Phenomenological Six-Pion Amplitude*

Richard Brower† and Gilbert Chu
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 31 August 1972)

To test the multiparticle, zero-width bootstrap, we have constructed a six-pion amplitude by modifying the general chiral pion amplitudes of Neveu, Schwarz, and Thorn. We show that imposing the chiral constraints to the six-pion dual amplitude does not fix the mass of the ω. The masses and widths of the low-mass resonances (e.g., ρ, ω, f, A_2, g, A_1, H, σ, and π') are given in terms of two masses (m_π and m_ω), one coupling ($g^2 \sim \Gamma_\rho$), and a universal trajectory slope (α'). We estimate $\Gamma(\omega \to 3\pi) = 6$ MeV, $\Gamma(A_1 \to \rho\pi) = 154$ MeV, and $\Gamma(A_2 \to \rho\pi) = 7.5$ MeV. Further phenomenological investigations are recommended.

I. INTRODUCTION

Shortly after Veneziano[1] wrote down a dual amplitude for $\pi\pi \to \pi\omega$, Lovelace and Shapiro[2] (LS) extended the idea to $\pi\pi \to \pi\pi$, producing an elegantly simple narrow-resonance model that embodied many of the characteristics that would please a phenomenologist: a realistic mass spectrum for the particles π, ρ, f, g, σ, etc.[3]; reasonable decay widths for the ρ, f, g, and σ; and even a surprisingly good fit to the data for $p\bar{n} \to 3\pi$.

One could suppose that the reasonable properties of the LS amplitude were a consequence of the following bootstrap constraints (a)–(d) combined with the chiral constraints[4] (e), (f):

(a) At low energies, factorizable resonances with no ghosts ($g^2 < 0$) and no tachyons ($m^2 < 0$).

(b) At high energies, Regge behavior.

(c) Leading trajectory duality [i.e., finite-energy sum rules (FESR's)][5] connecting (a) and (b).

(d) Exact crossing.

(e) Adler zeros.[6]

(f) No exotics ($I \geq 2$).

Since the early success with four pions, there have been many attempts to write down higher-order pion amplitudes bearing a resemblance to hadronic reality. Brower[7] took the initial step by combining chirality and duality in an N-pion amplitude with an A_1 at $m^2 \sim 1$ GeV^2 and a σ at $m^2 \sim \frac{1}{2}$ GeV^2 but was unable to eliminate the tachyons while preserving the other bootstrap constraints.

Neveu and Schwarz[8] (NS) constructed a prototype N-pion amplitude with a mechanism which eliminated the tachyon on the leading ρ trajectory and which introduced an ω at $m^2 = \frac{1}{2}$ GeV^2. Also there were enough gauge conditions to cancel all the ghosts[9]; the mass spectrum did not have either an A_1 or a σ and required unrealistic masses $m_\rho^2 = 0$ and $m_\pi^2 = -\frac{1}{2}$. Subsequently, a way of modifying the Neveu-Schwarz model was discovered[10] which would permit physical pion and ρ masses and possess the Adler zeros. Both an A_1 and σ reappeared as a concomitant of chirality,[7] but the price paid was the introduction of ghosts, with masses at $m^2 \geq \frac{3}{2}$ GeV^2. The model of Neveu and Thorn, and Schwarz, (NTS) satisfied all the constraints (a) through (f) but possessed a glaring defect in its spectrum: As in the NS model, the ω-A_2 trajectory was a half unit below the ρ-f trajectory.

In the past, it was thought that such a defect was inevitable. In fact it was conjectured by Ademollo, Veneziano, and Weinberg[11] that the constraints (c), (e), and (f) implied that in a transition $a + \pi \rightharpoonup b$, $\alpha' m_a^2$ and $\alpha' m_b^2$ must be separated by half integers. The conjecture was based on the assumption that the Adler zero must arise as a "dynamical" zero as in the LS amplitude where the Γ function in the denominator blows up at the Adler point,

$$\frac{\Gamma(1-\alpha_\rho(s))\,\Gamma(1-\alpha_\rho(t))}{\Gamma(1-\alpha_\rho(s)-\alpha_\rho(t))}.$$

However, observe that the Adler zeros in the $\omega \to 3\pi$ amplitude of Veneziano,

$$\epsilon_{\alpha\beta\gamma\delta} k_1^\beta k_2^\gamma k_3^\delta B(1-\alpha_\rho(s), 1-\alpha_\rho(t)),$$

are "kinematical" in origin, and the mass of the ω is unconstrained. In this paper, by a suitable modification of the NTS model, we show that in a six-pion amplitude the ω mass is still unconstrained. Furthermore, if it is assumed that there is no ω' degenerate with the A_1, and if the leading trajectory duality of the NTS model (see Fig. 1) is maintained, then the coupling of the ω is determined by the bootstrap.

Our model has a reasonable spectrum (see Fig. 2), where all the masses are calculated from three parameters, m_π^2, m_ω^2, and α', and where

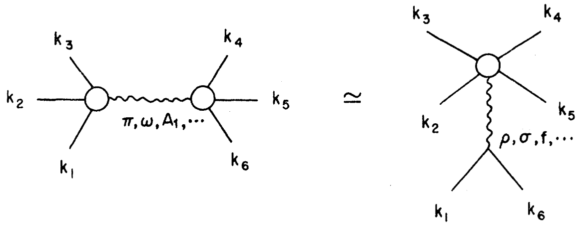

FIG. 1. Leading-trajectory bootstrap. The low-mass resonances in the three-pion channel (π, ω, A_1, ...) are dual to leading Regge poles (ρ, f, σ, ...). This requirement of (inclusive) finite-energy sum rules is imposed by putting the leading trajectories into the same B_6 functions.

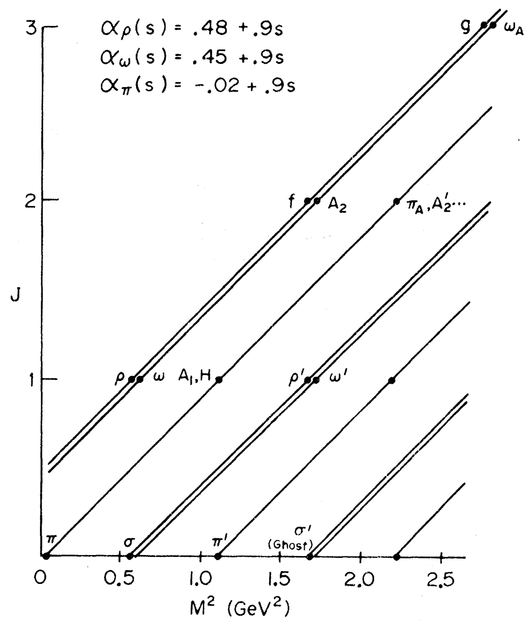

FIG. 2. Particle spectrum. All the states at or below $m_{A_2}{}^2$ which occur in our six-pion amplitude are indicated. The only ghost below 2 GeV^2 is the σ' at $m^2 = m_f{}^2$, which also occurs in the LS amplitude (for $m_\pi{}^2 > 0$).

the widths are related to a single coupling constant, $g^2 \sim \Gamma_\rho$ (see Table I). The residue of our amplitude in a three-pion channel on the π pole is a product of LS (4π) amplitudes, and the residue on the ω pole is a product of Veneziano ($\omega 3\pi$) amplitudes.

There are obvious weaknesses in our approach. It is not clear that the six-pion amplitude is consistent with a generalization to N pions. More important, it has not been demonstrated that our amplitude is the unique solution to the leading trajectory bootstrap [conditions (a) to (f)] for the six-pion amplitude. However, at present the equivalent program of Weinberg using effective Lagrangians and FESR's is difficult to implement for multibody processes.[12]

In Sec. II, the amplitude is expressed in a simple form, indicating how it arises as a modification of the NTS model, and there is a discussion of how the bootstrap constraints are preserved. In Sec. III, the decay amplitudes are listed for the lowest-lying odd-G-parity states in the model: ω, A_1, H, π', and A_2 together with the factorization on the π pole. In Sec. IV, we suggest some phenomenological tests of our amplitude. Finally, the Appendix describes alternate ways of expressing the amplitude that may make it more amenable to some calculations.

II. THE AMPLITUDE

Our chiral six-pion amplitude is written as follows:

$$A_N(k_1, a_1; \ldots ; k_N, a_N) = \frac{g^{N-2}}{\alpha'} \sum_P \mathrm{Tr}(\tau_{a_1} \cdots \tau_{a_N}) B_N(k_1, \ldots , k_N) \tag{1}$$

by setting $N=6$. The ith pion has momentum p_i and isospin label a_i. The coupling constant g has units of inverse mass, and α' is the universal slope for the Regge trajectories. Isospin has been introduced via the familiar Chan-Paton procedure,[13] which guarantees factorization in isospin and the absence of exotic $I=2$ states.[14] The sum is over permutations P of the order of the external pion legs, where it is understood that cyclic and anticyclic permutations are not to be included.

The dynamical factor B_6 is written in Koba-Nielson variables $\{z_i\}$ (Ref. 15):

$$B_6(k_1 \cdots k_6) = \int \frac{\prod_{i=1}^6 dz_i}{d^3\omega} F \prod_{i\neq j} |z_i - z_j|^{\alpha' k_i \cdot k_j - c^2(\delta_{i+1,j} + \delta_{i,j+1})} ,$$

$$d^3\omega = \frac{dz_a dz_b dz_c}{|z_a - z_b||z_b - z_c||z_c - z_a|} . \tag{2}$$

The metric used is $g_{ii} = -g_{00} = 1$ ($k^2 = -m_\pi^2$), the variables z_i are ordered on the real line, $z_i < z_{i+1}$, in one-to-one correspondence with the external pions, and $\{z_a, z_b, z_c\}$ is an arbitrarily chosen subset of $\{z_i\}$. The variable c^2 is related to the pion mass, $2c^2 = \frac{1}{2} + \alpha' m_\pi^2$.

When $F = 1$, (2) is simply the Koba-Nielson generalization of the beta function for six legs with the internal trajectories parametrized as $\alpha(s) = 1 - 2c^2 + \alpha' s$. Our factor F may be expressed most simply as a sum of three terms involving determinants,[16,17]

$$F(a, b, c; \lambda) = \| a_{ij}(0, 0, c) \|^{1/2} \tag{3a}$$

$$+ \lambda [\| a_{ij}(0, b, 0) \|^{1/2} - \| a_{ij}(a, b, 0) \|^{1/2}]$$

$$= F^{(0)}(c) + \lambda \delta F(a, b) + \cdots , \tag{3b}$$

where the matrices $a_{ij}(a, b, c)$ are defined by

$$a_{ij}(a, b, c) = \frac{\rho_{ij}(a, b)[\alpha' k_i \cdot k_j - c^2(\delta_{i+1,j} + \delta_{i,j+1})]}{|z_i - z_j|} , \tag{3c}$$

the factors $\rho_{ij}(a, b)$ are products of projectively invariant cross ratios which have the effect of moving trajectories in selected three-pion channels,

$$\rho_{i,i+1}(a, b) = \frac{(w_{i-1,i} w_{i+1,i+2})^{(a+3b)/6}}{(w_{i,i+1})^{a/3}} , \tag{3d}$$

$$\rho_{i,i+2}(a, b) = \frac{(w_{i-1,i})^{(a+3b)/6}}{(w_{i,i+1} w_{i+1,i+2})^{a/3}} , \tag{3e}$$

$$\rho_{i,i+3}(a, b) = \frac{1}{(w_{i,i+1} w_{i+1,i+2} w_{i+2,i+3})^{a/3}} , \tag{3f}$$

and where the cross ratios are defined as

$$w_{i,i+1} = \left| \frac{(z_i - z_{i-2})(z_{i+3} - z_{i+1})}{(z_{i+3} - z_i)(z_{i+1} - z_{i-2})} \right| . \tag{3g}$$

The determinants $\| a_{ij} \|^{1/2}$ may be expanded as the sum of 15 terms,

$$\| a_{ij} \|^{1/2} = [a_{12} a_{34} a_{56} + P_c(1)] + [a_{14} a_{26} a_{35} + P_c(2)]$$

$$+ [a_{12} a_{36} a_{45} + P_c(2)] - [a_{12} a_{35} a_{46} + P_c(5)]$$

$$- a_{14} a_{25} a_{36} , \tag{4}$$

where $P_c(n)$ means to add the n other cyclic permutations of the preceding expression. When $\lambda = c = 0$, the amplitude becomes the NS dual-pion model. When $\lambda = 0$ and $c^2 = \frac{1}{4}(1 + 2\alpha' m_\pi^2)$, the amplitude acquires Adler zeros and becomes the NTS chiral model. The first two terms in the expansion (4),

$$a_{12} a_{34} \cdots a_{N-1,N} + a_{23} a_{34} \cdots a_{N1} ,$$

are equivalent to the chiral model of Ref. 8, but with tachyon-eliminating factors added: $a_{i,i+1}$ is proportional to $\alpha(s_{i,i+1})$. By themselves, these two terms introduce ancestors, which are neatly canceled by the remaining terms in the determinant.

In our six-pion model

$$\begin{aligned} &\lambda = (2 + 2\alpha' m_\pi^2)^{-1} , \\ &c^2 = \tfrac{1}{4}(1 + 2\alpha' m_\pi^2) , \\ &a = 1 - \alpha'(m_\omega^2 - m_\pi^2) , \\ &b > 1 . \end{aligned} \tag{5}$$

Recapitulating, all three terms in $F(a, b, c, \lambda)$ are simple modifications of the NS amplitude for six pions. In the NTS term, $F^{(0)}(c)$, when the pion mass is moved by taking $c^2 > 0$, additional particles are produced: A_1, H, and π' at $\alpha_\pi(s) = \frac{1}{2} - 2c^2 + \alpha' s = 1$ and a σ at $\alpha_\rho(s) = 1 - 2c^2 + \alpha' s = 1$. However, the ω is also at $\alpha_\pi = 1$, and the ω-A_2 trajectory continues to lie a half unit below the ρ-f^0 trajectory.

The other two terms which give a contribution proportional to λ are expressed as a difference, $\delta F(a, b)$. Each term separately vanishes at the

TABLE I. Meson spectrum. The experimental masses and widths are in square brackets, and are taken from the Review of Particle Properties.[a]

Name	$I^G(J^P)$	Mass M (MeV)	Partial width Γ (MeV)	Mode
π	$1^-(0^-)$	137[b][137]		
σ	$0^+(0^+)$	760 [~800]	756[>>100]	2π
ρ	$1^+(1^-)$	760 [765]	145[b][145]	2π
ω	$0^-(1^-)$	784[b][784]	6[9]	3π
π'	$1^-(0^-)$	1060[NE][c]	1100[NE]	3π
H	$0^-(1^+)$	1060[990]	180[?]	3π
A_1	$1^-(1^+)$	1060[1070]	154	$\rho\pi$
			2	$\sigma\pi$
			165[100–200]	3π [d]
σ'	$0^+(0^+)$	1300[e]	−17[e]	2π
ρ'	$1^+(1^-)$	1300[NE]	145[NE]	2π
f	$0^+(2^+)$	1300[1269]	124[125]	2π
			[12][f]	4π
A_2	$1^+(2^+)$	1315[1310]	7.5[75]	$\rho\pi$
π_A	$1^-(2^-)$	1500[1640]	[108][f]	3π
g	$1^+(3^-)$	1670[1680]	48[64]	2π
			[80][f]	4π

[a] Particle Data Group, Phys. Letters 39B, 1 (1972).

[b] Free parameter of model fixed by experiment.

[c] NE indicates no experimental evidence for that meson.

[d] The ρ bands account for almost all the events, but interference should be accounted for.

[e] The σ' is a ghost which is also present in the LS model.

[f] The authors have not yet calculated the 4π decay widths of the f and g, or the 3π decay width of the π_A, although such widths *are* calculable in the model.

Adler point, and the difference of the two, $\delta F(a,b)$, satisfies the bootstrap constraints of no tachyon and no ancestors. Because $c^2=0$ in $\delta F(a,b)$, there is no contribution to the A_1, H, and π'. There is, however, an ω pole with negative coupling at $\alpha_\pi=1$ and a ω pole with positive coupling at $\alpha_\pi+a=1$.[18] The constant λ is adjusted to exactly cancel the ω pole in $F^{(0)}(c)$ leaving a single ω pole at $\alpha_\pi+a=1$.

The correction term $\delta F(a,b)$ also introduces a $J^P(I^G)=1^+(1^-)$ particle and a $J^P(I^G)=1^+(0^-)$ ghost at $\alpha_\pi-b=1$. The parameter b does not affect the two- and three-pion couplings discussed in Sec. III.

Therefore, the six-pion amplitude possesses an A_1, H, and π' at $\alpha_\pi=1$, and an ω at $\alpha_\omega=\alpha_\pi+a=1$ nondegenerate with the A_1.

III. DECAY AMPLITUDES

The full six-pion amplitude (2) may be factorized in a three-pion channel at either $\alpha_\pi(s_{13})=N$ for poles on the π-A_1 trajectory and its daughters, or at $\alpha_\pi(s_{13})+a=N$ for poles on the ω-A_2 trajectory and its daughters,

$$A_6(k_1,a_1;\ldots;k_6,a_6)\cong\frac{g^2}{\alpha'}\sum_{I=0}^{4}\sum_{P}\mathrm{Tr}(\tau_{a_1}\tau_{a_2}\tau_{a_3}\tau_I)B(k_1k_2k_3-k)\,\frac{1}{s_{13}-m^2}\,\frac{g^2}{\alpha'}\sum_{P'}\mathrm{Tr}(\tau_I\tau_{a_4}\tau_{a_5}\tau_{a_6})B(kk_4k_5k_6)\,, \tag{6}$$

where

$$\tau_0=1,\quad k=(k_1+k_2+k_3)=-(k_4+k_5+k_6),\quad m^2=-k^2\,,$$

and the sums over P and P' exclude cyclic and anticyclic permutations. At $\alpha_\pi(s_{13})=0$, there is only a $I^G(J^P)=1^-(0^-)$ pole with the residue function of the (123) permutation proportional to the $\pi\pi\to\pi\pi$ amplitude of LS,

$$B(p_1p_2p_3-k)=L(s,t)\equiv\frac{1}{2\sqrt2}\,\frac{\Gamma(1-\alpha_\rho(s))\Gamma(1-\alpha_\rho(t))}{\Gamma(1-\alpha_\rho(s)-\alpha_\rho(t))}\,, \tag{7}$$

where the Mandelstam variables are $s=-(p_1+p_2)^2$, $t=-(p_2+p_3)^2$ and the ρ trajectory is $\alpha_\rho(s)=1-2c^2+\alpha's$. Thus our model reproduces the 2π decay amplitudes and widths listed by Shapiro.

At $\alpha_\pi(s_{13})=1$, factorization yields three particle states: an $I^G(J^P)=1^-(1^+)$ pole identified with the A_1, a $0^-(1^+)H$, and a $1^-(0^-)\pi'$. The residue functions corresponding to the (123) permutation in (6) are

$$B_{A_1\to3\pi}(s,t)=\left(\frac{\alpha'c^2}{4}\right)^{1/2}\epsilon_{A_1}\cdot k_2B(1-\alpha_\rho(s),1-\alpha_\rho(t))\,, \tag{8}$$

$$B_{H\to3\pi}(s,t)=\alpha'\left(\frac{\alpha'c^2}{1+4c^2}\right)^{1/2}\epsilon_{H\mu}k_\nu[\,k_2^\mu(k_1-k_3)^\nu-(k_1-k_3)^\mu k_2^\nu\,]B(1-\alpha_\rho(s),1-\alpha_\rho(t))\,, \tag{9}$$

$$B_{\pi'\to3\pi}(s,t)=\left(\frac{c^6}{1+4c^2}\right)^{1/2}L(s,t)\,. \tag{10}$$

At $\alpha_\pi(s_{13})+a=0$, the residue vanishes. At $\alpha_\pi(s_{13})+a=1$, there is a single particle $0^-(1^-)$ identified with the ω. The residue function is proportional to the $\pi\pi\to\pi\omega$ amplitude proposed by Veneziano:

$$B_{\omega\to3\pi}(s,t)=\alpha'\left(\frac{\alpha'}{1+4c^2}\right)^{1/2}\epsilon_\omega^\mu\epsilon_{\mu\alpha\beta\gamma}k_1^\alpha k_2^\beta k_3^\gamma B(1-\alpha_\rho(s),1-\alpha_\rho(t))\,. \tag{11}$$

At $\alpha_\pi(s_{13})+a=2$, there is an A_2 meson on the leading trajectory $I^G(J^P)=1^-(2^+)$ and an ω' below it (see Fig. 2):

$$B_{A_2\to3\pi}(s,t)=\frac{(\alpha')^2}{(1+4c^2)^{1/2}}\epsilon_{A_2}^{\mu\nu}\epsilon_{\mu\alpha\beta\gamma}k_1^\alpha k_2^\beta k_3^\gamma\left(\frac{-k_{1\nu}(2k_1\cdot k_2+1-2c^2)+k_{3\nu}(2k_2\cdot k_3+1-2c^2)}{2k_1\cdot k_3+1}\right)B(1-\alpha_\rho(s),1-\alpha_\rho(t))\,. \tag{12}$$

Of course, the full amplitude for the decay of these resonances must include isospin factors and the appropriate sum over permutations as indicated by Eq. (1) for $N=4$.

IV. MASSES AND WIDTHS

All the masses in Table I are calculated by fixing m_π^2 and m_ω^2 at their experimental values and using a universal slope parameter $\alpha' = 0.90$ GeV^{-2}. The model contains many sum rules:

$$\alpha'(m_\rho^2 - m_\pi^2) = \tfrac{1}{2}, \tag{13a}$$

$$2(m_\rho^2 - m_\pi^2) = (m_{A_1}^2 - m_\pi^2) \tag{13b}$$

$$= (m_f^2 - m_\rho^2) \tag{13c}$$

$$= (m_{A_2}^2 - m_\omega^2). \tag{13d}$$

The first sum rule (13a) is equivalent to half integral spacing between the π and ρ trajectories, $\alpha_\rho(s) - \alpha_\pi(s) = \frac{1}{2}$, which is inherited from the algebra of the NS model, and appears to be a consequence of the NTS implementation of chirality and no tachyons in a dual model. It should be emphasized that there is no such constraint on the ρ mass in the four-pion amplitude, since

$$B_4(s,t) = [2\alpha_\rho(m_\pi^2) - \alpha_\rho(s) - \alpha_\rho(t)] \times B(1-\alpha_\rho(s), 1-\alpha_\rho(t)) \tag{14}$$

has Adler zeros for any m_ρ^2, and reduces to LS only if the sum rule holds. Our model indicates that for $N=6$, there is a constraint on the ρ mass, but none on the ω mass. For $N>6$, the bootstrap may indeed fix the ω mass. (It is certainly fixed in the NTS model.)

The second sum rule (13b) reduces for $m_\pi^2 = 0$ to Weinberg's mass ratio $m_{A_1}/m_\rho = \sqrt{2}$.[19] The last two sum rules are consequences of exchange degeneracy (no exotics) and a universal slope parameter (α').

The widths given in Table I are all normalized to the ρ width. Factorizing the LS amplitude (7) on the ρ pole yields the decay amplitude for $\rho \to 2\pi$,

$$A_{\rho\to 2\pi} = \frac{g}{(\alpha')^{1/2}(2)^{1/4}} (m_\rho^2 - 4m_\pi^2)^{1/2} \epsilon_{Iab} \cos\theta, \tag{15}$$

expressed in the center-of-mass system of the ρ. The width is then calculated from the general formula,

$$d\Gamma = \frac{1}{2M} \frac{1}{\prod_i m_i!} (2\pi)^4 \delta^4\left(P - \sum_{i=1}^{n} k_i\right) \times |A_{\text{decay}}|^2 \frac{d^3k_1}{(2\pi)^3 2\omega_1} \cdots \frac{d^3k_n}{(2\pi)^3 2\omega_n}, \tag{16}$$

where M is the mass of the decaying particle and m_i are the multiplicities of identical particles produced in the decay. The ρ width is easily calculated from the two-body phase-space integral,

$$\Gamma_{\rho\to 2\pi} = \frac{g^2(m_\rho^2 - 4m_\pi^2)^{3/2}}{48\sqrt{2}\,\pi\alpha' m_\rho^2}.$$

Thus, the experimental value for the ρ width can be used to fix the coupling constant g^2, determining all the widths in the model.

The two-pion partial widths calculated for f, g, σ, ρ', and σ' are necessarily those found by Shapiro.[2] However, this model does give the four-pion partial widths for the f, g, and ρ', although we have not calculated them in this paper.

The partial width for $\omega \to 3\pi$ was calculated from the amplitude (11) on the computer. In the computation, the ρ trajectory was parametrized by adding on imaginary part, to account for the widths of the ρ and f,

$$\alpha_\rho(s) = 1 - 2c^2 + \alpha' s + i\gamma(s - 4m_\pi^2)^{1/2},$$

where

$$\gamma = \frac{\alpha' m_\rho \Gamma_\rho}{(m_\rho^2 - 4m_\pi^2)^{1/2}}.$$

The width predicted by our model, 6 MeV, is slightly smaller than the experimental value. Actually, one would expect threshold effects (e.g., the presence of cuts) to add to the width, increasing the predicted value.

The three-pion widths for the A_1, A_2, H, and π' were also calculated on the computer. In the cases of the A_1 and A_2, the decay widths for $\rho\pi$ and $\sigma\pi$ may be calculated analytically by factorizing the amplitudes at $\alpha_\rho(s) = 1$. For example, the $A_1 \to \rho\pi$ amplitude is

$$A_{A_1\to\rho\pi} = \frac{g}{\alpha'} c(2)^{1/4} \epsilon_{Iab} \epsilon(\lambda, k_{A_1}) \cdot \epsilon(\lambda', k_\rho). \tag{17}$$

We are investigating the viability of the ρ-A_1 spin correlation given in (17).[20] At first glance, the absence of the $(\epsilon_A \cdot p_\rho)(\epsilon_\rho \cdot p_A)$ term (D wave) appears inconsistent with experimental data,[21] but the data for the 3π system must be reanalyzed in view of the background, particular to our model, arising from exchanged π poles (Deck effect)[22] and from a broad resonance π'.

The $A_2 \to \rho\pi$ amplitude is

$$A_{A_2\to\rho\pi} = \frac{g}{(2)^{1/4}} \left(\frac{3 + 4c^2 - 2a}{1 + 4c^2}\right)^{1/2} \epsilon_{Iab} \times [(\vec{q} \times \vec{\epsilon}_\rho)^i q^j + q^i (\vec{q} \times \vec{\epsilon}_\rho)^j] \epsilon_{A_2}^{ij} \tag{18}$$

as expressed in the rest frame of the A_2, where $\vec{q}$ is the momentum of the ρ.

The parameters in the model have been fixed at the following values:

$m_\pi^2 = 0.019$ GeV^2,

$m_\omega^2 = 0.614$ GeV^2,

$\Gamma_\rho = 0.15$ GeV,

$\alpha' = 0.90$ GeV^{-2}.

These values determine the other constants used in this paper:

$a = 0.47$,

$c^2 = 0.26$,

$g^2 = 46.6$ GeV^{-2}.

The principal defects in the spectrum of the six-pion amplitude at low energies ($m^2 < 2$ GeV^2) are essentially those of the four-pion amplitude: (a) a scalar ghost (σ') at $m = 1300$ MeV whose coupling to two pions vanishes as the pion mass approaches zero; and (b) a strongly coupled vector particle (ρ') at $m = 1300$ which has not been detected experimentally.

Some negative-G-parity states introduced by the model are of questionable experimental stature, but there are no glaring deficiencies: (c) a pseudoscalar (π') at $m = 1060$ MeV, whose predicted width is so large that its experimental existence is easily overlooked.

Finally the model makes some statements about a few controversial aspects of the experimental spectrum: (d) an axial-vector (H) at $m = 1060$ MeV (vs no resonance at all); (e) an axial-vector (A_1) at $m = 1060$ MeV (vs a threshold enhancement in the $\pi\rho$ system); (f) an unsplit tensor (A_2) at $m = 1315$ MeV (vs the possibility of a split A_2); and (g) a pseudotensor (π_A) at $m = 1500$ MeV (vs a threshold enhancement in the πf system).

V. CONCLUSION

Clearly more detailed comparison of the six-pion amplitude with experiment should be carried out. The four-body "vertices" in the model have been examined in the calculations of the three-pion decays ($\omega \to 3\pi$, $A_1 \to 3\pi$, $A_2 \to 3\pi$, etc.) indicating reasonable agreement with experiment. In addition, the Reggeon -3π vertex can be extracted from production data $\pi N \to \pi\pi N$ using factorization on the π and ω Regge exchanges.

However, to explore the rich many-body ($N > 4$) structure, one should at least investigate five-particle processes. For example, the four-pion decays (ρ', f, g, etc. $\to 4\pi$) may be calculated, and at least in one case ($g \to 4\pi$), there is enough experimental evidence for an unambiguous comparison.

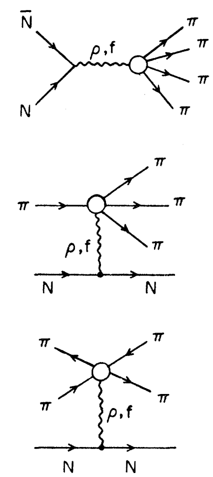

FIG. 3. Regge-4π vertex. The coupling of the natural-parity sequence ($\rho, f, \sigma, \ldots$) to four pions can be tested in three separate kinematical regions through (a) partial-wave analysis of annihilation ($N\bar{N} \to 4\pi$), (b) Regge analysis of production ($\pi N \to \pi\pi\pi N$), and (c) Mueller analysis of inclusive reactions ($\pi N \to \pi X$).

More promising is a detailed study of the natural parity-4π "vertex" through the following processes: (a) annihilation $N\bar{N} \to 4\pi$, (b) exclusive $\pi N \to \pi\pi\pi N$, and (c) inclusive $\pi N \to \pi +$ anything. (See Fig. 3.) Although the annihilation process (a) requires extrapolations from the s-, and p-, and d-wave resonances (σ, ρ, and f), a similar comparison of the LS amplitude to $p\bar{n} \to 3\pi$ data has proved useful.[2] Factorization of $\pi N \to \pi\pi\pi N$ (b) gives the Regge $(\rho, f) + \pi \to 3\pi$ vertex for three outgoing pions, and the Mueller analysis on the Regge-4π vertex for two ingoing and two outgoing pions can be compared to the inclusive process (c). These three separate kinematical regions provide a severe test of the four-pion coupling to the natural-parity sequence.

Finally it is tempting to account for the diffractive component by simply displacing the f^0 intercept to unity $[\alpha(0) \approx 1]$. In particular, the $\pi\rho$ mass enhancement in $\pi N \to \pi\rho N$ can be studied in a full Reggeized model. The idea of an f^0-dominated Pomeranchukon[23] gives some justification for this procedure, and also suggests similar applications to the diffractive or scaling part of inclusive reactions.[24]

ACKNOWLEDGMENTS

We wish to thank Carleton DeTar and Ron Waltz for helpful discussions, and especially Charles

Thorn for many conversations and detailed correspondence on all aspects of chiral dual models.

APPENDIX

The expression for our six-pion amplitude (2) and (3) is relatively simple and elegant, but often not suitable for calculations.

It is sometimes convenient to express the amplitude in Chan variables $\{x_i\}$ rather than the Koba-Nielson variables $\{z_i\}$. The change of variables is defined by taking

$$
\begin{aligned}
&z_a = z_1 \to 0, \quad x_1 = z_2/z_3, \\
&z_b = z_5 \to 1, \quad x_2 = z_3/z_4, \\
&z_c = z_6 \to \infty, \quad x_3 = z_4 .
\end{aligned} \tag{A1}
$$

The volume element transforms as

$$
\begin{aligned}
\frac{\prod dz_i}{d^3\omega} &= dz_2 dz_3 dz_4\, z_6^2 \\
&= dx_1 dx_2 dx_3\, z_3 z_4 z_6^2
\end{aligned} \tag{A2}
$$

and the integrals over the variables x_1, x_2, and x_3 are over the unit interval $[0, 1]$. If we define the shorthand notation,

$$
\begin{aligned}
&\bar{k}_i \cdot \bar{k}_j = \alpha' k_i \cdot k_j - c^2(\delta_{i+1,j} + \delta_{i,j+1}) \quad \text{with } \bar{k}^2 = \tfrac{1}{2}, \\
&\bar{t}_j = (\bar{k}_1 + \cdots + \bar{k}_j)^2 ,
\end{aligned} \tag{A3}
$$

then it is a simple matter to transform the product in (2):

$$
\prod_{i \neq j} |z_i - z_j|^{\bar{k}_i \cdot \bar{k}_j} = x_1^{-\bar{t}_2 - 1} x_2^{-\bar{t}_3 - 1} x_3^{-\bar{t}_4 - 1} (x_3 z_6)^{-1} \prod_{i<j} \left| 1 - \frac{z_i}{z_j} \right|^{2\bar{k}_i \cdot \bar{k}_j} . \tag{A4}
$$

The determinant terms are easily modified, and contribute z_6^{-1}. Since the volume element contributed z_6^2 and the product contributed z_6^{-1}, all the terms involving z_6 cancel.

The final result, reexpressing the six-point beta function[2] in Chan variables, is

$$
\begin{aligned}
B_6(k_1 \cdots k_6) = \int_0^1 & dx_1 dx_2 dx_3\, x_1^{-\alpha_{12}} (1 - x_1)^{-\alpha_{23}} x_2^{-\alpha_{13}^{\pi} - 1} (1 - x_2)^{-\alpha_{34}} x_3^{-\alpha_{56}} (1 - x_3)^{-\alpha_{45}} \\
& \times (1 - x_1 x_2)^{-\alpha_{24}^{\pi} + \alpha_{23} + \alpha_{34} - 2} (1 - x_2 x_3)^{-\alpha_{35}^{\pi} + \alpha_{34} + \alpha_{45} - 2} (1 - x_1 x_2 x_3)^{-\alpha_{16} - \alpha_{34} + \alpha_{24}^{\pi} + \alpha_{35}^{\pi} + 2}\, Y(k, x) ,
\end{aligned} \tag{A5}
$$

where the trajectories are defined as

$$
\alpha_{i,i+1} = 1 - 2c^2 - (k_i + k_{i+1})^2
$$

and

$$
\alpha_{i,i+2}^{\pi} = \tfrac{1}{2} - 2c^2 - (k_i + k_{i+1} + k_{i+2})^2 ,
$$

and $Y(k, x)$ is the sum of three terms corresponding to the three terms in (3),

$$
Y(k, x) = Z(k, x; 0, 0, c) + \lambda [Z(k, x; 0, b, 0) - Z(k, x; a, b, 0)] . \tag{A6}
$$

Each Z function is the sum of 15 terms corresponding to the expansion of the determinant $||a_{ij}(a, b, c)||^{1/2}$,

$$
\begin{aligned}
Z(k, x; a, b, c) = & \left(\bar{k}_1 \cdot \bar{k}_2\ \bar{k}_3 \cdot \bar{k}_4\ \bar{k}_5 \cdot \bar{k}_6 \frac{(u_{13} u_{24} u_{35})^b}{u_{12} u_{14} u_{34}} + P_c(1) \right) + \left(\bar{k}_1 \cdot \bar{k}_2\ \bar{k}_3 \cdot \bar{k}_6\ \bar{k}_4 \cdot \bar{k}_5 \frac{u_{24}^{1-a} (u_{13} u_{35})^b}{u_{12} u_{45}} + P_c(2) \right) \\
& + [\bar{k}_1 \cdot \bar{k}_4\ \bar{k}_2 \cdot \bar{k}_6\ \bar{k}_3 \cdot \bar{k}_5 (u_{13} u_{24})^{1-a} u_{35}^b + P_c(2)] - \left(\bar{k}_1 \cdot \bar{k}_2\ \bar{k}_3 \cdot \bar{k}_5\ \bar{k}_4 \cdot \bar{k}_6 \frac{u_{24}^{1-a} (u_{13} u_{35})^b}{u_{12}} + P_c(5) \right) \\
& - \bar{k}_1 \cdot \bar{k}_4\ \bar{k}_2 \cdot \bar{k}_5\ \bar{k}_3 \cdot \bar{k}_6 (u_{13} u_{24} u_{35})^{1-a} ,
\end{aligned}
$$

where the cross ratios u_{ij} prove convenient since a factor of $(u_{ij})^{\gamma}$ in the factor Z corresponds to a lowering of the trajectory α_{ij} by an amount γ,

$$
\begin{aligned}
&u_{12} = x_1, \quad u_{23} = (1 - x_1)(1 - x_1 x_2)^{-1} , \\
&u_{13} = x_2, \quad u_{24} = (1 - x_1 x_2)(1 - x_1 x_2 x_3)^{-1} , \\
&u_{14} = x_3, \quad u_{35} = (1 - x_2 x_3)(1 - x_1 x_2 x_3)^{-1} , \\
&u_{25} = (1 - x_1 x_2 x_3), \quad u_{45} = (1 - x_3)(1 - x_2 x_3)^{-1} , \\
&u_{34} = (1 - x_2)(1 - x_1 x_2 x_3)(1 - x_2 x_3)^{-1} (1 - x_1 x_2)^{-1} .
\end{aligned}
$$

*This work is supported in part through funds provided by the Atomic Energy Commission under Contract No. AT(11-1)-3069.

† Address after September 1, 1972: Department of Physics, California Institute of Technology, Pasadena, Calif.

[1]G. Veneziano, Nuovo Cimento 64A, 190 (1968).

[2]C. Lovelace, Phys. Letters 28B, 265 (1968); J. Shapiro, Phys. Rev. 179, 1345 (1969).

[3]Of course the pleasure would be somewhat tempered: viz., one would be unable to find a ρ' at $M^2=\frac{3}{2}$ in the Particle Data Tables although such a particle is predicted by LS; and one would notice a tiny pseudoscalar ghost σ' at $M^2=\frac{3}{2}$ predicted by LS.

[4]J. Ellis, Nucl. Phys. B21, 217 (1970). On the basis of Ellis's work, Ref. 7 notes that *any* dual model with no exotics, Adler zeros, and factorization on the pion pole *must* be equivalent at low energies to the nonlinear σ model with $(\frac{1}{2}, \frac{1}{2})$ symmetry breaking to $O(m_\pi^2)$. That is, the constraints (a), (e), and (f) imply chirality.

[5]R. Dolen, D. Horn, and C. Schmid, Phys. Rev. 166, 1768 (1968).

[6]S. Adler, Phys. Rev. 137, B1022 (1965).

[7]R. Brower, Phys. Letters 34B, 143 (1971).

[8]A. Neveu and J. Schwarz, Nucl. Phys. B31, 86 (1971).

[9]P. Goddard and C. Thorn, Phys. Letters 40B, 235 (1972); J. Schwarz, Nucl. Phys. B46, 61 (1972); R. Brower and K. Friedman, Phys. Rev. D (to be published).

[10]A. Neveu and C. Thorn, Phys. Rev. Letters 27, 1758 (1971); J. Schwarz, Phys. Rev. D 5, 886 (1972).

[11]M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters 22, 83 (1969).

[12]S. Weinberg, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by Stanley Deser *et al.*, (MIT Press, Cambridge, Mass., 1971); Phys. Rev. D 5, 900 (1972).

[13]Chan Hong-Mo and J. Paton, Nucl. Phys. B10, 519 (1969).

[14]The uniqueness of the procedure in Ref. 14 is shown by N. A. Törnqvist, Nucl. Phys. B26, 104 (1971).

[15]Z. Koba and H. B. Nielson, Nucl. Phys. B10, 633 (1969).

[16]D. B. Fairlie, Nucl. Phys. B42, 253 (1972). This clever way of combining terms was first discovered by Fairlie for the NS model and extended by him in an attempt to move the ω-A_2 trajectory in an N-pion model.

[17]The dots in (3b) indicate additional terms which modify *only* daughter trajectories and are unconstrained by the leading trajectory bootstrap. An optimist may hope that these terms will cancel the ghosts.

[18]In any given channel, nine of the terms ("π-like") in the expansion (4) of $||a_{ij}(a, b, c,)||^{1/2}$ have a pion pole at $\alpha_\pi - b = 0$. The other six terms ("ω-like") have an ω pole with *negative* coupling at $\alpha_\pi + a = 1$. It is crucial to notice that the π-like terms contribute an ω pole with *positive* coupling at $\alpha_\pi - b = 1$. When $a = b = 0$, the contribution of the π-like terms to the ω pole is exactly twice as large as that of the ω-like terms.

[19]S. Weinberg, Phys. Rev. Letters 18, 507 (1967).

[20]Our value of $g_T/g_L = 2m_A^2/(m_A^2 + m_\rho^2 - m_\pi^2) \approx \frac{4}{3}$ is the solution to the superconvergent sum rules of M. Ademollo, H. R. Rubenstein, G. Veneziano, and M. A. Virasoro, Nuovo Cimento 51A, 227 (1967).

[21]J. Ballam *et al.*, Phys. Rev. Letters 21, 934 (1968).

[22]R. Deck, Phys. Rev. Letters 13, 169 (1964). E. Berger, Phys. Rev. 166, 1525 (1968); 173, 1567 (1969).

[23]R. Carlitz, M. Green, and A. Zee, Phys. Rev. Letters, 26, 1515 (1971); C. Lovelace, Phys. Letters 34B, 500 (1971).

[24]R. Brower, Phys. Letters 37B, 121 (1971).

PHYSICAL REVIEW D VOLUME 7, NUMBER 1 1 JANUARY 1973

Heavy-Lepton Production in Inclusive Neutrino Reactions*

Carl H. Albright
Department of Physics, Northern Illinois University, DeKalb, Illinois 60115
(Received 5 June 1972)

Production of heavy μ-type leptons in inclusive neutrino reactions is studied phenomenologically by means of a Monte Carlo calculation. The single-particle inclusive spectrum is found to be distorted in a characteristic fashion when lepton locality is violated by the production of a heavy (excited) lepton. Various tests are presented which are sensitive to heavy leptons with masses up to 3 GeV.

I. INTRODUCTION

Since the theoretical relationship among the known leptons is completely unclear, in contrast to the hadrons, which fit neatly into the familiar SU(3) internal symmetry scheme, the existence of other yet unknown leptons is a real possibility. These so-called "heavy leptons," if discovered, can fall into two distinct categories[1]: (1) They may have their own characteristic quantum number, being neither electron- nor muon-type, and occur in pairs much like e^- and ν_e, μ^- and ν_μ,