A generalization of this theorem is suggested if we introduce the factor $Q^V/2J(2J-1)$. We therefore conjecture

$$A_3 = \frac{Q^V}{2J(2J-1)M^2}, \qquad (11)$$

for arbitrary spin.

The generalized Cabibbo-Radicati theorem, Eq. (9), can also be written as

$$A_1 = -2\omega\left(\frac{dG_0^V}{dt}\right)_{t=0}, \qquad (12)$$

where G_0^V is the isovector part of the physical form factor G_0 introduced by Gourdin,[2] extending for higher spins the definition of the charge form factor first introduced by Yennie, Lévy, and Ravenhall[8] for the nucleon case.

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[6]Had we defined Eq. (3) with

$$F_3 q_\rho q_\sigma \to F_3(q_\rho q_\sigma - \tfrac{1}{3}q^2\delta_{\rho\sigma})$$

similar to the case (Ref. 4) $J=1$, the quadrupole moment would be absent in the expressions of A_1 and of the charge radius $\langle r^2\rangle^V$, but the relation between them would be maintained,

$$A_1 = \omega(6\mu^V - 7 - 24M^2F_1^{V\prime})/12M^2$$
$$= \tfrac{1}{3}\omega\,\langle r^2\rangle^V.$$

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Bethe-Salpeter Equation for Nucleon-Nucleon Scattering Matrix Padé Approximants*

J. L. Gammel and M. T. Menzel
Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87544
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We first show that the so-called "kernel subtraction" technique which we used in a previous paper on the Bethe-Salpeter equation is unnecessary. The simplified equations which result make it possible to introduce matrix Padé approximants in a straightforward way. These are found to converge to the same solution that we found in our previous work and more rapidly than the ordinary Padé approximants.

I. ELIMINATION OF THE KERNEL-SUBTRACTION TECHNIQUE

If we do not use the kernel-subtraction technique,[1] introduced in Sec. II E in Ref. 2, everything goes as in Ref. 2 with the following minor and simplifying exceptions. The last term in Eq. (29) is unnecessary. Equations (32) and (34) remain valid, but they are equations for $\phi(p, ip_4, \alpha)$ and $\phi(p, E-E(p), \alpha)$; that is, the primed ϕ's are not introduced as they are in Eq. (24) of Ref. 2. Also, in place of Eqs. (37)–(41), there is a single, simpler equation,

$$\tan\delta = \frac{E}{2p}\,\phi(\hat{p}, 0, 1; \hat{p}, 0, 1), \qquad (1)$$

provided that, in the integrals over q in Eqs. (32) and (34), the principal part is taken.

The principal part is taken in this way. The integrals over q have two contributions, one from the double integral and one from the single integral on the right-hand side of Eqs. (32) and (34). The single integral with $\hat{p}$ as upper limit contributes to one side of the q integration (that is, the side q

$< \hat{p}$), and the double integral contributes to both sides ($q < \hat{p}$ and $q > \hat{p}$). The two parts go together in such a way that a principal-part integral occurs with the singularity at $q = \hat{p}$ (this fact is explained at greater length in connection with the reduction of the Bethe-Salpeter equation to a Schrödinger equation immediately below). The principal part is taken by choosing a mesh symmetrical about this singular point and omitting the singular point. [In order to make this last remark perfectly precise, we note that the integral,

$$\int_{-1}^{1} \frac{dx}{1-2x} , \tag{2}$$

can be calculated numerically by using a mesh with equally spaced points, one point being $x = \frac{1}{2}$, and omitting the infinite contribution of the point $x - \frac{1}{2}$.]

Therefore, Eq. (42) has to be replaced by

$$q = \frac{1+u}{1-u} \hat{p} , \tag{3}$$

so that the singular point is $u = 0$, which we always take to be a mesh point. Equation (43) has to be replaced by the same thing, since the double and single integrals combine to make the principal part and must be treated in the same way, that is, subjected to the same transformations. Finally, after the variables are transformed according to the new Eqs. (42) and (43), that is, according to Eq. (3) above [and Eqs. (44)–(46) of Ref. 2 for the q_4 integrations] the principal part is taken very simply by omitting the point $u = 0$.

It is possible in certain approximations[3] to reduce the Bethe-Salpeter equation to a Schrödinger equation with a potential, and it may help explain the above discussion if this is done in a particularly simple approximation as follows.

The propagator $S(q, iq_4, 1, 1)$ is

$$S(q, iq_4, 1, 1) = \frac{1}{[E(q) - E]^2 + q_4^2} . \tag{4}$$

A Schrödinger equation results if (1) all negative-energy states are ignored, (2) it is assumed that $\phi(q, iq_4, 1)$ varies slowly with q_4 when compared to this propagator, and (3) $K(p, 0, 1; q, iq_4, 1)$ varies slowly with q_4 when compared to this propagator. Then only $\phi(q, 0, 1)$ need be taken into account, and the integrals over q_4 in Eqs. (32) and (34) may be done analytically. These integrals are

$$\int_0^\infty dq_4 \frac{1}{[E(q) - E]^2 + q_4^2} = \tfrac{1}{2}\pi \frac{1}{|E(q) - E|} . \tag{5}$$

This equality is the basis of the often used approximation[4]

$$\frac{1}{[E(q) - E]^2 + q_4^2} - \tfrac{1}{2}\pi \frac{1}{|E(q) - E|} \delta(q_4) . \tag{6}$$

Equations (32) and (34) of Ref. 2 become identical, and both are

$$\begin{aligned} \phi(p, 0, 1) &= G(p, 0, 1; \hat{p}, 0, 1) + \frac{1}{2\pi} \int_0^\infty dq\, K(p, 0, 1; q, 0, 1) \frac{1}{|E(q) - E|} \phi(q, 0, 1) \\ &\quad + \frac{1}{\pi} \int_0^{\hat{p}} dq\, K(p, 0, 1; q, 0, 1) \frac{1}{E(q) - E} \phi(q, 0, 1) \\ &= G(p, 0, 1; \hat{p}, 0, 1) + \frac{\mathrm{P}}{\pi} \int_0^\infty dq\, G(p, 0, 1; q, 0, 1) \frac{1}{E(q) - E} \phi(q, 0, 1) , \end{aligned} \tag{7}$$

which is a Schrödinger equation.

From this result it is clear how the double and single integrals over q_4 in Eqs. (32) and (39) of Ref. 2 go together to make a principal-part integral. Furthermore, it is clear that the singularity dealt with is not too severe to be taken into account in the manner described above. The kernel-subtraction technique is unnecessary.

II. MATRIX PADÉ APPROXIMANTS

We go back to the full Eqs. (32) and (34) of Ref. 2 (for unprimed ϕ's). We define $\phi(p, ip_4, \alpha; \hat{p}, 0, \beta)$ and $\phi(p, E - E(p), \alpha; \hat{p}, 0, \beta)$ by using in the inhomogeneous terms in Eqs. (32) and (34) $G(p, ip_4, \alpha; \hat{p}, 0, \beta)$ and $G(p, E - E(p), \alpha; \hat{p}, 0, \beta)$. Then we define a tangent matrix,

$$(\tan\delta)_{\alpha\beta} = \frac{E}{2\hat{p}} \phi(\hat{p}, 0, \alpha; \hat{p}, 0, \beta) . \tag{8}$$

We claim that this procedure is a straightforward extension suggested by analogy with coupled-channel Schrödinger equations with potentials.[5]

By iterating Eqs. (32) and (34) of Ref. 2 as explained in Ref. 2 using the meshes described in Sec. I above, we obtain the expansion

$$(\tan\delta)_{\alpha\beta} = T_{\alpha\beta}^{(1)} \left(\frac{g^2}{4\pi}\right) + T_{\alpha\beta}^{(2)} \left(\frac{g^2}{4\pi}\right)^2 + \cdots . \tag{9}$$

The matrices T may be computed rapidly to any

TABLE I. Some elements of the symmetric tangent matrix, $^1S_0$ state, $E_{\text{lab}} = 100$ MeV. The $^3P_0^0$ state ($\alpha = 4$) has been neglected in all calculations.

n	$(\tan\delta)^{(n)}_{11}$	$(\tan\delta)^{(n)}_{12}$	$(\tan\delta)^{(n)}_{13}$	$(\tan\delta)^{(n)}_{22}$	$(\tan\delta)^{(n)}_{23}$	$(\tan\delta)^{(n)}_{33}$
1	-4.26×10^{-2}	-2.61	-2.61×10^{-1}	-4.26×10^{-2}	2.61×10^{-1}	1.05
2	9.90×10^{-3}	7.38×10^{-2}	8.14×10^{-3}	4.54×10^{-1}	-8.69×10^{-2}	-2.96×10^{-2}
3	-2.11×10^{-3}	-1.80×10^{-2}	-1.35×10^{-3}	-1.03×10^{-1}	-5.63×10^{-3}	3.30×10^{-4}
4	6.59×10^{-4}	4.28×10^{-3}	3.08×10^{-4}	3.12×10^{-2}	1.92×10^{-3}	7.61×10^{-5}
5	-1.97×10^{-4}	-1.30×10^{-3}	-8.65×10^{-5}	-8.02×10^{-3}	-5.11×10^{-4}	-2.70×10^{-5}
6	6.10×10^{-5}	3.83×10^{-4}	2.47×10^{-5}	2.45×10^{-3}	1.53×10^{-4}	8.77×10^{-6}
7	-1.85×10^{-5}	-1.17×10^{-4}	-7.35×10^{-6}	-7.23×10^{-4}	-4.51×10^{-5}	-2.68×10^{-6}
8	5.62×10^{-6}	3.51×10^{-5}	2.19×10^{-6}	2.20×10^{-4}	1.36×10^{-5}	8.22×10^{-7}

desired order (we have computed through 16th order in the present work). Some elements of the T matrices are given in Table I.

A [2, 2] matrix Padé approximant (for example, the generalization to higher orders is obvious) was defined by requiring that

$$(\tan\delta) = \left[N^{(1)}\left(\frac{g^2}{4\pi}\right) + N^{(2)}\left(\frac{g^2}{4\pi}\right)^2\right] \times \frac{1}{1 + D^{(1)}(g^2/4\pi) + D^{(2)}(g^2/4\pi)^2} \equiv N_2 \times \frac{1}{D_2} \quad (10)$$

agree with the series expansion of ($\tan\delta$) through fourth order. By cross multiplying one gets

$$\begin{aligned} &T^{(1)} = N^{(1)}\,,\\ &T^{(1)}D^{(1)} + T^{(2)} = N^{(2)}\,,\\ &T^{(2)}D^{(1)} + T^{(1)}D^{(2)} + T^{(3)} = 0\,,\\ &T^{(3)}D^{(1)} + T^{(2)}D^{(2)} + T^{(4)} = 0\,, \end{aligned} \quad (11)$$

which can be solved for the N's and D's. It is necessary to have the D's on the right of the T's in Eq. (11) since matrix multiplication is not commutative; care must be taken to make the order of all products correct. We could have put the denominator on the left in Eq. (10). The order of factors in Eq. (11) would then be reversed, and the resulting N's and D's would have been different. However, the result is exactly the same as we prove now. Let n_M and d_M be the numerator and denominator [both of order M in $(g^2/4\pi)$] when d_M stands on the left, and let N_M and D_M be the numerator and denominator when D_M stands on the right [as D_2 does in Eq. (10)]. Then we ask whether the equality

$$\frac{1}{d_M}\, n_M = N_M \frac{1}{D_M} \quad (12)$$

is correct. By cross multiplication we see that this Eq. (12) is equivalent to

$$n_M D_M = d_M N_M\,. \quad (13)$$

In Eq. (13), no terms of order greater than $2M$ in $(g^2/4\pi)$ occur. But the left- and right-hand sides of Eq. (12) agree to order $2M$ since both agree with the series expansion Eq. (9) through order $2M$. Therefore, Eq. (12) is exact. This proof differs in no way from the proof that diagonal Padé approximants to the S matrix are unitary.[6]

III. RESULTS OF CALCULATION

Calculations were done for the $^1S_0$ state, at $E_{\text{lab}} = 100$ MeV, with pseudoscalar-pion exchange, as in Ref. 2. Only the $^1S_0^+$, $^1S_0^-$, and $^3P_0^e$ states were included (see Ref. 2) because the $^3P_0^o$-state contribution is known to be negligible.[2]

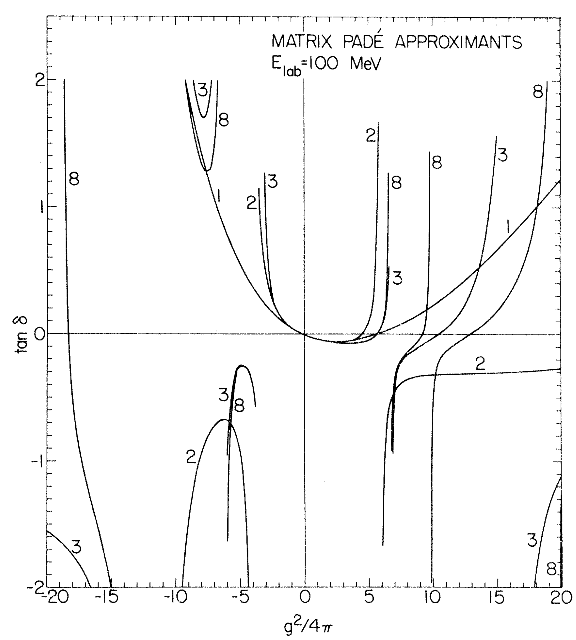

FIG. 1. Matrix Padé approximants. $\tan\delta(^1S_0)$ vs $g^2/4\pi$, $E_{\text{lab}} = 100$ MeV.

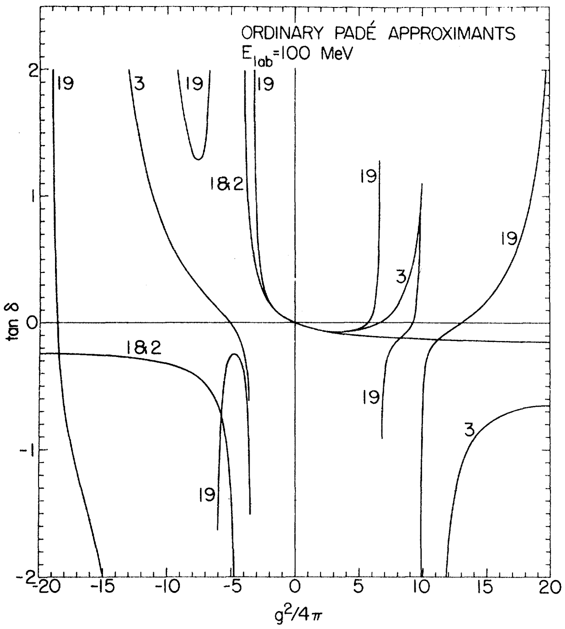

FIG. 2. Ordinary Padé approximants. $\tan\delta(^1S_0)$ vs $g^2/4\pi$, $E_{\rm lab}=100$ MeV.

The convergence of the $[M, M]$ matrix Padé approximants is shown in Fig. 1. The result to which they converge is the same as the result found in Ref. 2.

By applying the method of Padé approximants to the (1, 1) element of the tangent matrix with ordinary Padé approximants, the results shown in Fig. 2 are obtained.

We conclude that the matrix Padé approximants converge more rapidly than the ordinary Padé approximants. The [1, 1] matrix Padé approximant is far more reasonable than the ordinary Padé approximant, and the [2, 2] matrix Padé approximant locates the nearest singularities of $\tan\delta(^1S_0)$ as a function of $(g^2/4\pi)$.

[We conclude in Ref. 2 that these nearest singularities are branch points of $\tan\delta(^1S_0)$ as a function of $(g^2/4\pi)$. This has been disputed,[7] but we think incorrectly in view of the arguments of Mandelstam.[8] It is not our purpose to pursue this discussion.]

ACKNOWLEDGMENTS

We acknowledge conversations and correspondence with Daniel Bessis and William Wortman. The work "matrix" spoken by Bessis in conversations in Cargêse, Corsica in 1970 is sufficient to set off the train of thought reported in this paper. Bessis, Turchetti, and Wortman[9] have calculated all graphs through order $(g^2/4\pi)^2$ and formed a [1, 1] matrix Padé approximant. We have not yet seen the details of their calculation, although we have some graphs and numbers which they have sent us. Their results, which they have calculated for all partial waves, agree with the phase shifts deduced from experiment with precision for $l \geq 2$, and at least qualitatively for $l=0$ and 1. They also calculated what they would get when only the ladder graph is kept in fourth order, so that we might compare the result with our own result. The agreement is excellent and reinforces the idea that their calculation must be the same as ours in its details.

Since our [1, 1] matrix Padé approximant is not accurate for $(g^2/4\pi)$ as large as 15, we withhold judgement about the meaning of Bessis and Wortman's success. For $l \geq 2$, the Mandelstam branch point is much further out on the real axis than it is for the $^1S_0$ state, so that their results are more probably meaningful for these states, and even for $l=0$ and 1 the [1, 1] matrix Padé approximant may be accurate for field theory, which hopefully does not have the Mandelstam singularities.

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