) .$$

VI. SPACE-TIME AUTOMORPHISMS OF $\mathfrak{A}$

The main theorem[10] that relates the algebraic structure to the usual field theory expressed in terms of partial differential equations asserts that every automorphism of $\mathfrak{A}=C(\omega)$ can be implemented by a unique homeomorphism of ω.

Theorem 5: For every automorphism Q: $\mathfrak{A} \to \mathfrak{A}$, i.e., for

$$f(\psi) \rightarrowtail (Qf)(\psi) ,$$

there exists a unique homeomorphism $\omega \to \omega$ ($\psi \rightarrowtail g\psi$) such that

$$(Qf)\psi = f(g\psi) .$$

The proof is found in Ref. 10.

The causal structure of $\mathfrak{A}$ imposes constraints on the homeomorphisms q_t that implement time-translation automorphisms.

Theorem 6: If $P=(\vec{x}, t)$ is in the causal shadow of the hyperplane segment $(V, 0)$, and χ_V is the characteristic function of V, then

$$(q_t\psi)(\vec{x}) = (q_t\chi_V\psi)(\vec{x}) .$$

That is, the homeomorphism associates the same function $(q_t\psi)(\vec{x})$ to all functions ψ that agree within V.

Proof: By Theorem 4, $f_{i,P}(\psi) \equiv (Q_t f_{i,\vec{x}})(\psi)$ is in $\mathfrak{A}_c(V)$, so that

$$(f_{i,P})(\psi) = f_{i,P}(\chi_V\psi) .$$

From Theorem 5, we have

$$f_{i,P}(\chi_V\psi) = f_{i,\vec{x}}(q_t\chi_V\psi) = (q_t\chi_V\psi)(\vec{x}) .$$

On the other hand, by definition,

$$f_{i,\vec{x}}(q_t\psi) = (q_t\psi)(\vec{x}) .$$

A sufficient implementation of this homeomorphism is obtained by requiring that $(q_t\psi)(\vec{x}) \equiv \psi(\vec{x}, t)$ be the solution of a hyperbolic partial differential equation with the light cones as characteristics. However, it is not necessary that these homeomorphisms be diffeomorphisms. The mathematical theory of classical relativistic fields is properly a theory of a class of homeomorphisms, not of partial differential equations.

VII. NEWTON'S SECOND LAW

By assumption, the N linearly independent functionals

$$F_{i,x}(\psi) = \psi_i(x) \quad (i=1, \ldots, N)$$

in $\mathcal{O}_{cx}$ span a linear representation space of the Lorentz group L. In addition, the canonical algebra

$$\mathcal{O}_{cx} = (\{F_{i,x}, i=1, \ldots, N\})_{cl}$$

of point observation procedures generated by the functionals $F_{i,x}$, being a subalgebra of $\mathcal{O}_x$, has a group of acceleration automorphisms

$$V_g: \ \mathcal{O}_{cx} \to \mathcal{O}_{cx}, \quad g \in M_x$$

induced by the group M_x of meromorphisms of Minkowski space. We wish to prove that these automorphisms are trivial, i.e., that the acceleration group acts as a unit operator on $\mathcal{O}_{cx}$.

By virtue of the assumption made in Sec. III, the linear manifold $\mathfrak{M}_x$ generated by the generating canonical point-observation procedures $\{F_{i,x}\}$ ($i=1, \ldots, N$) is stable under the automorphisms V_g. It suffices to show that the linear representation of the acceleration group on the manifold $\mathfrak{M}_x$ is trivial.

As noted in Sec. III, $\mathfrak{M}_x$ is a representation space for the Lorentz group L. Since, according to the assumption of Sec. III, each irreducible subspace $\mathfrak{M}_x(\gamma)$ of $\mathfrak{M}_x$ is identified with a particular collection of blueprints $b^{(\gamma)}$, the acceleration group acting through $(b^{(\gamma)}, \{P_n\}) \rightarrowtail (b^{(\gamma)}, \{gP_n\})$ ($g \in M_x$) leaves each irreducible subspace $\mathfrak{M}_x(\gamma)$ stable. The mathematical problem is then: Can an irreducible finite-dimensional representation of the Lorentz group be extended nontrivially to a representation of the acceleration group M_x?

The acceleration group M_x defined by Eq. (4.9) cannot be faithfully represented by finite matrices. This is clear, since M_x contains elements that depend on an arbitrarily large number of independent parameters, whereas in an n-dimensional matrix representation, the number of independent parameters is at most n^2. Hence, any finite-dimensional representation of M_x is a proper homomorphism.

A homomorphism maps an invariant subgroup onto the unit, and we must determine the invariant subgroups of M_x. Such invariant subgroups are the groups generated by the subgroups $g_{i,n}$, $n > m$, where m is an integer not less than 1. It can be shown that these are *all* the invariant subgroups.

The group to be represented faithfully on the irreducible representation space of L is then the quotient group M_x/G_m that contains L as a subgroup. From the Lie algebra (4.10) one sees that a subgroup H_m generated by the generators $\{a_{i,n}\}$ $(0<n<m)$ is an invariant subgroup of M_x/G_m. Hence, this group is a semidirect product

$$M_x/G_m = H_m \otimes L \ .$$

Let V_H be the representation subduced on H_m by the representation V of $H_M \otimes L$. V is, of course, an irreducible representation, since its restriction to L is irreducible. Therefore[11] V_H is completely reducible and all conjugate representations must appear in its decomposition. Now the quotient group M_x/G_m is a Lie group, the algebra of which is given by (4.10) with $a_{i,n}=0$ for $n>m$. It is seen from this equation that there are infinitely many distinct elements of the form $r^{-1}ur$ where $r\in L$ and $u\in H_m$.[12] Hence, either the representation space is infinite-dimensional, contrary to our assumption, or the irreducible representation of H_m is trivial.

The same argument holds for every invariant subgroup except that generated by all subgroups G_n, $n>0$. Hence, the acceleration group is mapped onto the unit.

This is the generalized form of Newton's second law: The observation procedures $\{\mathcal{O}_{xc}\}$ that have full predictive power are acceleration invariant.

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[1]Recently, efforts to make Lagrangian theories consistent have had encouraging results [W. Zimmermann, in *Lectures in Elementary Particles and Quantum Field Theory,* edited by S. Deser *et al.* (M.I.T. Press, Cambridge, Mass., 1971)]. The correct form of renormalized field equations can be guessed by arguing as if the canonical commutation relations were correct.

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