PHYSICAL REVIEW D VOLUME 7, NUMBER 4 15 FEBRUARY 1973

Gravitational Radiation from a Mass Projected into a Schwarzschild Black Hole*

Remo Ruffini
Joseph Henry Physical Laboratories, Princeton, New Jersey 08540
(Received 14 July 1972)

Gravitational radiation emitted by a particle projected with nonzero kinetic energy from infinite distance into a Schwarzschild black hole is examined. Direct comparison between a semirelativistic approach and the fully relativistic approach in the Regge-Wheeler-Zerilli formalism gives an insight into the nature of the results. Detailed spectral distributions are given. Contrary to the case in which the particle falls in with zero kinetic energy, the spectrum does not vanish any more at low frequencies and a considerably larger amount of radiation is emitted.

I. INTRODUCTION

In recent works Ruffini and Wheeler,[1] Davis and Ruffini,[2] Davis, Ruffini, Press, and Price,[3] and Davis, Ruffini, and Tiomno[4] have examined the details of a burst of gravitational radiation emitted by a particle falling radially into a Schwarzschild black hole. Different aspects of this problem were separately analyzed. In Ref. 1 two apparently contradictory assumptions are made: (a) The particle follows a geodetic in the curved Schwarzschild background, and (b) the system radiates as if it were in flat space, the energy radiated being simply proportional to the square of the third time derivative of the quadrupole moment of the system (see, e.g., Landau and Lifshitz[5]). The reason for adopting these approximations is simply explained: We can by an easy computational and analytic analysis put in evidence some of the main qualitative features of the spectrum of the radiation.

We leave to more detailed analysis the search for the exact numerical values, as well as the details of the polarization of the radiation and the study of the shape of the burst.

The subsequent analysis by Davis and Ruffini[2] and Davis, Ruffini, Press, and Price[3] was approached with the help of a different formalism and was mainly directed to analyze these details. Here the Regge-Wheeler formalism[6] as developed by Zerilli[7] is adopted. The spectrum is found to be qualitatively similar to the one given in Ref. 1, but the amount of radiation is approximately four times as large. Also, more details are given of the multipole distribution of the radiation. Nearly 90% of the total energy emitted is contained in the quadrupole mode ($l=2$), 9% in the octupole mode ($l=3$), and the remaining part in higher multipoles. In the subsequent paper by Davis, Ruffini, and Tiomno[4] the analysis is further generalized and the shape of the burst of the radiation is obtained for the first time.

In this paper it is our aim to examine the spectral distribution of the radiation emitted by a particle falling radially inward, starting its motion from infinite distance ($r=+\infty$) with a *finite* value of the kinetic energy. In the previous analysis we always supposed that the particle was falling inward starting *at rest* at infinity. That the spectrum of the radiation should indeed have completely new features can be deduced from the treatment given in Ref. 1. On the basis of these preliminary qualitative results we have developed the full Regge-Wheeler treatment. The spectrum, as expected using the approximation in Ref. 1, does not go to zero at low frequencies. A considerably larger amount of energy is emitted by the system consisting of the particle and the black hole. Detailed results of the total energy emitted as well as spectral and multipole distribution are given here.

II. LOW-FREQUENCY RADIATION IN SEMIRELATIVISTIC APPROACH

Following the formalism developed in Refs. 1 and 5, we have that the gravitational radiation emitted by a system with quadrupole moment Q_{rs} is given by

$$-\frac{dE}{dt}=\tfrac{1}{45}\dddot{Q}^{rs}\dddot{Q}_{rs}\ , \qquad (1)$$

and the total amount of radiation emitted is given by

$$-\Delta E=\tfrac{1}{45}\int_{-\infty}^{+\infty}\dddot{Q}^{rs}(t)\dddot{Q}_{rs}(t)\,dt$$

$$=\tfrac{1}{45}\int_{-\infty}^{+\infty}\dddot{Q}^{rs}(\omega)\dddot{Q}_{rs}(\omega)\,d\omega\ . \qquad (2)$$

Here and in the following we assume geometrical units with $G=c=1$, and the dot represents differentiation with respect to time. In Eq. (2) $Q^{rs}(\omega)$ is the Fourier transform of the quadrupole moment, namely

$$\dddot{Q}^{rs}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\dddot{Q}^{rs}(t)e^{i\omega t}\,dt\,. \tag{3}$$

We are mainly concerned with the features of the spectrum at low frequencies. We then have

$$\lim_{\omega\to 0}\dddot{Q}^{rs}(\omega)=\frac{[\ddot{Q}^{rs}(t)]_{-\infty}^{+\infty}}{\sqrt{2\pi}}\quad, \tag{4}$$

and for the emission of energy at low frequencies

$$\begin{aligned}-\frac{dE}{d\nu}&=-2\pi\frac{dE}{d\omega}\\&=\tfrac{4}{45}\pi\dddot{Q}^{rs}(\omega\to 0)\,\dddot{Q}_{rs}(\omega\to 0)\\&=\tfrac{2}{45}[\ddot{Q}^{rs}(t)]_{-\infty}^{+\infty}[\ddot{Q}_{rs}(t)]_{-\infty}^{+\infty}\,.\end{aligned} \tag{5}$$

Turning now from these general arguments to the case of a particle of mass m falling into a larger mass M, we have for the quadrupole moment of the system

$$Q^{rs}=m(3x^rx^s-\delta^{rs}|x|^2)+M(3X^rX^s-\delta^{rs}|X|^2)\,, \tag{6}$$

where with $x^r(t)$ and $X^r(t)$ we indicate respectively the coordinates of the masses m and M. We then clearly have

$$\ddot{Q}^{rs}=m(3\ddot{x}^rx^s+6\dot{x}^r\dot{x}^s+3x^r\ddot{x}^s-2\delta^{rs}\ddot{x}^px_p-2\delta^{rs}\dot{x}^p\dot{x}_p)$$
$$+\text{analogous term for the other mass.} \tag{7}$$

In the asymptotic regime $(r\to+\infty)$, we also have

$$\ddot{x}^r=M(X^r-x^r)/r^3\,, \tag{8a}$$

$$\ddot{x}^s=m(X^s-x^s)/r^3\,. \tag{8b}$$

In the other asymptotic direction $(r\to 2m)$ the particle, in the Ruffini-Wheeler[1] approximation, is "frozen" at $r\sim 2m$ in the final evolution of the implosion. From (7), (8a), and (8b) we can, therefore, immediately conclude that:

(a) Quite apart from the details of the implosion, uniquely from the asymptotic regimes, we can predict that the intensity of the gravitational radiation emitted has to vanish in the limit of low frequencies if the particle starts its implosion from rest. This effect was, indeed, clearly confirmed in the detailed analysis in Ref. 3.

(b) If the implosion takes place with a nonzero initial velocity $(\dot{x}^r\neq 0)$ then the spectrum will not be mainly concentrated around frequencies ~0.024/M (see Ref. 1) but will have sizable low-frequency components.

(c) The energy radiated at low frequencies should be proportional to the fourth power of the velocity, in the limit that the initial velocity of the particle at infinity is small.

From these general conclusions let us proceed to a detailed treatment.

III. RADIATION IN THE REGGE-WHEELER FORMALISM

The radiation as well as the particle of mass m are treated in the Regge-Wheeler formalism as a small perturbation of the given Schwarzschild background metric generated by the larger mass M:

$$g_{\mu\nu}=(g^0_{\mu\nu})_{\text{Schwarzschild}}+h_{\mu\nu}\,. \tag{9}$$

Using the symmetry properties of the background space the perturbations $h_{\mu\nu}$ have been expanded in tensorial spherical harmonics. The equations governing their radial dependence have been obtained by requiring that the metric $g_{\mu\nu}$ fulfill Einstein's equations with a source term given by the tensor energy-momentum of a pointlike particle of mass m.

Details of the treatment with associated gauge conditions are given by Zerilli in Ref. 7. For our purpose it is enough to know that the magnetic parity terms in the Zerilli formalism do not give any contribution in the case of radial fall. The solution for the electric parity terms reduces to the integration of equations governing the radial part $R_l(r)$ of the perturbation, the angular part being automatically described by the orthonormal set of spherical harmonics. We have

$$\frac{d^2R_l}{dr^{*2}}+[\omega^2-V_l(r)]R_l(r)=S_l(\omega,r)\,. \tag{10}$$

Here

$$r^*=r+2M\ln(r/2M-1)\,; \tag{11}$$

$$V_l(r)=\frac{(1-2M/r)[2\lambda^2(\lambda+1)r^3+6\lambda^2Mr^2+18M^2r\lambda+18M^3]}{r^3(\lambda r+3M)^2}\,, \tag{12}$$

λ being given by $\lambda=\frac{1}{2}(l-1)(l+2)$; and

$$S_l(\omega,r)=-\frac{8\pi(r-2M)}{r}\,\frac{\lambda r+3M}{\omega}\,\frac{d}{dr}\left[\frac{r(r-2M)}{(\lambda r+3M)^2}\,\frac{A_l^{(1)}(\omega,r)}{\sqrt{2}}\right]+\frac{r(r-2M)}{\lambda r+3M}A_l(\omega,r)\,. \tag{13}$$

We have

$$A_{lm}(r,t)=m\frac{dt}{d\tau}\left(\frac{dR}{dt}\right)^2\frac{\delta(r-R(t))Y^*_{lm}(\Omega(t))}{(r-2m)^2}\ , \quad (14a)$$

$$A^{(1)}_{lm}(r,t)=\sqrt{2}\,im\frac{dt}{d\tau}\frac{dR}{dt}\frac{\delta(r-R(t))Y^*_{lm}(\Omega(t))}{r^2}\ , \quad (14b)$$

τ being the proper time of the inward-falling particle.

For a particle projected radially from infinity with nonzero initial velocity we have for the Fourier-transformed quantities (14a) and (14b)

$$A_l(\omega,r)=\left(\frac{m}{2\pi}\right)\left[l+\tfrac{1}{2}\left(\gamma^2-1+\frac{2m}{r}\right)\right]^{1/2}\frac{e^{i\omega T(r)}}{(r-2m)^2}\ , \quad (15a)$$

$$A^{(1)}_l(\omega,r)=-i\left(\frac{m}{2\pi}\right)\gamma(2l+1)^{1/2}\frac{e^{i\omega T(r)}}{r(r-2m)}\ , \quad (15b)$$

the quantity γ being

$$\gamma=\left(1-\frac{2m}{r}\right)\frac{dt}{d\tau}\ . \quad (16)$$

We have to integrate Eq. (10) with the usual boundary conditions of purely ingoing waves at the surface of the black hole and purely outgoing waves at infinity:

$$A_l(\omega)e^{i\omega r^*},\quad r^*\to+\infty \quad (17a)$$

$$B_l(\omega)e^{-i\omega r^*},\quad r^*\to-\infty\ . \quad (17b)$$

The integration of the equations has been carried out by using numerical Green's function techniques. More details will be given elsewhere (see Ref. 8).

IV. RESULTS AND CONCLUSIONS

In Fig. 1 we compare and contrast the results of the present analysis for $\gamma=1.2$ and $\gamma=1.4$ with the ones presented in Ref. 3 for $\gamma=1.0$.

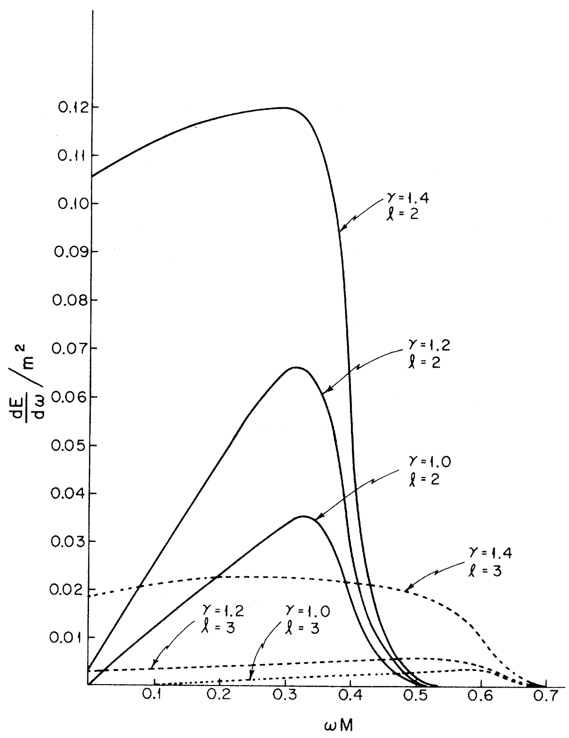

FIG. 1. Compared and contrasted are the spectra of the gravitational energy emitted by a particle radially projected into a Schwarzschild black hole. The solution for $\gamma=1.0$ refers to the case in which the particle starts at rest at infinity. The solid lines give the quadrupole radiation ($l=2$) for selected values of γ ($\gamma=1.0$, $\gamma=1.2$, $\gamma=1.4$). The dashed lines show the octupole radiation ($l=3$) for the corresponding selected values of γ.

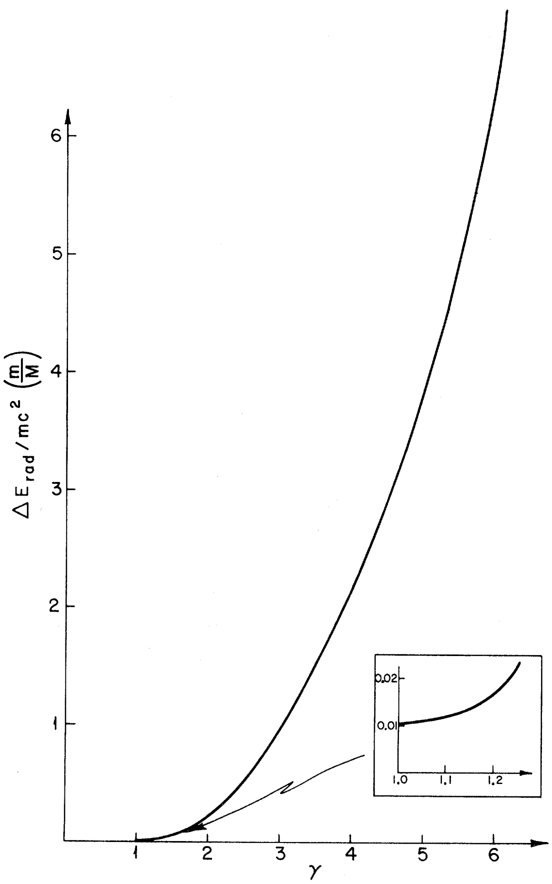

FIG. 2. Total energy radiated outward by a particle projected from infinite distance into a Schwarzschild black hole as a function of γ. The radiation for $\gamma\sim 1$ is here magnified to allow a direct comparison with the treatment presented in Sec. III.

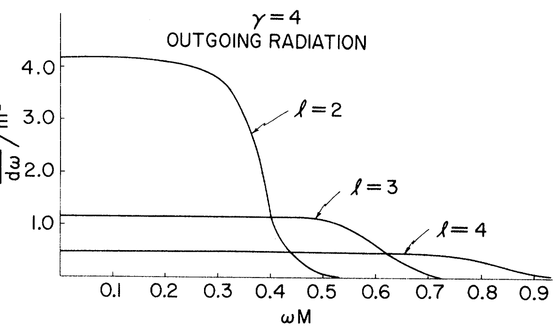

FIG. 3. Gravitational radiation emitted outward by a particle projected from infinity with $\gamma = 4$. The contribution due to lowest multipoles as well as the total energy emitted is clearly enhanced.

As we expected already on the basis of the semi-relativistic approach presented in Sec. III, we can reach the following conclusions:

(1) The spectral distribution does not go to zero at low frequencies.

(2) The amount of energy increases substantially with γ, and at least for values of γ near unity the qualitative behavior suggested in Sec. III is, indeed, found to be correct (see Fig. 2).

(3) The contribution of higher-order multipoles becomes more and more significant for larger values of γ.

(4) To the sizable change in the amount of energy radiated there does not correspond a sizable change in the *maximum* frequency of the spectrum. The bulk of the radiation is still emitted at values of $\omega \lesssim 0.7/M$.

These results are made even more transparent and enhanced by going to larger values of γ. In Figs. 3 and 4 we summarize the results, respectively, for $\gamma = 4$ and $\gamma = 6$. Finally, Fig. 2 summarizes the results for the total energy radiated summed over all multipoles for values of $\gamma > 1$. The solution for $\gamma < 1$ corresponding to the motion of a particle falling inward from rest at *finite* distance presents difficulties in handling the Fourier transforms properly. We do not, however, expect any new peculiar property in this regime apart from a reduction in the amount of radiated energy.

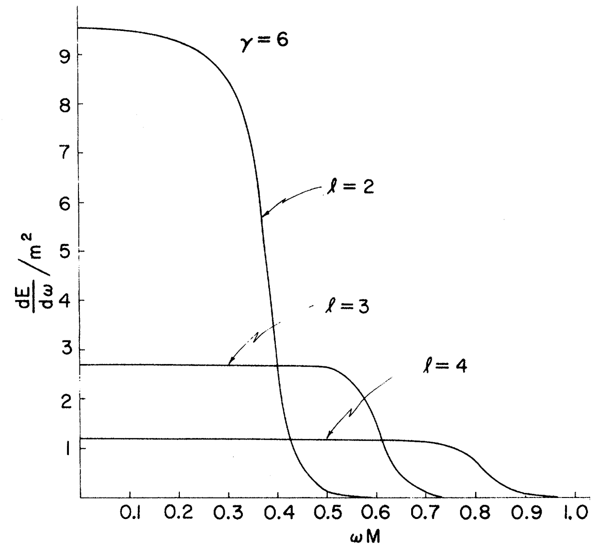

FIG. 4. Gravitational radiation emitted outward by a particle projected from infinity with $\gamma = 6$. The $l = 3$ mode is approximately one half of the $l = 2$; the $l = 4$ one fourth. The contribution due to the $l > 2$ modes is much larger than the one given in Ref. 3 for the case $\gamma = 1.0$.

The violent dependence of the energy emitted on the value of the initial kinetic energy (see Fig. 2), while predictable on the ground of the work of Ruffini and Wheeler,[1] is of the greatest interest and importance for the understanding of the nature of gravitational radiation.

On the other hand, from a realistic astrophysical point of view we find it hard to conceive such a relativistic source. Details of the polarization and gravitational radiation going *into* the black hole will be presented elsewhere.[8]

ACKNOWLEDGMENT

It is a pleasure to acknowledge discussions with J. A. Wheeler on some aspects of this problem.

---

*Work partially supported by NSF Grant No. 30799X.

[1]R. Ruffini and J. A. Wheeler, in *The Significance of Space Research for Fundamental Physics*, edited by A. F. Moore and V. Hardy (European Space Research Organization, Paris, 1971).

[2]M. Davis and R. Ruffini, Lett. Nuovo Cimento 2, 1165 (1971).

[3]M. Davis, R. Ruffini, W. Press, and R. Price, Phys. Rev. Letters 27, 1466 (1971).

[4]M. Davis, R. Ruffini, and J. Tiomno, Phys. Rev. D 5, 2932 (1972).

[5]L. Landau and E. Lifshitz, *Théorie des Champs* (MIR, Moscow, 1970).

[6]T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063

(1957).
[7]F. Zerilli, J. Math. Phys. 11, 2203 (1970); Phys. Rev. Letters 24, 737 1970; Phys. Rev. D 2, 2141, 1970.

[8]R. Ruffini, in Proceedings of the Les Houches Summer School "Les Astres Noire," edited by C. DeWitt (Gordon and Breach, to be published).

PHYSICAL REVIEW D VOLUME 7, NUMBER 4 15 FEBRUARY 1973

Conformal Energy-Momentum Tensor in Riemannian Space-Time*

Leonard Parker
University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201†
(Received 24 May 1972; revised manuscript received 17 August 1972)

We consider the scalar field with quartic self-interaction in Riemannian space-time. Identities are proved which connect the modified energy-momentum tensors of Callan, Coleman, and Jackiw in different conformally related space-times. We consider the quantized scalar field in a conformally flat metric, and show that our identities relate the matrix elements of the modified energy-momentum tensor to corresponding matrix elements in Minkowski space. We show further that when the mass can be neglected in the conformal wave equation there is no gravitationally induced particle creation in conformally flat space-times, thus generalizing a result proved earlier in the free-field case. The influence of additional fields and interactions on that result is briefly discussed.

I. INTRODUCTION

A modification of the conventional energy-momentum tensor of the scalar field with quartic self-interaction has been proposed by Callan, Coleman, and Jackiw.[1] Their modified energy-momentum tensor has interesting properties at both the classical and quantum levels. It evidently has, at least in Minkowski space-time, finite matrix elements, in the sense that they possess a finite limit in every order of renormalized perturbation theory, as the cutoff approaches infinity. Furthermore, the currents associated with conformal coordinate transformations are simply expressed in terms of the modified energy-momentum tensor, so that it is sometimes referred to as the conformal energy-momentum or stress tensor. Callan, Coleman, and Jackiw also showed how to alter the general relativistic action functional to make the fully covariant form of the conformal stress tensor the source of the gravitational field. A similar gravitational theory has also been considered by Chernikov and Tagirov.[2]

The physical distinction between the conventional and conformal stress tensors is most evident in strong gravitational fields. For example, the consequences of the theories involving the two tensors are quite different near the cosmological singularity in an isotropically expanding universe.[3] It is therefore of interest to consider the properties of the conformal energy-momentum tensor in Riemannian space-time. This paper will be concerned with those properties, both for a classical and a quantized scalar field.[4]

The basis of our treatment will be a number of identities involving fields and energy-momentum tensors in metrics which are related by conformal transformation.[5] Those identities are proved in Sec. II. Section III is concerned with the quantized scalar field with quartic self-interaction in conformally flat space-time. The class of conformally flat metrics includes the fundamentally significant Robertson-Walker metrics.[6] The canonically quantized scalar field in the curved space-time is related by an unquantized conformal factor to a corresponding canonically quantized scalar field in Minkowski space. The identities proved earlier relate the matrix elements of the modified energy-momentum tensor in conformally flat space-time to corresponding matrix elements in Minkowski space. Finally, we show that for the massless scalar field obeying the conformal wave equation with quartic self-interaction there is no gravitationally induced particle creation in conformally flat metrics.

II. MODIFIED STRESS TENSOR IN CONFORMALLY RELATED METRICS

By replacing ordinary derivatives ∂_μ by covariant derivatives ∇_μ in the modified energy-momentum tensor, one obtains the following tensor[7]: