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Conformal Energy-Momentum Tensor in Riemannian Space-Time*

Leonard Parker
University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201†
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We consider the scalar field with quartic self-interaction in Riemannian space-time. Identities are proved which connect the modified energy-momentum tensors of Callan, Coleman, and Jackiw in different conformally related space-times. We consider the quantized scalar field in a conformally flat metric, and show that our identities relate the matrix elements of the modified energy-momentum tensor to corresponding matrix elements in Minkowski space. We show further that when the mass can be neglected in the conformal wave equation there is no gravitationally induced particle creation in conformally flat space-times, thus generalizing a result proved earlier in the free-field case. The influence of additional fields and interactions on that result is briefly discussed.

I. INTRODUCTION

A modification of the conventional energy-momentum tensor of the scalar field with quartic self-interaction has been proposed by Callan, Coleman, and Jackiw.[1] Their modified energy-momentum tensor has interesting properties at both the classical and quantum levels. It evidently has, at least in Minkowski space-time, finite matrix elements, in the sense that they possess a finite limit in every order of renormalized perturbation theory, as the cutoff approaches infinity. Furthermore, the currents associated with conformal coordinate transformations are simply expressed in terms of the modified energy-momentum tensor, so that it is sometimes referred to as the conformal energy-momentum or stress tensor. Callan, Coleman, and Jackiw also showed how to alter the general relativistic action functional to make the fully covariant form of the conformal stress tensor the source of the gravitational field. A similar gravitational theory has also been considered by Chernikov and Tagirov.[2]

The physical distinction between the conventional and conformal stress tensors is most evident in strong gravitational fields. For example, the consequences of the theories involving the two tensors are quite different near the cosmological singularity in an isotropically expanding universe.[3] It is therefore of interest to consider the properties of the conformal energy-momentum tensor in Riemannian space-time. This paper will be concerned with those properties, both for a classical and a quantized scalar field.[4]

The basis of our treatment will be a number of identities involving fields and energy-momentum tensors in metrics which are related by conformal transformation.[5] Those identities are proved in Sec. II. Section III is concerned with the quantized scalar field with quartic self-interaction in conformally flat space-time. The class of conformally flat metrics includes the fundamentally significant Robertson-Walker metrics.[6] The canonically quantized scalar field in the curved space-time is related by an unquantized conformal factor to a corresponding canonically quantized scalar field in Minkowski space. The identities proved earlier relate the matrix elements of the modified energy-momentum tensor in conformally flat space-time to corresponding matrix elements in Minkowski space. Finally, we show that for the massless scalar field obeying the conformal wave equation with quartic self-interaction there is no gravitationally induced particle creation in conformally flat metrics.

II. MODIFIED STRESS TENSOR IN CONFORMALLY RELATED METRICS

By replacing ordinary derivatives ∂_μ by covariant derivatives ∇_μ in the modified energy-momentum tensor, one obtains the following tensor[7]:

$$\Theta_{\mu\nu}(g_{\alpha\beta},\phi,\lambda_0,\mu_0)=\partial_\mu\phi\partial_\nu\phi-\tfrac{1}{2}g_{\mu\nu}g^{\lambda\sigma}\partial_\lambda\phi\partial_\sigma\phi-\tfrac{1}{6}\nabla_\mu\partial_\nu(\phi^2)+\tfrac{1}{6}g_{\mu\nu}g^{\lambda\sigma}\nabla_\lambda\partial_\sigma(\phi^2)+\tfrac{1}{2}g_{\mu\nu}\mu_0{}^2\phi^2+g_{\mu\nu}\lambda_0\phi^4. \tag{1}$$

Later we will consider a further generalization of the modified energy-momentum tensor. Since various metrics and fields will be discussed, we have found it convenient to display the dependence of $\Theta_{\mu\nu}$ on those quantities, as well as on λ_0 and μ_0. When the dependence on λ_0 or μ_0 is not included, it will mean that the parameter not appearing is equal to zero. Our present considerations are for a classical scalar field. The quantized field will be discussed in the next section.

In this paper, we are mainly concerned with conformally related metrics. A conformal transformation of the metric is any transformation of the form

$$g_{\alpha\beta}\to\tilde{g}_{\alpha\beta}=\Omega^{-2}g_{\alpha\beta}, \tag{2}$$

where Ω is a function of the space-time coordinates. The relation between conformal coordinate transformations and conformal metric transformations is briefly discussed in Appendix A.

An identity proved by Synge,[8] which relates the Einstein tensors in conformally related metrics, can be written in terms of the modified stress tensor as follows:

$$G_{\mu\nu}(\phi^2g_{\alpha\beta})=G_{\mu\nu}(g_{\alpha\beta})-6\phi^{-2}\Theta_{\mu\nu}(g_{\alpha\beta},\phi), \tag{3}$$

where $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor, formed using the metric indicated. Writing

$$\phi=\Omega^{-1}\tilde{\phi} \tag{4}$$

and applying the same identity as in (3), one obtains

$$G_{\mu\nu}(\phi^2g_{\alpha\beta})=G_{\mu\nu}(\tilde{g}_{\alpha\beta})-6\tilde{\phi}^{-2}\Theta_{\mu\nu}(\tilde{g}_{\alpha\beta},\tilde{\phi}), \tag{5}$$

where $\tilde{g}_{\alpha\beta}$ is given by Eq. (2). Hence

$$G_{\mu\nu}(\tilde{g}_{\alpha\beta})-6\tilde{\phi}^{-2}\Theta_{\mu\nu}(\tilde{g}_{\alpha\beta},\tilde{\phi})=G_{\mu\nu}(g_{\alpha\beta})-6\phi^{-2}\Theta_{\mu\nu}(g_{\alpha\beta},\phi). \tag{6}$$

This symmetrical identity has an interesting interpretation, which will be discussed later.

By replacing $G_{\mu\nu}(\Omega^{-2}g_{\alpha\beta})$ in (6) by $G_{\mu\nu}(g_{\alpha\beta})-6\Omega^2\Theta_{\mu\nu}(g_{\alpha\beta},\Omega^{-1})$, one obtains the further identity

$$\tilde{\phi}^{-2}\Theta_{\mu\nu}(\tilde{g}_{\alpha\beta},\tilde{\phi})=\phi^{-2}\Theta_{\mu\nu}(g_{\alpha\beta},\phi)-\Omega^2\Theta_{\mu\nu}(g_{\alpha\beta},\Omega^{-1}). \tag{7}$$

Putting $\phi=1$ (and replacing $\tilde{\phi}$ by ϕ) yields

$$\phi^{-2}\Theta_{\mu\nu}(\phi^{-2}g_{\alpha\beta},\phi)=-\phi^2\Theta_{\mu\nu}(g_{\alpha\beta},\phi^{-1}). \tag{8}$$

We emphasize that Eqs. (3)–(8) are identities, independent of what equations of motion ϕ and $g_{\alpha\beta}$ may satisfy.

As noted by Synge,[8] one can use Eq. (3) to transform one gravitational problem into another conformally related problem. Thus, if $g_{\alpha\beta}$ describes a gravitational field for some distribution of matter, then the metric $\phi^2g_{\alpha\beta}$ is the solution of Einstein's field equations with an energy-momentum tensor given by the right-hand side of (3) multiplied by $-(8\pi G)^{-1}$. Given ϕ, one must confirm that the energy-momentum tensor obtained has reasonable properties. The identification of the second term on the right of (3) with $\Theta_{\mu\nu}$ should be useful in that respect. For example, if one requires that ϕ satisfy

$$(\nabla^\mu\nabla_\mu+\tfrac{1}{6}R)\phi=0, \tag{9}$$

then

$$g^{\mu\nu}\Theta_{\mu\nu}(g_{\alpha\beta},\phi)=-\tfrac{1}{6}R\phi^2, \tag{10}$$

so that the right-hand side of (3) has vanishing trace. Of course, it must still be checked that the solution has physically reasonable density and pressure.

As Callan, Coleman, and Jackiw show,[1] the gravitational equations incorporating their modified stress tensor are

$$G_{\mu\nu}(g_{\alpha\beta})=-\left(\frac{1}{8\pi G}-\tfrac{1}{6}\phi^2\right)^{-1}\Theta_{\mu\nu}(g_{\alpha\beta},\phi,\lambda_0,\mu_0). \tag{11}$$

This result can be written in the form

$$G_{\mu\nu}=-8\pi G\Lambda_{\mu\nu}, \tag{12}$$

where

$$\Lambda_{\mu\nu}=\Theta_{\mu\nu}-\tfrac{1}{6}\phi^2G_{\mu\nu}. \tag{13}$$

For Eq. (12) to be consistent one must require that

$$(\nabla^\mu\nabla_\mu+\mu_0{}^2+\tfrac{1}{6}R)\phi+4\lambda_0\phi^3=0. \tag{14}$$

Then

$$\nabla_\nu\Lambda^{\mu\nu}=0 \tag{15}$$

and

$$\Lambda_\mu{}^\mu=\mu_0{}^2\phi^2. \tag{16}$$

The energy-momentum tensor $\Lambda_{\mu\nu}$ is the generalization of the modified stress tensor to Riemannian space-time, acting as the source of the gravitational field. It was used (with $\lambda_0=0$) by Chernikov and Tagirov in Ref. 2. Equation (6) can now be written in the very simple form

$$\Lambda_{\mu\nu}(\tilde{g}_{\alpha\beta}, \tilde{\phi}) = \Omega^2 \Lambda_{\mu\nu}(g_{\alpha\beta}, \phi), \tag{17}$$

where $\Lambda_{\mu\nu}$ is defined in terms of the $\Theta_{\mu\nu}$ with the corresponding arguments by means of (13), and $\tilde{g}_{\alpha\beta}$ and $\tilde{\phi}$ are given by (2) and (4), respectively. The identity (17) holds regardless of the equations satisfied by $g_{\alpha\beta}$, ϕ, and $\tilde{\phi}$.

Since

$$\Lambda_{\mu\nu}(g_{\alpha\beta}, \phi, \lambda_0) = \Lambda_{\mu\nu}(g_{\alpha\beta}, \phi) + \lambda_0 g_{\mu\nu}\phi^4, \tag{18}$$

one finds that (17) can be generalized to

$$\Lambda_{\mu\nu}(\tilde{g}_{\alpha\beta}, \tilde{\phi}, \lambda_0) = \Omega^2 \Lambda_{\mu\nu}(g_{\alpha\beta}, \phi, \lambda_0). \tag{19}$$

If one sets $\tilde{\phi} = 1$ (i.e., $\Omega = \phi^{-1}$), and uses (13) and (18), then the last identity can be written in the form

$$G_{\mu\nu}(\tilde{g}_{\alpha\beta}) - 6\lambda_0 \tilde{g}_{\mu\nu} = -6\phi^{-2}\Lambda_{\mu\nu}(g_{\alpha\beta}, \phi, \lambda_0). \tag{20}$$

One can relate solutions of Einstein's equations with a cosmological constant by means of (20), in analogy to the use of Eq. (3) discussed above. When ϕ satisfies (14) with vanishing μ_0, then the trace of the right-hand side of (20) vanishes.

When μ_0 is nonzero,

$$\Lambda_{\mu\nu}(g_{\alpha\beta}, \phi, \lambda_0 \mu_0) = \Lambda_{\mu\nu}(g_{\alpha\beta}, \phi, \lambda_0) + \tfrac{1}{2}\mu_0{}^2 g_{\mu\nu}\phi^2, \tag{21}$$

and one finds that

$$\Omega^{-2}\Lambda_{\mu\nu}(\tilde{g}_{\alpha\beta}, \tilde{\phi}, \lambda_0, \Omega\mu_0) = \Lambda_{\mu\nu}(g_{\alpha\beta}, \phi, \lambda_0, \mu_0). \tag{22}$$

Finally, we give an alternate approach based on an action functional. Equation (19) and a related identity can be obtained by considering the action functional

$$S = \tfrac{1}{2}\int_V d^4x\sqrt{-g}\,(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{6}R\phi^2 - 2\lambda_0\phi^4), \tag{23}$$

where V is an arbitrary four-volume. Let

$$\tilde{S} = \tfrac{1}{2}\int_V d^4x\sqrt{-\tilde{g}}\,(\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} - \tfrac{1}{6}\tilde{R}\tilde{\phi}^2 - 2\lambda_0\tilde{\phi}^4), \tag{24}$$

where $\tilde{g}_{\mu\nu}$ and $\tilde{\phi}$ are given by (2) and (4), respectively, and $\tilde{R} = R(\tilde{g}_{\alpha\beta})$. Using the identity[8]

$$\tilde{R} = \Omega^2[R - 6g^{\mu\nu}\nabla_\mu\nabla_\nu \ln\Omega + 6g^{\mu\nu}(\partial_\mu \ln\Omega)(\partial_\nu \ln\Omega)], \tag{25}$$

one finds that

$$\tilde{S} = S - \tfrac{1}{2}\int_{\partial V} d^3x\sqrt{-g}\,\eta_\mu g^{\mu\nu}(\partial_\nu \ln\Omega)\phi^2, \tag{26}$$

where η_μ is the outward normal to the three-dimensional hypersurface ∂V.

Vary $g_{\mu\nu}$ such that $\delta g_{\mu\nu}$ and $\partial_\lambda \delta g^{\mu\nu}$ vanish on ∂V. Then

$$\delta\tilde{g}^{\mu\nu} = \Omega^2 \delta g^{\mu\nu}, \tag{27}$$

and it follows from (26) that

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{\delta\tilde{S}}{\delta\tilde{g}^{\mu\nu}}\Omega^2. \tag{28}$$

Calculation shows that

$$\frac{\delta S}{\delta g^{\mu\nu}} = \tfrac{1}{2}\Lambda_{\mu\nu}\sqrt{-g}\,, \tag{29}$$

so that (28) is equivalent to (19), when one notes that

$$(g/\tilde{g})^{1/2} = \Omega^4. \tag{30}$$

Similarly, one finds that

$$\frac{\delta S}{\delta\phi} = \frac{\delta\tilde{S}}{\delta\tilde{\phi}}\Omega. \tag{31}$$

Using

$$\frac{\delta S}{\delta\phi} = -(\nabla^\mu\nabla_\mu\phi + \tfrac{1}{6}R\phi + 4\lambda_0\phi^3)\sqrt{-g}\,, \tag{32}$$

Eq. (31) yields the known identity[9]

$$\tilde{\nabla}^\mu\tilde{\nabla}_\mu\tilde{\phi} + \tfrac{1}{6}\tilde{R}\tilde{\phi} + 4\lambda_0\tilde{\phi}^3 = \Omega^3(\nabla^\mu\nabla_\mu\phi + \tfrac{1}{6}R\phi + 4\lambda_0\phi^3). \tag{33}$$

III. QUANTIZED SCALAR FIELD IN CONFORMALLY FLAT SPACE-TIME

We now suppose that the metric under consideration is conformally flat. Then the original scalar field ϕ is conformally related to a corresponding scalar field $\tilde{\phi}$ in Minkowski space. We prove that canonical quantization of $\tilde{\phi}$ in the Minkowski space is equivalent to canonical quantization of ϕ in the curved space-time. The metric and conformal factor are not quantized. It follows that the quantized energy-momentum tensor $\Lambda_{\mu\nu}$ is simply related by a power of the conformal factor to the corresponding energy-momentum tensor in Minkowski space. Therefore, the corresponding matrix elements are related by an unquantized factor, so that one would expect properties of the matrix elements in Minkowski space to be carried over to the corresponding matrix elements in the conformally flat space-time. The dynamical consistency and general covariance of the present quantization procedure is demonstrated in Appendix B. We are mainly concerned here with the algebraic structure of the quantized theory, rather than with the problem of constructing the Hilbert space of state vectors. Finally, we give an application of the present formalism, making use of $\tilde{\phi}$ as an asymptotic field.

The theory given in Ref. 1 is based on the action

$$S=\frac{1}{2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{6}R\phi^2-\mu_0{}^2\phi^2-2\lambda_0\phi^4+\frac{1}{8\pi G}R\right). \tag{34}$$

The equation of motion for the field ϕ is given by Eq. (14), namely,

$$\nabla^\mu\nabla_\mu\phi+\frac{1}{6}R\phi+\mu_0{}^2\phi+4\lambda_0\phi^3=0. \tag{35}$$

By definition, conformal flatness of the metric implies that there exists a coordinate system and a function Ω such that $\tilde{g}_{\mu\nu}$ given by Eq. (2) takes the Minkowskian form $\eta_{\mu\nu}$, i.e.,

$$\begin{aligned}g_{\mu\nu}(x)&=\Omega^2(x)\tilde{g}_{\mu\nu}(x)\\&=\Omega^2(x)\eta_{\mu\nu}.\end{aligned} \tag{36}$$

The field ϕ will be quantized, while the metric $g_{\mu\nu}$ and conformal factor Ω are unquantized. The source of the gravitational field, in this semiclassical approximation, can be taken to be a suitable expectation value of $\Lambda_{\mu\nu}$. Then our assumption that the metric is conformally flat implies certain limitations on the state vectors with respect to which the expectation value of $\Lambda_{\mu\nu}$ is taken. Alternatively, we may regard $g_{\mu\nu}$ as a prescribed metric, produced by classical sources, with respect to which the gravitational influence of the quantized scalar field is negligible.

Let $\tilde{\phi}$ be defined in accordance with Eq. (4) as

$$\tilde{\phi}=\Omega\phi. \tag{37}$$

It follows from Eqs. (33), (35), and (36) that

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\tilde{\phi}+4\lambda_0\tilde{\phi}^3+(\Omega\mu_0)^2\tilde{\phi}=0. \tag{38}$$

Thus, $\tilde{\phi}$ obeys the special relativistic equation of motion, but with a space-time-dependent "mass." Thus, with regard to $\tilde{\phi}$, Ω plays the role of an external potential acting in Minkowski space. The Lagrangian corresponding to Eq. (38) is

$$\tilde{\mathfrak{L}}=\frac{1}{2}[\eta^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi}-2\lambda_0\tilde{\phi}^4-(\Omega\mu_0)^2\tilde{\phi}^2]. \tag{39}$$

Therefore the momentum conjugate to $\tilde{\phi}$ is

$$\begin{aligned}\tilde{\pi}&=\frac{\partial\tilde{\mathfrak{L}}}{\partial(\partial_0\tilde{\phi})}\\&=\partial_0\tilde{\phi}.\end{aligned} \tag{40}$$

We show that the canonical commutation relations on $\tilde{\phi}$ and $\tilde{\pi}$ are equivalent to those on ϕ and π.

The momentum conjugate to ϕ follows from the action (34), and is

$$\pi=\sqrt{-g}\,g^{0\mu}\partial_\mu\phi. \tag{41}$$

Equation (36) implies that

$$\sqrt{-g}=\Omega^4 \tag{42}$$

and

$$g^{\mu\nu}=\Omega^{-2}\eta^{\mu\nu}, \tag{43}$$

so that

$$\pi=\Omega^2\partial_0\phi. \tag{44}$$

Then (37) and (40) yield

$$\pi=\Omega\tilde{\pi}-\tilde{\phi}\partial_0\Omega. \tag{45}$$

Now suppose that we are given

$$[\tilde{\phi}(x),\tilde{\pi}(x')]_t=i\delta(\vec{x}-\vec{x}') \tag{46}$$

and

$$\begin{aligned}[\tilde{\phi}(x),\tilde{\phi}(x')]_t&=[\tilde{\pi}(x),\tilde{\pi}(x')]_t\\&=0.\end{aligned} \tag{47}$$

Then, as a consequence of (45) and (47), we find

$$[\phi(x),\pi(x')]_t=\Omega^{-1}(\vec{x},t)\Omega(\vec{x}',t)[\tilde{\phi}(x),\tilde{\pi}(x')]_t,$$

and from (46),

$$[\phi(x),\pi(x')]_t=i\delta(\vec{x}-\vec{x}'). \tag{48}$$

Similarly,

$$\begin{aligned}[\pi(x),\pi(x')]_t=&-\Omega(\vec{x},t)\partial_0\Omega(\vec{x}',t)[\tilde{\pi}(x),\tilde{\phi}(x')]_t\\&-\Omega(\vec{x}',t)\partial_0\Omega(\vec{x},t)[\tilde{\phi}(x),\tilde{\pi}(x')]_t,\end{aligned}$$

which gives

$$[\pi(x),\pi(x')]_t=0. \tag{49}$$

Finally, it is obvious that

$$[\phi(x),\phi(x')]_t=0. \tag{50}$$

Conversely, it is straightforward to show that the canonical commutators of ϕ and π imply those of $\tilde{\phi}$ and $\tilde{\pi}$, so that the equivalence has been demonstrated.

It is clear that the commutation relations of $\tilde{\phi}$ and $\tilde{\pi}$ yield the Minkowski space equation of motion (38) as a consequence of the Heisenberg equation involving $\tilde{\pi}$ and the Hamiltonian obtained from $\tilde{\mathfrak{L}}$. Dynamical consistency in the curved space-time is not immediately evident, so that we demonstrate it in Appendix B. There we show that the canonical commutators of ϕ and π (which are equivalent to those of $\tilde{\phi}$ and $\tilde{\pi}$), and the Heisenberg equation involving π and the Hamiltonian obtained from the action (34), yield the field equation (35) for ϕ. We also point out in Appendix B that the canonical commutators are consistently propagated by the equation of motion, and that the canonical quantization scheme in the curved space-time is generally covariant.

For our present purposes, the mathematical adjunct field $\tilde{\phi}$ need only be defined over the class of coordinate systems which are related by Lorentz transformation to a particular set of coordinates in which Eq. (36) holds. The fields $\tilde{\phi}$ and Ω will be Lorentz scalars. Then, under Lorentz transformation applied to $g_{\mu\nu}$ in Eq. (36), we have

$$g'_{\mu\nu}(x')=\Omega^2(x)\eta_{\mu\nu} \\ =\Omega'^2(x')\eta_{\mu\nu}, \tag{51}$$

and, since $\phi(x)$ is a scalar under arbitrary coordinate transformation, and in particular under Lorentz transformation, it follows that

$$\tilde{\phi}'(x')=\Omega'(x')\phi'(x') \\ =\Omega(x)\phi(x) \\ =\tilde{\phi}(x). \tag{52}$$

Thus, the generally covariant canonically quantized theory involving the physical field ϕ induces a Lorentz-covariant canonically quantized theory in the Minkowski space corresponding to a given conformal factor Ω. We can, for our present purposes, regard $\tilde{\phi}$ as a mathematical adjunct field defined over a limited set of coordinates, which is nevertheless useful for deducing certain properties and consequences of the physical theory involving ϕ.

Returning to the energy-momentum tensor, we note that Eq. (22) continues to hold when ϕ and $\tilde{\phi}$ are quantized in the manner described above, since Ω is an ordinary function, and the derivation of the identity does not require that noncommuting quantities be interchanged. In this case, $\tilde{g}_{\alpha\beta}$ is the Minkowski metric $\eta_{\alpha\beta}$, so that (22) becomes

$$\Lambda_{\mu\nu}(g_{\alpha\beta},\phi,\lambda_0,\mu_0)=\Omega^{-2}\Theta_{\mu\nu}(\eta_{\alpha\beta},\tilde{\phi},\lambda_0,\Omega\mu_0). \tag{53}$$

Regardless of how the Hilbert space of physical state vectors is constructed, it follows that matrix elements of $\Lambda_{\mu\nu}$ in the curved space-time are related to those of $\Theta_{\mu\nu}$ in Minkowski space by means of the unquantized factor Ω^{-2}, as in Eq. (53). Thus, properties of $\Theta_{\mu\nu}$ or its matrix elements in Minkowski space which follow purely from the field algebra and equation of motion of $\tilde{\phi}$ must yield corresponding properties of the physical energy-momentum tensor $\Lambda_{\mu\nu}$ or its matrix elements via Eq. (53). This seems to indicate that problems involving regularization or renormalization of $\Lambda_{\mu\nu}$ can be treated by applying special relativistic methods to $\Theta_{\mu\nu}$, as in Ref. 1. Some limited assumptions concerning the construction of the physical Hilbert space of state vectors and behavior of Ω may also be necessary, in addition to the algebraic structure which has been discussed up to now.

As an application of the foregoing formalism, we consider the question of the production of particles or quanta of the self-interacting scalar field by a conformally flat gravitational field, when the rest mass μ_0 vanishes, or is assumed to be negligible (as one would expect at sufficiently high energy). We will find that particle production by the gravitational field is absent under the above conditions. Our derivation uses the device of making $\Omega(x)$ statically bounded, or asymptotically static. That is, we assume that Ω is a smooth function of x^0 which is constant for $x^0<-T$ and $x^0>T$, where T can be arbitrarily large, and Ω need not be equal to the same constant initially and finally. Our result concerning the absence of gravitationally induced particle creation holds for otherwise arbitrary Ω, and is independent of the precise manner in which Ω is statically bounded (it can even approach the constant asymptotic values rapidly). We therefore conclude that the result does not depend on the static bounding, and is valid for arbitrary conformally flat metrics. The use of a convenient set of coordinate systems, which exist in a conformally flat space-time, is not a violation of the principle of general covariance. Admittedly, a manifestly covariant derivation, which does not make use of relations that hold only in a subset of coordinate systems, would be desirable, but the present derivation should nevertheless be valid.

When Ω is statically bounded, it follows from Eq. (37) that for $x^0<-T$ and $x^0>T$, $\tilde{\phi}$ is equal to the Heisenberg field ϕ to within a constant factor. Therefore, $\tilde{\phi}$ plays the role of an in-field when $x^0<-T$ and an out-field when $x^0>T$. During these periods $\tilde{\phi}$ has physical significance, and it makes sense to use it to construct the physical Hilbert space of state vectors. Since $\tilde{\phi}$ is a Lorentz-covariant canonically quantized field in Minkowski space, it is plausible to construct the Hilbert space for the in- and out-fields exactly as in special relativity.[10] When μ_0 vanishes or can be neglected in Eq. (38), the conformal factor Ω does not appear in the resulting equation,

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\tilde{\phi}+4\lambda_0\tilde{\phi}^3=0. \tag{54}$$

Thus, $\tilde{\phi}$ satisfies the special relativistic equation of motion with no external potential for all x^0, so that the gravitational field has no influence on $\tilde{\phi}$. Since $\tilde{\phi}$ is the in-field as $x^0\to-\infty$ and the out-field as $x^0\to+\infty$, it follows that there can be no gravitationally induced particle creation or instability of the vacuum and one-particle states when μ_0 can be neglected.

The present result generalizes an earlier result concerning the conformal scalar field with no self-interaction in an isotropic, homogeneous universe. It has been shown[11,12] that if the scalar field satisfies the conformally-invariant, massless, free-field equation [i.e., Eq. (35) with $\mu_0 = \lambda_0 = 0$], then in a Robertson-Walker metric there will be no particle creation induced by the expansion; and that when μ_0 has a value small with respect to the Planck mass (10^{-5} gm), then, even near the cosmological singularity, the energy-density of the created particles will be small with respect to the energy-density of the matter which must be present to act as the source of the Robertson-Walker expansion in Einstein's equation. According to the present considerations, those results should continue to hold in the Robertson-Walker metrics, when the field obeys Eq. (35) with nonvanishing λ_0, and with μ_0 small compared to the Planck mass,[13] as for the known elementary-particle rest masses. The Robertson-Walker metrics are known to be conformally flat.[6]

It seems evident that, by means of the same method of reasoning, one can reach the same conclusion regarding the absence of gravitationally induced particle creation for a theory involving several kinds of interacting particles of various spins, provided the following conditions are met: (1) The rest masses of the particles can be neglected. (2) The free-particle fields obey equations of motion invariant under conformal metric transformation, when the rest masses are neglected. (3) The interactions are invariant under conformal metric transformation, at least in the same approximation that the rest masses are neglected. (4) The metric is conformally flat. Condition (3) is analogous to the requirement that

$$\sqrt{-g}\,\lambda_0 \phi^4 = \sqrt{-\tilde{g}}\,\lambda_0 \tilde{\phi}^4 . \tag{55}$$

More precisely, suppose that the generalization of the special relativistic interaction between two fields $\psi^{(1)}$ and $\psi^{(2)}$ is the interaction term

$$F(g_{\alpha\beta}, \psi^{(1)}, \psi^{(2)}, \nabla_\mu \psi^{(1)}, \nabla_\nu \psi^{(2)}),$$

and that $\psi^{(1)}$ and $\psi^{(2)}$ must be transformed into $\tilde{\psi}^{(1)}$ and $\tilde{\psi}^{(2)}$ to make the massless free-field equations form invariant under conformal metric transformation. Then the interaction is invariant under conformal metric transformation if

$$\sqrt{-g}\,F(g_{\alpha\beta}, \psi^{(1)}, \psi^{(2)}, \nabla_\mu \psi^{(1)}, \nabla_\nu \psi^{(2)}) = \sqrt{-\tilde{g}}\,F(\tilde{g}_{\alpha\beta}, \tilde{\psi}^{(1)}, \tilde{\psi}^{(2)}, \tilde{\nabla}_\mu \tilde{\psi}^{(1)}, \tilde{\nabla}_\nu \tilde{\psi}^{(2)}) . \tag{56}$$

The invariance given by Eq. (56) should be related to invariance under conformal and scale transformations of the coordinates, which has been suggested as an approximate symmetry of elementary-particle interactions.

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APPENDIX A: CONFORMAL COORDINATE AND METRIC TRANSFORMATIONS

An infinitesimal coordinate scale transformation is given by

$$x'^{\mu} = x^{\mu} + \epsilon x^{\mu} , \tag{A1}$$

and an infinitesimal coordinate conformal transformation by

$$x'^{\mu} = x^{\mu} + 2\epsilon_\nu x^\mu x^\nu - \epsilon_\nu g^{\mu\nu} g_{\lambda\sigma} x^\lambda x^\sigma . \tag{A2}$$

Under coordinate transformations the metric satisfies the equation

$$g'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g^{\mu\nu} . \tag{A3}$$

For the scale transformation (A1), one finds that

$$g'^{\alpha\beta} = (1 + 2\epsilon) g^{\alpha\beta} , \tag{A4}$$

while for the conformal transformation (A2), one obtains

$$g'^{\alpha\beta} = (1 + 4\epsilon_\nu x^\nu) g^{\alpha\beta} . \tag{A5}$$

The infinitesimal scale and conformal coordinate transformations leave the velocity of light unchanged, since the condition $ds^2 = 0$ is not affected by any multiplicative factor on $g^{\alpha\beta}$.

A conformal metric transformation leaves the coordinates unchanged, but replaces the metric $g_{\mu\nu}$ by $\tilde{g}_{\mu\nu}$, where

$$\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu} , \tag{A6}$$

and Ω is a function of the coordinates. Thus, the effect of scale and conformal coordinate transformations on the metric is the same as the particular conformal metric transformations with $\Omega = 1 - \epsilon$ and $\Omega = 1 - 2\epsilon_\nu x^\nu$, respectively.

APPENDIX B: DYNAMICAL CONSISTENCY AND COVARIANCE

We show that the Heisenberg equation of motion is equivalent to

$$\nabla^\mu \nabla_\mu + \tfrac{1}{6} R\phi + \mu_0{}^2\phi + 4\lambda_0\phi^3 = 0 . \tag{B1}$$

Our derivation makes use of the following assumptions, which are satisfied by the Lagrangian density $\mathcal{L}$ appearing in the action (34) (note that $\mathcal{L}$ includes the $\sqrt{-g}$ factor): namely, that

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}, \tag{B2}$$

that the equation of motion is

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0, \tag{B3}$$

and that ϕ and π obey the canonical commutation relations, Eqs. (48)–(50), which we have already shown are consistent with the canonical commutators of $\tilde{\phi}$ and $\tilde{\pi}$.

We have

$$-[\pi(x), \partial_i'\phi(x')]_t = \partial_i'[\pi(x), \phi(x')]_t = i\partial_i'\delta(\vec{x} - \vec{x}'), \tag{B4}$$

for $i = 1, 2, 3$. From Eqs. (48)–(50) and Eq. (B4) it follows that if F can be written as a power series in ϕ, $\partial_i\phi$, and π, then

$$-[\pi(x), F(x')]_t = i\delta(\vec{x} - \vec{x}')\left[\left(\frac{\partial F}{\partial\phi}\right) - \partial_i\left(\frac{\partial F}{\partial(\partial_i\phi)}\right)\right]_t, \tag{B5}$$

where summation over i is from 1 to 3. The Hamiltonian is given by

$$H = \int d^3x(\pi\partial_0\phi - \mathcal{L}). \tag{B6}$$

The Heisenberg equation of motion is

$$[\pi(x), H(t)] = i\partial_0\pi(x). \tag{B7}$$

We show that (B7) is equivalent to (B3),

$$\begin{aligned}[\pi(x), H(t)] &= \int d^3x'\left[\pi(x), \frac{1}{[-g(x')]^{1/2}}\pi^2(x') - \mathcal{L}(\phi(x'), \partial_i'\phi(x'), \pi(x'))\right]_t \\ &= i\int d^3x'\delta(\vec{x} - \vec{x}')\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\right)\right]_t \\ &= i\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_i\left(\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\right)\right]_x.\end{aligned} \tag{B8}$$

But

$$\partial_0\pi = \partial_0\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\right). \tag{B9}$$

Therefore, Eq. (B7) is indeed equivalent to (B3), so that the method is dynamically consistent. Furthermore, the canonical commutation relations are propagated consistently by the equation of motion. This follows from the propagation of the commutators of $\tilde{\phi}$ and $\tilde{\pi}$ in the Minkowski space, or can be directly demonstrated for the commutators of ϕ and π in the curved space-time.[14]

The canonical quantization procedure for ϕ is generally covariant, although that is not obvious. The point is that, although the introduction of $\tilde{\phi}$ and $\tilde{\pi}$ involved a particular coordinate system in which the conformal flatness of the metric was manifest, the induced canonical structure involving ϕ and π in the curved space-time is independent of the coordinate system.[15] This covariance follows from the consistent propagation of the canonical commutation relations by the equation of motion. For if the canonical commutation relations hold on one spacelike hypersurface, then they hold on any spacelike hypersurface (as a consequence of the equation of motion). Therefore, they hold in any coordinate system with spacelike constant time hypersurfaces, so that the field algebra and equation of motion are generally covariant.

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[1]C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970); see also S. Coleman and R. Jackiw, *ibid.* 67, 552 (1971).

[2]N. A. Chernikov and E. A. Tagirov, Ann. Inst. Henri Poincaré 9, 109 (1968).

[3]L. Parker, Phys. Rev. Letters 28, 705 (1972); 28, 1497(E) (1972).

[4]The metric tensor is unquantized throughout this paper.

[5]Conformal transformation of the metric is defined in Sec. II. Its relation to conformal transformation of the coordinates is briefly discussed in Appendix A.

[6]L. Infeld and A. Schild, Phys. Rev. 68, 250 (1945); G. E. Tauber, J. Math. Phys. 8, 118 (1967).

[7]Our conventions are the same as those in Ref. 1; metric signature −2, and $\hbar = c = 1$.

[8]J. L. Synge, *Relativity, The General Theory* (North-Holland, Amsterdam, 1960), pp. 318 and 319. In Eq. (46) one replaces Synge's ψ by $2\ln\phi$ to obtain our result.

[9]R. Penrose, in *Relativity, Groups and Topology,*

edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964), p. 565.

[10]One can proceed nonrigorously by means of perturbation theory. However, even when the mass is not zero, there exist unsolved problems in the $\lambda\phi^4$ theory. See, for example, J. Glimm and A. Jaffe, in *Statistical Mechanics and Quantum Field Theory,* edited by C. DeWitt and R. Stora (Gordon and Breach, New York, 1971), p. 1.

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[12]Ya. B. Zel'dovich and A. A. Starobinsky, Zh. Eksp. Teor. Fiz. 61, 2161 (1971) [Sov. Phys. JETP 34, 1159 (1972)].

[13]Dimensional considerations for the Friedmann universes, as well as the earlier results, indicate that the Planck mass is the mass with which μ_0 must be compared in order to determine when the gravitationally induced particle creation will yield an energy density which is small with respect to the energy density acting as the source of the isotropic expansion.

[14]The direct demonstration is given in S. A. Fulling, Ph.D. thesis, Princeton University, 1972 (unpublished).

[15]Canonical quantization in curved space-time is discussed by the following: R. Utiyama, Phys. Rev. 125, 1727 (1962); R. Utiyama and B. S. DeWitt, J. Math. Phys. 3, 608 (1962); and C. Møller, *Royaumont Conference Proceedings* (National Center for Scientific Research, Paris, 1962), who noted the consistency and general covariance of the method. A recent discussion of the method is given in Ref. 14, where further references are given.

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Presymmetry of Classical Relativistic Fields*

Y. Avishai and H. Ekstein
Argonne National Laboratory, Argonne, Illinois 60439
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The physical act of accelerating observation instruments has an obvious counterpart in Newtonian physics: It is the transformation induced by $(\vec{x}, t) \mapsto (\vec{x} + \frac{1}{2}\vec{a}t^2, t)$. In special relativity, a theoretical counterpart for physical acceleration has been introduced only for a small subset of measurement procedures, such as clocks and yardsticks, but not for such instruments as accelerometers. We propose a general theoretical counterpart for physical accelerations in Minkowski space. In the algebra $\mathcal{O}$ of observation procedures, it induces automorphisms, not of $\mathcal{O}$ but of a point subalgebra $\mathcal{O}_x \subset \mathcal{O}$ that is associated to a single point x. From the postulates of presymmetry, we deduce a generalized version of Newton's second law; an acceleration-invariant subset $\bigcup_{(x|x_0=0)} \mathcal{O}_{xc}$ of instant observation procedures provides complete predictive power. The main technical contribution of the paper is the introduction of a topological algebra $\mathcal{O}$ in which local subsets associated to a single point are proper (although unbounded) subalgebras, whose automorphisms are discussed.

I. INTRODUCTION

The motivation for the present study is the desire to obtain clues in classical field theory for a more complete and physically plausible axiom system of quantum field theory. The canonical commutation relations of the early field theories had to be rejected because they produced inconsistencies, but the new mathematically consistent axiomatic field theories of Wightman and Araki-Haag lack an important element: kinematics.[1] This deficiency is exemplified by the lack of limitation on the order of the equations of motion. In a nonrelativistic n-particle problem, the equations of motion are $3n$ second-order differential equations for purely kinematic reasons. The Lagrangian field theories have analogous properties, but no such limitation is contained in axiomatic field theories as they stand. Yet a limitation of this kind is indispensable for a physical theory with effective predictive power. Is it possible to add simple and physically cogent assumptions to the axioms of quantum field theory such that a generalized kind of Newton's second law follows as a theorem? For the simpler case of nonrelativistic quantum mechanics, this question was answered positively in a previous paper.[2]

Classical relativistic field theory may provide another clue for answering the main question at hand. The present paper establishes what we hope to be physically reasonable basic assumptions from which a generalized form of Newton's second law follows as a theorem. Couched in abbreviated terms, it states that an acceleration-independent subset of instant observation procedures provides full predictive power for the future.

For the systematic comparison of two theories, it is at least necessary that they purport to answer