Properties of Polarization Density Matrix in Regge-Pole Models

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We study two-body (or quasi-two-body) parity-conserving reactions with unpolarized beam and target. From the general considerations of angular momentum and parity conservation we derive the structure of the density matrices of the final particles. This structure becomes conspicuous when one uses transversity quantization with a special reordering for the lines and columns. We study the predictions of a single Regge trajectory exchange model. We derive the consequences of reality and factorization of residue functions, and of parity conservation at each vertex. The results are summarized in two tables. For the cases of low-spin particles we give these results in a geometrical form. As an example we plot some experimental data on Q-meson polarization.

I. INTRODUCTION

There is no need to emphasize the importance of the experimental determination of spins and parities of new particles or resonant states in high-energy physics. From the theoretical point of view, the principles involved are the well-known ones, namely, those that originate from the requirement of relativistic invariance. In practice, however, this is not an easy task. One has to make various selections in the data, and also often one has to make assumptions regarding the production mechanism, and so on. Probably the most important problem is the right choice of variables so that one can extract the maximum amount of information with a given set of data. Towards this end, three of the present authors have been engaged in an extensive study[1,2] of the "polarization density matrix and how to present its measurement." The emphasis is on the intrinsic geometrical properties of the density matrix arising from relativistic invariance and the allowed domain for the values of the elements of the density matrix. The experimentally measured values must fall in such a domain to be consistent with the conservation of angular momentum and parity if the particle has a definite spin and parity.

The main purpose of the present paper is to study the implications of the general considerations of Refs. 1 and 2 in a specific model. The model we choose is one in which a single Regge pole with factorizable residues dominates the production mechanism. The results can be generalized to slightly more complicated models which we shall discuss in a later section. The plan of the paper is as follows: In Sec. II, we shall briefly review some general results of Refs. 1 and 2 in the case of a two-particle (or a quasi-two-particle) reaction. Of special importance are the symmetry properties and the rank conditions on the density matrix in helicity and transversity frames. In Sec. III, we specialize to Regge-pole models and discuss in detail two special cases in which the produced particle is (i) a boson with arbitrary spin and parity, the beam particle being a pseudoscalar meson, and (ii) a fermion isobar, the target being a spin-$\frac{1}{2}$ particle. In Sec. IV, we discuss a few specific examples of (i) and (ii). We shall also discuss in this section the recent experimental results[3] on Q production. The final section is devoted to a summary of the main results.

II. MEASUREMENT AND DESCRIPTION OF POLARIZATION

Consider a reaction between four hadrons

$$1+2\to 3+4 \quad (s \text{ channel}),$$

$$4^*+2\to 3+1^* \quad (t \text{ channel}),$$

with p_i and j_i representing the four-momenta and spins of the particles. Particles 3 and/or 4 can be unstable. In the present paper we are interested in the information concerning the polarization state of 3 and/or 4 obtained through the study of angular distributions in parity-conserving decays. It is well known that the polarization state of a particle with positive mass, spin j, and energy-momentum p, is completely specified by a density operator ρ. By an appropriate choice of an orthonormal basis in a $(2j+1)$-dimensional Hilbert space $\mathcal{H}_{2j+1}$,

$$J^2|jm\rangle = j(j+1)|jm\rangle,$$
$$J_3|jm\rangle = m|jm\rangle, \qquad (2.1)$$
$$\langle jm'|jm\rangle = \delta^{m'}{}_m,$$

the density operator can be represented by a

Hermitian, nonnegative, and trace-1 matrix

$$\rho^\dagger = \rho \geqslant 0, \quad \mathrm{tr}\rho = 1 \tag{2.2}$$

whose matrix elements are

$$\rho^{m'}{}_m = \langle jm' | \rho | jm \rangle \ . \tag{2.3}$$

Conditions (2.2) imply that the eigenvalues of ρ are nonnegative and the diagonal elements satisfy

$$0 \leqslant \rho^m{}_m \leqslant 1 \ . \tag{2.4}$$

The joint polarization of particles 3 and 4, with known energy momenta, is described by an $n \times n$ Hermitian, nonnegative, trace-1 matrix $\rho(3,4)$ with $n = (2j_3+1)(2j_4+1)$.

A. Polarization Domain

The set of all $n \times n$ Hermitian matrices form a $(N+1)$-dimensional real Euclidean space which we shall call $\mathcal{E}_{N+1}$. The set of all trace-1, Hermitian matrices form a hyperplane $\mathcal{E}_N$ in this vector space, where $N = n^2 - 1$. The set of the density matrices which satisfy (2.2) is a subdomain of $\mathcal{E}_N$. This subdomain is called the polarization domain $\mathfrak{D}_j$:

$$\mathfrak{D}_j \subset \mathcal{E}_N \subset \mathcal{E}_{N+1} \ .$$

$\mathfrak{D}_j$ is a *convex domain,* i.e., if

$$\rho_1, \rho_2 \subset \mathfrak{D}_j \ , \tag{2.5}$$

then

$$\alpha\rho_1 + \beta\rho_2 \subset \mathfrak{D}_j \ ,$$

where

$$0 \leqslant \alpha, \beta, \quad \alpha + \beta = 1 \ .$$

B. Multipole Expansion

1. One-Particle Polarization

The density matrix ρ can be expanded in terms of a set of Hermitian matrices $Q_M^{(L)}$ and a set of real parameters $r_M^{(L)}$ as follows:

$$\rho = \rho_0 + \sum_{L=1}^{2j} \rho^{(L)} , \tag{2.6}$$

where

$$\rho_0 = \frac{1}{2j+1} \underline{1} \qquad [\underline{1}: (2j+1)\times(2j+1) \text{ identity matrix}]$$

represents the unpolarized state, and

$$\rho^{(L)} = \frac{2j}{2j+1} \sum_{M=-L}^{M=L} Q_M^{(L)} r_M^{(L)} \ . \tag{2.7}$$

The $Q_M^{(L)}$'s and $r_M^{(L)}$'s are related to the more customary expansion of ρ in terms of the multipoles $T_M^{(L)}$'s and the multipole-expansion parameters $t_M^{(L)}$'s by

$$Q_0^{(L)} = \left(\frac{2L+1}{2j}\right)^{1/2} T_0^{(L)} \ ,$$

$$Q_M^{(L)} = (-1)^M \left(\frac{2L+1}{j}\right)^{1/2} \tfrac{1}{2}(T_M^{(L)} + T_M^{(L)\dagger}), \quad M > 0 \ , \tag{2.8}$$

$$Q_{-M}^{(L)} = (-1)^M \left(\frac{2L+1}{j}\right)^{1/2} \frac{1}{2i}(T_M^{(L)} - T_M^{(L)\dagger}), \quad M > 0 \ ,$$

and

$$r_0^{(L)} = \left(\frac{2L+1}{j}\right)^{1/2} t_0^{(L)} ,$$

$$r_M^{(L)} = (-1)^M \left(\frac{2L+1}{j}\right)^{1/2} \mathrm{Re}\, t_M^{(L)}, \quad M > 0, \tag{2.9}$$

$$r_{-M}^{(L)} = (-1)^M \left(\frac{2L+1}{j}\right)^{1/2} \mathrm{Im}\, t_M^{(L)}, \quad M > 0 \ .$$

We recall that

$$(T_M^{(L)})^m{}_n = (2j+1)^{1/2} \begin{pmatrix} m & L & j \\ j & M & n \end{pmatrix} , \tag{2.10}$$

where $\left(\begin{smallmatrix} m & L & j \\ j & M & n \end{smallmatrix}\right)$ denotes Wigner's $3j$ symbol. The $T_M^{(L)}$'s provide a real non-Hermitian orthonormal basis for the expansion of ρ. The expansion coefficients $t_M^{(L)}$'s in this basis are in general complex, but the Hermiticity of ρ requires

$$t_M^{(L)*} = (-1)^M t_{-M}^{(L)} \ . \tag{2.11}$$

The parameters $r_M^{(L)}$ are more convenient. They are real (proportional to the real and imaginary part of the $t_M^{(L)}$) and they are independent of each other. Furthermore their normalization is such that the degree of polarization of the state is

$$d(\rho) = \left(\sum_{L=1}^{2j} \sum_{M=-L}^{L} (r_M^{(L)})^2 \right)^{1/2} \ . \tag{2.12}$$

The parameters $r_M^{(L)}$ are very helpful in plotting the experimental results and to see whether the results are consistent with the requirements of conservation of angular momentum. For details we refer the reader to Refs. 1 and 2.

2. Two-Particle Polarization

For joint polarization of a two-particle state, the density operator can be expanded in double multipoles:

$$\rho(3,4) = \sum_{L=0}^{2j_3} \sum_{L'=0}^{2j_4} \rho^{(L,L')} , \tag{2.13}$$

with

$$\rho^{(L,L')} = \frac{(2L+1)(2L'+1)}{(2j+1)(2j'+1)} \sum_{M=-L}^{L} \sum_{M'=-L'}^{L'} t_{MM'}^{(L,L')*} \times T(j_3)_M^{(L)} \otimes T(j_4)_{M'}^{(L')} \ . \tag{2.14}$$

The density matrix of each particle when the polarization of the other is not observed is

$$\rho_3 = \mathrm{tr}_4\, \rho(3,4)\ ,$$
$$\rho_4 = \mathrm{tr}_3\, \rho(3,4)\ . \tag{2.15}$$

The polarization correlation between the two particles 3 and 4 is

$$K(3,4) = \rho(3,4) - \rho_3 \otimes \rho_4\ . \tag{2.16}$$

C. Decay Angular Distribution

1. *Two-Body Decay*

Let $\mathcal{G}_a(\theta, \varphi)$ denote the normalized angular distribution of one of the decay products, in a two-body parity-conserving decay mode a, of a particle of spin j. If the decaying particle is described by the density matrix ρ, then[4]

$$\mathcal{G}_a(\theta,\varphi) = \frac{1}{4\pi} + \sum_{L=1}^{2j} \left(\frac{2L+1}{2j}\right)^{1/2} \lambda_a(L,j) \sum_{M=-L}^{L} t_M^{(L)*}\, Y_M^{(L)}(\theta,\varphi)\ , \tag{2.17}$$

$$\mathcal{G}_a(\theta,\varphi) = \frac{1}{4\pi} + \sum_{L=1}^{2j} \lambda_a(L,j)\left(r_0^{(L)} Y_0^{(L)}(\theta,\varphi) + \sum_{M=1}^{L} (-)^M \sqrt{2}\,[r_M^{(L)}\ \mathrm{Re}Y_M^{(L)}(\theta,\varphi) + r_{-M}^{(L)}\ \mathrm{Im}Y_M^{(L)}(\theta,\varphi)]\right)\ , \tag{2.18}$$

where $Y_M^{(L)}(\theta,\varphi)$ are the usual spherical harmonics, and $t_M^{(L)}$ and $r_M^{(L)}$ are the expansion parameters already defined for the density matrix. The $\lambda_a(L,j)$'s are scalar coefficients (decay coefficients) which depend on L, j and on the particular decay mode a. Their value may depend on the dynamics of the decay mode a, e.g., $1^+ \to 1^- + 0^-$, $j \to \frac{3}{2} + 0$; however, for the most usual two-body hadronic decays,

$$1^- \to 0^- + 0^-,\quad 1^- \to 1^- + 0^-,\quad \tfrac{3}{2}^\eta \to \tfrac{1}{2}^+ + 0^- \quad (\eta = \pm), \tag{2.19}$$

they are independent of the dynamics. Indeed the corresponding angular distribution can be written

$$\mathcal{G}_a(\theta,\varphi) = \frac{1}{4\pi}[1 + \lambda_a'(2,j) A^{(2)}(\theta,\varphi)] \tag{2.20}$$

with

$$A^{(2)}(\theta,\varphi) = \left(\frac{4\pi}{2j}\right)^{1/2} \sum_{M=-2}^{M=+2} t_M^{(2)*}\, Y_M^{(2)}(\theta,\varphi)$$
$$= \tfrac{1}{2} r_0^{(2)}(3\cos^2\theta - 1) + \tfrac{1}{2}\sqrt{3}\,\sin^2\theta\,(r_2^{(2)}\cos 2\varphi + r_{-2}^{(2)}\sin 2\varphi) + \tfrac{1}{2}\sqrt{3}\,\sin 2\theta\,(r_1^{(2)}\cos\varphi + r_{-1}^{(2)}\sin\varphi)\ , \tag{2.21}$$

and

$$\lambda_a'(2,j) = (20\pi)^{1/2}\lambda_a(2,j)\ .$$

The $\lambda_a'(2,j)$'s are numerical coefficients independent of dynamics. Their value is [cf. Ref. 1 and Ref. 2b with tables giving $\sqrt{4\pi}\,C(L) = \lambda_a'(L,j)/\sqrt{2j}$]:

$$\lambda_a'(2,j) = \begin{cases} -2 & \text{for } 1^- \to 0^- + 0^- \\ 1 & \text{for } 1^- \to 1^- + 0^- \\ -\sqrt{3} & \text{for } \tfrac{3}{2}^\eta \to \tfrac{1}{2}^+ + 0^-. \end{cases} \tag{2.22}$$

Parity conservation in a two-body decay implies that

$$\lambda_a(L,j) = 0 \quad \text{for } L \text{ odd}\ . \tag{2.23}$$

Therefore, by observing a parity-conserving two-body decay, one obtains information concerning only $r_M^{(L)}$'s with L even; i.e., one measures $\rho^{(L)}$, cf. Eqs. (2.6) and (2.7), where L is even. We shall refer to this as the measurement of *even* polarization.

2. *Three-Body Decay*

For three-body decays, the probability distribution is a function of five variables. One can choose for them two of the s, t, u invariants and three angles, θ, φ, ψ, where θ, φ fix the orientation of the normal to the decay plane (in the rest system of the decaying particle) and ψ fixes the rotation of the decay around the normal, e.g., ψ is the azimuth of a chosen one of the final particles. In general both even and odd polarization can be measured; let us be more specific for two usual decays:

(i) $1^- \to 0^- + 0^- + 0^-$. If the dependence of $\mathcal{G}$ on s, t, u is not observed, then we find

$$\mathcal{G}(\theta,\varphi,\psi) = \frac{1}{4\pi}[1 - 2A^{(2)}(\theta,\varphi)]\ . \tag{2.24}$$

This is independent of ψ and of the dynamics. It yields the value of the even polarization (i.e., the coefficients $r^{(2)}$'s).

(ii) $1^+ \to 0^- + 0^- + 0^-$. If only the dependence in

θ, φ is observed, one obtains the even polarization by

$$\mathcal{G}(\theta,\varphi)=\frac{1}{4\pi}[1+A^{(2)}(\theta,\varphi)] \tag{2.25}$$

with $A^{(2)}(\theta,\varphi)$ given in (2.20).

In such a decay it is also possible to measure the odd polarization (i.e., the $r_M^{(1)}$'s) up to a sign, from the complete observation of $\mathcal{G}(\theta,\varphi,\psi,s,t)$.[4]

3. *Correlated Decays*

To observe joint polarization of 3 and 4 one has to measure the joint angular distribution. For instance, for two two-body decays, this is $\mathcal{G}(\theta_3,\varphi_3,\theta_4,\varphi_4)$; it can be expanded in a manner similar to (2.18) with the basis $Y_M^{(L)}(\theta_3,\varphi_3)\times Y_{M'}^{(L')}(\theta_4,\varphi_4)$. If $K(3,4)=0$, i.e., no polarization correlation, then $\mathcal{G}$ factorizes:

$$\mathcal{G}(\theta_3,\varphi_3;\theta_4,\varphi_4)=\mathcal{G}_3(\theta_3,\varphi_3)\mathcal{G}_4(\theta_4,\varphi_4)\ . \tag{2.26}$$

The explicit form of the decay distribution function (and also the explicit form of the ρ matrix) depends on the choice of the reference frame. We deal with this question next.

D. Choice of Reference Frames

To choose an orthonormal basis in $\mathcal{H}_{2j+1}$, one needs to specify a reference frame and an axis of quantization. Following Ref. 2, we associate with each particle a tetrad, i.e., a set of four vectors $n^{(\alpha)}$, $\alpha=0,1,2,3$, which are orthonormal:

$$n^{(\alpha)}\cdot n^{(\beta)}=g^{\alpha\beta}\ . \tag{2.27}$$

(We use the metric $g^{00}=-g^{ii}=1$.) By convention, $n^{(3)}$ is the quantization axis. In a four-particle reaction of the type under consideration, the normal to the reaction plane n (which is defined up to a sign) can be chosen to be an axis common to all particle tetrads. In a given channel, *transversity* quantization is a choice of tetrad ${}^Tn_i^{(\alpha)}$ for each particle i, with ${}^Tn_i^{(3)}=n$ and a conventional choice for ${}^Tn_i^{(2)}$ (then ${}^Tn_i^{(1)}$ is fixed) which is universal but for a sign (see below). In *helicity quantization* ${}^Hn_i^{(2)}=n$; the quantization axis ${}^Hn_i^{(3)}$ is chosen in the following way: In the rest frame of the particle i the space part of ${}^Hn_i^{(3)}$ is collinear to the momentum of the associated particle (e.g., 1 with 3, 2 with 4 in the t channel, 1 with 2, 3 with 4 in the s channel). This defines ${}^Hn_i^{(3)}$ up to a sign. There are different sign conventions in the literature, so every experimental or theoretical paper should be explicit about these conventions. Our conclusions are of course independent of conventions, but our computation will be done with the (channel-independent) convention for every particle:

$$ {}^Tn^{(3)}=n={}^Hn^{(2)},\quad {}^Tn^{(1)}={}^Hn^{(1)},\quad {}^Tn^{(2)}=-{}^Hn^{(3)}\ . \tag{2.28}$$

That is, the helicity frame is transformed into the transversity frame by a rotation $\tilde{R}$ of $-\frac{1}{2}\pi$ around the first axis: In terms of Euler angles

$$\tilde{R}=(\tfrac{1}{2}\pi,\tfrac{1}{2}\pi,-\tfrac{1}{2}\pi)\ ,$$

and consequently

$$|jm\rangle^T=|jm'\rangle^H D^{(j)}(\tilde{R})^{m'}{}_m\ . \tag{2.29}$$

$D^{(j)}(\tilde{R})$ represents the $(2j+1)$-dimensional, unitary, irreducible representation of the rotation group. In the present case

$$D^{(j)}(\tilde{R})^m{}_{m'}=e^{-i\pi(m-m')/2}d^{(j)}(\tfrac{1}{2}\pi)^m{}_{m'}\ . \tag{2.30}$$

From the symmetry properties[5]

$$\begin{aligned} d^{(j)}(\theta)^m{}_{m'}&=d^{(j)}(\theta)^m{}_{m'}{}^*\\ &=(-1)^{m-m'}d^{(j)}(\theta)^{m'}{}_m\\ &=(-1)^{m-m'}d^{(j)}(\theta)^{-m}{}_{-m'}\ ,\\ d^{(j)}(\tfrac{1}{2}\pi)^m{}_{m'}&=(-1)^{j-m}d^{(j)}(\tfrac{1}{2}\pi)^m{}_{-m'}\ , \end{aligned} \tag{2.31}$$

it is easy to obtain for the rotation matrix (2.30)

$$\begin{aligned} D^{(j)}(\tilde{R})^m{}_{m'}&=D^{(j)}(\tilde{R})^{m'}{}_m\\ &=D^{(j)}(\tilde{R})^{-m}{}_{-m'}\\ &=(-)^{m-m'}D^{(j)}(\tilde{R})^m{}_{m'}{}^*\\ &=e^{i\pi j}D^{(j)}(\tilde{R})^m{}_{-m'}{}^*\ . \end{aligned} \tag{2.32}$$

In either transversity or helicity frame, to go from one channel to the other, one has to use crossing matrices which are functions of the variables s, t, and u. Henceforth we shall confine our attention to the t channel. The helicity frame then is the Gottfried-Jackson frame. The density matrices ${}^T\rho$ and ${}^H\rho$ are related by

$${}^T\rho=D^{(j)}(\tilde{R}^{-1})\,{}^H\rho\, D^{(j)}(\tilde{R})\ . \tag{2.33}$$

E. Restrictions on the Density Matrix Due to the Mode of Observation or to the Nature of the Production Reaction

1. *Even Polarization*

As we have seen in Sec. II C, in many cases only the even part (even-L multipoles) of the one-particle density matrix can be measured. What is observed in these cases is the projection ${}^{(E)}\rho$ of ρ on a subspace which is a symmetry plane of $\mathfrak{D}_j$. One can prove[2] that

$${}^{(E)}\rho=\sum_{L\text{ even}}\rho^{(L)}$$

is also a Hermitian, positive, and trace-1 matrix. One also shows[2] that

$$(\rho^{(L)})^T = \rho^{(L)*} = (-)^L \Gamma_j \rho^{(L)} \Gamma_j^{-1} , \qquad (2.34)$$

where Γ_j is the $(2j+1)\times(2j+1)$ matrix defined, independently of the choice of the quantization axis, by

$$(\Gamma_j)^m{}_n = (-)^{j-m} \delta^{-m}{}_n . \qquad (2.35)$$

From (2.34) one easily observes that ${}^{(E)}\rho$ satisfies

$$({}^{(E)}\rho)^T = {}^{(E)}\rho^* = \Gamma_j({}^{(E)}\rho)\Gamma_j^{-1} . \qquad (2.36)$$

Γ_j is the representation of a rotation of $-\pi$ around the $n^{(2)}$ axis,

$$\Gamma_j = D^{(j)}(0,\pi,0) . \qquad (2.37)$$

As we shall see, some models are specially interesting by predicting only an even polarization, that is,

$${}^{(E)}\rho = \rho .$$

2. *B Symmetry*

Although in general the polarization domain $\mathfrak{D}_j$ of the density matrix is completely determined by conditions (2.2), in practice, there are further restrictions depending on the nature of the reaction which is responsible for the production of the particle of interest. In this paper, we shall be mainly concerned with a quasi-two-particle reaction which is parity conserving and in which the beam and the target particles are unpolarized. In such a reaction there are only three linearly independent energy momenta. The space-time hyperplane which contains the three observed energy momenta is a symmetry plane for the reaction in which the normal to this hyperplane is the common quantization axis Oz for the tetrads of the four particles. In each tetrad, the B symmetry is represented by the space reflection through the XY plane. Under such a reflection,

$$|jm\rangle^T \to |jm\rangle^{T'} = \eta |jm'\rangle^T D^{(j)}(0,0,\pi)^{m'}{}_m = \eta e^{i\pi m}|jm\rangle^T , \qquad (2.38)$$

where η is the intrinsic parity of the particle. Consequently,

$$({}^T\rho)^{m'}{}_m = e^{i\pi(m'-m)}({}^T\rho)^{m'}{}_m , \qquad (2.39)$$

which implies that

$$({}^T\rho)^{m'}{}_m = 0 \text{ if } m'-m \text{ is odd.} \qquad (2.40)$$

On any one-particle density matrix, with the natural order for the indices (i.e., $m', m = j, j-1, \ldots, -j$) there is a "checkerboard" pattern when we call "white" the elements with $(m'-m)$ odd and "black" those with $(m'-m)$ even. The main diagonal is always black, the second diagonal is black for integer spins and white for half-integer ones. Equation (2.39) means that all the white elements of a B-symmetric density matrix are zero in transversity quantization.

It is clear that by reordering the indices, the nonvanishing (black) elements can be brought together and ${}^T\rho$ can be written in a block form. For this purpose it is convenient to discuss two cases separately:

a. When j is integral. Choose the separation order of indices which separates the black and the white squares: $j, j-2, \ldots, 2-j, -j;\ j-1, j-3, \ldots, 3-j, 1-j$. See for example Fig. 1(a). Then

$${}^T\rho = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} , \qquad (2.41)$$

where A and B are Hermitian matrices of dimensions $(j+1)\times(j+1)$ and $j\times j$, respectively, with

$$\mathrm{tr}A + \mathrm{tr}B = 1 . \qquad (2.42)$$

Let Δ_k be a $k\times k$ matrix which has 1 on the second diagonal and zero elsewhere,

$$(\Delta_k)^\alpha{}_\beta = \delta^\alpha{}_{-\beta} , \quad \Delta_k{}^2 = \underline{1} . \qquad (2.43)$$

If ρ contains only even polarization, A and B are symmetric through the second diagonal:

$$A^* = A^T = \Delta_{j+1} A \Delta_{j+1}, \quad B^* = B^T = \Delta_j B \Delta_j . \qquad (2.44)$$

The density matrix ${}^H\rho$ is given by (2.33). To study the structure of ${}^H\rho$ we must first study that of the rotation matrix $D^{(j)}(\tilde{R})$ in the separation order of indices. Then it can be written in the block form:

$$D^{(j)}(\tilde{R}) = \begin{bmatrix} X & iW \\ iW^T & Y \end{bmatrix}, \qquad (2.45)$$

where X and Y are square matrices of the same dimensions as A and B, respectively, and W is a rectangular matrix of dimension $(j+1)\times j$. From (2.32), one can write symmetry properties for X, Y, and W in the following way:

$$X = X^T = X^* = \Delta_{j+1} X \Delta_{j+1} = (-)^j \Delta_{j+1} X = (-)^j X \Delta_{j+1} ,$$

$$Y = Y^T = Y^* = \Delta_j Y \Delta_j = (-)^j \Delta_j Y = (-)^j Y \Delta_j , \qquad (2.46)$$

$$W = W^* = \Delta_{j+1} W \Delta_j = -(-)^j \Delta_{j+1} W = -(-)^j W \Delta_j .$$

From the unitarity of $D^j(\tilde{R})$ and the property

$$[D^{(j)}(\tilde{R})]^2 = (-)^j \Delta , \qquad (2.47)$$

where

$$\Delta = \Delta_{j+1} \oplus \Delta_j ,$$

we have

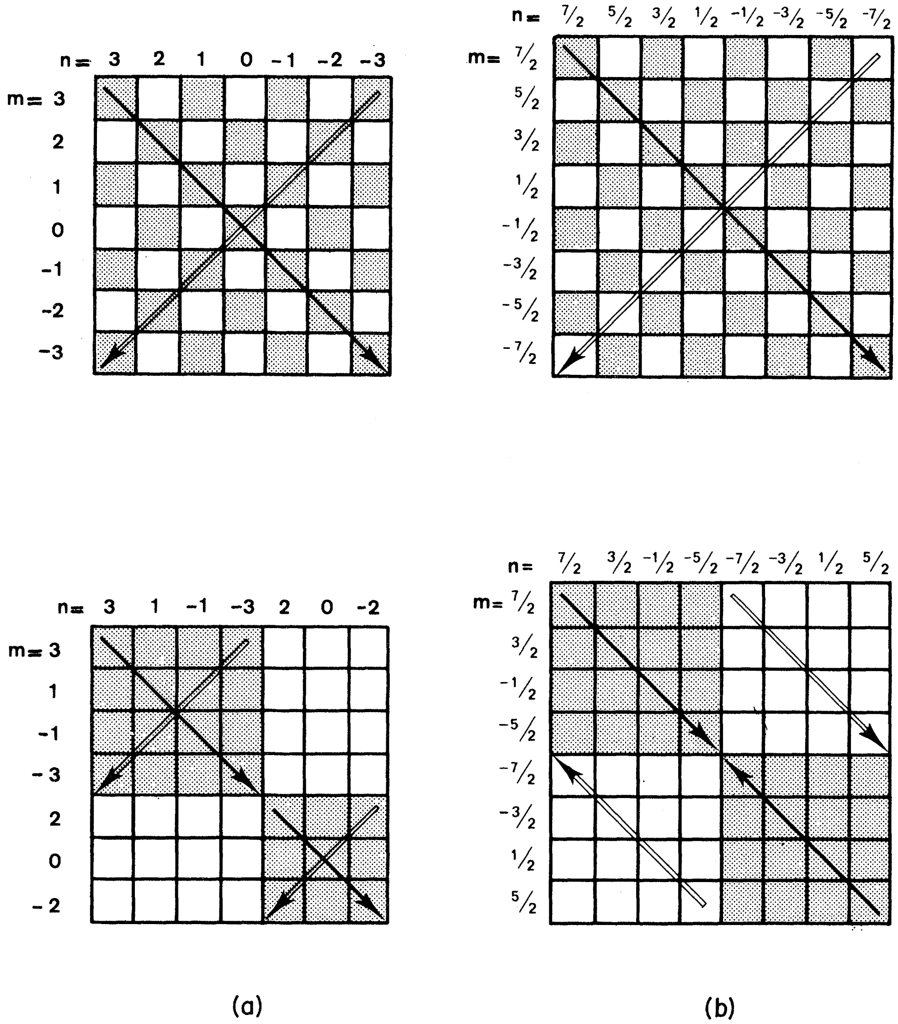

FIG. 1. Structure of Bohr-symmetric density matrices in transversity quantization for the natural order and for the separation order of indices. Part (a) illustrates the case of integer spin ($j=3$) and part (b) the case of half-integer spin ($j=\frac{7}{2}$). The arrows show the transformation of the main and second diagonal in each case.

$$\begin{aligned} &X^2+WW^T=\underline{1}\,,\\ &Y^2+W^TW=\underline{1}\,,\\ &X^2-WW^T=(-)^j\Delta_{j+1}\,,\\ &Y^2-W^TW=(-)^j\Delta_j\,,\end{aligned} \tag{2.48}$$

and

$$XW=WY=0\,.$$

In the separation order of indices, the density matrix ${}^H\rho$ can be written

$${}^H\rho=\begin{bmatrix} C & D\\ D^\dagger & F\end{bmatrix}, \tag{2.49}$$

with

$$\begin{aligned} &C=C^\dagger=\Delta_{j+1}C\Delta_{j+1}\,,\\ &F=F^\dagger=\Delta_jF\Delta_j\,,\\ &D=-\Delta_{j+1}D\Delta_j\,,\\ &\mathrm{tr}C+\mathrm{tr}F=1\,.\end{aligned} \tag{2.50}$$

The relation between the submatrices C, D, F and A and B is given by

$$\begin{aligned} &C=XAX+WBW^T\,,\\ &F=W^TAW+YBY\,,\\ &D=-i(XAW-WBY)\,,\end{aligned} \tag{2.51}$$

or, conversely,

$$A = XCX + WFW^T + i(XDW^T - WD^\dagger X),$$
$$B = YFY + W^T CW + i(YD^\dagger W - W^T DY). \tag{2.52}$$

Furthermore if ρ contains only even polarization, we get

$$C = C^* = C^T = \Delta_{j+1} C \Delta_{j+1},$$
$$F = F^* = F^T = \Delta_j F \Delta_j, \tag{2.53}$$
$$D = D^* = -\Delta_{j+1} D \Delta_j.$$

The number of parameters necessary to specify ${}^T\rho$ or ${}^H\rho$ is given by k, where

$$k = 2j(j+1) \text{ if } \rho \text{ is } B \text{ symmetric},$$
$$k = j(j+2) \text{ if } \rho \text{ is } B \text{ symmetric and even}. \tag{2.54}$$

b. When j is half-integral. Choose another separation order of indices: $j, j-2, \ldots, -j+1;\ -j, -j+2, \ldots, j-1$. See for example Fig. 1(b). Then a B-symmetric density matrix ${}^T\rho$ can be written in the form

$${}^T\rho = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \tag{2.55}$$

where A and B are $(j+\frac{1}{2})\times(j+\frac{1}{2})$ Hermitian matrices with

$$\mathrm{tr}A + \mathrm{tr}B = 1. \tag{2.56}$$

When ρ contains only even polarization, we get furthermore $B = A^T$ so that

$${}^T\rho = \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}. \tag{2.57}$$

The unitary transformation $D^{(j)}(\tilde{R})$ in this case can be written in the form

$$D^{(j)}(\tilde{R}) = \frac{1}{\sqrt{2}} \begin{bmatrix} V & e^{i\pi j} V \\ e^{i\pi j} V & V \end{bmatrix}, \tag{2.58}$$

where V is a real, symmetrical, orthogonal transformation:

$$V = V^* = V^T, \quad V^2 = \underline{1}. \tag{2.59}$$

The density matrix ${}^H\rho$ in the helicity frame for a B-symmetric reaction has the form

$${}^H\rho = \begin{bmatrix} C & D \\ -D & C \end{bmatrix}, \tag{2.60}$$

with

$$C = C^\dagger,$$
$$D = -D^\dagger, \tag{2.61}$$
$$\mathrm{tr}C = \tfrac{1}{2}.$$

The submatrices C and D are related to the submatrices A and B by

$$C = \tfrac{1}{2} V(A+B)V,$$
$$D = -\tfrac{1}{2} e^{i\pi j} V(A-B)V, \tag{2.62}$$

or, conversely,

$$A = V(C + e^{i\pi j} D)V,$$
$$B = V(C - e^{i\pi j} D)V. \tag{2.63}$$

If ρ contains only even polarization by putting $B = A^T$ we get the same form (2.60) for the density matrix in the helicity frame but with

$$C = C^* = C^T = \tfrac{1}{2} V(A + A^T)V,$$
$$D = D^* = -D^T = -\tfrac{1}{2} e^{i\pi j} V(A - A^T)V. \tag{2.64}$$

The number of free parameters to specify the density matrix is k, where

$$k = 2j(j+1) - \tfrac{1}{2} \text{ if } \rho \text{ is } B \text{ symmetric},$$
$$k = (j+\tfrac{3}{2})(j-\tfrac{1}{2}) \text{ if } \rho \text{ is } B \text{ symmetric and even}. \tag{2.65}$$

These results are summarized in Table I.

3. *Rank Condition*

The final-state density matrix ρ_f and the cross section σ are related to the initial-state density matrix ρ_i by

$$\sigma\rho_f = T\rho_i T^\dagger, \tag{2.66}$$

where T is the transition matrix. Since the rank of the product of matrices is smaller than or equal to the smallest rank of the matrices of the product, the rank of ρ_f cannot exceed the rank of ρ_i. For example in the reaction

$$\pi + N \to \pi + N^*,$$
$$\text{rank } \rho(N^*) \leq 2, \tag{2.67}$$

no matter what N^* spin is. Thus if the spin j of the isobar N^* is $>\frac{1}{2}$ (2.67) imposes constraints on the measured elements of the density matrix. Also we note that the rank of the sum of matrices cannot be larger than the sum of their ranks. Hence in (2.64), for example

$$\text{rank } C \leq 2 \text{ rank } A, \tag{2.68}$$

but from (2.63)

$$\text{rank } (C + e^{i\pi j} D) = \text{rank } A. \tag{2.69}$$

As we see, in some reactions angular momentum conservation may impose restriction on the rank of the density matrix (for more examples see Refs. 1 and 2).

Independently of any model, if only the even part ${}^{(E)}\rho$ of the density matrix is measured, one gets, for the rank of this even part,[1]

$$\text{rank } {}^{(E)}\rho \leq 2 \text{ rank } \rho. \tag{2.70}$$

In models which predict only an even polarization ${}^{(E)}\rho = \rho$ implies of course rank ${}^{(E)}\rho = $ rank ρ.

TABLE I. Structure of B-symmetric even-polarization density matrices in the separation order of indices.

	j = integer		j = half integer	
	Separation order of indices: $j, j-2, \ldots, 2-j, -j;$ $j-1, j-3, \ldots, 3-j, 1-j$		Separation order of indices: $j, j-2, \ldots, -j+1;$ $-j, -j+2, \ldots, j-1$	
	B symmetry	B symmetry and even polarization	B symmetry	B symmetry and even polarization
In any transversity frame	${}^T\rho = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ $A = A^\dagger$ $B = B^\dagger$ $\mathrm{tr}A + \mathrm{tr}B = 1$	${}^T\rho = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ $A^* = A^T = \Delta_{j+1} A \Delta_{j+1}$ $B^* = B^T = \Delta_j B \Delta_j$ $\mathrm{tr}A + \mathrm{tr}B = 1$	${}^T\rho = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ $A = A^\dagger$ $B = B^\dagger$ $\mathrm{tr}A + \mathrm{tr}B = 1$	${}^T\rho = \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}$ $A = A^\dagger$ $\mathrm{tr}A = \frac{1}{2}$
In any helicity frame	${}^H\rho = \begin{bmatrix} C & D \\ D^\dagger & F \end{bmatrix}$ $C = C^\dagger = \Delta_{j+1} C \Delta_{j+1}$ $F = F^\dagger = \Delta_j F \Delta_j$ $D = -\Delta_{j+1} D \Delta_j$ $\mathrm{tr}C + \mathrm{tr}F = 1$	${}^H\rho = \begin{bmatrix} C & D \\ D^T & F \end{bmatrix}$ $C = C^* = C^T = \Delta_{j+1} C \Delta_{j+1}$ $F = F^* = F^T = \Delta_j F \Delta_j$ $D = D^* = -\Delta_{j+1} D \Delta_j$ $\mathrm{tr}C + \mathrm{tr}F = 1$	${}^H\rho = \begin{bmatrix} C & D \\ -D & C \end{bmatrix}$ $C = C^\dagger$ $D = -D^\dagger$ $\mathrm{tr}C = \frac{1}{2}$	${}^H\rho = \begin{bmatrix} C & D \\ -D & C \end{bmatrix}$ $C = C^* = C^T$ $D = D^* = -D^T$ $\mathrm{tr}C = \frac{1}{2}$
Number k of independent parameters	$k = 2j(j+1)$	$k = j(j+2)$	$k = 2j(j+1) - \frac{1}{2}$	$k = (j+\frac{3}{2})(j-\frac{1}{2})$
Unitary transformation from transversity to helicity, $\tilde{R} = [\frac{1}{2}\pi, \frac{1}{2}\pi, -\frac{1}{2}\pi]$	$D^{(j)}(\tilde{R}) = \begin{bmatrix} X & iW \\ iW^T & Y \end{bmatrix}$ with: $X = X^T = X^* = \Delta_{j+1} X \Delta_{j+1} = (-)^j \Delta_{j+1} X = (-)^j X \Delta_{j+1}$ $Y = Y^T = Y^* = \Delta_j Y \Delta_j = (-)^j \Delta_j Y = (-)^j Y \Delta_j$ $W = W^* = \Delta_{j+1} W \Delta_j = -(-)^j \Delta_{j+1} W = -(-)^j W \Delta_j$ $X^2 + WW^T = \underline{1};$ $X^2 - WW^T = (-)^j \Delta_{j+1}$ $Y^2 + W^T W = \underline{1};$ $Y^2 - WW^T = (-)^j \Delta_j$ $XW = WY = 0$		$D^{(j)}(\tilde{R}) = \frac{1}{\sqrt{2}} \begin{bmatrix} V & e^{i\pi j}V \\ e^{i\pi j}V & V \end{bmatrix}$ $V = V^* = V^T$ $V^2 = \underline{1}$	
Relations between submatrices in helicity and transversity frames	$A = XCX + WFW^T + i(XDW^T - WD^\dagger X)$ $B = YFY + W^T CW + i(YD^\dagger W - W^T DY)$ $C = XAX + WBW^T$ $F = W^T AW + YBY$ $D = -i(XAW - WBY)$		$A = V(C + e^{i\pi j}D)V$ $B = V(C - e^{i\pi j}D)V$ $C = \frac{1}{2}V(A+B)V$ $D = -\frac{1}{2}e^{i\pi j}V(A-B)V$	
Notations	Δ_k is a $k \times k$ matrix whose elements are $(\Delta_k)^m{}_n = \delta^m{}_{-n}$			

III. POLARIZATION AND REGGE-POLE MODELS

The results discussed in the previous section give the general structure that the density matrices must satisfy from conservation of angular momentum and parity. Any specific model which conserves angular momentum and parity will give density matrices with this structure, but, in addition, if the model has any predictive power on the polarizations, it will give further conditions on the density matrices.

Regge-pole models are such specific models and they are extensively used in the analysis of high-energy scattering. In Sec. III A we shall be mainly concerned with two-body reactions in which a single bosonic Regge trajectory of definite signature is exchanged. At the end of the section we discuss the exchange of several trajectories with the same signature. In Sec. III B we study many-body reactions within the model of a multi-Regge exchange.

A. Two-Body Reactions

Let $G_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)$ and $F_{\lambda_3\lambda_1;\lambda_4\lambda_2}(s,t)$ represent the corresponding s- and t-channel helicity amplitudes for the four-particle reactions mentioned in the previous sections. From the well-known Trueman-Wick crossing relation, we have

$$G_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \sum_{\lambda_4'\lambda_3'\lambda_2'\lambda_1'} d^{j_1}_{\lambda_1'\lambda_1}(\chi_1) d^{j_2}_{\lambda_2'\lambda_2}(\chi_2) \times d^{j_3}_{\lambda_3'\lambda_3}(\chi_3) d^{j_4}_{\lambda_4'\lambda_4}(\chi_4) F_{\lambda_3'\lambda_1';\lambda_4'\lambda_2'}, \tag{3.1}$$

where χ_i is the crossing angle of the ith particle. If the high-s, fixed-t behavior is governed by a single Regge trajectory, we can write

$$F_{\lambda_3\lambda_1;\lambda_4\lambda_2}(s,t) = \beta^{(1)}_{\lambda_3\lambda_1}(t)\beta^{(2)}_{\lambda_4\lambda_2}(t) \times \frac{1+\tau\exp[-i\pi\alpha(t)]}{\sin\pi\alpha(t)} s^{\alpha(t)}, \tag{3.2}$$

where $\beta^{(1)}$ and $\beta^{(2)}$ are the factorized Regge residues, $\alpha(t)$ is the Regge trajectory function, and τ is the signature. The residue functions are real and due to parity conservation at each vertex, they obey the symmetry relations

$$\beta^{(1)}_{-\lambda_3-\lambda_1} = \sigma_E \eta_3 \eta_1 * (-1)^{j_3+j_1}(-1)^{\lambda_3-\lambda_1}\beta^{(1)}_{\lambda_3\lambda_1}, \tag{3.3}$$

$$\beta^{(2)}_{-\lambda_4-\lambda_2} = \sigma_E \eta_4 * \eta_2 (-1)^{j_4+j_2}(-1)^{\lambda_4-\lambda_2}\beta^{(2)}_{\lambda_4\lambda_2}, \tag{3.4}$$

where σ_E is the naturality of the trajectory ($\sigma_E = \tau P$ where P is the parity of the trajectory: $\sigma_E = +1$ or -1 if the trajectory has natural or unnatural parity), and η_i is the intrinsic parity of the ith particle.

Let us consider the polarization state of particle 3 (or 4) when the polarization of 4 (or 3) is not observed. The density matrix in the t-channel helicity frame (or more popularly known as Gottfried-Jackson frame) is given by

$${}^H\rho(s,t)^{m'}{}_{m} = \frac{\sum_{\lambda_1\lambda_2\lambda_4} F_{m'\lambda_1;\lambda_4\lambda_2}(s,t) F^*_{m\lambda_1;\lambda_4\lambda_2}(s,t)}{\sum_{\lambda_1\lambda_2\lambda_3\lambda_4} F_{\lambda_3\lambda_1;\lambda_4\lambda_2}(s,t) F^*_{\lambda_3\lambda_1;\lambda_4\lambda_2}(s,t)} \tag{3.5}$$

with a similar expression for the density matrix of particle 4. If we use (3.2), we see that the density matrix of particle 3 does not depend on s and has the form

$${}^H\rho(t)^{m'}{}_{m} = \sum_{\lambda_1} \gamma^{(1)}_{m'\lambda_1}\gamma^{(1)}_{m\lambda_1}, \tag{3.6}$$

where

$$\gamma^{(1)}_{m\lambda_1} = \beta^{(1)}_{m\lambda_1}\left(\sum_{m\lambda_1}(\beta^{(1)}_{m\lambda_1})^2\right)^{-1/2}, \tag{3.7}$$

so that

$$\sum_{m\lambda_1}\gamma^{(1)}_{m\lambda_1}\gamma^{(1)}_{m\lambda_1} = 1.$$

Likewise, the density matrix for particle 4 does not depend on s and can be written as

$${}^H\rho(t)^{m'}{}_{m} = \sum_{\lambda_2} \gamma^{(2)}_{m'\lambda_2}\gamma^{(2)}_{m\lambda_2}, \tag{3.8}$$

with

$$\sum_{m\lambda_2}\gamma^{(2)}_{m\lambda_2}\gamma^{(2)}_{m\lambda_2} = 1.$$

It is clear that the $\gamma^{(i)}$'s obey the symmetry properties of the corresponding $\beta^{(i)}$'s expressed in Eqs. (3.3) and (3.4).

Before entering into any detail concerning the spin of the particles involved into the reaction, one may derive some general consequences from the expressions (3.6) and (3.8) of the density matrices in terms of the *factorized* residue functions $\gamma_{m\lambda}$.

(i) The reality of the residue functions implies the reality of the B-symmetric density matrix in helicity quantization which, as we have seen in the previous section, means that we have only even polarization.

(ii) Because of assumed factorizability of the residues at the two vertices, the ranks of the density matrices of particles 3 and 4 are determined only by the spins of their associated particles in the t channel, i.e., particles 1 and 2, respectively. Thus the density matrices satisfy

$$\begin{aligned} \text{rank}\ (\rho_3) &\le 2j_1+1, \\ \text{rank}\ (\rho_4) &\le 2j_2+1. \end{aligned} \tag{3.9}$$

(iii) Furthermore, if we consider the joint polarization between the two particles 3 and 4, then factorization implies that there is no polarization correlation between the two particles, i.e.,

$$K(3,4) = 0 \Longleftrightarrow \rho(3,4) = \rho(3)\otimes\rho(4). \tag{3.10}$$

(iv) Finally, consider several reactions which have one common vertex, i.e., they are dominated by the same trajectory and particles 1 and 3 (or 2 and 4) are respectively the same for all reactions. Then the density matrix of the final particle 3 (or 4) is the same for all these reactions, i.e., it is independent of the nature of the particles at the other vertex.

It should be noted here that although the exchange of a single Regge pole dominates a number of processes, it is by no means sufficient to account for all the features of the data. Generally one modifies the simple pole model by including absorptive corrections on contributions due to Regge cuts. There is considerable arbitrariness in introducing such modifications and consequently there is a variety of recipes and models[6] in the literature. Very few general model-independent remarks can be made concerning the density matrices in this context. In general corrections to pole models lead to nonreal residue functions and consequently property (i) disappears. In many models the factorization of the residue functions is also not preserved. But it may be possible to introduce the necessary modifications without destroying the factorization property, in which case the properties (ii), (iii), and (iv) discussed above are still valid.

1. *Production of a Boson*

Let us consider the production of a boson of spin and parity j^η using a pseudoscalar meson as the beam particle:

$$0^- + j_2^{\eta_2} \to j^\eta + j_4^{\eta_4},$$

where the target particle 2 is unpolarized and the polarization state of 4 is not observed.

The density matrix for the observed boson must have rank 1 according to (3.9). It is clear from (3.6) that we can write ${}^H\rho$ in the form (since $\lambda_1 = 0$)

$$\begin{aligned} {}^H\rho &= \begin{bmatrix} c \\ d \end{bmatrix} [c^T \; d^T] \\ &= \begin{bmatrix} cc^T & cd^T \\ dc^T & dd^T \end{bmatrix}, \end{aligned} \tag{3.11}$$

where

$$c = \begin{bmatrix} \gamma_j \\ \gamma_{j-2} \\ \vdots \\ \gamma_{2-j} \\ \gamma_{-j} \end{bmatrix}, \quad d = \begin{bmatrix} \gamma_{j-1} \\ \vdots \\ \gamma_{1-j} \end{bmatrix} \tag{3.12}$$

are two column vectors with $(j+1)$ and j elements, respectively. The dyadic form (3.11) of ${}^H\rho$ exhibits clearly the rank-1 condition which, expressed in terms of density-matrix elements, reads

$${}^H\rho^m{}_n \, {}^H\rho^{m'}{}_{n'} = {}^H\rho^m{}_{n'} \, {}^H\rho^{m'}{}_n . \tag{3.13}$$

The symmetry property (3.3) in this special case can be written as

$$\gamma_{-\lambda} = -\eta\sigma_E(-1)^{j+\lambda}\gamma_\lambda \tag{3.14}$$

or

$$\begin{aligned} \Delta_{j+1}c &= -\eta\sigma_E c , \\ \Delta_j d &= \eta\sigma_E d . \end{aligned} \tag{3.15}$$

Note that (3.11) has the structure (2.49) with the identification

$$C = cc^T, \quad D = cd^T, \quad \text{and } F = dd^T .$$

From (3.14) we get

$$\begin{aligned} \Delta_{j+1}C &= C\Delta_{j+1} = -\eta\sigma_E C , \\ \Delta_j F &= F\Delta_j = +\eta\sigma_E F , \\ \Delta_{j+1}D &= -D\Delta_j = -\eta\sigma_E D . \end{aligned} \tag{3.16}$$

These relations expressed in terms of the matrix elements of ${}^H\rho$ read

$$\begin{aligned} {}^H\rho^m{}_n &= -\sigma_E\eta(-)^{j+n}\, {}^H\rho^m{}_{-n} \\ &= -\sigma_E\eta(-)^{j+m}\, {}^H\rho^{-m}{}_n . \end{aligned} \tag{3.17}$$

The above relations represent the generalization of those found by Ader *et al.*[7] for the diagonal elements of ${}^H\rho$.

The density matrix in the transversity frame takes an even simpler form. As noted in (2.41), ${}^T\rho$ is the direct sum of two matrices A and B. The rank of the total matrix has to be unity. Hence the rank of A or B has to be zero, i.e., A or B has to vanish identically. We show that vanishing of A or B depends on the signature σ_E of the exchanged trajectory and the naturality or unnaturality of the produced boson. Equation (2.52) can be written

$$\begin{aligned} A &= X\Delta_{j+1}{}^2 CX + W\Delta_j{}^2 FW^T \\ &\quad + i(X\Delta_{j+1}{}^2 DW^T - WD^T\Delta_{j+1}{}^2 X), \end{aligned}$$

and using (2.46) and (3.16) one gets

$$A = -(-)^j \eta\sigma_E A . \tag{3.18}$$

In the same way, one can write

$$\begin{aligned} B &= Y\Delta_j{}^2 FY + W^T C\Delta_{j+1}{}^2 W \\ &\quad + i(YD^T\Delta_{j+1}{}^2 W - W^T\Delta_{j+1}{}^2 DY) \end{aligned}$$

and get

$$B = +(-)^j\eta\sigma_E B . \tag{3.19}$$

Therefore parity and rank conditions imply

$$\sigma_E\eta(-1)^j = -1 \Rightarrow {}^T\rho = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{rank } A = 1 \tag{3.20}$$

and

$$\sigma_E\eta(-1)^j = +1 \Rightarrow {}^T\rho = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}, \quad \text{rank } B = 1 . \tag{3.21}$$

Expressed in terms of density-matrix elements (3.20) and (3.21) read

$${}^T\rho^m{}_n = 0 \text{ if } (-)^m \text{ or } (-)^n = \sigma_E\eta , \tag{3.22}$$

$${}^T\rho^m{}_n \, {}^T\rho^{m'}{}_{n'} = {}^T\rho^m{}_{n'} \, {}^T\rho^{m'}{}_n . \tag{3.23}$$

Equations (3.22) and (3.23) are the translation, in the transversity frame, of Eqs. (3.17) and (3.13).

The usefulness of (3.20) and (3.21) in practice should be apparent. We shall discuss a few specific cases in the next section.

We note that when several trajectories of the same naturality dominate the production mechanism, Eqs. (3.17), (3.18), (3.19), and (3.22) still hold while in general the absence of odd polarization and the rank condition disappear.

2. Production of a Fermion Isobar

Consider first the case when the fermion is produced from a spin-$\frac{1}{2}$ target

$$j_1+\tfrac{1}{2}\to j_3+j\,.$$

In this case,

$$^{H}\rho^{m'}{}_{m}=\sum_{\lambda}\gamma^{(2)}_{m'\lambda}\gamma^{(2)}_{m\lambda}\,,\quad \lambda=\tfrac{1}{2},-\tfrac{1}{2}, \tag{3.24}$$

with the symmetry properties given by Eq. (3.4).

Equation (3.24) exhibits that the rank of $^H\rho$ must be 2, independently of the spin and parity of the produced isobar. From (2.57) it follows that, in the transversity frame, the rank of A is unity, i.e., the elements of $^T\rho$ satisfy the conditions

$$^{T}\rho^{m}{}_{n}\,^{T}\rho^{m'}{}_{n'}={}^{T}\rho^{m}{}_{n'}\,^{T}\rho^{m'}{}_{n}\,, \tag{3.25}$$

where

$$m,n,m',n'=j,j-2,\ldots,-j+1\,. \tag{3.26}$$

The equivalent condition in helicity quantization is, from (2.69), that the rank of $(C+e^{i\pi j}D)$ is unity. This condition is equivalent to the vanishing of the determinants of every 2×2 submatrix of $(C+e^{i\pi j}D)$. With the identification

$$^{H}\rho^{m'}{}_{m}=C^{m'}{}_{m}={}^{H}\rho^{-m'}{}_{-m}$$

and (3.27)

$$^{H}\rho^{n'}{}_{-n}=D^{n'}{}_{n}=-{}^{H}\rho^{-n'}{}_{n}$$

for m,n,m',n' given by (3.26), the rank condition reads

$$^{H}\rho^{m'}{}_{m}\,^{H}\rho^{n'}{}_{n}+{}^{H}\rho^{m'}{}_{-n}\,^{H}\rho^{n'}{}_{-m}={}^{H}\rho^{m'}{}_{n}\,^{H}\rho^{n'}{}_{m}+{}^{H}\rho^{m'}{}_{-m}\,^{H}\rho^{n'}{}_{-n}\,, \tag{3.28}$$

$$^{H}\rho^{m'}{}_{-m}\,^{H}\rho^{n'}{}_{n}+{}^{H}\rho^{m'}{}_{m}\,^{H}\rho^{n'}{}_{-n}={}^{H}\rho^{m'}{}_{-n}\,^{H}\rho^{n'}{}_{m}+{}^{H}\rho^{m'}{}_{n}\,^{H}\rho^{n'}{}_{-m} \tag{3.29}$$

for the values of the indices given by (3.26).

The above relations are those found by Ringland and Thews[8] for $m=m'$ and $n=n'$, i.e., for (3.28),

$$({}^{H}\rho^{m}{}_{n})^2+({}^{H}\rho^{m}{}_{-n})^2={}^{H}\rho^{m}{}_{m}\,^{H}\rho^{n}{}_{n}\,, \tag{3.30}$$

while (3.29) becomes an identity.

Note that the above conditions are also valid if the spin-$\frac{1}{2}$ particle associated with the fermion isobar is the beam particle instead of the target.

B. Many-Body Reactions

Up to now we have been interested only in two-body (or quasi-two-body) reactions. We would like to recall some results about multiparticle reactions when the production mechanism is dominated by a multi-Regge exchange and show how our analysis can be extended to such reactions.

Let us first consider the simplest case of a three-particle reaction:

$$1+2\to3+5+4\,, \tag{3.31}$$

where the production mechanism is dominated by a double Regge exchange described by Fig. 2(a).

We define

$$s_{ij}=(p_i+p_j)^2\,,$$
$$t_{ij}=(p_i-p_j)^2\,,$$

where p_i represents the four-momentum of the ith particle.

There are five independent kinematical variables. In order to preserve the symmetry of Fig. 2(a), they can be chosen as follows:

$$s'=s_{35},\quad s''=s_{45},\quad t'=t_{13},\quad t''=t_{24},\quad \varphi\,, \tag{3.32}$$

where φ is the angle between the normals to the three planes defined by p_1,p_3,p_5 and p_2,p_4,p_5.

The amplitude for the crossed channel,

$$4^*+2\to3+5+1^*\,, \tag{3.33}$$

can be written assuming factorization as[9]

$$F_{\lambda_3\lambda_4\lambda_1,\lambda_5\lambda_2}=\beta^{(1)}_{\lambda_3\lambda_1}(t')\beta^{(2)}_{\lambda_4\lambda_2}(t'')\beta^{(5)}_{\lambda_5}(t',t'',\varphi)\,\frac{1+\tau_1e^{i\pi\alpha_1(t')}}{\sin\pi\alpha_1(t')}\,\frac{1+\tau_2e^{i\pi\alpha_2(t'')}}{\sin\pi\alpha_2(t'')}\,s'^{\alpha_1(t')}s''^{\alpha_2(t'')}\,, \tag{3.34}$$

where τ_1 (τ_2) is the signature of the Regge trajectory α_1 (α_2); $\beta^{(1)}(t')$ [$\beta^{(2)}(t'')$] represents the coupling of the Regge trajectory α_1 [α_2] to the particles 1, 3 [2, 4]; and $\beta^5(t',t'',\varphi)$ represents the coupling of particle 5 to the Regge trajectories α_1 and α_2.

Parity conservation at each vertex gives, for $\beta^{(1)}$ and $\beta^{(2)}$, symmetry properties analogous to those given in (3.3) and (3.4), while we get for $\beta^{(5)}$

$$\beta^{(5)}_{-\lambda_5}=\sigma_{E1}\sigma_{E2}\eta_5(-)^{j+\lambda}\beta^{(5)}_{\lambda_5}\,, \tag{3.35}$$

where σ_{E1} and σ_{E2} are the naturalities of the α_1 and α_2 trajectories.

As we have already seen, factorization implies that the polarization of a particle is independent of the vertices where it does not appear. So one gets the following relation:

$$_{(a)}\rho_3(t')={}_{(b)}\rho_3(t')\,,$$
$$_{(a)}\rho_4(t'')={}_{(c)}\rho_4(t'')\,, \tag{3.36}$$

where $_{(a)}\rho_3$ ($_{(a)}\rho_4$) is the density matrix of particle

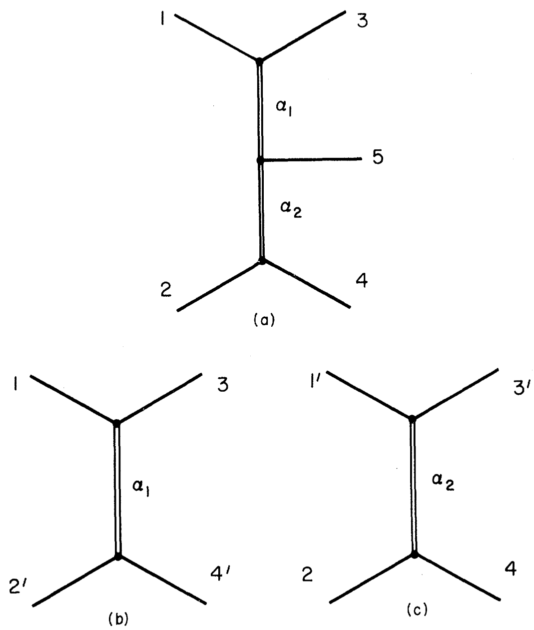

FIG. 2. Trajectory-exchange diagrams for (a) double Regge exchange, (b) and (c) single Regge exchange. The model predicts the same polarization for particles 3 in (a) and (b). It also predicts the same polarization for particles 4 in (a) and (c).

3 (4) for the process described by (3.31), $_{(b)}\rho_3$ is the density matrix for particle 3 in the following process:

$$1+2'\rightarrow 3+4',$$

with the single trajectory α_1 exchanged in the t channel [see Fig. 2(b)]. In this process we call p_i' the four-momentum of particle i and

$$t'=(p_1'-p_3')^2\,. \tag{3.37}$$

In a similar way $_{(c)}\rho_4$ is the density matrix for particle 4 in the following process:

$$1'+2\rightarrow 3'+4,$$

when the single trajectory α_2 is exchanged in the t channel [see Fig. 2(c)]. In this process, we call p_i'' the four-momentum of particle i and

$$t''=(p_2''-p_4'')^2\,. \tag{3.38}$$

Relations (3.36) have been obtained by Capella and Ranft.[10] Our analysis of two-body reactions can be extended immediately to three-body reactions when the production mechanism is dominated by a double Regge exchange.

Furthermore if one looks at the polarization of particle 5, it follows from (3.35) that relations (3.17) and (3.22) hold with the following substitutions:

$$\begin{aligned}&-\sigma_E\rightarrow\sigma_{E1}\sigma_{E2}\,,\\ &\eta\rightarrow\eta_5\,,\\ &j\rightarrow j_5\,.\end{aligned} \tag{3.39}$$

One obtains also the rank condition

$$\text{rank}\ \rho_5(t',t'',\varphi)=1\,. \tag{3.40}$$

This pure state is not easily detected since in order to measure the polarization of particle 5 (if it exists) corresponding to a small bin of t', t'', and φ, a very good statistics is required.

This analysis can be extended in an obvious way to the multiparticle production,

$$1+2\rightarrow 3+5+\cdots+n+4\,.$$

All the results of this section are summarized in Table II.

IV. APPLICATIONS TO $j=1,\,2,\,\frac{3}{2},\,\frac{5}{2}$

We shall now discuss in greater detail the results of the previous sections in a few specific cases of current experimental interest. Prior to that we would like to make some general remarks, concerning polarization measurements and polarization domains, which are independent of production mechanisms (see Refs. 1 and 2).

A. Polarization Measurement

The quantities which are determined directly from the experimental study of a normalized decay distribution $\mathcal{I}(\theta,\varphi)$ are the coefficients $y_M^{(L)}$ of the spherical harmonics $Y_M^{(L)}(\theta,\varphi)$:

$$\mathcal{I}(\theta,\varphi)=\frac{1}{4\pi}+\sum_{L=1}^{2j}\sum_{M=-L}^{L}y_M^{(L)*}\,Y_M^{(L)}(\theta,\varphi)\,. \tag{4.1}$$

(i) If the decay is parity-conserving, $y_M^{(L)}=0$ for odd L. One must check this first experimentally and if this is not satisfied, it means either parity is not conserved or more likely there are interference effects between the resonance channel and the background; clearly for further analysis of polarization measurements, one must account for the background (see Sec. II B).

(ii) If the reaction is B symmetric, one must check that the components $y_M^{(L)}$ satisfy this symmetry; i.e., in any transversity frame

$$y_M^{(L)}=0\quad\text{for } M \text{ odd}\,, \tag{4.2}$$

and in any helicity frame

$$y_M^{(L)}=(-1)^L y_M^{(L)*}\,. \tag{4.3}$$

Again, if conditions (4.2) or (4.3) are not satisfied, one is not studying the decay of a particle with definite spin and parity; more likely this is due to interferences with the background.

TABLE II. Predictions on polarization from Regge-pole models.

1	For two-body reactions $1+2\to 3+4$ independently of any model, if beam 1 and target 2 are unpolarized.	
	Parity conservation in the reaction $\Rightarrow$	B symmetry for ρ_3, ρ_4; ${}^T\rho^m{}_n=0$ if $(-)^{m-n}=-1 \Longleftrightarrow {}^H\rho^m{}_n=(-)^{m-n}\,{}^H\rho^{-m}{}_{-n}$
2	The single trajectory exchange model adds three relevant hypotheses, whose implications are given separately.	
	2.1 Relative reality of the amplitudes $\Rightarrow$	(a) Reality of ${}^H\rho_3$ and ${}^H\rho_4$ (b) Assuming 1, even character of ρ_3, ρ_4: $\rho_3={}^{(E)}\rho_3$, $\rho_4={}^{(E)}\rho_4 \Longleftrightarrow \rho^m{}_n=(-)^{m-n}\rho^{-n}{}_{-m}$
	2.2 Factorization of the amplitudes $\Rightarrow$	(a) Rank restrictions: rank $\rho_3\le 2j_1+1$, rank $\rho_4\le 2j_2+1$ (b) No correlation in the joint matrix: $\rho(3,4)=\rho_3\otimes\rho_4$ (c) Vertex dependence of the polarization: for reactions (a) $1+2\to 3+4$, (b) $1+2'\to 3+4'$, and (c) $1'+2\to 3'+4$, with the same trajectory exchange, we have ${}_{(a)}\rho_3(t)={}_{(b)}\rho_3(t)$, ${}_{(a)}\rho_4(t)={}_{(c)}\rho_4(t)$
	2.3 Parity conservation at each vertex $\Rightarrow$	For $j_1=0$, and assuming 1, signature constraints for $\rho(3)$: ${}^T\rho^m{}_n=0$ if $(-)^m$ or $(-)^n=-\sigma_E\,\eta_1\eta_3$ $\Longleftrightarrow {}^H\rho^m{}_n=\sigma_E\,\eta_1\eta_3(-)^{j_3+n}\,{}^H\rho^m{}_{-n}$
	These hypotheses can be combined, e.g.:	
	2.1 (b) 2.2 (a) for $j_2=\frac{1}{2}$ $\Rightarrow$	Stronger rank restrictions for ρ_4: ${}^T\rho^{m'}{}_m\,{}^T\rho^{n'}{}_n={}^T\rho^{m'}{}_n\,{}^T\rho^{n'}{}_m \Longleftrightarrow$ $(m, m', n, n' = j, j-2, \ldots, -j+1)$ $\Longleftrightarrow \begin{cases} {}^H\rho^{m'}{}_m\,{}^H\rho^{n'}{}_n+{}^H\rho^{m'}{}_{-n}\,{}^H\rho^{n'}{}_{-m}={}^H\rho^{m'}{}_n\,{}^H\rho^{n'}{}_m+{}^H\rho^{m'}{}_{-m}\,{}^H\rho^{n'}{}_{-n} \\ {}^H\rho^{m'}{}_{-m}\,{}^H\rho^{n'}{}_n+{}^H\rho^{m'}{}_m\,{}^H\rho^{n'}{}_{-n}={}^H\rho^{m'}{}_{-n}\,{}^H\rho^{n'}{}_m+{}^H\rho^{m'}{}_n\,{}^H\rho^{n'}{}_{-m} \end{cases}$
	The implications of these hypotheses can be used separately in other models.	
3	For several-trajectory exchange with the same naturality σ_E the prediction 2.3 still holds.	
4	For reaction $1+2\to 3+5+4$ with the multi-Regge exchange of Fig. 1(a), all the predictions in 2 hold for particles 1, 2, 3, 4. For particles 5, we have the rank restriction, rank $\rho_5=1$ and the signature constraints, as given in 2.3 with the substitutions $\sigma_E\eta_1\eta_3\to\sigma_{E_1}\sigma_{E_2}\eta_5$, $j_3\to j_5$.	

(iii) We note that

$$y_M^{(L)}=\left(\frac{2L+1}{2j}\right)^{1/2}\lambda_a(L,j)\,t_M^{(L)}\,, \tag{4.4}$$

and as discussed in Sec. II C 1, $\lambda_a(L,j)$ are "dynamics" independent for the decays listed in (2.19). If they depend on the dynamics of the decay mode one only obtains the $t_M^{(L)}$'s of the density matrix from the $y_M^{(L)}$'s up to an L-dependent factor. A simple example of such a case is

$$j^\eta\to 1^-+0^-\,,\quad \text{where } \eta=-(-1)^j\,.$$

Indeed, it is easy to see that there are two orbital angular momenta involved ($l=j-1$, $l=j+1$) in the decay and $\lambda_a(L,j)$ will depend on the ratio of the two corresponding amplitudes. Thus one cannot construct the density matrix for A_1 ($j^\eta=1^+$) by studying the angular distribution of $\pi^\pm$ in $A_1^\pm\to\rho^0+\pi^\pm$ without further dynamical assumptions (relative strength of s and d waves in the decay). We shall remark later concerning the decay of A_1 into three pseudoscalar mesons.

B. Polarization Domains

The geometrical plots are for polarization what Dalitz plots are for energy momenta. Each polarization state is represented by a point of the polarization domain. We give below for spin 1 how to obtain this point from the polarization parameters, either the $\rho^m{}_n$'s or the $t_M^{(L)}$'s or better the $r_M^{(L)}$'s. Values of the polarization parameters which yield points outside the allowed domain are physically meaningless (in some cases they even require negative angular distribution). Such plots which use the intrinsic properties of the density matrix are also very useful in testing special mechanisms and special properties such as whether the s- or t-channel helicities are conserved.

For example, the even polarization domain of a B-symmetric spin-1 particle[2] is shown in Fig. 3. It is an axially symmetric cone whose meridian section is an equilateral triangle. Each point of this domain represents a polarization state. For a given channel it is possible to define a correspondence independent of the quantization frame and of

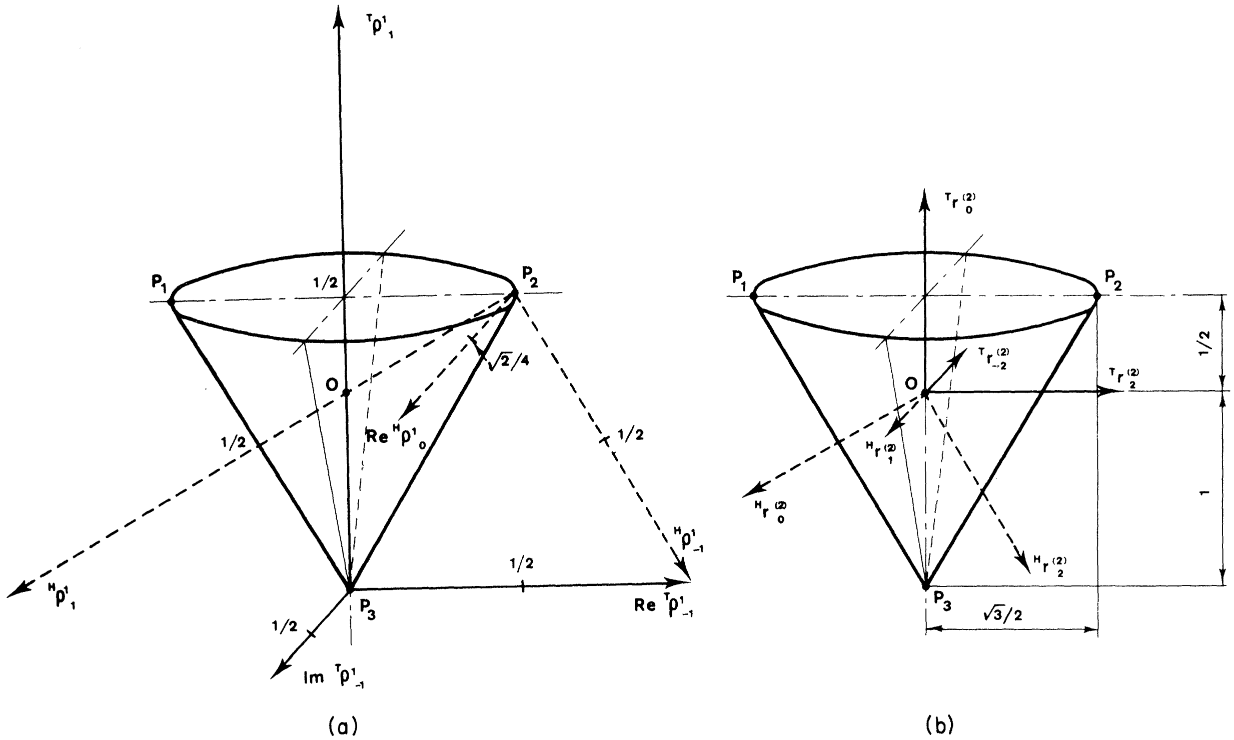

FIG. 3. Domain of B-symmetric even polarization for spin-1 particle. In (a) the orthogonal coordinate system with origin at P_2 (P_3) corresponds to parametrization of the density-matrix element in the transversity (helicity) frame. The length of the coordinate axes represents the unity for the corresponding parameters. In these coordinate systems the equations of the cone are: $|{}^T\rho^1{}_{-1}| \le {}^T\rho^1{}_1 \le \frac{1}{2}$; $1 \ge {}^H\rho^1{}_1 + {}^H\rho^1{}_{-1} \ge 0$, ${}^H\rho^0{}_0({}^H\rho^1{}_1 - {}^H\rho^1{}_{-1}) \ge 2(\mathrm{Re}\,{}^H\rho^1{}_0)^2$. In (b) the two orthonormal coordinate systems with origin at O correspond to multipole parameter $r_M^{(2)}$ parametrization in transversity and in helicity frames. In these coordinate systems the equations of the cone are: $-1 \le {}^Tr_0^{(2)} \le \frac{1}{2}$, $({}^Tr_2^{(2)})^2 + ({}^Tr_{-2}^{(2)})^2 \le \frac{1}{3}({}^Tr_0^{(2)}+1)^2$; $|{}^Hr_2^{(2)}| \le \sqrt{\frac{1}{3}}\,({}^Hr_0^{(2)}+1)$, $({}^Hr_1^{(2)})^2 \le \frac{1}{3}(1-2\,{}^Hr_0^{(2)})(1+{}^Hr_0^{(2)} - \sqrt{3}\,{}^Hr_2^{(2)})$.

the coordinate system of the polarization space; for instance, if the matrix elements of ${}^H\rho$ are used as a set of coordinates, the corresponding axes are drawn on Fig. 3(a). Note that they do not form an orthonormal system of coordinates, the drawn length of each axis represents the value unity.

In Fig. 3(b), the three nonvanishing $r_M^{(2)}$ multipole parameters form an orthonormal system of coordinates in any frame; they are drawn for transversity and helicity quantization. Let us suppose we have chosen the reference frame in the t channel. The axes and coordinates for the s channel are deduced from those in the t channel by a rotation of twice the crossing angle (a function of s and t) around the axis of the cone.

For forward reactions or when the production is helicity-conserving in a specific channel, the density matrix in the helicity frame of this channel must be diagonal, and the representative point must be in the interval of the ${}^H\rho^1{}_1$ axis which is inside the cone.

The point O represents the unpolarized state. The pure states (i.e., totally polarized $\Longleftrightarrow$ rank $\rho = 1$) are at a distance unity from O. For even polarization they are only P_3, the vertex of the cone and the points of the circle bounding the basis of the cone.

The point P_3 represents the state of longitudinal polarization along the normal n, i.e., the state $|1, 0\rangle^T$, and hence ${}^T\rho^0{}_0 = 1$.

The points of the circle represent the transverse polarization state in transversity quantization. For such state the polarization vector e is real (up to a phase) and orthogonal to n, the normal to the reaction plane: $e \cdot n = 0$. So there is a helicity quantization with $n^{(3)} = e$, where such a state is longitudinal and the only nonvanishing matrix element is ${}^H\rho^0{}_0 = 1$.

The even polarization domain for spin $\frac{3}{2}$ is three-dimensional. Its boundary is the two-dimensional sphere drawn in Fig. 4. The matrices which are represented by points lying on this sphere have rank 2. For further details concerning the polarization domain for spin $\frac{3}{2}$ see Ref. 2.

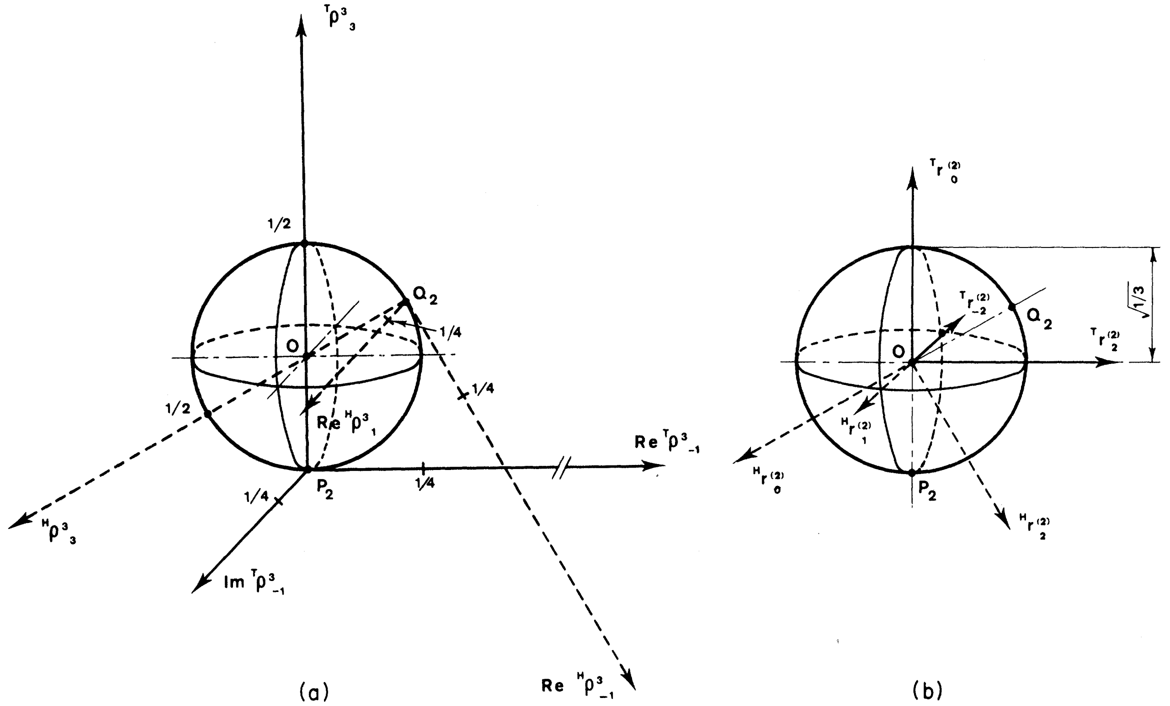

FIG. 4. Domain of B-symmetric even polarization for spin-$\frac{3}{2}$ particle. In (a) the orthogonal coordinate system with origin at P_2 (Q_2) corresponds to density-matrix-element parametrization in the transversity (helicity) frame. The length of the coordinate axes represents the unity for the corresponding parameter. In these coordinate systems the equations of the sphere are: $|{}^T\rho^3_{-1}|^2 \le {}^T\rho^1_1\,{}^T\rho^3_3$; $(\mathrm{Re}\,{}^H\rho^3_{-1})^2 + (\mathrm{Re}\,{}^H\rho^3_1)^2 \le {}^H\rho^1_1\,{}^H\rho^3_3$. In (b) the two orthonormal coordinate systems with origin at O correspond to multipole parameter $r^{(2)}_M$ parametrization in the transversity and in helicity frame. In these coordinate systems the equations of the sphere are: $({}^Tr^{(2)}_2)^2 + ({}^Tr^{(2)}_{-2})^2 + ({}^Tr^{(2)}_0)^2 \le \frac{1}{3}$; $({}^Hr^{(2)}_2)^2 + ({}^Hr^{(2)}_1)^2 + ({}^Hr^{(2)}_0)^2 \le \frac{1}{3}$.

For spin 2, the polarization domain of B-symmetric even polarization states is eight-dimensional. It is convenient to use its projection on the two-dimensional plane of density matrices which are diagonal in any transversity frame. This projection is an isosceles triangle $P_1P_2P_3$ represented in Fig. 5. The angle at the P_1 vertex is

$$\chi = \cos^{-1}(\tfrac{2}{3}) \cong 48.2^\circ .$$

The unpolarized state is projected in the point O ($\rho^2_2 = \rho^1_1 = \rho^0_0 = \frac{1}{5}$). The set of pure states (i.e., rank 1) contains two disconnected parts, a one-dimensional circle whose projection is the point P_2, and a two-dimensional surface whose projection is the segment P_1P_3.

In any helicity quantization the diagonal matrices form another triangle, whose projection in the plane of Fig. 2 is $C_0\,C_1\,C_2$. The point C_0 is the projection of the pure state with ${}^H\rho^0_0 = 1$; the points C_1 and C_2 are the projections of the rank-2 states with ${}^H\rho^1_1 = \frac{1}{2}$ and ${}^H\rho^2_2 = \frac{1}{2}$, respectively.

For a complete study of the spin-2 even polarization domain we refer the reader to Ref. 2a or 2b, and for the study of the spin-$\frac{5}{2}$ even polarization domain to Ref. 2a.

C. Production of Bosons with $j^\eta = 1^+$ When the Natural-Parity Trajectory Dominates

Recently there has been considerable interest[3] in studying the polarization properties of particles with $j^\eta = 1^+$ (A_1, Q mesons) in high-energy reactions. Since the Pomeranchuk trajectory can dominate, such a study provides information concerning the nature of this singularity which is one of the least understood problems in hadronic interactions. As stated in Sec. II C 1, if one studies only the angular distribution $\mathcal{J}(\theta, \varphi)$ of the normal to the decay plane of $1^+ \to 0^- + 0^- + 0^-$, one measures only even multipoles and hence only even polarization. Also such an angular distribution is independent of the decay dynamics.

If we assume that the production of the 1^+ particle is dominated by a Pomeranchuk singularity which can be represented by a single Regge pole with factorizable residues, then from the parity relations (3.14) and (3.15) since $\eta = +1$ and $\sigma_E = +1$,

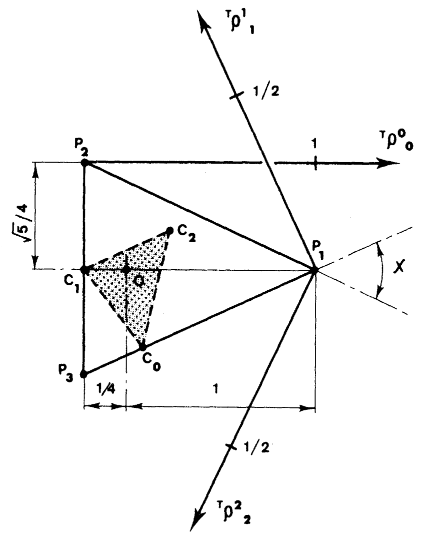

FIG. 5. Projection of the domain of B-symmetric even polarization for spin 2, on the plane of matrices which are diagonal in transversity quantization. The triangle C_0, C_1, C_2 is the projection of the domain of density matrices which are diagonal in any helicity quantization. The three coordinate axes which are drawn correspond to parametrization of the density-matrix element in the transversity frame. Note that these three coordinates are not independent (they satisfy ${}^T\rho^0{}_0 + 2\,{}^T\rho^1{}_1 + 2\,{}^T\rho^2{}_2 = 1$). The coordinate $({}^T\rho^0{}_0,\ {}^T\rho^1{}_1,\ {}^T\rho^2{}_2)$ of the points are $O = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5})$, $P_1 = (1, 0, 0)$, $P_2 = (0, \frac{1}{2}, 0)$, $P_3 = (0, 0, \frac{1}{2})$, $C_0 = (\frac{1}{4}, 0, \frac{3}{8})$, $C_1 = (0, \frac{1}{4}, \frac{1}{4})$, and $C_2 = (\frac{3}{8}, \frac{1}{4}, \frac{1}{16})$.

the amplitudes c and d are

$$c = \begin{bmatrix} \gamma_1 \\ \\ -\gamma_1 \end{bmatrix}, \quad d = [\gamma_0]\,. \tag{4.5}$$

In helicity quantization, the density matrix ${}^H\rho$ defined by (3.11) is

$${}^H\rho = \begin{bmatrix} \gamma_1 \\ -\gamma_1 \\ \gamma_0 \end{bmatrix} [\gamma_1,\ -\gamma_1,\ \gamma_0] = \begin{bmatrix} \gamma_1^2 & -\gamma_1^2 & \gamma_1\gamma_0 \\ -\gamma_1^2 & \gamma_1^2 & -\gamma_1\gamma_0 \\ \gamma_0\gamma_1 & -\gamma_0\gamma_1 & \gamma_0^2 \end{bmatrix}. \tag{4.6}$$

We note that the order of indices is $m, m' = 1, -1, 0$, and that the trace-1 condition implies

$$\gamma_0^2 + 2\gamma_1^2 = 1\,. \tag{4.7}$$

For $j = 1$, the unitary transformation $D^{(j)}(\tilde{R})$ has the form (2.45)

$$D^{(1)}(\tilde{R}) = \begin{bmatrix} X & iW \\ iW & Y \end{bmatrix}$$

with

$$X = \tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad Y = 0, \quad \text{and} \quad W = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{4.8}$$

In transversity quantization, the density matrix ${}^T\rho$ deduced from ${}^H\rho$ by (2.33) is

$$\begin{aligned} {}^T\rho &= \begin{bmatrix} \gamma_1 - (i/\sqrt{2})\gamma_0 \\ -\gamma_1 - (i/\sqrt{2})\gamma_0 \\ 0 \end{bmatrix} [\gamma_1 + (i/\sqrt{2})\gamma_0,\ -\gamma_1 + (i/\sqrt{2})\gamma_0,\ 0] \\ &= \begin{bmatrix} \frac{1}{2} & -(\gamma_1 - (i/\sqrt{2})\gamma_0)^2 & 0 \\ -(\gamma_1 + (i/\sqrt{2})\gamma_0)^2 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{4.9}$$

Equations (4.6) and (4.9) exhibit clearly the rank 1 of the density matrix. Note that the matrix ${}^T\rho$ has the general form (3.20)

$${}^T\rho = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

where the 2×2 matrix A has rank 1. Furthermore, note that because of (4.7) there is only one free parameter to describe the density matrix.

In transversity quantization the rank-1 condition of A reads

$$|{}^T\rho^1{}_{-1}| = {}^T\rho^1{}_1 = {}^T\rho^{-1}{}_{-1} = \tfrac{1}{2}\,. \tag{4.10}$$

Therefore in Fig. 3 the representative points should lie on the circle which bounds the basis of the cone. Likewise, for the matrix elements of ${}^H\rho$ we have

$$\begin{aligned} &{}^H\rho^1{}_1 + {}^H\rho^1{}_{-1} = 0\,, \\ &({}^H\rho^1{}_0)^2 = {}^H\rho^1{}_1\,{}^H\rho^0{}_0\,. \end{aligned} \tag{4.11}$$

Let us assume that t-channel coordinates are used (e.g., the so-called Gottfried-Jackson frame which is the t-helicity quantization frame). If t-channel helicity is conserved, in addition one has

$${}^H\rho^1{}_{-1} = 0, \quad \text{i.e.,} \quad \gamma_1 = 0 \quad \text{and} \quad \gamma_0^2 = 1\,. \tag{4.12}$$

Then all the points should be confined to the point P_2. If s helicity is conserved, the points should be in the transformed of P_2 by the s- and t-dependent rotation of crossing.

In Figs. 6 and 7 we have plotted for the Q meson, results from Ref. 3a for even polarization. In this experiment the odd part of the polarization has also been measured and the results are compatible with zero. We see immediately from these plots that some experimental points are compatible with the exchange of a single trajectory of natural parity and with t-helicity conservation.

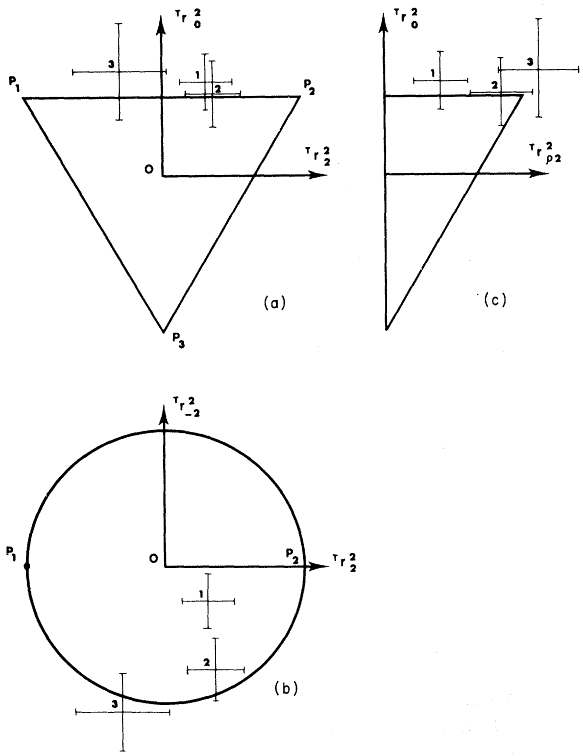

FIG. 6. Plot of experimental data of Q-meson polarization measured in s-channel frame. (a) Projection of the the cone of Fig. 3 on the meridian plane containing P_1 and P_2. (b) Projection of the same cone on a plane perpendicular to its axis. (c) Meridian triangle generating the cone, with ${}^T r^{(2)}_{\rho 2} = [({}^T r^{(2)}_2)^2 + ({}^T r^{(2)}_{-2})^2]^{1/2}$. The experimental points 1, 2, 3, represented in (a), (b), and (c) are taken from Ref. 3a. They correspond to a beam momentum of 8.25 GeV/c and successive intervals of momentum transfer. The points barely satisfy positivity conditions [to be inside the triangle (c)]. The three points are compatible with the prediction of the exchange of natural-parity trajectories [to be in the circle P_1P_2 projected on (a) and (b)], and the points 2 and 3 with the predictions of a single-trajectory exchange of natural parity (to be on the boundary of the circle). They are not compatible with the prediction of s-helicity conservation [to be on the straight line P_2O of (a) and (b)].

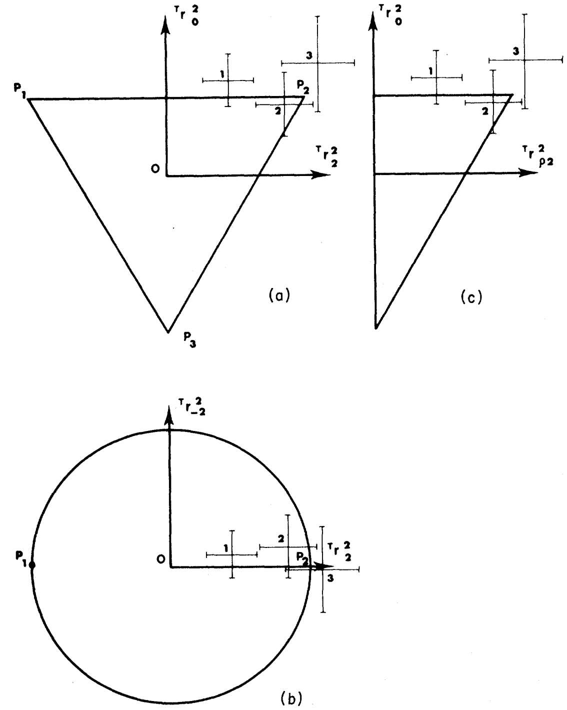

FIG. 7. Plot of the same experimental data as in Fig. 6, measured in the t-channel frame. For the comparison with Fig. 6, recall that both parts (c) have to be identical, and that the points in both parts (b) are rotated from each other by twice the crossing angle χ, which is (Ref. 3a) $0° < \chi < 30°$ for point 1, $30° < \chi < 60°$ for point 2, and $45° < \chi < 70°$ for point 3. Points 2 and 3 are compatible with the prediction of the exchange of a single natural-parity trajectory (to be on the boundary of circle P_1P_2) and moreover with the prediction of t-helicity conservation (to be on the straight line P_2O).

D. Production of a Boson with $j^{\eta} = 1^-$ When the Natural-Parity Trajectory Dominates

In this case, since $\eta = -1$ and $\sigma_E = +1$, the parity relations (3.14) and (3.15) imply for the amplitudes c and d

$$c = \begin{bmatrix} \gamma_1 \\ \gamma_1 \end{bmatrix}, \quad d = [0]. \tag{4.13}$$

So, in helicity quantization, the density matrix ${}^H\rho$ defined by (3.11) is

$${}^H\rho = \begin{bmatrix} \gamma_1 \\ \gamma_1 \\ 0 \end{bmatrix} [\gamma_1, \ \gamma_1, \ 0] = \begin{bmatrix} \gamma_1{}^2 & \gamma_1{}^2 & 0 \\ \gamma_1{}^2 & \gamma_1{}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.14}$$

The trace-1 condition implies $\gamma_1{}^2 = \frac{1}{2}$. The whole polarization is then fixed. This is the pure polarization state represented by the vertex P_3 of the cone:

$${}^H\rho^1{}_0 = 0, \quad {}^H\rho^1{}_1 = {}^H\rho^{-1}{}_{-1} = {}^H\rho^1{}_{-1} = \tfrac{1}{2}. \tag{4.15}$$

In this case there is no possible helicity conservation in any channel.

In transversity quantization the density matrix has the general form (3.21)

$${}^T\rho = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.16}$$

i.e., the only nonvanishing element is

$$^{T}\rho^{0}{}_{0}=1\,. \tag{4.17}$$

If unnatural-parity trajectory dominates it is clear that all the calculations made for $1^{\pm}$ (with $\sigma_E=+1$) becomes true for $1^{\mp}$ (with $\sigma_E=-1$).

E. Production of a Boson with $j^{\eta}=2^{+}$ When the Natural-Parity Trajectory Dominates

From the parity relations (3.14) and (3.15), since $\eta=+1$ and $\sigma_E=+1$, the amplitudes c and d are

$$c=\begin{bmatrix}\gamma_2\\0\\-\gamma_2\end{bmatrix},\quad d=\begin{bmatrix}\gamma_1\\\gamma_1\end{bmatrix}, \tag{4.18}$$

so that the matrix $^{H}\rho$ is

$$^{H}\rho=\begin{bmatrix}\gamma_2\\0\\-\gamma_2\\\gamma_1\\\gamma_1\end{bmatrix}[\gamma_2,\ 0,\ -\gamma_2,\ \gamma_1,\ \gamma_1]$$

$$=\begin{bmatrix}\gamma_2^2 & 0 & -\gamma_2^2 & \gamma_1\gamma_2 & \gamma_1\gamma_2\\ 0&0&0&0&0\\ -\gamma_2^2 & 0 & \gamma_2^2 & -\gamma_1\gamma_2 & -\gamma_1\gamma_2\\ \gamma_1\gamma_2 & 0 & -\gamma_1\gamma_2 & \gamma_1^2 & \gamma_1^2\\ \gamma_1\gamma_2 & 0 & -\gamma_1\gamma_2 & \gamma_1^2 & \gamma_1^2\end{bmatrix} \tag{4.19}$$

with the trace-1 condition

$$\gamma_1^2+\gamma_2^2=\tfrac{1}{2}\,. \tag{4.20}$$

For $j=2$, the unitary transformation $D^{(2)}(\tilde{R})$ has the form (2.45) with

$$X=\tfrac{1}{4}\begin{bmatrix}1 & -\sqrt{6} & 1\\ -\sqrt{6} & -2 & -\sqrt{6}\\ 1 & -\sqrt{6} & 1\end{bmatrix},$$

$$Y=-\tfrac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}, \tag{4.21}$$

$$W=\tfrac{1}{2}\begin{bmatrix}1&-1\\0&0\\-1&1\end{bmatrix}.$$

In transversity quantization, the matrix $^{T}\rho$ deduced from $^{H}\rho$ by (2.33) is

$$^{T}\rho=\begin{bmatrix}0\\0\\0\\-\gamma_1-i\gamma_2\\-\gamma_1+i\gamma_2\end{bmatrix}[0,\ 0,\ 0,\ -\gamma_1+i\gamma_2,\ -\gamma_1-i\gamma_2]\,, \tag{4.22}$$

i.e., $^{T}\rho$ has the general form (3.21)

$$^{T}\rho=\begin{bmatrix}0&0\\0&B\end{bmatrix},$$

where the 2×2 matrix B has rank 1:

$$B=\begin{bmatrix}\tfrac{1}{2} & (\gamma_1+i\gamma_2)^2\\ (\gamma_1-i\gamma_2)^2 & \tfrac{1}{2}\end{bmatrix}. \tag{4.23}$$

From (4.19), (4.20), and (4.23) one can see immediately the whole structure of the density matrix when a single natural-parity trajectory is exchanged. In helicity quantization we get

$$^{H}\rho^{m}{}_{n}+(-)^{n}\,{}^{H}\rho^{m}{}_{-n}=0\ \forall\ m,n\,,$$
$$(^{H}\rho^{2}{}_{1})^2={}^{H}\rho^{2}{}_{2}\,\rho^{1}{}_{1}\,, \tag{4.24}$$

and in transversity quantization

$$^{T}\rho^{1}{}_{1}={}^{T}\rho^{-1}{}_{-1}=|{}^{T}\rho^{1}{}_{-1}|=\tfrac{1}{2}\,,$$
$$^{T}\rho^{m}{}_{n}=0\ \ \text{for}\ m\ \text{or}\ n=\pm2,0\,. \tag{4.25}$$

Equation (4.24) or (4.25) gives the circle of pure state which is projected on point P_2 of Fig. 5. In this case there is no possible helicity conservation in any channel.

F. Production of a Boson with $j^{\eta}=2^{-}$ When the Unnatural-Parity Trajectory Dominates

In this case, since $\eta=-1$ and $\sigma_E=+1$, the parity relations (3.14) and (3.15) imply for the amplitudes c and d,

$$c=\begin{bmatrix}\gamma_2\\\gamma_0\\\gamma_2\end{bmatrix},\quad d=\begin{bmatrix}\gamma_1\\-\gamma_1\end{bmatrix}, \tag{4.26}$$

so that the matrix H_ρ is

$$H_\rho = \begin{bmatrix} \gamma_2 \\ \gamma_0 \\ \gamma_2 \\ \gamma_1 \\ -\gamma_1 \end{bmatrix} [\gamma_2,\ \gamma_0,\ \gamma_2,\ \gamma_1,\ -\gamma_1] = \begin{bmatrix} \gamma_2^2 & \gamma_0\gamma_2 & \gamma_2^2 & \gamma_1\gamma_2 & -\gamma_1\gamma_2 \\ \gamma_0\gamma_2 & \gamma_0^2 & \gamma_0\gamma_2 & \gamma_0\gamma_1 & -\gamma_0\gamma_1 \\ \gamma_2^2 & \gamma_0\gamma_2 & \gamma_2^2 & \gamma_1\gamma_2 & -\gamma_1\gamma_2 \\ \gamma_1\gamma_2 & \gamma_0\gamma_1 & \gamma_1\gamma_2 & \gamma_1^2 & -\gamma_1^2 \\ -\gamma_1\gamma_2 & -\gamma_0\gamma_1 & -\gamma_1\gamma_2 & -\gamma_1^2 & \gamma_1^2 \end{bmatrix} \tag{4.27}$$

with the trace-1 condition

$$\gamma_0^2 + 2(\gamma_1^2 + \gamma_2^2) = 1 \ . \tag{4.28}$$

In transversity quantization, the matrix ${}^T\rho$ has the general form (3.20)

$${}^T\rho = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with rank } A = 1,$$

where the 3×3 matrix A is given by

$$A = \tfrac{1}{16} \begin{bmatrix} 2\gamma_2 - \sqrt{6}\,\gamma_0 - 4i\gamma_1 \\ -2\gamma_0 - 2\sqrt{6}\,\gamma_2 \\ 2\gamma_2 - \sqrt{6}\,\gamma_0 + 4i\gamma_1 \end{bmatrix} [2\gamma_2 - \sqrt{6}\,\gamma_0 + 4i\gamma_1,\ -2\sqrt{6}\,\gamma_2 - 2\gamma_0,\ 2\gamma_2 - \sqrt{6}\,\gamma_0 - 4i\gamma_1] \ . \tag{4.29}$$

From (4.27), (4.28), and (4.29) one can easily get all the relations between spin-density-matrix elements. For instance in helicity quantization one gets

$$\begin{aligned} &{}^H\rho^m{}_n - (-)^n\, {}^H\rho^m{}_{-n} = 0 \quad \forall\, m, n, \\ &({}^H\rho^2{}_0)^2 = {}^H\rho^2{}_2\, {}^H\rho^0{}_0 \ , \\ &({}^H\rho^2{}_1)^2 = {}^H\rho^2{}_2\, {}^H\rho^1{}_1 \ , \\ &({}^H\rho^1{}_0)^2 = {}^H\rho^1{}_1\, {}^H\rho^0{}_0 \ , \end{aligned} \tag{4.30}$$

and in transversity quantization,

$$\begin{aligned} &|{}^T\rho^2{}_0|^2 = {}^T\rho^2{}_2\, {}^T\rho^0{}_0 \ , \\ &|{}^T\rho^2{}_{-2}|^2 = ({}^T\rho^2{}_2)^2 \ , \\ &\arg {}^T\rho^2{}_{-2} = 2 \arg {}^T\rho^2{}_0 \ , \\ &{}^T\rho^m{}_n = 0 \quad \text{for } m \text{ or } n = \pm 1 \ . \end{aligned} \tag{4.31}$$

Equations (4.30) or (4.31) give the surface of pure states which is projected on the segment P_1P_3 of Fig. 5. Only one point of this surface is compatible with helicity conservation in any channel. It is projected on C_0 and has as coordinates in the transversity and helicity quantization corresponding to this channel:

$$\begin{aligned} &{}^T\rho^2{}_2 = {}^T\rho^2{}_{-2} = \sqrt{\tfrac{3}{2}}\ {}^T\rho^2{}_0 = \tfrac{3}{8} \ , \\ &{}^H\rho^0{}_0 = 1, \quad \text{other } {}^H\rho^m{}_n = 0 \ . \end{aligned}$$

As in the spin-1 case, it is clear that if unnatural trajectory dominates all results for $2^\pm$ (with $\sigma_B = +1$) are true for $2^\mp$ (with $\sigma_B = -1$).

G. Production of a Fermion Isobar

Let us consider B-symmetric reactions of the type

$$j_1 + \tfrac{1}{2} \to j_3 + j \ ,$$

such as, for example, N^* production from a nucleon target.

As we have seen in Sec. III A 1, if the reaction is dominated by the exchange of a simple trajectory, the density matrix $\rho(j)$ has the following structure:

(i) $\rho(j)$ has only even polarization, i.e., in the separation order of indices, the density matrices ${}^T\rho$ and ${}^H\rho$ are

$${}^T\rho = \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}, \quad {}^H\rho = \begin{bmatrix} C & D \\ -D & C \end{bmatrix}, \tag{4.32}$$

with

$$A = V(C + e^{i\pi j} D)V \ . \tag{4.33}$$

(ii) Rank $\rho(j) = 2$, hence

$$\text{rank } A = 1 \ , \tag{4.34}$$

and from (4.33)

$$\text{rank } (C + e^{i\pi j} D) = 1 \ . \tag{4.35}$$

Let us now apply these results to spin $\frac{3}{2}$ and $\frac{5}{2}$. Note that in these applications the matrix elements are denoted by $\rho^{2m}{}_{2n}$ instead of $\rho^m{}_n$.

1. *Spin* $\frac{3}{2}$

In this case the matrix A is a 2×2 matrix. Hence the rank condition (4.34) reads $\det A = 0$, i.e., the matrix elements of ${}^T\rho$ satisfy

$$|{}^T\rho^3{}_{-1}|^2 = {}^T\rho^3{}_3\, {}^T\rho^{-1}{}_{-1} \tag{4.36}$$

with

$${}^T\rho^3{}_3 + {}^T\rho^{-1}{}_{-1} = \tfrac{1}{2}.$$

The corresponding condition on the density-matrix elements of ${}^H\rho$ is obtained from the rank condition (4.35) which reads $\det(C - iD) = 0$, i.e.,

$$({}^H\rho^3{}_1)^2 + ({}^H\rho^3{}_{-1})^2 = {}^H\rho^3{}_3\, {}^H\rho^{-1}{}_{-1} \tag{4.37}$$

with

$${}^H\rho^m{}_n = \text{real} \quad \forall m, n, \quad {}^H\rho^3{}_3 + {}^H\rho^{-1}{}_{-1} = \tfrac{1}{2}.$$

This result has been already noted by Ringland and Thews.[8]

Relations (4.36) or (4.37) mean that the point which represents the polarization density matrix (see Fig. 4) must be on the sphere, i.e., on the boundary of the polarization domain.

2. *Spin* $\frac{5}{2}$

In this case the matrix A is a 3×3 matrix. By writing the rank-1 condition for the 3×3 matrix A, we get in transversity quantization

$$\begin{aligned} &|{}^T\rho^5{}_{-3}|^2 = {}^T\rho^5{}_5\, {}^T\rho^{-3}{}_{-3},\\ &|{}^T\rho^5{}_1|^2 = {}^T\rho^5{}_5\, {}^T\rho^1{}_1,\\ &|{}^T\rho^1{}_{-3}|^2 = {}^T\rho^{-3}{}_{-3}\, {}^T\rho^1{}_1,\\ &{}^T\rho^5{}_1\, {}^T\rho^1{}_{-3} = {}^T\rho^1{}_1\, {}^T\rho^5{}_{-3}, \end{aligned} \tag{4.38}$$

with

$${}^T\rho^5{}_5 + {}^T\rho^1{}_1 + {}^T\rho^{-3}{}_{-3} = \tfrac{1}{2}.$$

In helicity quantization the rank conditions (3.28) and (3.29) give the relations

$$\begin{aligned} &({}^H\rho^5{}_3)^2 + ({}^H\rho^5{}_{-3})^2 = {}^H\rho^5{}_5\, {}^H\rho^{-3}{}_{-3},\\ &({}^H\rho^5{}_1)^2 + ({}^H\rho^5{}_{-1})^2 = {}^H\rho^5{}_5\, {}^H\rho^1{}_1,\\ &({}^H\rho^1{}_3)^2 + ({}^H\rho^1{}_{-3})^2 = {}^H\rho^{-3}{}_{-3}\, {}^H\rho^1{}_1,\\ &{}^H\rho^5{}_1\, {}^H\rho^1{}_{-3} = {}^H\rho^5{}_{-3}\, {}^H\rho^1{}_1 + {}^H\rho^5{}_{-1}\, {}^H\rho^1{}_3,\\ &{}^H\rho^5{}_{-1}\, {}^H\rho^1{}_{-3} + {}^H\rho^5{}_1\, {}^H\rho^1{}_3 = {}^H\rho^5{}_3\, {}^H\rho^1{}_1, \end{aligned} \tag{4.39}$$

with

$${}^H\rho^m{}_n = \text{real} \quad \forall m, n \quad \text{and} \quad {}^H\rho^5{}_5 + {}^H\rho^1{}_1 + {}^H\rho^{-3}{}_{-3} = \tfrac{1}{2}.$$

The first three relations are those of Ringland and Thews.[8] The number of relations increases with the spin value but the corresponding relations can be computed in the same way for any half-integer spin.

V. CONCLUSION

We have studied two-body (or quasi-two-body) parity-conserving reactions

$$1 + 2 \to 3 + 4$$

with unpolarized beam and target.

From the general consideration of angular momentum and parity conservation we have derived the structure of the density matrices of the final particles 3 and 4. We have emphasized the fact that the structure of the density matrices is more simple in transversity quantization than in helicity quantization. We have also noticed that it is very convenient to choose the separation order for the lines and columns instead of the usual order. All these results on the strucutre of the density matrices have been summarized in Table I.

We have studied the predictions of a single Regge trajectory exchange model for the density matrices. Our aim was to derive all the consequences of reality and factorization of the residue functions, and of parity conservation at each vertex. The results have been summarized in Table II.

The particular form of energy dependence of Regge amplitudes is irrelevant for our conclusions thus, more generally, other models can yield some of these conclusions. For instance any model in which all helicity amplitudes have the same phase [e.g., exchange of several particle members of same multiplet of a symmetry group, e.g.,[11] $SU(6)_W$] will also predict a lack of odd polarization in a B-symmetric reaction.

As an example, we have applied these considerations to experimental results in the case of 1^+ particle and, on this example, we have discussed the usefulness of the geometrical plots. When experimental results are plotted in this way, one can see clearly to what extent the results are meaningful (i.e., if they represent the properties of a particle of given mass, spin, and parity), and to what extent they are consistent with the model in which a single Regge-pole exchange mechanism dominates. With experiments in progress for the determination of density matrices for high spin isobars, it should be clear that such considerations will play an important role.

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