Dynamics of the Friedmann Universe Using Regge Calculus

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(Received 29 March 1972)

Models for the Friedmann universe are constructed from 5, 16, or 600 dust-filled tetrahedrons connected so as to form a closed space. Using the techniques of Regge calculus the time development of these model universes is determined and the results compared with the standard analytic solution for an isotropic dust-filled universe.

I. INTRODUCTION

In cosmology one of the basic assumptions usually made in constructing models of the universe is that at a particular time and apart from local inhomogeneity, the universe looks the same to any observer in it. This postulate that the universe is homogeneous is known as the cosmological principle.[1] In addition to homogeneity it is usually assumed that the universe is isotropic. If one also assumes that the universe is spatially closed[2] then its dynamics is uniquely determined by Einstein's equations providing the equation of state of the mass-energy within the universe is known. Two well-known examples of this type of universe are the Friedmann and Tolman universes.[3] The Friedmann universe is a closed space filled with dust at zero pressure whereas the Tolman universe is filled with radiation with the pressure equaling one third of the energy density. In this paper we shall consider the Friedmann universe; our aim is to compare the description of it in conventional analytic terms and that obtained using Regge calculus.[4]

Regge calculus is a method of approximating curved 4-dimensional space-times by a collection of 4-dimensional blocks (4-simplexes), the geometry inside each block being Euclidean. These blocks will not, in general, fit together in 4-dimensional Euclidean space – there will be gaps or overlaps ("rattle") between the blocks. Regge has shown[4] that Einstein's equations may be reexpressed in terms of constraints on the lengths of the edges of the blocks. The techniques of Regge calculus have already been applied[5] to the relatively simple problem of finding solutions to the initial-value problem at the moment of time symmetry. This consists of determining the geometry of space at a particular time and reduces to a problem in 3 dimensions. In this paper we wish to study the time dependence of a particular space and shall therefore have to cope with the difficulties of visualizing 4-dimensional blocks.

II. THE MODELS

Our analysis will be based on three models first suggested by Wheeler.[6] These models approximate the Friedmann universe by a set of equilateral tetrahedrons each containing an equal amount of dust and connected together so as to form a closed space. The isotropy of the Friedmann universe requires that the number of tetrahedrons meeting at any edge or vertex must always be the same. This is only possible for spaces constructed from 5, 16, or 600 tetrahedrons.[7] As the analyses for these three possibilities only differ in minor details we shall only give the derivation of the constraint equations for the 5-tetrahedron model. The corresponding equations for the other two models will be given in Sec. V.

The closed space of the 5-tetrahedron model is constructed in the following way: Choosing one of the tetrahedrons we attach one of the others to each of its four faces. The points at the apexes of these four tetrahedrons are now identified as a single point. This leaves no tetrahedron face exposed – we have a closed universe whose volume is determined by the edge length l. This five-block universe provides a model for a homogeneous and isotropic closed space. Expansions and contractions of the universe are expressed in terms of the variation of l with time. In order to investigate this time dependence we divide the time axis into intervals of length η_i where the interval η_i starts at time t_i and finishes at t_{i+1}; $\eta_i = t_{i+1} - t_i$. We do *not* assume all the η_i are equal. At time t_i the corresponding value of l is l_i. Thus, corresponding to each t_i, we may construct a 3-dimensional hypersurface (i.e., a 5-tetrahedron universe with $l = l_i$) in 4-dimensional space-time. By joining the vertices of the tetrahedrons at time t_i to the corresponding ones at time t_{i+1} we divide the space-time contained between the two hypersurfaces into five 4-dimensional blocks. One of these blocks is illustrated diagrammatically in Fig. 1. By repeating this procedure for all other intervals we may di-

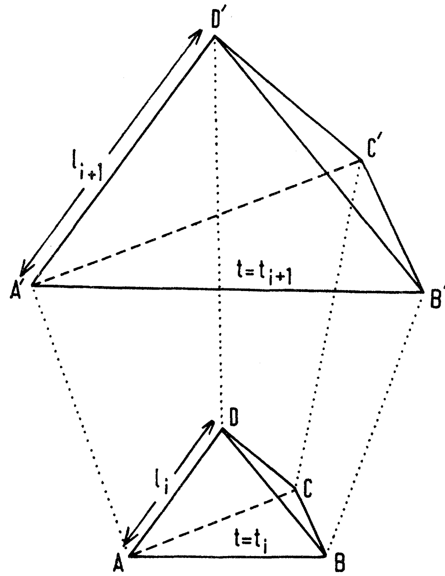

FIG. 1. Diagram illustrating a 4-dimensional block.

vide all the space-time traced out by our model universe into slices of five 4-dimensional blocks separated by 3-dimensional spacelike hypersurfaces. We must now apply Regge calculus to this set of blocks in order to find the dependence of l_i upon t_i.

III. EINSTEIN'S EQUATIONS

One of the standard methods of deriving Einstein's field equations is by means of the principle of stationary action. This may be expressed as

$$\delta\left(\int \mathcal{L}\sqrt{-g}\, d^4x - \frac{c^4}{16\pi G}\int R\sqrt{-g}\, d^4x\right) = 0\,, \quad (1)$$

where $\mathcal{L}$ is the matter Lagrangian, R is the curvature scalar, and the integrals are over all space-time. The variation is with respect to the metric elements $g_{\mu\nu}$.

We wish to rewrite Eq. (1) in a form appropriate to our model. In order to do this we note the following properties of our blocks. Each 4-dimensional block has six 3-dimensional faces. These comprise "upper" and "lower" tetrahedral faces and four side faces which are truncated tetrahedrons. These faces meet on triangles or trapeziums. For example, in Fig. 1 the faces $ABCA'B'C'$ and $BCDB'C'D'$ meet on the trapezium $BB'C'C$ and the faces $ABCA'B'C'$ and $ABCD$ meet on the triangle ABC. The 2-dimensional surfaces on which the faces of blocks meet are called hinges. In general more than one block will "hinge" on one of these surfaces. If our set of blocks represented a Euclidean space-time then the sum of the angles between the two faces of each block meeting on a hinge would be 2π. However, for a curved space-time the angles between blocks do not add up to 2π. If the sum of the angles at the hinge j is $2\pi-\delta_j$, then we call δ_j the deficit angle at that hinge. It has been shown by Regge[4] that the integral

$$\int R\sqrt{-g}\, d^4x \quad (2)$$

may be reexpressed in terms of the areas L_j of the hinges and the deficit angles δ_j according as

$$2\sum_{\text{all hinges}} L_j\delta_j\,. \quad (3)$$

We now have to rewrite the integral

$$\int \mathcal{L}\sqrt{-g}\, d^4x \quad (4)$$

in terms of the l_i and t_i of our model universe. Noting that the geometry inside each block is Euclidean we may immediately rewrite (4) as

$$\sum_{\text{all blocks } j}\int_{\text{block } j} \rho c^2 d^4x\,, \quad (5)$$

where ρ is the density of matter at a point inside the block, measured in the rest frame at that point. At the moment of maximum expansion of the universe all the dust is at rest and is uniformly distributed throughout the tetrahedrons. Therefore, providing the blocks are assumed to contract uniformly as we move away from this point, ρ will remain constant throughout the tetrahedrons. We may therefore write (5) as[8]

$$5\sum_{\text{slices } i} i\int_{t_i}^{t_{i+1}} V^{(3)}(t)\rho(t)c^3dt\,, \quad (6)$$

where $V^{(3)}(t)$ is the volume of one of the tetrahedrons at time t. Since ρ, the invariant density, is not varied in applying the action principle and since we shall also be taking $\eta_i\to 0$ in the final stages of our calculation, we may simplify (6) to

$$5\sum_{\text{slices } i} V_i^{(4)}\rho_i c^2\,, \quad (7)$$

where $V_i^{(4)}$ is the 4-volume of one of the blocks in slice i, and ρ_i is the density at the "center" of the block.

Equation (1) may now be rewritten as

$$\delta\left(5\sum_{\text{slices } i} V_i^{(4)}\rho_i c^2 - \frac{c^4}{8\pi G}\sum_{\text{hinges } j} L_j\delta_j\right) = 0\,, \quad (8)$$

where the variation is now with respect to the edge lengths of the blocks.[4] We shall make use of the general result proved by Regge that it is not necessary to consider variation of the angles since

$$\sum_{\text{all hinges}} L_j\delta(\delta_j) = 0\,. \quad (9)$$

IV. CALCULATION OF AREAS, ANGLES, AND VOLUMES

We must now calculate the hinge areas L_j, deficit angles δ_j, and block volumes $V_i^{(4)}$. In order to do this it is convenient to consider one block and to set up Cartesian coordinates (x, y, z, ict) as follows (see Fig. 1): Taking the center of the tetrahedrons as the origin we adopt a right-handed set of coordinates with the x axis parallel to AB and the z axis passing through D. In this coordinate system the coordinates of the vertices of the tetrahedron of edge l_i at time t_i are given by

$$\begin{aligned} A &= \left(-\frac{l_i}{2}, -\frac{l_i}{2\sqrt{3}}, -\frac{l_i}{2\sqrt{6}}, ict_i\right), \\ B &= \left(\frac{l_i}{2}, -\frac{l_i}{2\sqrt{3}}, -\frac{l_i}{2\sqrt{6}}, ict_i\right), \\ C &= \left(0, \frac{l_i}{\sqrt{3}}, -\frac{l_i}{2\sqrt{6}}, ict_i\right), \\ D &= \left(0, 0, \frac{\sqrt{3}\, l_i}{2\sqrt{2}}, ict_i\right). \end{aligned} \tag{10}$$

Similar expressions give the coordinates of the vertices A', B', C', and D' of the tetrahedron of edge l_{i+1} which forms the upper face of the block shown in Fig. 1.

First of all we consider the hinge areas, which are given by the standard formulas for the areas of a triangle and a trapezium. In particular, for the triangular hinge ABC and the trapezoidal hinge $ABB'A'$, the areas L_i^{tri} and L_i^{trap} are given by

$$L_i^{\text{tri}} = \tfrac{1}{4}\sqrt{3}\, l_i^2, \tag{11}$$

$$L_i^{\text{trap}} = \tfrac{1}{2} ic\,(l_i + l_{i+1})(t_{i+1} - t_i) \times \left[1 - \frac{1}{8c^2}\left(\frac{l_{i+1} - l_i}{t_{i+1} - t_i}\right)^2\right]^{1/2}. \tag{12}$$

In applying the variational principle in Regge calculus, it is the lengths of the block edges which have to be varied. Therefore it is convenient to rewrite L_i^{trap} in terms of l_i, l_{i+1}, and m_i, where m_i is the length of the timelike edges of the block joining the upper and lower faces (e.g., AA' in Fig. 1). m_i is given by

$$m_i^2 = \tfrac{3}{8}(l_{i+1} - l_i)^2 - c^2(t_{i+1} - t_i)^2. \tag{13}$$

L_i^{trap} may then be written as

$$L_i^{\text{trap}} = \tfrac{1}{2} i\,(l_i + l_{i+1})(1 - \kappa_i)^{1/2} \times \left[\tfrac{3}{8}(l_{i+1} - l_i)^2 - m_i^2\right]^{1/2}, \tag{14}$$

where

$$\kappa_i = \frac{(l_{i+1} - l_i)^2}{3(l_{i+1} - l_i)^2 - 8m_i^2}. \tag{15}$$

Since in our calculation we choose to vary the edge length m_i and since L_i^{tri} does not depend on m_i, we shall not need to know the explicit form of the deficit angle δ_i^{tri} at a triangular hinge. Hence the only deficit angle we need to calculate is δ_i^{trap}, the deficit angle at a trapezoidal hinge. Three identical 4-dimensional blocks meet on each of these hinges. These blocks have as their upper and lower faces those tetrahedrons which meet on the upper and lower ends of the hinge. Therefore the deficit angle is given by

$$\delta_i^{\text{trap}} = 2\pi - 3\theta_i, \tag{16}$$

where θ_i is the angle between the faces of a block meeting on the hinge. To calculate this angle, we shall consider the faces $\alpha \equiv ABCA'B'C'$ and $\beta \equiv ABDA'B'D'$ which meet on the hinge $ABB'A'$. If we define unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$, and $\vec{l}$, in the x, y, z, and ict directions then the unit normals to the faces α and β are

$$\begin{aligned} \hat{n}_\alpha &= \frac{\vec{k} + (\epsilon/2\sqrt{6})\vec{l}}{(1+\epsilon^2/24)^{1/2}}, \\ \hat{n}_\beta &= \frac{-2\sqrt{2}\,\vec{j} + \vec{k} - (\epsilon\sqrt{3}/2\sqrt{2})\vec{l}}{3(1+\epsilon^2/24)^{1/2}}, \end{aligned} \tag{17}$$

where

$$\epsilon = \frac{l_{i+1} - l_i}{ic(t_{i+1} - t_i)}.$$

As defined above, $\hat{n}_\alpha$ points into the block and $\hat{n}_\beta$ points out. The angle θ_i between the two faces is given by

$$\begin{aligned} \cos\theta_i &= \hat{n}_\alpha \cdot \hat{n}_\beta \\ &= \frac{8 - \epsilon^2}{24 + \epsilon^2} \\ &= \frac{1 + \kappa_i}{3 - \kappa_i}. \end{aligned} \tag{18}$$

Finally we have to calculate the volume $V_i^{(4)}$ of a 4-dimensional block. Since the timelike edges joining the lower tetrahedral face of the block to the upper face are assumed to be straight lines, the volume of a block is given by

$$V_i^{(4)} = \int_{t_i}^{t_{i+1}} V_i^{(3)}(l(t))\, icdt, \tag{19}$$

where

$$l(t) = l_i + (l_{i+1} - l_i)\,\frac{t - t_i}{t_{i+1} - t_i}.$$

The volume $V^{(3)}(l)$ of an equilateral tetrahedron of edge l is given by

$$V^{(3)}(l) = l^3/6\sqrt{2}. \tag{20}$$

We therefore obtain

$$V_i^{(4)} = ic\,\frac{t_{i+1}-t_i}{24\sqrt{2}}\,\frac{l_{i+1}{}^4 - l_i{}^4}{l_{i+1}-l_i} = \frac{i}{24\sqrt{2}}\,\frac{l_{i+1}{}^4 - l_i{}^4}{l_{i+1}-l_i}\left[\tfrac{3}{8}(l_{i+1}-l_i)^2 - m_i{}^2\right]^{1/2}. \tag{21}$$

V. VARIATION OF THE EDGE LENGTHS

By considering variations of the action with respect to m_j, we shall now obtain an expression relating the time separation $\eta_i = t_{i+1} - t_i$ of two spacelike hypersurfaces to the edge lengths l_i and l_{i+1} characterizing them. Equation (8), varied with respect to m_j, becomes

$$5\sum_{\text{slices } i} \frac{\partial V_i^{(4)}}{\partial m_j}\rho_i c^2 - \frac{c^4}{8\pi G}\sum_{\text{hinges } i}\left(\frac{\partial L_i^{\text{tri}}}{\partial m_j}\delta_i^{\text{tri}} + \frac{\partial L_i^{\text{trap}}}{\partial m_j}\delta_i^{\text{trap}}\right) = 0. \tag{22}$$

Noting that there are 10 trapezoidal and 10 triangular hinges in each slice and using the equations

$$\frac{\partial L_i^{\text{tri}}}{\partial m_j} = 0, \tag{23}$$

$$\frac{\partial L_i^{\text{trap}}}{\partial m_j} = \begin{cases} \dfrac{-i(l_i + l_{i+1})m_i}{2c(t_{i+1}-t_i)(1-\kappa_i)^{1/2}}, & i=j \\ 0, & i \neq j \end{cases} \tag{24}$$

$$\frac{\partial V_i^{(4)}}{\partial m_j} = \begin{cases} \dfrac{-i(l_{i+1}{}^4 - l_i{}^4)m_i}{24\sqrt{2}\,c(l_{i+1}-l_i)(t_{i+1}-t_i)}, & i=j \\ 0, & i\neq j \end{cases} \tag{25}$$

we obtain

$$(1-\kappa_i)^{1/2}\rho_i c^2\,\frac{l_{i+1}{}^4 - l_i{}^4}{l_{i+1}-l_i} = \frac{6c^4}{\pi G}(2\pi - 3\theta_i)(l_i + l_{i+1}). \tag{26}$$

This equation provides the required relationship between l_i, l_{i+1}, and m_i. As a check on our analysis we may compare this equation with the initial-value equation at the moment of time symmetry ($t = t_0$). To obtain the form of Eq. (26) at this time we let $l_{i+1} = l_i \to l_0$, the edge length at the moment of symmetry. Equation (26) then reduces to

$$\sqrt{2}\,\rho l_0{}^2 = \frac{3c^2}{\pi G}\left[2\pi - 3\cos^{-1}(\tfrac{1}{3})\right]. \tag{27}$$

This is to be compared with the initial-value equation[6]:

$${}^3R = \frac{16\pi G}{c^4}(\text{energy density}), \tag{28}$$

where ${}^3R$ is the intrinsic curvature of the spacelike hypersurface $t = t_0$. This hypersurface is a 3-dimensional space comprising five tetrahedral blocks of edge l_0. ${}^3R$ is given by[6]

$${}^3R = \frac{\sum_i l_i \delta_i}{\text{“volume per vertex”}}, \tag{29}$$

where the sum is over all edges i meeting at a given vertex and δ_i is the deficit angle between 2π and the sum of the dihedral angles between block faces meeting at the edge i. The dihedral angle between the faces of an equilateral tetrahedron is $\cos^{-1}(\frac{1}{3})$. In our case the “volume per vertex” equals the volume of one tetrahedron and so the initial-value equation becomes

$$4l_0\,\frac{2\pi - 3\cos^{-1}(\frac{1}{3})}{\sqrt{2}\;l_0{}^3/12} = \frac{16\pi G\rho}{c^2} \tag{30}$$

which is identical to Eq. (27).

We now make Eq. (26) into a differential equation by letting $\eta_i \equiv t_{i+1} - t_i \to 0$ so that $l_{i+1} \to l_i$. In this limit

$$\kappa_i \to \frac{1}{8c^2}\left(\frac{dl}{dt}\right)^2$$

and we obtain

$$\sqrt{2}\,\rho l^2\left[1 - \frac{1}{8c^2}\left(\frac{dl}{dt}\right)^2\right]^{1/2} = \frac{3c^2}{\pi G}(2\pi - 3\theta), \tag{31}$$

where

$$\cos\theta = \left[8c^2 + \left(\frac{dl}{dt}\right)^2\right]\Big/\left[24c^2 - \left(\frac{dl}{dt}\right)^2\right]. \tag{32}$$

A second expression of the kind given in Eq. (26) may be obtained by varying the edge length l_i. However, in the limit $t_{i+1} \to t_i$ we find that this equation is always trivially satisfied as both sides vanish.

Using the fact that

$$\rho = \frac{6\sqrt{2}\,M}{5l^3} \tag{33}$$

where M is the mass of the universe we may rearrange Eqs. (31) and (32) to obtain

$$l = \frac{4\sqrt{2}\,\pi G M\tan\frac{1}{2}\theta}{5(2\pi - 3\theta)c^2}, \tag{34a}$$

$$\left(\frac{dl}{dt}\right)^2 = 8c^2\left[1 - 2\tan^2(\tfrac{1}{2}\theta)\right]. \tag{35}$$

These equations provide a parametric solution for the dynamics of our model universe.

Similar equations may be derived for the 16- and 600-tetrahedron models. In both cases all details of the 4-dimensional blocks are the same as for the 5-tetrahedron model. The only differences between the three models are in the number of tetrahedrons meeting at an edge and in the total number of tetrahedrons and trapezoidal hinges. Equation (35) remains the same for all three models but the edge lengths, l_{16} and l_{600}, for the 16- and 600-tetrahedron models are now given by

$$l_{16}=\frac{\sqrt{2}\;\pi GM\tan\frac{1}{2}\theta}{3(2\pi-4\theta)c^2}\,, \tag{34b}$$

$$l_{600}=\frac{\pi GM\tan\frac{1}{2}\theta}{45\sqrt{2}\;(2\pi-5\theta)c^2}\,. \tag{34c}$$

VI. COMPARISON WITH THE ANALYTIC SOLUTION

The Friedmann universe has the same curvature in all directions at all points. If we embed it in a Euclidean 4-dimensional space, it has the geometry of a 3-dimensional sphere which expands and contracts with time. The exact solution to Einstein's equations for the Friedmann universe may be written in the form of parametric equations for the radius, a, of the 3-sphere and τ, the proper time as measured at a point on the sphere[7]:

$$a=\tfrac{1}{2}a_0(1-\cos\mu)\,, \tag{36}$$

$$\tau=\frac{a_0}{2c}(\mu-\sin\mu)\,, \tag{37}$$

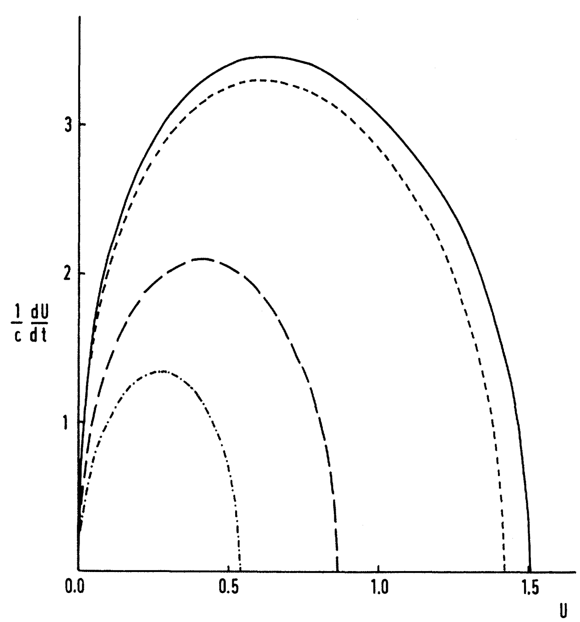

FIG. 2. Rate of change of the volume of the universe plotted against the volume for $MG/c^2=1$; analytic solution ———— , 600-tetrahedron model -------- , 16-tetrahedron model — — — — , 5-tetrahedron model —·—·—· .

where a_0 is related to the mass of the universe by

$$a_0=\frac{4MG}{3\pi c^2}\,. \tag{38}$$

Equations (36) and (37) are the parametric equations of a cycloid.

In order to compare our models with the exact solution it is convenient to consider U, the volume of the universe. For a 3-sphere of radius a the volume is given by[9]

$$U=2\pi^2a^3\,.$$

For our models the volume is

$$U_n=\frac{nl^3}{6\sqrt{2}}\,,\quad n=5,\,16,\,600\,.$$

In Fig. 2 we show graphs of dU/dt against U for $MG/c^2=1$. For the 600-tetrahedron model we have integrated Eqs. (34) and (35c) numerically to obtain the graph of U against t shown in Fig. 3. We see that as the number of tetrahedrons into which the space is divided increases, so the agreement with the exact solution improves. The most striking improvement is seen in the maximum volume of the universe. For the 5-tetrahedron model this is only about a third of the value for the exact solution. For the 600-tetrahedron model the maximum volumes are very nearly the same.

In making this comparison of our model with the analytic solution we have used the time coordinate t. This is the proper time as measured by a test particle situated at the center of a tetrahedron. The proper time measured at test particles situated at other points in the tetrahedron will differ from t. We shall now investigate this problem in detail for the 5-tetrahedron model.

In order to gain a better understanding of the na-

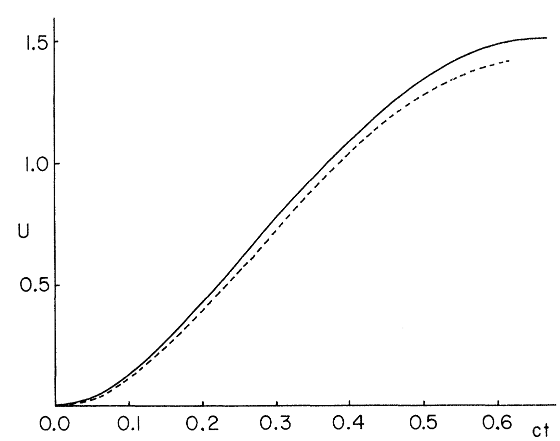

FIG. 3. Time dependence of the volume of the universe for $MG/c^2=1$; analytic solution ———— , 600-tetrahedron model ------- .

ture of the approximation involved in our model we embed it in a 4-dimensional Euclidean space. To do this we take a set of rectangular coordinates u_1, u_2, u_3, and u_4. The five vertices of the tetrahedrons in our model will lie in a 3-sphere such that the distance between any pair is l. We now find five such points using the fact that the coordinates of a 3-sphere of radius b may be written as[9]

$$\begin{aligned} u_1 &= b\sin\chi\sin\theta\sin\phi\,, \\ u_2 &= b\sin\chi\sin\theta\cos\phi\,, \\ u_3 &= b\sin\chi\cos\theta\,, \\ u_4 &= b\cos\chi\,. \end{aligned} \tag{39}$$

If, for simplicity, we assume that one of the points lies on the u_4 axis, a second in the u_3u_4 plane and a third has $u_1 = 0$ then we obtain the coordinates

$$\begin{aligned} P_1 &= (0, 0, 0, b)\,, \\ P_2 &= \left(0, 0, \frac{\sqrt{15}b}{4}, \frac{-b}{4}\right), \\ P_3 &= \left(0, \frac{\sqrt{5}b}{\sqrt{6}}, \frac{-\sqrt{5}b}{4\sqrt{3}}, \frac{-b}{4}\right), \\ P_4 &= \left(\frac{\sqrt{5}b}{2\sqrt{2}}, \frac{-\sqrt{5}b}{2\sqrt{6}}, \frac{-\sqrt{5}b}{4\sqrt{3}}, \frac{-b}{4}\right), \\ P_5 &= \left(\frac{-\sqrt{5}b}{2\sqrt{2}}, \frac{-\sqrt{5}b}{2\sqrt{6}}, \frac{-\sqrt{5}b}{4\sqrt{3}}, \frac{-b}{4}\right), \end{aligned} \tag{40}$$

where $b^2 = \frac{2}{5}l^2$. The five tetrahedrons are now constructed by joining all these vertices with straight lines so that our model has the geometry of a 3-dimensional pentahedron. The 3-sphere of the exact solution may be expected to intersect this pentahedral hypersurface on 2-dimensional spheres with the centers of the tetrahedrons as their centers. It is fairly straightforward to show that, if these spheres intersect the faces of the tetrahedrons they do so on circles of radius $\bar{a}$ related to a and b by

$$\bar{a}^2 = a^2 - \tfrac{1}{6}b^2\,. \tag{41}$$

We shall consider three possible choices for the test particle at which the proper time, τ, is to be measured in our model:

(1) Test particle at the center of a tetrahedron

$$d\tau_1 = dt\,. \tag{42}$$

(2) Test particle at a tetrahedron vertex

$$d\tau_2{}^2 = dt^2 - \tfrac{3}{8}dl^2/c^2\,. \tag{43}$$

(3) In the embedding space choice 1 corresponds to measuring the proper time on a sphere which is inscribed in the pentahedral hypersurface of our model just touching this surface at the center of each face. Choice 2 corresponds to taking the other extreme – i.e., measuring the proper time at a point on a sphere enclosing the pentahedron and just touching it at the tetrahedron vertices. All other possible choices for the test particle lie between these spheres. As our third choice we take a test particle which, at the moment of time symmetry, lies on the intersection of the 3-sphere of the exact solution and the pentahedron of our model. Using Eq. (41) we obtain

$$d\tau_3{}^2 = dt^2 - \left(\frac{a_0{}^2}{l_0{}^2} - \frac{1}{40}\right)\frac{dl^2}{c^2}\,. \tag{44}$$

Using Eqs. (42)–(44) we may write

$$\begin{aligned} \frac{dU}{d\tau_1} &= \frac{5l^2}{2\sqrt{2}}\frac{dl}{dt}\,, \\ \frac{dU}{d\tau_2} &= \frac{5l^2}{2\sqrt{2}}\left[1 - \frac{3}{8c^2}\left(\frac{dl}{dt}\right)^2\right]^{-1/2}\frac{dl}{dt}\,, \\ \frac{dU}{d\tau_3} &= \frac{5l^2}{2\sqrt{2}}\left\{1 - \frac{1}{c^2}\left[\left(\frac{a_0}{l_0}\right)^2 - \frac{1}{40}\right]\left(\frac{dl}{dt}\right)^2\right\}^{-1/2}\frac{dl}{dt}\,, \end{aligned} \tag{45}$$

where l and dl/dt are given in Eqs. (34) and (35).

In Fig. 4 we show graphs of $dU/d\tau$ against U for these three choices. We see that the curves are only slightly different for large U. However, as U decreases, we find that $dU/d\tau_2$ and $dU/d\tau_3$ become infinite at a value of U corresponding to the time at which the test particle is moving with the speed

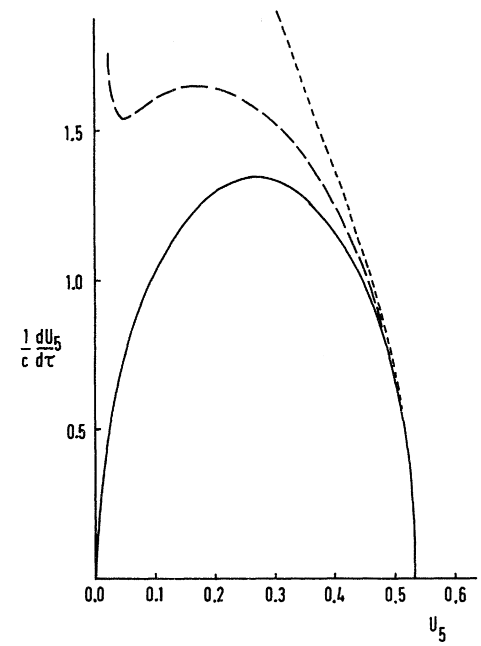

FIG. 4. Rate of change of the volume of the universe plotted against the volume for $MG/c^2 = 1$, with the test particle at the center of the block ———— , at a vertex - - - - - - - , and, at the moment of time symmetry, on the intersection of the 3-sphere of the analytic solution and the pentahedron of our model — — — — .

of light in our original coordinate frame. This never occurs for test particle (1). ($dU/d\tau$ will become infinite before U reaches zero if τ is measured at any test particle situated outside a sphere passing through the middle of the edges of a tetrahedron.) Thus we see that, away from the moment of time symmetry, $dU/d\tau$ is very dependent on the choice of the point at which τ is measured. Similar conclusions may be drawn for the 16- and 600-tetrahedron models.

These problems connected with the choice of test particle used to measure the proper time in our model universes demonstrate the difficulty in interpreting the models rather than any particular failure in the techniques used.

In this paper we have seen how Regge calculus may be used to trace the time development of the Friedmann universe. Although difficulties in interpretation are encountered, it has been shown that the techniques do provide a method of obtaining numerical solutions to Einstein's equations. For problems with less symmetry than the one considered here the practical difficulties of constructing successive spacelike hypersurfaces will increase. However, with the availability of high speed computers, it is possible to contemplate the construction of programs to solve the constraint equations once the conceptual difficulties of setting up a model for a particular physical problem have been overcome.

ACKNOWLEDGMENT

One of us (P. A. C.) wishes to thank the Science Research Council for a Postdoctoral Research Fellowship.

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[1]See, for example, H. Bondi, *Cosmology* (Cambridge Univ. Press, Cambridge, England, 1961).

[2]See A. Einstein, *The Meaning of Relativity* (Princeton Univ. Press, Princeton, N. J., 1950), p. 107.

[3]A. A. Friedmann, Z. Physik 10, 377 (1922); 21, 326 (1924); R. C. Tolman, Proc. Nat. Acad. Sci. U. S. 20, 169 (1934).

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[6]J. A. Wheeler, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).

[7]See, for example, J. A. Wheeler, *Geometrodynamics* (Academic, New York, 1962), p. 60; or Ref. 6.

[8]We use one imaginary coordinate, $x_4 = ict$.

[9]R. W. Lindquist and J. A. Wheeler, Rev. Mod. Phys. 29, 432 (1957). In this paper a similar problem of comparing an approximate model with the Friedmann universe is considered.