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# Pion Condensation in Superdense Nuclear Matter

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Previous work, based on the interaction of only one mode of the pion field with nucleons, indicated the possibility that the ground state of nuclear matter at densities above some onset density would consist of a coherent mixture of protons, neutrons, and pions, the latter condensed in a plane-wave state of momentum ≈170 MeV/$c$. The model is now made more realistic by inclusion of ordinary nuclear forces, emission and absorption of noncondensed pion modes, $S$-wave pion-nucleon interactions, and pion-pion interactions. A pion condensation is again predicted, with onset between $\rho = 0.5$ baryons/F$^3$ and $\rho = 1$ baryon/F$^3$. Some possible consequences for neutron stars are discussed.

## I. INTRODUCTION

Superdense nuclear matter, or infinite matter at densities higher than those of nuclei, is of interest in connection with neutron-star theory, supernova theory, and the theory of the early stages of the big bang. Of particular interest is the equation of state for cold superdense matter (where the equation of state is concerned the temperature must be in the MeV region in order to be a significant factor). To find the equation of state it is necessary to find the energy density as a function of the baryon density in the ground state of an electrically neutral system with a certain number of baryons confined in a volume $V$.[1] The equation of state then can be used to determine neutron-star parameters, most notably the mass-radius relationship and the maximum allowable mass.[2]

In the neutron-star regime the baryon densities will range from nuclear densities, 0.15 baryons/F$^3$, to perhaps 6 baryons/F$^3$. In almost all theoretical treatments the matter has been assumed to consist entirely of fermions, that is, baryons and leptons.[3] In the lower ranges of density, $0.15 < \rho < 1$ F$^{-3}$, it turns out, in these theories, that the composition is largely neutrons, with a small fraction of protons neutralized by electrons. It is at least conceivable that the successful theory of nuclear matter, based on phenomenological nucleon-nucleon potentials, can be extended reliably into this density regime, and this has been the aim of considerable recent work.[4-6]

Above a density of about 1 baryon/F$^3$, other baryons, $p$, $\Lambda$, $\Sigma^-$, $\Sigma^0$, etc., are thought to enter in appreciable fractions.[4,5] In this regime it is clear that theoretical calculations become shakier. The forces between most of the baryon pairs are unknown experimentally. Important contributions of many-body forces become a likelihood as densities are increased. There is also always the question of whether the list of constituents is complete.

In the present work we investigate the possibility that pions are a significant constituent of the ground state of superdense matter, even in the lower density region, $\rho < 1$ F$^{-3}$, in which neutrons are presently thought to be the dominant constituent. It will turn out that $\pi^-$ particles are the only serious contender (they are a natural possibility since they can neutralize proton charge without costing electron Fermi energy).

Clearly the realization of such possibilities will depend on the strong interaction of pions with nucleons. It would be natural to begin an investigation by asking what is known experimentally and theoretically about the interactions of $\pi^-$ particles in nuclei. For $A=2Z$ nuclei the optical potential of Auerbach *et al.*[7] fits the medium-energy $\pi^-$-nucleus scattering data quite well. The real part of this potential is a strongly attractive velocity-dependent potential derived from the low-energy $P$-wave pion-nucleon phase shifts. It can be expressed as a modification of the energy-momentum relation for a $\pi^-$ in the medium,

$$\omega^2 = k^2(1-6\rho)+m_\pi^{\ 2}, \qquad (1.1)$$

where $\rho$ is measured in nucleons/$\mathrm{F}^3$.[8] This potential has the rather strange property that above a certain nucleon density ($\rho \approx 0.16$ nucleons/$\mathrm{F}^3$) the energy is lower for a moving pion than for a $k=0$ pion.

This raises the question of whether at some density the matter can lower its energy by creating a condensed phase of $\pi^-$ particles, all in a state with some momentum $\vec{k}$. However, this question cannot be answered on the basis of the above $\pi^-$ potential in nuclear matter, basically because when $\omega \to 0$, for the $\pi^-$, degenerate perturbation theory must be used to treat the problem. Direct use of the optical potential corresponds roughly to use of nondegenerate second-order perturbation theory. It is necessary instead to go back to the source of the $P$-wave scattering in the emission and absorption of single pions.

We formulate the problem by writing down a Hamiltonian for the system of protons, neutrons, and pions in the form of the free Hamiltonian plus the standard trilinear pion-nucleon Hamiltonian plus the Hamiltonian for those nucleon-nucleon forces which we shall not calculate explicitly from pion exchange.[9] On the basis of this Hamiltonian we wish to solve for the difference between the lowest energy of a pure neutron state and the state of the same nucleon density with a pion condensation. Through a lucky circumstance this energy difference can be calculated reliably even in the presence of strong nuclear forces (such as hard cores) which lead to a highly correlated wave function for the nucleons, as long as these forces are reasonably spin- and isospin-independent. The energy difference will turn out to be almost independent of the nuclear wave functions. Since the single-pion-exchange potential *is* extremely spin-dependent we separate it out and estimate its contribution to the energy difference perturbatively.

The Hamiltonian will be written as

$$H = H_C + H_{NN} + H_\pi + H_S \,.$$

$H_C$ is to include the kinetic energy of the nucleons, treated as nonrelativistic, and of the mesons; and in addition the interaction of nucleons with a single $\pi^-$ mode, which we take as having momentum $k$ in the $-z$ direction,[10,11]

$$H_C = H_0 + \frac{if k\sqrt{2}}{m_\pi(2\omega_k V)^{1/2}} \int d^3x [\bar{n}(\vec{x})\sigma_3 p(\vec{x}) a_{(-)}(-k\hat{z})e^{-ikz} + \bar{p}(\vec{x})\sigma_3 n(\vec{x}) a^\dagger_{(-)}(-k\hat{z})e^{ikz}]\,. \qquad (1.2)$$

Here $a_{(-)}(-k\hat{z})$ is the destruction operator for a $\pi^-$ of momentum $\vec{k}=-k\hat{z}$. The constant $f=1.1 = [4\pi(0.088)]^{1/2}$. $H_0$ contains the kinetic-energy terms.

The next term, $H_{NN}$, will contain the conventional nucleon-nucleon interaction with the exclusion of single-pion-exchange terms. $H_\pi$ will contain the interaction of nucleons with all the other modes of the pion field. $H_S$ will contain pion-nucleon and pion-pion $S$-wave effects.

In Sec. II we solve the problem characterized by $H_C$ exactly, in the $V\to\infty$ limit, for both the energy and the wave function of the ground state and all excited states. In Sec. III we show how the result of Sec. II, for the energy difference between the condensed and noncondensed states, survives even in the presence of the strong nucleon-nucleon forces in $H_{NN}$, as long as they are isospin- and spin-independent. In Sec. IV the second-order energy perturbations due to noncondensed pions are evaluated, using $H_C$ as an unperturbed Hamiltonian. In Sec. V the effects of $S$-wave $\pi N$ and $\pi\pi$ interactions are included.

The results taken together will argue strongly for the onset of a pion condensation at some density between $\rho=0.25\ \mathrm{F}^{-3}$ and $\rho=1\ \mathrm{F}^{-3}$. In Sec. VI we discuss the effect of electromagnetic forces and some of the possible consequences of the condensation for neutron stars.

## II. THE MEAN-FIELD SOLUTION

Phase changes associated with condensed modes are well-known phenomena in many-body systems. For example, superfluid He, superconducting metals, and lasers are all characterized by the existence of a finite field amplitude for a particle mode. This so-called condensate requires that the mode be macroscopically and coherently occupied so that it has both a well-defined amplitude and phase. For three-dimensional systems, a self-consistent mean-field approach provides a useful starting point because of phase-space suppression of the low-lying long-wavelength fluctuations of the condensate. Once the mean-field ground state and excitations are obtained, the question of the modification of the normal-state correlations can be considered. Therefore, in this section, we will study the mean-field problem of $N$ baryons interacting with a condensed $\pi^-$ mode. Our solution of this problem shows that at a critical baryon density the normal neutron Fermi sea becomes unstable with respect to the formation of an infinitesimal $\pi^-$ condensate, and that at larger densities the ground state has a $\pi^-$ condensate. The elementary excitations in this new state as well as its physical properties will be discussed.

Replacing the $\pi^-$ field by the mean field describing the condensate mode and neglecting the noncondensed pion interactions, the Hamiltonian consists of the kinetic energy of the protons and neutrons, the pion energy, and the $P$-wave part of the pion-nucleon interaction,

$$H_C=\sum_{qs}\frac{q^2}{2M}(\bar{p}_{qs}p_{qs}+\bar{n}_{qs}n_{qs})+XN\omega_k +\sum_q\sqrt{XN}\,M_k(\bar{p}_{q+k\uparrow}n_{q\uparrow}-\bar{p}_{q+k\downarrow}n_{q\downarrow})+\text{H.c.} \tag{2.1}$$

Here $\bar{p}_{qs}$ and $\bar{n}_{qs}$ are creation operators for protons and neutrons of momentum $q$ and spin $s$, and $\omega_k$ is the free pion energy. We take nonrelativistic kinematics for the nucleons; $M$ is the nucleon mass. The mean field amplitude $\sqrt{XN}$ corresponds to a condensate which has on the average $XN$ negative pions in a mode of momentum $\vec{k}=-k\hat{z}$. It is convenient to represent the occupation of this mode as a fraction $X$ of the total baryons $N$. The matrix element $M_k$ has the form implied by (1.2):

$$M_k=\frac{-ifk}{m_\pi(\omega_k V)^{1/2}}, \tag{2.2}$$

with $f=1.1$, and $V$ the volume.

In order to maintain over-all charge neutrality, the number of protons must be equal to the number of pions. Actually, as we mentioned, in order to have a well-defined phase for the pion amplitude, the ground state must be a linear combination of states with different number of pions. This implies that the proton and neutron numbers also vary. Then since $XN$ represents the average pion occupation, charge conservation and baryon conservation imply that

$$XN=\sum_{qs}\langle\bar{p}_{qs}p_{qs}\rangle, \qquad N(1-X)=\sum_{qs}\langle\bar{n}_{qs}n_{qs}\rangle. \tag{2.3}$$

Finally, in order to balance momentum we demand that the protons recoil so that[12]

$$XNk=\sum_{qs}q_z\langle\bar{p}_{qs}p_{qs}\rangle. \tag{2.4}$$

The constraints on the state, Eqs. (2.3) and (2.4), are conveniently dealt with by introducing Lagrange multipliers and writing an effective Hamiltonian

$$\mathcal{H}_C=H_C-\mu_1\sum_{qs}\bar{p}_{qs}p_{qs}-\mu_2\sum_{qs}\bar{n}_{qs}n_{qs}-\vec{\lambda}\cdot\sum_{qs}\vec{q}\,\bar{p}_{qs}p_{qs}. \tag{2.5}$$

Here the chemical potential $\mu_1$ and $\mu_2$ for the protons and neutrons are fixed by the average-number requirements Eq. (2.3), while it follows from the constraint Eq. (2.4) that $\lambda=\vec{k}/M$. Collecting the terms in Eq. (2.5), $\mathcal{H}_C$ takes the simple form

$$\mathcal{H}_C=\sum_{qs}\left[\left(\frac{q^2}{2M}-\mu_1'\right)\bar{p}_{q+k,s}p_{q+k,s}+\left(\frac{q^2}{2M}-\mu_2\right)\bar{n}_{qs}n_{qs}+(-1)^s(\sqrt{XN}M_k\bar{p}_{q+k,s}n_{qs}+\sqrt{XN}M_k^*\bar{n}_{qs}p_{q+k,s})\right]+XN\omega_k, \tag{2.6}$$

with $\mu_1'=\mu_1+k^2/2M$.

The quadratic form Eq. (2.6) is easily diagonalized by making the canonical transformation

$$\bar{u}_q=(1-\theta^2)^{1/2}\bar{n}_q+i\bar{p}_{q+k}\sigma_z\theta, \qquad \bar{v}_q=(1-\theta^2)^{1/2}\bar{p}_{q+k}+i\bar{n}_q\sigma_z\theta, \tag{2.7}$$

with

$$\frac{\theta(1-\theta^2)^{1/2}}{1-2\theta^2}=\frac{\sqrt{XN}\,|M_k|}{\mu_2-\mu_1'}. \tag{2.8}$$

Here we have introduced a spinor notation in which, for example, $\bar{n}_q=(\bar{n}_{q\uparrow},\bar{n}_{q\downarrow})$. Expressing $\mathcal{H}_C$ in terms of the $u$ and $v$ operators yields the diagonal form

$$\mathcal{H}_C=\sum_q\left[\left(\frac{q^2}{2M+\Omega_-}\right)\bar{u}_qu_q+\left(\frac{q^2}{2M+\Omega_+}\right)\bar{v}_qv_q\right]+XN\omega_k, \tag{2.9}$$

with

$$\Omega_\pm=\tfrac{1}{2}(\mu_1'+\mu_2)\pm\{\tfrac{1}{2}[(\mu_1'-\mu_2)]^2+XN|M_k|^2\}^{1/2}. \tag{2.10}$$

Now, the ground-state energy is determined by occupying the lowest-$N$ single-particle baryon states. For the densities of interest, the splitting between the $u$ and $v$ states exceeds the Fermi energy $\mu$ of the normal-state neutron sea

$$2\{\tfrac{1}{2}[(\mu_1' - \mu_2)]^2 + XN|M_k|^2\}^{1/2} > \mu \,. \qquad (2.11)$$

This important feature implies that the ground state is obtained by filling the single-particle $u$ states inside the usual Fermi sphere of radius $p_F$ set by the total baryon density $(3\pi^2 N/V)^{1/3}$:

$$|\psi_0\rangle = \prod_{q \leq p_F} \bar{u}_q |\mathrm{vac}\rangle \,. \qquad (2.12)$$

In this ground state, the neutron and proton occupation numbers are, respectively,

$$\langle\psi_0|\bar{n}_{qs} n_{qs}|\psi_0\rangle = \begin{cases} 1-\theta^2, & q < p_F \\ 0, & p_F < q \end{cases}$$

and (2.13)

$$\langle\psi_0|\bar{p}_{qs} p_{qs}|\psi_0\rangle = \begin{cases} \theta^2, & |q+k| < p_F \\ 0, & p_F < |q+k| \end{cases}.$$

The charge-neutrality constraint, Eq. (2.3), becomes

$$XN = \sum_{qs} \langle\psi_0|\bar{p}_{qs} p_{qs}|\psi_0\rangle = N\theta^2 \,. \qquad (2.14)$$

The relation between the nucleon chemical potentials and the parameter $X$ (the fraction of protons) follows from (2.8):

$$\frac{\sqrt{X}(1-X)^{1/2}}{1-2X} = \frac{\sqrt{N}M_k\sqrt{X}}{\mu_2 - \mu_1'} \,. \qquad (2.15)$$

This is trivially satisfied when $X=0$. However, we shall see that when the baryon density exceeds a critical value, a finite value of $X$ satisfying (2.15) will lead to a lower energy than the normal-state ($X=0$) solution.

Taking the expectation value of $H_C$ in the ground state and using the constraint Eq. (2.14) to eliminate $\theta$ in favor of $X$ we find

$$E_0 = \tfrac{3}{5}\mu N + NX\bar{\omega}_k - 2\sqrt{N}|M_k|X(1-X)^{1/2}N \,, \qquad (2.16)$$

where

$$\bar{\omega}_k = \omega_k + k^2/2M \,, \qquad (2.17)$$

and $E_0$ is the energy of the entire system, not including nucleon rest energy. The first term on the right-hand side of Eq. (2.16) is just the normal-state energy of a free neutron Fermi sea. The remaining part of the energy, the condensation energy, is negative when

$$\alpha_k = \frac{2\sqrt{N}|M_k|}{\bar{\omega}_k} \geq 1 \,. \qquad (2.18)$$

The maximum value of $\alpha_k$ occurs for $k^* \simeq 1.26 m_\pi$. Using this, and taking the equality in Eq. (2.18), leads to a critical baryon density

$$\rho_c = \left(\frac{m_\pi \sqrt{\omega_{k^*}}\,\omega_{k^*}}{2fk^*}\right)^2 . \qquad (2.19)$$

When $\rho$ exceeds $\rho_c$, the condensation energy per baryon is

$$\frac{E_0 - \frac{3}{5}\mu N}{N} = \bar{\omega}_k X[1 - \alpha_k(1-X)^{1/2}] \,. \qquad (2.20)$$

Minimizing this with respect to $X$ gives the condensate amplitude

$$X = 1 - \left(\frac{1}{3\alpha_k}\right)^2 [1 + (1 + 3\alpha_k^2)^{1/2}]^2 \,. \qquad (2.21)$$

The value of $k$ which minimizes the ground-state energy is a function of density. In looking for the lowest-energy ground state we must substitute the condensate amplitude $X$, Eq. (2.21), into the expression for the energy and minimize it with respect to $k$. However, in the homogeneous problem treated here, once a $k$ value is selected, we expect that the condensate will not be able to adjust the $k$ value as the density increases adiabatically even though a new $k$ value might correspond to a yet lower energy. Thus in analogy to the case of current-carrying superfluid states, we will be dealing with a metastable state. Therefore, in determining a property, such as the pressure versus density, of the homogeneous condensed state, we will calculate with fixed $k$. On a more macroscopic scale, the condensate will have a structure governed by the interplay of the condensate energy density and the electromagnetic energy density. When the macroscopic structure of the condensate is taken into account, an applied force can, for example, compress the macrostructure, distorting the mode of the $\pi^-$ condensate.

Taking $k$ equal to its value at $\rho_c$, the condensation energy per baryon becomes

$$E_0 - \tfrac{3}{5}\mu = \bar{\omega}_{k^*} X[1 - (\rho/\rho_c)^{1/2}(1-X)^{1/2}] \,, \qquad (2.22)$$

where

$$X = 1 - \frac{\rho_c}{9\rho}[1 + (1 + 3\rho/\rho_c)^{1/2}]^2 \,. \qquad (2.23)$$

For densities slightly greater than $\rho_c$, the percentage $X$ of protons varies as $\frac{1}{2}[(\rho/\rho_c) - 1]$, and the condensation energy per baryon is given by $-0.125\bar{\omega}_{k^*}(\rho/\rho_c - 1)^2$. In the high-density limit where $\rho/\rho_c \gg 1$, it follows from Eq. (2.23) that the percentage of protons approaches 67% while the condensation energy is $-0.38N\bar{\omega}_{k^*}(\rho/\rho_c)^{1/2}$.

Although this state is anisotropic because of the geometry of the $\pi^-$ mode, the pressure calculated formally by varying the energy with respect to the baryon density is still of interest. Note that, as previously mentioned, we do not allow $k^*$ to vary. Then

$$P=\rho^2\frac{\partial(E_0/N)}{\partial\rho}. \qquad (2.24)$$

Using the previous expressions we find that the fractional pressure loss due to the condensation is

$$\frac{P_N-P}{P_N}=-\frac{5\overline{\omega}_k}{2\mu_c}\left(\frac{\rho}{\rho_c}\right)^{1/3}\frac{\partial}{\partial(\rho/\rho_c)}\times\{X[1-(\rho/\rho_c)^{1/2}(1-X)^{1/2}]\}, \qquad (2.25)$$

with $X$ given as a function of $(\rho/\rho_c)$ by Eq. (2.23), and $\mu_c$ equal to the Fermi energy at the critical density $\rho_c$. $P_N$ is the pressure of the free Fermi gas of neutrons. This pressure difference is plotted as a function of $(\rho/\rho_c)$ in Fig. 1. Near the critical density and for densities large compared to $\rho_c$, we can use the expressions for the energy previously discussed to obtain approximate expressions for the pressure

$$\frac{P_N-P}{P_N}\cong\begin{cases}\left(\frac{5\overline{\omega}_k}{2\mu_c}\right)\left(\frac{1}{4}\right)\left(\frac{\rho}{\rho_c}-1\right), & \frac{\rho}{\rho_c}\gtrsim 1\\ \left(\frac{5\overline{\omega}_k}{2\mu_c}\right)(0.19)\left(\frac{\rho}{\rho_c}\right)^{-1/6}, & \frac{\rho}{\rho_c}\gg 1\end{cases} \qquad (2.26)$$

Having discussed the ground-state properties of the mean-field solution, we turn to the excited states. These correspond to quasiholes in the $u$ sea and excited $u$ or $v$ states. Making use of the charge-neutrality constraint, we replace the $\theta$ factors in the canonical transformation Eq. (2.7) by $\sqrt{X}$ and also absorb the factors of $i$ in the phase of the $\pi^-$ field amplitude. Then

$$\overline{u}_q=(1-X)^{1/2}\overline{n}_q+\sqrt{X}\overline{p}_{q+k}\sigma_z,$$
$$\overline{v}_q=(1-X)^{1/2}\overline{p}_{q+k}\sigma_z-\sqrt{X}\,\overline{n}_q. \qquad (2.27)$$

Note that in the limit in which $X$ vanishes, $u$ goes over to the neutron operator and $v$ becomes the proton operator. In this limit, the ground state Eq. (2.12) is simply a filled Fermi sea of neutrons.

At a finite condensate density, $u$ and $v$ are linear combinations of neutron and proton operators. Clearly they do not conserve charge. The origin of this is the simple form of the mean-field theory in which the creation and destruction operators of the $\pi^-$ field were replaced by the $c$-number amplitude $\sqrt{XN}$. Here average charge conservation was obtained using the constraint, (2.3), relating the total proton number to the total pion number $XN$. In dealing with the excitations, as we will in Sec. IV, where we examine the effect of the noncon-

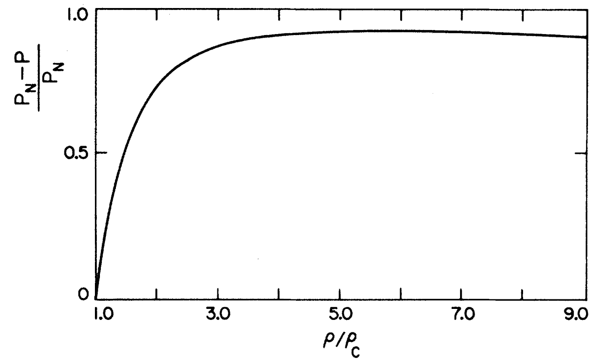


FIG. 1. The fractional pressure loss of the condensed state as compared to the normal state. In (2.25), $\overline{\omega}_k=1.66m_\pi c^2$, $\mu_c=0.57m_\pi c^2$.

densed pions, it is convenient to modify the mean-field formalism slightly to obtain an explicitly charge-conserving theory. In addition this will allow us to obtain a clearer picture of the coherent nature of the new ground state.

Suppose that instead of simply replacing the pion field operators $b_{-k}^\dagger$ and $b_{-k}$ for the mode $-k$ by $\sqrt{XN}$, we introduce the formal condensate field operators

$$b_{-k}^\dagger\rightarrow\sqrt{XN}\tilde{\Pi}^\dagger b_{-k}\rightarrow\sqrt{XN}\tilde{\Pi}. \qquad (2.28)$$

Here the normalized operator $\tilde{\Pi}^\dagger$ acting on a state with $N$ condensed pions transforms it to the corresponding state with $N+1$ condensed pions, while $\tilde{\Pi}$ reduces the number of condensed pions by one. Furthermore, to order $(1/\sqrt{NX})$,

$$\tilde{\Pi}^\dagger\tilde{\Pi}\simeq\tilde{\Pi}\tilde{\Pi}^\dagger\simeq 1. \qquad (2.29)$$

Using the operator replacement Eq. (2.28) and the relations given by Eq. (2.29), the diagonalization of $\mathcal{H}_C$ proceeds just as before. The eigenvalues $q^2/2M+\Omega_\mp$ remain the same; however, the eigenoperators (2.27) now take the explicitly charge-conserving forms

$$\overline{u}_q'=(1-X^2)^{1/2}\overline{n}_q+\sqrt{X}\tilde{\Pi}^\dagger\overline{p}_{q+k}\sigma_z,$$
$$\overline{v}_q'=(1-X^2)^{1/2}\tilde{\Pi}^\dagger\overline{p}_{q+k}\sigma_z-\sqrt{X}\overline{n}_q. \qquad (2.30)$$

A proton creation operator is accompanied by a $\pi^-$ condensate creation operator. Using this notation, the ground state takes the form[13]

$$|\psi_0\rangle=\prod_{q<p_F}[(1-X^2)^{1/2}\overline{n}_q+\sqrt{X}\tilde{\Pi}^\dagger\overline{p}_{q+k}\sigma_z]\,|\text{vac}\rangle \qquad (2.31)$$

and we see that it is a coherent state of neutrons, protons, and $\pi^-$'s.

## III. INCLUSION OF SHORT-RANGE NUCLEAR FORCES

In this section we show how our result for the energy difference between the pure neutron case and the condensed pion case survives the inclusion of at least a considerable number of nucleon-nucleon interaction effects. We shall assume that the Hamiltonian term $H_{NN}$ (the nuclear force minus the single-pion exchange force) can be written as

$$H_{NN}=\tfrac{1}{2}\int(\overline{N}(x)N(x))\,V_1(x-x')(\overline{N}(x')N(x'))d^3x\,d^3x'+\sum_{i=1}^{15}\int(\overline{N}(x)\Gamma_iN(x))\,V_2(x-x')(\overline{N}(x')\Gamma_iN(x'))\,d^3x\,d^3x'\,, \tag{3.1}$$

where we have introduced the isospin notation for nucleon fields, $(\overline{N}(x)N(x))=\overline{p}(x)p(x)+\overline{n}(x)n(x)$, and the $\Gamma_i$ are the matrices;

$$\Gamma_i=\sigma_j,\ \tau_k,\ \sigma_m\tau_n\,,\quad i=1,2,\ldots,15\,.$$

This is the most general nucleon-nucleon potential invariant under Wigner $SU_4$ (the transformations generated by the matrices $\Gamma_i$). We have hopes that for the purposes of determining the nuclear wave functions the most important part of the force (the short-range repulsion) is of type $V_1$ in Eq. (3.1). The $V_2$ term is probably an unnecessary luxury. Note that the most dramatically $SU_4$-noninvariant force, the tensor force arising from single-pion exchange, was not included in $H_{NN}$ and will be treated separately.

In a lowest-order perturbation-theoretic calculation it is easy to see why an interaction of the $V_1$ form in (3.1) gives the same energy shift for a pure neutron state and for our condensed state. In the condensed pion state found in Sec. II the density-density correlation function $\langle\rho(x)\rho(x')\rangle$ can be calculated, where $\rho$ is the baryon density operator $(\overline{N}(x)N(x))$, and it has the same value as for a free neutron gas. We can develop a much farther reaching result, however, in which all orders of $V_1$, $V_2$ and distortions of the wave function are taken into account.

Using the $u$ and $v$ fields introduced in (2.27) we have

$$\begin{aligned} n(\vec{x})&=(1-\theta^2)^{1/2}u(\vec{x})-\theta v(\vec{x})\,,\\ p(\vec{x})&=[\theta\sigma_3u(\vec{x})+(1-\theta^2)^{1/2}\sigma_3v(\vec{x})]\,e^{+ikz}\,. \end{aligned} \tag{3.2}$$

Note first that $(\overline{N}(\vec{x})N(\vec{x}))=\overline{u}(\vec{x})u(\vec{x})+\overline{v}(\vec{x})v(\vec{x})$. Thus the $V_1$ term in $H_{NN}$ has the identical form in the fields $u$ and $v$ as in the fields $n$ and $p$. As for the $V_2$ term, we note that except for the $e^{+ikz}$ factor, the transformation from $(n,p)$ to $(u,v)$ is an $SU_4$ transformation. Thus if the potential $V_2$ has a range $a$ and $ka\ll 1$, the $V_2$ term in (3.1) has very nearly the same form in the fields $u$ and $v$ as in the fields $n$ and $p$.

Now we consider the Hamiltonian $H_C+H_{NN}$ expressed in terms of the new fields $u$ and $v$, with the classical substitution for the meson operators in $H_C$. We form an $\mathcal{H}$ exactly as in (2.5),

$$\begin{aligned} \mathcal{H}&=\mathcal{H}_C+H_{NN}\\ &=\sum_{q}\frac{q^2}{2m}(\overline{u}_qu_q+\overline{v}_qv_q)+H_{NN}(u,v)+\Omega_-(\theta)\int\overline{u}u\,d^3x\\ &\quad+\Omega_+(\theta)\int\overline{v}v\,d^3x+\theta^2N\omega_k\,. \end{aligned} \tag{3.3}$$

The ground state for the pure neutron system under the influence of the forces contained in $H_{NN}$ is the lowest eigenstate of $\mathcal{H}$ for the choice $\theta=0$. This ground state will consist of some correlated state of $u$ particles (neutrons in this case). For the condensed case, $\theta>0$, we see that the eigenfunctions of $\mathcal{H}$ are the same states of $u$ particles (which are now mixtures of neutron and proton) as those of the $\theta=0$ case. This observation follows from (a) the fact that $H_{NN}(u,v)$ is independent of $\theta$, and (b) the fact that the term $\Omega_-(\theta)\int\overline{u}u$, which *is* $\theta$-dependent, measures only the number of $u$ particles and is independent of the correlations.

The energies of the two states differ through the $\Omega_-(\theta)$ and $\theta^2N\omega_k$ terms as well as through the Lagrange multiplier terms which must be added to $\mathcal{H}$ to give $H$. At given densities of protons and neutrons all of these terms are the same as for the case with $H_{NN}=0$. Thus the difference in energy per baryon between the pure neutron state and the state with condensed pions is again given by (2.20). We note that the proton density $\langle\overline{p}(x)p(x)\rangle$ is given by $\theta^2\rho_{\text{baryon}}$, as before, independently of the wave function, in any state of $u$ particles only.

## IV. NONCONDENSED-PION CORRECTION

In the previous section it was seen that it is the spin- or isospin-dependent nucleon-neucleon forces which might lead to significant changes in our results. It is therefore important to examine the effects of the most prominent of such forces, the single-pion-exchange force. However, in what follows it will become clear that it would be a mistake to reduce the single-pion exchange to a static potential and calculate the expectation value in our new ground state, because the intermediate states implicit in the potential itself are modified by the condensation. Instead we shall begin with the Hamiltonian for the emission and absorption of pions (for all the uncondensed modes) and apply second-order perturbation theory directly, using intermediate energies derived from $H_C$, rather than from $H_0$.

We begin by expressing the pion-nucleon interaction in terms of the new fields $u'$ and $v'$ of (2.30):

$$H_\pi = \int \mathcal{H}_\pi d^3x ,$$

$$\mathcal{H}_\pi = \frac{if}{m_\pi}[\vec{\nabla}\pi^0 \cdot (\bar{p}\vec{\sigma}p - \bar{n}\vec{\sigma}n) + \sqrt{2}\,\vec{\nabla}\pi^+ \cdot (\bar{n}\vec{\sigma}p) + \sqrt{2}\,\vec{\nabla}\pi^- \cdot (\bar{p}\vec{\sigma}n)]$$

$$= \frac{if}{m_\pi}\{\vec{\nabla}\pi^0 \cdot [\bar{u}'(-\vec{\sigma} + \theta^2\vec{s})u' + \bar{v}'\vec{s}\theta(1-\theta^2)^{1/2}u' + \bar{v}'\vec{s}\theta(1-\theta^2)^{1/2}u' + \bar{v}'(-\vec{s}\theta^2 + \sigma_3\vec{\sigma}\sigma_3)v']$$

$$+ \sqrt{2}\vec{\nabla}\pi^+ \cdot [\theta(1-\theta^2)^{1/2}\bar{u}'\vec{\sigma}\sigma_3 u' - \theta^2\bar{v}'\vec{\sigma}\sigma_3 u' + (1-\theta^2)\bar{u}'\vec{\sigma}\sigma_3 v' - \theta(1-\theta^2)^{1/2}\bar{v}'\vec{\sigma}\sigma_3 v']e^{ikz}\tilde{\Pi}^\dagger$$

$$+ \sqrt{2}\,\vec{\nabla}\pi^- \cdot [\theta(1-\theta^2)^{1/2}\bar{u}'\sigma_3\vec{\sigma}u' - \theta^2\bar{u}'\sigma_3\vec{\sigma}v' + (1-\theta^2)\bar{v}'\sigma_3\vec{\sigma}u' - \theta(1-\theta^2)^{1/2}\bar{v}'\sigma_3\vec{\sigma}v']e^{-ikz}\tilde{\Pi}\} . \qquad (4.1)$$

The spin vector $\vec{s}$ here has components $s_1 = 0$, $s_2 = 0$, $s_3 = 2\sigma_z$.

For the unperturbed ground state $|\psi_0\rangle$ we take a sea of plane-wave $u$ particles filled up to the Fermi momentum $p_F$ corresponding to the ground state, (2.12), discussed in Sec. II. The intermediate states generated by applying $\mathcal{H}_\pi$ to $|\psi_0\rangle$ contain a hole in the $u$ sea, an excited $u$ or $v$ particle, and one noncondensed pion. In addition, if the noncondensed pion is negatively charged a pion is removed from the condensate, while if it is positively charged a pion is added to the condensate. The intermediate-state energies relative to the ground-state energies are obtained in the usual way using the $u$-$v$ single-particle energies given by $q^2/2M + \Omega_\pm$. The energy associated with adding a $\pi^-$ to the condensate is just the chemical potential of the $\pi^-$ condensate $\mu_\pi$. For example, consider the virtual processes corresponding to the second-to-last term in Eq. (4.1). Here a $u$ particle and a condensed $\pi^-$ are scattered to a $v$ particle and a noncondensed $\pi^-$,[14]

$$u(\vec{p}) + \tilde{\Pi}^- \to v(\vec{p} - \vec{q} + \vec{k}) + \pi^-(\vec{q}) . \qquad (4.2)$$

In second-order perturbation theory this process gives the energy correction

$$-\mathrm{Tr}\sum_{p<p_F}\sum_q \frac{|f|^2}{\omega_q V m_\pi^2}\frac{(1-\theta^2)|\vec{q}\cdot\vec{\sigma}\sigma_z|^2}{[(\vec{p}-\vec{q}+\vec{k})^2(2M)^{-1} + \Omega_-] + \omega_q - [p^2(2M)^{-1} + \Omega_+] - \mu_\pi} , \qquad (4.3)$$

with the trace being carried out over the spin variables. The energy denominator just corresponds to the energy of an excited $v$ particle of momentum $\vec{p} - \vec{q} + \vec{k}$, a noncondensed pion of momentum $q$, a hole in the $u$ sea of momentum $p$, and a hole in the pion condensate.

We can understand the energy denominators in, for example, Eq. (4.3), without the artifice of the $\tilde{\Pi}$ operator. If we returned to the original definition of $u$ and $v$ operators, (2.27) [instead of (2.30)], then the $\tilde{\Pi}^-$ operators would be deleted in (4.1) and we would consider the process $u(\vec{p}) \to v(\vec{p} - \vec{q} + \vec{k}) + \pi^-(\vec{q})$. The imbalance of momentum comes from the exponential factor in the $\pi^-$ emission part of (4.1). According to (2.9) we gain additional energy $\Omega_+ - \Omega_-$ by replacing a $u$ by a $v$ of the same momentum. But the energy of the whole medium has changed in another way; since charge is conserved, the emission of the noncondensed $\pi^-$ must be compensated by a change in the charge of the residual system of nucleons plus condensed pions. This can come about in either of two ways: The condensate can give up a $\pi^-$, or the parameter $\theta$ can change infinitesimally $[O(1/N)]$ in order to make one less unit of baryon electric charge. At the equilibrium density of condensed $\pi^-$'s the energy loss (or gain) in giving up one $\pi^-$ is exactly equal to the gain (or loss) in changing a proton to a neutron. Thus in either case we have only to calculate the energy change when one condensed pion is added, with the proton and neutron occupancy staying the same, that is, the chemical potential of the $\pi^-$, $\mu_\pi$.

To evaluate the second-order energy shift associated with the noncondensed pions we shall regulate the integrals by putting a sharp cutoff on the momentum of the pion, through a factor $\Lambda(\vec{q}) = \theta(k_{\max} - |\vec{q}|)$. Such a cutoff is conventional in meson theory and represents an effect which is almost certainly present, the suppression of single-meson emission at high momentum. In the present problem there is an additional reason for imposing a rather low cutoff. We are calculating the energy shift using plane waves for the unperturbed system. This calculation will probably overestimate the energy shift in any event, since the correlations which keep particles apart at short distances have been omitted. A cutoff on pion momentum at least suppresses the singularity of the pion exchange force at short distances, which should partially compensate for having omitted the hard-core effects from the wave functions.

We make two more simplifications: (1) Neglect of the nucleon kinetic-energy differences. Static kinematics, $M\to\infty$, for the nucleons will suffice in evaluating perturbations of this kind. (2) Neglect of the effect of $k\hat{z}$ on the integration regions. This is not a superb approximation since in the region of interest $p_F$ runs from 2.5 to 3.5 times $k$ (as it increases the approximation improves). However, it is good enough for the 20% to 30% accuracy we need in making our estimate.

Defining the momentum of the intermediate nucleon as $\vec{p}\,'$ and setting $\vec{p}-\vec{p}\,'=\vec{q}$, $\theta^2=X$, we obtain the following terms in the second-order energy shift.

$$\begin{aligned}
&u\to u+\pi^0\to u:\quad \delta E_a=C\int_{<p_F}d^3p\int_{>p_F}d^3p'\,\frac{q^2-4Xq_3{}^2+4X^2q_3{}^2}{\omega_q{}^2}\Lambda(q)\,,\\
&u\to u+\pi^+ +\tilde{\Pi}\to u:\quad \delta E_b=C\int_{<p_F}d^3p\int_{>p_F}d^3p'\,\frac{2X(1-X)q^2}{\omega_q(\omega_q+\mu_\pi)}\Lambda(q)\,,\\
&u+\tilde{\Pi}^-\to u+\pi^-\to u+\tilde{\Pi}^-:\quad \delta E_c=C\int_{<p_F}d^3p\int_{>p_F}d^3p'\,\frac{2X(1-X)q^2}{\omega_q(\omega_q-\mu_\pi)}\Lambda(q)\,,\\
&u\to v+\pi^0\to u:\quad \delta E_d=C\int_{<p_F}d^3p\int_{\text{all}}d^3p'\,\frac{4X(1-X)q_3{}^2}{\omega_q(\omega_q+\Omega_+-\Omega_-)}\Lambda(q)\,,\\
&u+\tilde{\Pi}^-\to v+\pi^-\to u+\tilde{\Pi}^-:\quad \delta E_e=C\int_{<p_F}d^3p\int_{\text{all}}d^3p'\,\frac{2(1-X)^2q^2}{\omega_q(\omega_q+\Omega_+-\Omega_--\mu_\pi)}\Lambda(q)\,,\\
&u\to v+\pi^+ +\tilde{\Pi}^-\to u:\quad \delta E_f=C\int_{<p_F}d^3p\int_{\text{all}}d^3p'\,\frac{2X^2q^2}{\omega_k(\omega_k+\Omega_+-\Omega_-+\mu_\pi)}\Lambda(q)\,.
\end{aligned}\tag{4.4}$$

Here the constant $C$ is given by

$$C=-f^2m_\pi{}^{-2}(2\pi)^{-6}\,.\tag{4.5}$$

It remains to evaluate $\mu_\pi$ and perform the integrals. $\mu_\pi$ is calculated by returning to the expectation value of the meson part of the Hamiltonian (2.1) in the ground state $\Pi\bar{u}|\text{vac}\rangle$,

$$\left\langle\sqrt{N_\pi}\sum_{qs}(-1)^sM_k\bar{p}_{q+k,s}n_{q,s}+\text{H.c.}+N_\pi\omega_k\right\rangle=-2N\left(\frac{N_\pi}{V}\right)^{1/2}\theta(1-\theta^2)^{1/2}\frac{kf}{(\omega_k m_\pi)^{1/2}}+N_\pi\omega_k\,,\tag{4.6}$$

and changing $N_\pi$ by +1, giving

$$\begin{aligned}
\mu_\pi&=\delta E'\\
&=-\frac{fK\rho}{m_\pi\sqrt{\omega_k}}\left(\frac{V}{N_\pi}\right)^{1/2}\theta(1-\theta^2)^{1/2}+\omega_k\\
&=-\frac{fK\sqrt{\rho}}{m_\pi\sqrt{\omega_k}}(1-\theta^2)^{1/2}+\omega_k\,.
\end{aligned}\tag{4.7}$$

In the second form for $\mu_\pi$ the equation of constraint $\theta^2\rho V=N_\pi=XN$ has been resubstituted (note that the variation of $\mu_\pi$ has to be performed before this equation is substituted, since we want the response to the addition of a condensed meson with no change in baryon occupancy). For small values of $X$ we note the limits

$$\begin{aligned}
&\mu_\pi\to\tfrac{1}{2}\omega_k\,,\\
&\Omega_+-\Omega_--\mu_\pi\to\omega_kX\,.
\end{aligned}\tag{4.8}$$

Here we have substituted into (4.5) the expression for onset density in terms of $\omega_k$. The energy shift of the pure neutron sea, calculated to second order in $f$, is found by evaluating the sum of terms of (4.4) for the case $X=0$, $\Omega_+-\Omega_-=0$, $\mu_\pi=0$.

It is interesting to note that except for the differences in denominators coming from nonvanishing $\Omega_+-\Omega_-$ and $\mu_\pi$, the sum of the contributions in which the variable $\vec{p}\,'$ was outside the Fermi sphere would be exactly equal to the free-neutron result, for any $X$. These represent contributions from ordinary single-nucleon self-energies which, being the same for a proton and a neutron, should not contribute to the difference in the two cases.

If the differences in the denominators were neglected the contributions in which $p'$ was below the Fermi surface would not equal those for the pure neutron case. The difference would be exactly that calculated from the static one-pion-exchange potential, which splits the energies because of its spin and isospin dependence. More than half of our answer will, in fact, come from this source. But the correction due to the denominators will not be negligible. This raises the question of to what extent the rest of the $NN$ potentials discussed in Sec. III are affected by the presence of the condensate. However, questions on a par with this one have not been settled in ordinary nuclear-matter theory.[15]

We have estimated

$$\delta E_1 = \delta E(\text{condensed case}) - \delta E(\text{neutron case})$$

near onset $\rho \approx \rho_c$, $X \approx 0$ retaining only the terms proportional to $X$. We obtain

$$\delta E_1 \approx 0.5 m_\pi X \quad \text{for } k_{\max} = 3m_\pi ,$$

$$\delta E_1 \approx 0.16 m_\pi X \quad \text{for } k_{\max} = 2m_\pi .$$

A comparison with the numbers in Table I reveals that for the case $k_{\max} = 2m_\pi$ the negative energy gained from the condensation in the absence of pion corrections exceeds the (positive) shift due to pion corrections for $\rho > 0.3$ F$^{-3}$. As the density is increased beyond this point the negative condensation energy per condensed pion grows roughly as $\sqrt{\rho}$ while the pion corrections, already limited by $k_{\max}$, stay more or less constant.

For the case $k_{\max} = 3m_\pi$ the equalization occurs at a density of about $\rho = 0.5$ F$^{-3}$. We actually predict onset before this point since at a given density, now that the repulsive term $\delta E_\pi$ proportional to $X$ is included, it will be advantageous to reduce $X$ below the best value for the previously solved case. We thus anticipate a second-order phase transition in the presence of the pion corrections, with much the same nature as the one which is tabulated in Table I, only with the onset somewhat delayed by the pion corrections.

Use of more realistic wave functions should strengthen this conclusion, since they would act to reduce the single-pion effects.

TABLE I. $\rho$ is density in baryons/F$^3$. $E_c$ is the condensation energy per baryon of Eq. (2.20) for the values of $k$ and $X$ which minimize $E_c$. The $k^*$ and $X$ given are these minimizing values.

| $\rho$ (F$^{-3}$) | $-E_c/m_\pi c^2$ | $k^*/m_\pi c$ | $X$ |
|---|---|---|---|
| 0.234 | $1.09 \times 10^{-3}$ | 1.27 | 0.034 |
| 0.261 | $6.85 \times 10^{-3}$ | 1.30 | 0.083 |
| 0.289 | $1.68 \times 10^{-2}$ | 1.34 | 0.124 |
| 0.319 | $3.05 \times 10^{-2}$ | 1.36 | 0.160 |
| 0.349 | $4.73 \times 10^{-2}$ | 1.40 | 0.191 |
| 0.416 | $8.88 \times 10^{-2}$ | 1.46 | 0.243 |
| 0.488 | 0.139 | 1.54 | 0.283 |
| 0.566 | 0.197 | 1.60 | 0.316 |
| 0.650 | 0.260 | 1.68 | 0.343 |
| 0.739 | 0.329 | 1.74 | 0.366 |
| 0.936 | 0.481 | 1.90 | 0.400 |
| 1.16 | 0.649 | 2.04 | 0.426 |
| 1.40 | 0.830 | 2.20 | 0.445 |
| 1.66 | 1.02 | 2.36 | 0.461 |
| 1.95 | 1.23 | 2.52 | 0.473 |
| 2.26 | 1.45 | 2.68 | 0.483 |
| 2.60 | 1.67 | 2.84 | 0.492 |
| 2.95 | 1.91 | 3.0 | 0.499 |

## V. PION-NUCLEON AND PION-PION S-WAVE EFFECTS

There is a well-known formula, often quoted as an argument against the existence of $\pi^-$'s in neutron-star interiors, for the effect of $S$-wave pion-nucleon interactions on the energy of a $\pi^-$ at rest in nuclear matter[16]:

$$\delta E = 219(\rho_n - \rho_p) \text{ MeV}, \tag{5.1}$$

where $\rho_n$ and $\rho_p$ are the neutron and proton densities measured in particles per F$^3$. Experimental $S$-wave $\pi^- p$ and $\pi^- n$ scattering lengths determine the coefficients in (5.1).

Superficially the result (5.1) argues against the gradual onset of a condensed phase as described in Sec. II, because of the large positive energies in a neutron-rich medium. It would clearly allow the fully developed $\rho_n \approx \rho_p$ condensed phase predicted in Table I at higher densities, however, and even push it towards greater proton occupancy. Thus one might anticipate a first-order phase transition to an $np\pi^-$ phase at some density around $\rho = 0.5$ F$^{-3}$. However, $S$-wave $\pi^-\pi^-$ interactions should also be taken into account before a conclusion is reached in this regard, since we might expect them to be of comparable importance.

There is no direct experimental information on the $I = 2$ ($\pi^-\pi^-$) $S$-wave scattering length. We shall take Weinberg's[17] prediction from current algebra for the $S$-wave scattering length ($-0.06 m_\pi^{-1}$) and scale it down by the same factor (0.85) which scales his predictions for the pion-nucleon scattering lengths to the experimental results. This leads to $S$-wave interaction-energy formulas analogous to (5.1) which are expressible to within the errors by the following simple expression for the energy per *baryon:*

$$\pi^- N \text{ interactions}: \quad \delta E = \frac{1}{2}\frac{\rho}{m_\pi^2} X(1-2X),$$
$$\pi^-\pi^- \text{ interactions}: \quad \delta E = \frac{1}{4}\frac{\rho}{m_\pi^2} X^2, \tag{5.2}$$

where $X=\theta^2$, the fraction of baryons that are protons.

Thus the ground-state parameters $(X, k)$ and energy will be determined by minimizing

$$E - E(\text{neutrons}) = X\left(\frac{2kf\sqrt{\rho}\,(1-X)^{1/2}}{\sqrt{\omega_k}} + \omega_k + \tfrac{1}{2}\rho - \tfrac{3}{4}X\rho\right). \tag{5.3}$$

Here $m_\pi$ has been set equal to unity.

It turns out that (5.3) predicts a second-order phase transition very similar to the one described in Sec. II. The onset is delayed by the repulsions to a density of $\rho \approx 0.36$ F$^{-3}$, the value of $k$ at onset is increased to about 1.9 $m_\pi$, and $X$ increases much more rapidly above onset than it did in the case of Sec. II (in order to reduce the repulsive $S$-wave energy).

Thus the pion condensation survives both the second-order noncondensed pion corrections and the inclusion of $S$-wave forces. Will it stand up under further corrections? It is difficult to say for sure. We are already deeply into the strong-interaction morass. As an example of one correction from an almost interminable list of corrections, let us return for a moment to the pion-nucleon $S$-wave interaction discussed above.

Had we written this interaction in terms of fields (in the form $\bar{\psi}\psi\phi^*\phi$) we would have retrieved the energy shift of (5.1) in a term in which both $\phi^*$ and $\phi$ were replaced by the condensed field, $\phi_c$. In addition there would be a new pion-nucleon vertex, $\bar{\psi}\psi\phi_c^*\phi_{\rm nc}$. This in turn should be used in a new calculation of the energy shift coming from emission and absorption of noncondensed pions (there is no interference with the old calculation since there is no $\vec{\sigma}$ matrix between $\bar{\psi}$ and $\psi$ here). The effect this time is clearly in the direction of favoring the condensation, the second-order self-energy parts being an energy-lowering effect in our systems.

Instead of continuing the list of theoretical omission and uncertainties, we would prefer at this point to underscore the relative cleanliness of the basic calculation of the condensation. The problem of one meson mode interacting with nucleons was solved without approximations. The demands made on the nucleon wave functions in order to sustain the pion condensation, $\langle \bar{p}(x)\sigma_3 n(x)\rangle \sim \text{const} \times e^{-ikz}$, turned out to be completely compatible with the demands made by short-range repulsive forces, if spin-independent.

The condensation persists through the inclusion of the two effects that we thought to be most dangerous to it, changing only in the onset density, which in any case remains below the value of $\rho = 1$ F$^{-3}$, at which point life may become more complicated through the creation of other species of baryons. Even these other baryons can be used to sustain the condensation, however, e.g., through the couplings $\Sigma^- \leftrightarrow \Lambda + \pi^-$, so we predict no tendency for the condensation to disappear at higher densities.

## VI. ELECTROMAGNETIC EFFECTS AND EQUATION OF STATE

The state we have described has been constrained to have average charge neutrality. As with other many-body systems, the fluctuations of the total charge have a negligible, $N^{-1/2}$, effect on the system. However, nothing has been said about local charge-density fluctuations. In addition, the state we have described carries an extraordinarily large electric current in the $z$ direction. Clearly magnetic energy will preclude the formation of a homogeneous system with such a current. Since the velocity of the pion condensate is near that of light, the strength of the current interactions and the charge-density Coulomb interactions are both set by $e^2/\hbar c$. The small size of $e^2/\hbar c$ means that the effects of these electromagnetic interactions will occur on a larger scale of distances than those of the characteristic condensate scale of length $(Xk^*)^{-1}$, giving rise to a macrostructure of the condensate.

In Ref. 11 two possible macrostructures were discussed:

(1) A structure in which a standing wave is used for the pion field. This will cost at most a factor of $2^{-1/2}$ in the interaction-energy term and could delay the onset until $\rho \approx 0.5$ F$^{-3}$ in the case without virtual-pion and $S$-wave corrections,[18] or until a correspondingly higher value of $\rho$ in the presence of these corrections. The cost in Coulomb energy, due to the charge-density wave in this case, is negligible due to the very short wavelength ($\sim m_\pi^{-1}$).

(2) A filamentary structure, which we presently favor, with current moving in opposite directions on neighboring filaments. An estimate for small $X$ shows that a pair of cylindrical filaments side by side with opposed currents can have a radius of order $(\hbar c/e^2)^{1/2}(k^*X)^{-1}$ before the positive magnetic energy exceeds in magnitude the negative amount gained from the condensation.[19] This filament radius is much larger than the character-

istic condensate length $(k^*X)^{-1}$.

A determination of the exact form of the macrostructure requires that the local condensate energy, including its variations to distortions of the condensate wave function, the electromagnetic field, and the interaction between the condensate and the electromagnetic field, all be treated on the same footing. This is analogous to the flux structure in a type-II superconductor. Our result for the condensate energy can be generalized to a nonuniform condensate by replacing $\theta\sqrt{N}$ by a slowly varying condensate amplitude $\pi(x)$. Including the electromagnetic field and its coupling to the condensate then leads to an energy functional dependent upon $\pi(x)$ and the electromagnetic field. We are presently investigating the self-consistent macrostructure generated in this way.

In the microscopic domain, then, we predict a highly directional form of matter. We do not know whether the directionality would persist over macroscopic distances (perhaps even appreciable fractions of the neutron star's diameter), or whether radomly oriented domains of a microscopic size will develop instead. In either event, since we have filled the greater part of the volume with the condensate, we can apply the macroscopic pressure formula, (2.24).

In Fig. 1 we plot the fractional pressure loss of the condensed state as compared to the normal state for the case in which $S$-wave effects and pion corrections were neglected, (2.22). Compared to the neutron case we note a 30% loss of pressure at $\rho = 0.30\ \mathrm{F}^{-3}$ shortly after onset and a 73% loss at a density of $\rho = 0.5\ \mathrm{F}^{-3}$. There is no qualitative difference in this conclusion if the free Fermi pressure is replaced by the results of one of the several calculations in which nuclear forces are included.[4]

The corrections to (2.22) discussed in this paper will probably not only delay the onset of the phase transition as the density is increased, but will reduce the loss of pressure after onset. We do not consider our treatment of the various corrections to be sufficiently complete at this time to justify deriving a numerical equation of state and computing the structure of a neutron star, although we hope to return to these calculations in the future. We still anticipate a substantial softening of the equation of state, and this can only result in a smaller maximum mass for the neutron star.[20]

There is also the possibility that the softening of the equation of state will lead to a region of instability of the star as a whole, analogous to that between white dwarf densities and neutron-star densities. If the equation of state hardens again at still higher densities this could lead to a new class of stable object, less massive and more dense than a neutron star, a "pion-nucleon" star.

*Note added in proof.* Contrary to the statement in the Introduction, $\pi^-$ particles are not the only serious contenders. There may be a $\pi^0$ condensation, but the nuclear physics questions will be much harder to resolve in this case. A rather small fraction of $\pi^+$ particles can be added to the model of the present paper and result in an earlier onset and a somewhat lower energy. Standing-wave solutions have been worked out in the absence of nuclear forces and have a critical density equal to that for the running-wave case (instead of twice that as conjectured in Sec. VI); however, the nuclear physics corrections to the standing-mode case are again hard to evaluate and probably deter the formation of the state. All these subjects are discussed in R. F. Sawyer and A. C. Yao (unpublished).

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[1]The number of leptons is not specified, since we are assuming that neutrinos can escape freely.

[2]For example, G. Baym, C. Pethick, and P. Sutherland, Astrophys. J. 170, 299 (1971); James B. Hartle and Kip S. Thorne, *ibid.* 153, 807 (1968); C. E. Rhoades, Jr., and R. Ruffini, *ibid.* 163, L83 (1971).

[3]However, see J. Bahcall and R. A. Wolf [Phys. Rev. 140, B1445 (1965); Astrophys. J. 142, 1254 (1965)] for a suggestion that pions may enter and for some of the consequences for neutron stars.

[4]V. R. Pandharipande, Nucl. Phys. A178, 123 (1971); A181, 33 (1972).

[5]S. Tsuruta and A. G. W. Cameron, Can. J. Phys. 44, 1895 (1966).

[6]G. Baym, H. A. Bethe, and C. J. Pethick, Nucl. Phys. A175, 225 (1971).

[7]E. H. Auerbach, D. M. Fleming, and M. M. Sternheim, Phys. Rev. 162, 1683 (1967).

[8]For pure neutron matter, using the same approach we estimate

$$\omega^2 = k^2(1-8.5\rho) + m_\pi{}^2 + 4.4\rho m_\pi \omega,$$

where the last term is due to repulsive $S$-wave $\pi^- n$ interactions.

[9]There will be a small number of electrons present

in the actual ground state of the matter, but they will have no significant effect on the results. In our solution with a pion condensate we can determine the number of electrons from the condition that the pion chemical potential is equal to the electron chemical potential in the ground state. For example, at the onset density $\rho = 0.25$ $F^{-3}$ in our calculation, neglecting nuclear forces, the $\pi^-$ chemical potential is given by Eq. (4.8) and is numerically equal to $0.77\ m_\pi \approx p_e^F$. This gives an electron density of 1.6% of the nucleon density, an absolutely negligible number as far as the pressure determination is concerned. For comparison, in the case without a pion condensation at a density of $\rho = 0.25\ F^{-3}$ the electron density is about 1% of the nucleon density; the reduction in pressure which results from including the electrons is about 0.5%.

[10]This is the nonrelativistic form of the conventional pseudoscalar meson interaction with nucleons. See, for example, G. Källén, *Elementary Particle Physics* (Addison-Wesley, Reading, Mass., 1964), Chaps. 3–5.

[11]The problem characterized by $H_C$, above, has been treated earlier from a variational standpoint by R. F. Sawyer, Phys. Rev. Letters 29, 382 (1972). The approach followed in the present paper, as applied to $H = H_C$, was reported by D. J. Scalapino, Phys. Rev. Letters 29, 386 (1972).

[12]This constraint, that the proton momentum should compensate the $\pi^-$ momentum in the frame of zero total momentum, is in fact artificial. Had we left the momentum distribution unconstrained, the transformation which later diagonalizes the interaction would somewhat undiagonalize the kinetic-energy terms. Taking only the diagonal terms, we would have found the momentum balance condition automatically obeyed in the resulting ground state; the off-diagonal terms could then be treated perturbatively and be shown to be unimportant. By imposing the constraint from the beginning we shall eliminate off-diagonal terms altogether. The lowest energy for the constrained system is in any event an upper bound on that for the unconstrained system.

[13]To be more precise, all matrix elements involving $|\psi_0\rangle$ or excited states with *nearly* the same number of condensed pions, $NX$, are correctly calculated from the product forms analogous to (2.31), using the properties (2.29) of $\tilde{\Pi}$.

[14]Here, as always, $\vec{k}$ stands for the momentum of the condensed pion mode, $\vec{k} = -k\hat{z}$.

[15]Modification of a nucleon-nucleon potential through a modification of the meson propagator in the medium corresponds to the inclusion of a many-nucleon force.

[16]H. A. Bethe, Ann. Rev. Nucl. Sci. 21, 93 (1971), p. 158.

[17]S. Weinberg, Phys. Rev. Letters 17, 168 (1966).

[18]The $2^{-1/2}$ factor is the result of the variational estimate given in Ref. 10, and not a diagonalization of the two-mode problem, which must give at least as low an energy as the variational estimate.

[19]The magnetic energy of a filament, per baryon, goes as the $(\text{radius})^2$ of the filament.

[20]In addition to the effect on the equation of state at zero temperature, a pion condensation could be expected to affect the transport properties of the material. See J. Bahcall and R. Wolf (Ref. 3), and J. Kogut and J. Manassah (unpublished).