Finally in the same way $D\mu^2$ is

$$D\mu^2 = 16Bg^2 - (8f_1 + 48f_2)A \ .$$

We give the coupling-constant renormalization constants to second order:

$$Z_{f_1} = 1 + \lambda^2 Df_1/f_1 \ ,$$

$$Z_{f_2} = 1 + \lambda^2 Df_2/f_2 \ ,$$

$$Z_g = 1 + \lambda^2 Dg/g \ .$$

[1]M. Gell-Mann, Physics 1, 63 (1964).
[2]S. Adler and R. Dashen, *Current Algebras* (Benjamin, New York, 1968).
[3]M. Lévy, Nuovo Cimento 52, 23 (1967).
[4]S. Gasiorowicz and D. Geffen, Rev. Mod. Phys. 41, 531 (1969).
[5]L.-H. Chan and R. W. Haymaker, preceding paper, Phys. Rev. D 7, 402 (1973).
[6]J. L. Basdevant and B. W. Lee, Phys. Rev. D 2, 1680 (1970).
[7]J. Zinn-Justin, Phys. Reports 1C, 55 (1971); L. Copley and D. Masson, Phys. Rev. 164, 2059 (1967); D. Bessis and M. Pusterla, Phys. Letters 25B, 279 (1967); Nuovo Cimento 54A, 243 (1968); J. Basdevant, D. Bessis, and J. Zinn-Justin, Phys. Letters 27B, 230 (1968); Nuovo Cimento 60A, 185 (1969).
[8]H. W. Crater, Phys. Rev. D 1, 3313 (1970).
[9]S. Coleman, J. Wess, and B. Zumino, Phys. Rev. 177, 2239 (1969).
[10]P. Carruthers and R. W. Haymaker, Phys. Rev. D 4, 1808 (1971).
[11]R. Olshansky, Phys. Rev. D 4, 2440 (1971).
[12]Similar relations can be written for f_K by using the divergence of the vector current. Since it is unstable we omit it.
[13]We must have a divergent mass counterterm in this model arising from the det(M) term unlike the SU_2 model. Further we do not use normal ordering which then gives rise to an additional divergent mass counterterm coming from the f_i couplings.
[14]B. W. Lee, Nucl. Phys. B9, 649 (1969).
[15]K. Symanzik, Commun. Math. Phys. 16, 48 (1970).

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Electrodynamics of Spin-0 Mesons at Small Distances*

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It is shown that if Z_3 (where Z_3 is the photon wave-function renormalization constant) is assumed finite and the nonasymptotic part h of the renormalized photon propagator vanishes with power-law behavior, then all the remaining renormalization constants in scalar electrodynamics can be made finite order by order, except the charged-meson self-mass δm^2. The condition that δm^2 be finite forces the asymptotic coupling α_0 to satisfy at least one eigenvalue equation. A second eigenvalue condition for α_0 emerges from the requirement that the theory have a Hermitian Lagrangian. Finally, on the basis of the renormalization group, we expect that the initial assumption of a finite value of Z_3 is self-consistent only if α_0 satisfies a third eigenvalue condition. Hence, we conjecture that a completely finite, closed theory of scalar electrodynamics is probably internally inconsistent. Assuming that h falls off sufficiently rapidly, we are able to show that the meson propagator has a very simple asymptotic form for momenta much greater than its physical mass.

I. INTRODUCTION AND SUMMARY OF RESULTS

The development of relativistic quantum field dynamics during the past quarter-century has been largely dominated by the recurrent question of whether a completely finite, pathology-free local field theory, with some claim of describing physical reality, exists. Attention in this regard has naturally focused on the one theory which has had the most quantitative success – quantum electrodynamics. One of the most systematic attempts to answer this question in quantum electrodynamics, considered as a closed theory, has been the series of papers by Johnson, Baker, and Willey[1] published over the past eight years. Their main conclusion is that all of the renormalization constants of quantum electrodynamics are finite provided (a) the electron bare mass m_0 is zero and (b) the equation $F^{[1]}(x)=0$ has a positive root, where $F^{[1]}(x)$ is the coefficient of the logarithmic divergence in Z_3 (the photon wave-function renormalization constant) obtained from the sum of all single-electron-loop vacuum-polarization graphs calculated with coupling constant x. Since $F^{[1]}(x)$ is known only to order x^3, the consistency of a finite, closed theory of quantum electrodynamics still remains an open question.[2]

This paper asks the same question of the electrodynamics of spin-0 mesons minimally coupled to the Maxwell field – scalar electrodynamics. Here there are five genuine primitive divergences encountered in the power-series expansion of the unrenormalized theory, as opposed to three in spinor electrodynamics. These are summarized by the meson self-mass δm^2, the meson-photon vertex renormalization constant Z_1 $(=Z_2)$, the Compton vertex renormalization constant Z_4 $(=Z_1=Z_2)$, Z_3, and the graphs with four external meson lines (M parts). The equality of Z_1, the meson wave-function renormalization constant Z_2, and Z_4 is a consequence of gauge invariance.

The approach we take to determine under what conditions, if any, the unrenormalized theory is finite was suggested by the previous work of Johnson, Baker, and Willey. It is a well-known fact that the divergences in the basic unrenormalized Green's functions are closely related to the behavior of the renormalized meson and photon propagators far off the mass shell. Following Johnson, Baker, and Willey we begin by postulating the asymptotic finiteness of the renormalized photon propagator, or, equivalently, the finiteness of Z_3. We are then able to make definite statements about the remaining divergences and the asymptotic behavior of the renormalized meson propagator. To complete our program we must show that the theory is indeed consistent with the assumed asymptotic behavior of the renormalized photon propagator and a finite value of Z_3.

Specifically, the results obtained here are based on the following assumptions:

(a) Scalar electrodynamics combined with a $\lambda\phi^{\dagger 2}\phi^2$ counterterm is a renormalizable field theory.

(b) The renormalized photon propagator $\tilde{D}_{\mu\nu}(k)$ is asymptotically finite,

$$\alpha\tilde{D}_{\mu\nu}(k) \underset{k^2\gg m^2}{\sim} \left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{\alpha_0}{k^2}+\alpha_0 G\frac{k_\mu k_\nu}{(k^2)^2}, \quad (1.1)$$

where α is the fine-structure constant or physical coupling, α_0 is the asymptotic coupling, m is the physical meson mass, and G is a gauge parameter.

(c) The bare (unrenormalized) meson mass m_0 is finite.

By the first assumption we mean that all of the divergences of the theory can be absorbed in a renormalization of mass, charge, and the coupling λ. Significant progress toward the proof of this difficult program was made by Salam[3] and by Matthews and Salam.[4] In the latter paper the authors claim to have completed the proof of the renormalizability of scalar electrodynamics, and to this author's knowledge, no paper has since appeared on the subject.[5] Perhaps due to lack of experimental stimulus[6] the actual systematic implementation of this program has not been carried beyond second order in perturbation theory.

The second assumption is based on the following considerations:

(1) When λ is expanded as a power series in α the divergences in the unrenormalized photon propagator $D_{\mu\nu}$ can be removed by a multiplicative renormalization in each order of perturbation theory,

$$D_{\mu\nu}=Z_3(\Lambda^2/m^2,\alpha)\tilde{D}_{\mu\nu}, \quad (1.2)$$

with Z_3 the photon wave-function renormalization constant, and Λ a suitably defined ultraviolet cutoff. The propagator $D_{\mu\nu}$ is calculated in terms of the canonical (bare) coupling α_c which is related to α by $\alpha=Z_3\alpha_c$.

(2) Define the function $d_c(k^2/m^2,\alpha)$ by

$$\alpha\tilde{D}_{\mu\nu}(k)=\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{\alpha d_c(k^2/m^2,\alpha)}{k^2}+\alpha_0 G\frac{k_\mu k_\nu}{(k^2)^2}. \quad (1.3)$$

Then the method of the renormalization group can be used to study the behavior of $\alpha d_c(k^2/m^2,\alpha)$ in the region $k^2\gg m^2$. In this case we obtain a functional equation for the asymptotic part d_c^∞ of d_c analogous to the Gell-Mann–Low equation[7] in spinor electrodynamics,

$$\ln(k^2/m^2)=\int_{q(\alpha)}^{\alpha d_c^\infty(k^2/m^2,\alpha)}\frac{dx}{\psi(x)}, \quad (1.4)$$

with

$$\psi(q(\alpha))=\alpha\frac{\partial}{\partial x}d_c^\infty(x,\alpha)\bigg|_{x=1}=\frac{\alpha^2}{12\pi}+\cdots,$$

$$q(\alpha)=\alpha d_c^\infty(1,\alpha)=\alpha+\cdots,$$

and where α is related to the physical charge e by $\alpha=e^2/4\pi$. The function $d_c^\infty(k^2/m^2,\alpha)$ is defined as the sum of the series obtained by dropping all terms in each order of the expansion of $d_c(k^2/m^2,\alpha)$ in powers of α that vanish as $k^2/m^2\to\infty$. Implicit here is the assumption that the neglected terms do not sum to a function that asymptotically dominates d_c^∞.

(3) For some finite value $x=\alpha_0$, $\psi(x)$ vanishes, and hence $\alpha d_c^\infty(k^2/m^2, \alpha)\to\alpha_0$ as $k^2/m^2\to\infty$ in Eq. (1.4). Therefore, our assumption of the asymptotic finiteness of $\tilde{D}_{\mu\nu}$ may be phrased more specifically as the assumption that $\psi(x)$ has a sufficiently strong zero to make the integral in (1.4) diverge when its upper limit reaches α_0.

In the following, we shall require a more detailed statement about the nonasymptotic part h of d_c which we define by

$$\alpha d_c(k^2/m^2, \alpha)=\alpha_0+h(k^2/m^2, \alpha), \tag{1.5}$$

with the boundary condition $h(k^2/m^2, \alpha)\to 0$ as $k^2/m^2\to\infty$. First, we note that if the value of the physical coupling α satisfies

$$q(\alpha)=\alpha_0, \tag{1.6}$$

we obtain from Eq. (1.4) the result

$$\alpha d_c^\infty(k^2/m^2, \alpha)=\alpha_0. \tag{1.7}$$

In this case, h makes no contribution to the asymptotic part of $d_c(k^2/m^2, \alpha)$, indicating that it decreases at least as fast as m^2/k^2 as $k^2/m^2\to\infty$. With Adler,[8] we call the case when Eq. (1.7) is satisfied Type 1 asymptotic behavior. The case when $d_c^\infty(k^2/m^2, \alpha)$ has a nontrivial k^2 dependence is called Type 2 asymptotic behavior. Here, we wish to point out that our results are consistent with Type 2 asymptotic behavior only if $h(k^2/m^2, \alpha)$ vanishes as a power of k^2/m^2 (essentially) as $k^2/m^2\to\infty$. This will become evident below.

To summarize, we expect on general grounds that our assumption of an asymptotically finite renormalized photon propagator is consistent only if the asymptotic coupling α_0 is fixed to be the first zero of $\psi(x)$ as x increases from zero along the real axis. The physical coupling α is a free parameter restricted only by the requirement that $\alpha<\alpha_0$. Moreover, if the zero of $\psi(x)$ is a simple one and if $\psi'(\alpha_0)<0$, then it follows from (1.4) that $h(k^2/m^2, \alpha)$ vanishes as a power of k^2/m^2, independently of the value of α.

We mention here that many of the techniques developed by Johnson, Baker, and Willey in their study of the small-distance behavior of the photon propagator in spinor electrodynamics[9] and in the proof of the equivalence of their approach to that of Gell-Mann and Low[10] are applicable to scalar electrodynamics. Consequently, it is at least thinkable that a simpler eigenvalue condition on α_0 than $\psi(\alpha_0)=0$ will emerge in future work as a necessary condition for an asymptotically finite $\tilde{D}_{\mu\nu}$.

Finally, the implication of Adler's recent important paper[8] for scalar electrodynamics is obscured by the presence of the boson-boson counterterm. In particular, his proof that the conjectured zero in $\psi(x)$ in spinor electrodynamics is of infinite order breaks down in the presence of this counterterm.[11] Moreover, his proof that there exists a unique, asymptotically finite solution for $\tilde{D}_{\mu\nu}$, in addition to the one expected on the basis of the renormalization group, provided α is suitably restricted, does not immediately carry through in scalar electrodynamics due to M-part divergences.[12]

The third assumption made above requires that the purely electrodynamic contribution (i.e., neglecting all couplings except the charged-meson–Maxwell-field coupling) to the mass splitting of an isotopic multiplet of spin-0 mesons be finite. As we will see in Sec. VI, this assumption is probably wrong.

With these assumptions we show in Sec. III that a gauge exists in which Z_1 $(=Z_2=Z_4)$ is finite to all orders of perturbation theory. An explicit calculation of the gauge that renders Z_1 finite through fourth order is given. It is also shown that the renormalized meson propagator $\Delta(p^2)$, calculated in this gauge, has the asymptotic behavior

$$\Delta(p^2)\underset{p^2\to\infty}{\sim}\frac{C(e_0^{\,2})}{p^2}, \tag{1.8}$$

where C is a numerical constant that can be calculated as a power series in $e_0^{\,2}$ $(=4\pi\alpha_0)$. Next, we show in Sec. IV that there is a unique power-series expansion of λ in terms of $e_0^{\,2}$ which is *finite* term by term and which removes all M-part divergences. The values of the expansion coefficients are, in general, complex. To regain a Hermitian Lagrangian, we have required that the asymptotic coupling be fixed to be a nonvanishing value $\alpha_0>0$ for which the imaginary part of λ vanishes. In Sec. V we begin a study of the asymptotic behavior of $\Delta(p^2)$, with finite m, in the region of large spacelike momenta. When $\tilde{D}_{\mu\nu}$ has Type 1 asymptotic behavior, then the sum of the renormalized perturbation theory series for $\Delta(p^2)$, in the gauge in which Z_2 is finite, asymptotically approaches the function

$$\Delta^{-1}(p^2)\underset{p^2\gg m^2}{\sim}C^{-1}(e_0^{\,2})\times\left\{p^2+\tfrac{1}{2}a(e_0^{\,2})m^2\left[\left(\frac{p^2}{m^2}\right)^{\epsilon_+}+\left(\frac{p^2}{m^2}\right)^{\epsilon_-}\right]\right\}. \tag{1.9}$$

Here $\epsilon_\pm$ are constants calculated from the power-series expansion of the renormalized Bethe-Salpeter kernel for meson-antimeson scattering. The constant $a(e_0^{\,2})$ may also be calculated from renormalized perturbation theory. The constants $\epsilon_\pm$ are calculated to order $e_0^{\,2}$ and are found to be

$$\epsilon_+ = \frac{e_0^2}{80\pi^2}\left[\left(\frac{5\sqrt{1953}+147}{2}\right)^{1/2} \pm i\left(\frac{5\sqrt{1953}-147}{2}\right)^{1/2}\right] + O(e_0^4) = -\epsilon_- . \quad (1.10)$$

The oscillating part of the asymptotic expression for Δ introduced by the imaginary parts of $\epsilon_\pm$ will drop out when $\epsilon_\pm$ are calculated to all orders with the value of α_0 for which $\mathrm{Im}\lambda = 0$. In Sec. VI we study the meson self-mass and find that the quadratic divergences present in its perturbation expansion are intrinsic to scalar electrodynamics unless the asymptotic coupling α_0 satisfies the eigenvalue condition

$$g(\alpha_0) = 0, \quad (1.11)$$

where g is the coefficient of the quadratic mass divergence. The gauge invariance of g is proved in the Appendix. Due to the inherent ambiguity in defining a quadratically divergent integral, the equation for g is not unique.

The two conditions on α_0, $\mathrm{Im}\lambda(\alpha_0) = 0$ for Hermiticity and $g(\alpha_0) = 0$ for a quadratic-divergence-free meson self-mass, are in obvious conflict. When these are combined with the conjectured additional condition $\psi(\alpha_0) = 0$ for an asymptotically finite value of $\tilde{D}_{\mu\nu}$, it appears probable that the asymptotic coupling is overdetermined. Hence, we conjecture, provided $g(\alpha_0)$ can be defined unambiguously, that *a completely finite theory of scalar electrodynamics, considered as a closed theory, is probably internally inconsistent.*

In the next section we write down an equation for the unrenormalized meson propagator that is the analog of the Schwinger-Dyson equation for the electron propagator in spinor electrodynamics. We then describe an expansion of the meson mass operator in terms of the exact unrenormalized meson and photon propagators. This expansion enables us to obtain a linear integral equation for the unrenormalized meson-photon vertex Γ_μ, which will be the starting point of our study of the ultraviolet divergences associated with Z_1.

We mention in closing an analysis of spin-0 electrodynamics similar in spirit to Johnson, Baker, and Willey's first paper[13] and the lowest-order calculations in this paper carried out by Flamm and Freund.[14] By replacing the unrenormalized photon propagator with its conjectured asymptotic value (1.1) and truncating the Bethe-Salpeter kernel at order α_0, they concluded that δm^2 cannot be rendered finite in this approximation. Since they did not extend their analysis to all orders, no eigenvalue condition on α_0 was obtained as a necessary condition for suppressing the quadratic divergences in δm^2. In addition, their analysis makes no mention of M-part divergences, which cannot be neglected in higher-order approximations to the Bethe-Salpeter kernel.

II. SCALAR ELECTRODYNAMICS

In this section we write down a functional differential equation for the unrenormalized meson propagator and discuss its expansion in terms of the exact unrenormalized meson and photon propagators. We then give a linear, gauge-covariant equation for the unrenormalized meson-photon vertex function.

A. Schwinger Equation in Scalar Electrodynamics

The Lagrange function operator describing the coupling of the charged-meson and Maxwell fields is defined here as

$$L = -\tfrac{1}{2}i\{\Pi_\mu^\dagger, [(1/i)\partial^\mu - e_c A^\mu]\phi\} + \text{H.c.} + \Pi_\mu^\dagger \Pi^\mu - m_0^2\phi^\dagger\phi + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{4}\{F^{\mu\nu}, \partial_\mu A_\nu - \partial_\nu A_\mu\} + \tfrac{1}{2}\lambda\phi^{\dagger 2}\phi^2 + A_\mu J^\mu + \phi^\dagger\phi J, \quad (2.1)$$

where $\{A, B\}$ signifies $AB + BA$, and J_μ and J are classical, external currents. The coupling e_c is the canonical (bare) charge, which is related to α_c by $e_c^2 = 4\pi\alpha_c$. For the present we leave the coupling λ arbitrary.

All dynamical information pertaining to this coupled field system is contained in its Green's functions. The basic Green's functions are $D_{\mu\nu}$, the meson Green's function Δ, the meson-photon vertex function Γ_μ, and the Compton vertex function $\Gamma_{\mu\nu}$. Other higher-order Green's functions are functionals of these four basic functions. Define the meson Green's function by

$$\Delta(x, x') = \frac{i\langle 0, \sigma_1 | (\phi(x)\phi^\dagger(x))_+ | 0, \sigma_2\rangle}{\langle 0, \sigma_1 | 0, \sigma_2\rangle}.$$

Then from Schwinger's action principle[15] we obtain a functional differential equation for Δ:

$$\left[\left(\frac{1}{i}\partial_\mu - e_c\langle A_\mu(x)\rangle + ie_c\frac{\delta}{\delta J^\mu(x)}\right)^2 + i\lambda\Delta(x, x) - J(x) + i\lambda\frac{\delta}{\delta J(x)} + m_0^2\right]\Delta(x, x') = \delta(x - x'), \quad (2.2)$$

where

$$\langle A_\mu(x)\rangle = \frac{\langle 0, \sigma_1 | A_\mu(x) | 0, \sigma_2\rangle}{\langle 0, \sigma_1 | 0, \sigma_2\rangle}.$$

Equation (2.2) is the scalar-electrodynamic counterpart of the Schwinger equation in spinor electrodynamics.

B. Integral Equation for Δ

Equation (2.2) can be cleared of functional derivatives by introducing the photon propagator

$$D_{\mu\nu}(x, x') = \frac{\delta}{\delta J^\nu(x')} \langle A_\mu(x)\rangle, \tag{2.3}$$

the meson-photon vertex

$$\Gamma_\mu(x, x'; \xi) = -\frac{\delta}{\delta\langle e_c A^\mu(\xi)\rangle} \Delta^{-1}(x, x'), \tag{2.4}$$

the Compton vertex

$$\Gamma_{\mu\nu}(x, x'; \xi, \xi') = \frac{\delta}{\delta\langle e_c A^\mu(\xi)\rangle} \frac{\delta}{\delta\langle e_c A^\nu(\xi')\rangle} \Delta^{-1}(x, x'), \tag{2.5}$$

and the four-meson vertex

$$\Gamma(x, x'; \xi) = -\frac{\delta}{\delta J(\xi)} \Delta^{-1}(x, x'), \tag{2.6}$$

where

$$\int d^4x'' \Delta(x, x'')\Delta^{-1}(x'', x') = \delta(x - x'). \tag{2.7}$$

With the help of the auxiliary quantities (2.3)–(2.6), Eq. (2.2) (in momentum space) becomes in the limit $J_\mu = 0$, $J = 0$

$$\begin{aligned}\frac{1}{\Delta(p^2)} = {} & p^2 + m_0^2 + i\lambda \int \frac{d^4s}{(2\pi)^4} \Delta(p+s) + i\lambda \int \frac{d^4s}{(2\pi)^4} \Delta(p+s)\Gamma(p+s, p) - ie_c^2 \int \frac{d^4s}{(2\pi)^4} D_\mu{}^\mu(s) \\ & + ie_c^2 \int \frac{d^4s}{(2\pi)^4} D^{\mu\nu}(s)(2p+s)_\mu \Delta(p+s)\Gamma_\nu(p+s, p) \\ & - e_c^4 \int \frac{d^4s}{(2\pi)^4} \frac{d^4s'}{(2\pi)^4} D^{\alpha\mu}(s) D^{\beta\nu}(s')(2g_{\alpha\beta})\Delta(p+s+s')\Gamma_\nu(p+s+s', p+s)\Delta(p+s)\Gamma_\mu(p+s, p) \\ & + e_c^4 \int \frac{d^4s}{(2\pi)^4} \frac{d^4s'}{(2\pi)^4} D^{\alpha\mu}(s) D_\alpha{}^\nu(s')\Delta(p+s+s')\Gamma_{\mu\nu}(p+s+s', p+s, p),\end{aligned} \tag{2.8}$$

which is the scalar-electrodynamic equivalent of the Schwinger-Dyson equation for the electron propagator.

In order to make effective use of our assumption regarding the asymptotic behavior of $\tilde{D}_{\mu\nu}$, we will expand Γ_μ, $\Gamma_{\mu\nu}$, and Γ in Eq. (2.8) in terms of the exact Δ and D functions. This is accomplished by making repeated use of the definitions (2.3)–(2.7) combined with Eq. (2.2). It turns out that, due to the $\phi^{\dagger 2}\phi^2$ counterterm, considerable care must be taken to functionally differentiate all graphs, including tadpoles. Otherwise, graphs will be generated with the wrong over-all numerical weighting factors. Substitution of the results in Eq. (2.8) gives

$$\begin{aligned}\frac{1}{\Delta(p^2)} = {} & p^2 + m_0^2 + 2i\lambda \int \frac{d^4s}{(2\pi)^4} \Delta(p+s) - ie_c^2 \int \frac{d^4s}{(2\pi)^4} D_\mu{}^\mu(s) + ie_c^2 \int \frac{d^4s}{(2\pi)^4} D_{\mu\nu}(s)(2p+s)^\mu \Delta(p+s)(2p+s)^\nu \\ & + e_c^4 \int \frac{d^4s}{(2\pi)^4} \frac{d^4s'}{(2\pi)^4} D^{\alpha\mu}(s) D_\alpha{}^\nu(s')\Delta(p+s+s')(2g_{\mu\nu}) + 2\lambda^2 \int \frac{d^4s}{(2\pi)^4} \frac{d^4s'}{(2\pi)^4} \Delta(p+s+s')\Delta(s)\Delta(s') \\ & + 2\lambda e_c^2 \int \frac{d^4s}{(2\pi)^4} \frac{d^4s'}{(2\pi)^4} D_{\mu\nu}(s)(2p+s)^\mu \Delta(p+s)\Delta(s+s')(2s'+s)^\nu \Delta(s') + \cdots .\end{aligned} \tag{2.9}$$

Equation (2.9) is depicted graphically in Fig. 1. Let the functional $\Sigma(p^2; \Delta, e_c{}^2 D)$ denote the sum of all meson proper self-energy graphs without meson or photon self-energy insertions and with all internal meson and photon lines replaced with the exact Δ and D functions. Then Eq. (2.9) may be written in the form

$$\frac{1}{\Delta(p^2)} = p^2 + m_0{}^2 + \Sigma(p^2; \Delta, e_c{}^2 D). \qquad (2.10)$$

The last entry in Σ is a reminder that $e_c{}^2$ and $D_{\mu\nu}$ always appear in the combination $e_c{}^2 D_{\mu\nu}$ in Eqs. (2.8) and (2.9). In Eqs. (2.8) and (2.9) the external momentum p is routed through the Δ functions. This is equivalent to the convention of routing p through the base line of meson self-energy graphs in the absence of λ vertices. We shall adhere to this convention throughout unless otherwise stated.

C. Equation for Γ_μ

A linear, gauge-covariant equation for Γ_μ can be obtained by transforming Eq. (2.10) to coordinate space, turning on an external field, and carrying out the functional differentiation indicated in Eq. (2.4). The result of this operation, when the external field is switched off, is summarized by the equation (in momentum space)

$$\begin{aligned}\Gamma_\mu(p_+, p_-) = 2p_\mu &+ \int \frac{d^4 s}{(2\pi)^4} K_\mu(p_+, p_-, s_+, s_+)\Delta(s_+) \\ &+ \int \frac{d^4 s}{(2\pi)^4} K_\mu(p_+, p_-, s_-, s_-)\Delta(s_-) \\ &+ \int \frac{d^4 s}{(2\pi)^4} K(p_+, p_-, s_+, s_-) \\ &\quad \times \Delta(s_+)\Gamma_\mu(s_+, s_-)\Delta(s_-), \qquad (2.11)\end{aligned}$$

where

$$p_+ = p + \tfrac{1}{2}q, \quad s_+ = s + p + \tfrac{1}{2}q,$$
$$p_- = p - \tfrac{1}{2}q, \quad s_- = s + p - \tfrac{1}{2}q.$$

Equation (2.11) is illustrated graphically in Fig. 2.

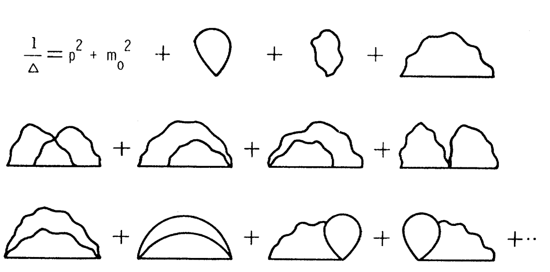

FIG. 1. Equation for the meson propagator, including all graphs through orders $e_c{}^4$, $e_c{}^2\lambda$, and λ^2. The solid and wavy lines represent the exact Δ and D functions, respectively.

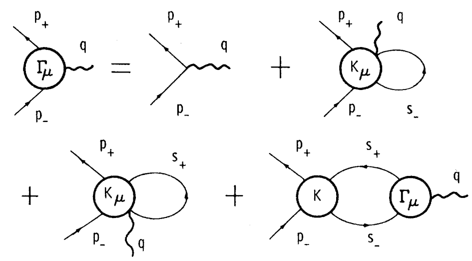

FIG. 2. Graphical representation of the linear integral equation for Γ_μ.

The kernel K is the spin-0 meson-antimeson Bethe-Salpeter scattering kernel which includes all scattering graphs with neither a single-photon or a π^+-π^- (Ref. 16) intermediate state. We also include in K all graphs with two-photon intermediate states although they are not directly obtainable from the functional differentiation of Eq. (2.10) (no photon self-energy insertions). All internal lines represent the exact Δ and D functions and have no photon or meson self-energy insertions. Some typical graphs in the series for K are shown in Fig. 3(a).

The second and third terms on the right-hand side of Eq. (2.11) are a result of the explicit field dependence of Σ. The functions $K_\mu(p_+, p_-, s_-, s_-)$ and $K_\mu(p_+, p_-, s_+, s_+)$ may be obtained from $K(p_+, p_-, s_+, s_-)$ by the following rules:

(1) Divide the graphs defining K into two groups: those that *cannot* be bisected by vertically cutting photon lines (group A) and those that can (group B).

(2) Select from group A only those graphs having a one-photon emission vertex (polarization ν) in the extreme upper [lower] right-hand corner that is *not* joined to the remainder of the graph by a D function ending on the upper [lower] charged-meson line. The function $K_\mu(p_+, p_-, s_-, s_-)$ [$K_\mu(p_+, p_-, s_+, s_+)$] is the sum of graphs obtained by replacing the vertex in the extreme upper [lower] right-hand corner by $-2g_{\mu\nu}$.

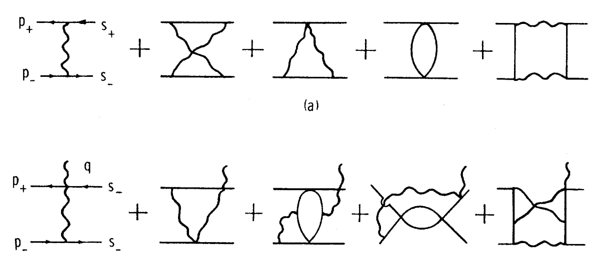

FIG. 3. (a) Typical diagrams in the series for $K(p_+, p_-, s_+, s_-)$. (b) Typical diagrams in the series for $K_\mu(p_+, p_-, s_-, s_-)$.

(3) Select from group B all graphs with a one-photon emission vertex (polarization ν) in the extreme upper right-hand corner. Their contribution to $K_\mu(p_+, p_-, s_-, s_-)$ is the sum of graphs obtained by replacing this vertex with $-2g_{\mu\nu}$. By convention, the graphs in group B give no contribution to $K_\mu(p_+, p_-, s_+, s_+)$.

A graphical illustration of these rules is given in Fig. 4. They were obtained empirically. It is important when applying them to keep the distinction between direct and exchange graphs in the series for K_μ. Otherwise, the cancellations among vertex graphs derived from K_μ that are necessary to give them their correct over-all numerical weight will not occur. This distinction is equally critical for graphs in the series for K. Figure 3(b) depicts some typical graphs in the series for $K_\mu(p_+, p_-, s_-, s_-)$.

Imagine for the moment that λ is represented as a power series in e_c^2. Then, because of Eq. (2.4), the approximate expressions for Δ and Γ_μ obtained by truncating Σ in Eq. (2.10) and the kernels K and K_μ in Eq. (2.11) at the same order in e_c^2 will satisfy the Ward identity

$$q^\mu \Gamma_\mu(p+\tfrac{1}{2}q, p-\tfrac{1}{2}q) = \Delta^{-1}(p+\tfrac{1}{2}q) - \Delta^{-1}(p-\tfrac{1}{2}q). \tag{2.12}$$

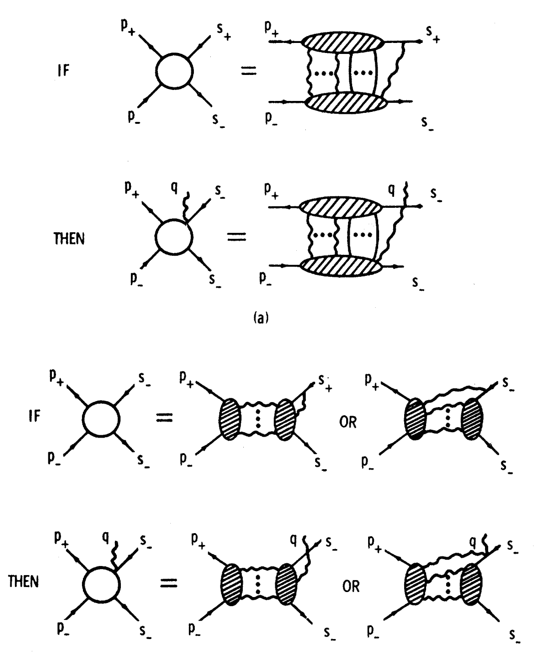

FIG. 4. (a) Illustration of the rule for obtaining the contribution to the series for $K_\mu(p_+, p_-, s_-, s_-)$ from a typical diagram in the series for $K(p_+, p_-, s_+, s_-)$ without a multiphoton intermediate state. (b) Illustration of the same rule when $K(p_+, p_-, s_+, s_-)$ has a multiphoton intermediate state.

The work in the following sections does not require an explicit equation for $\Gamma_{\mu\nu}$. Here it is sufficient to assume that $\Gamma_{\mu\nu}$ has been calculated from the truncated expression for Σ using definitions (2.5) and Eq. (2.10). This will ensure its gauge covariance and the validity of the two Ward identities relating $\Gamma_{\mu\nu}$ to Γ_μ.

III. Z_2 AND THE CALCULATION OF C

In this section we prove that a gauge exists in which Z_2 is finite. We then give the rules for calculating C in this gauge and conclude this section with an explicit calculation of the gauge that renders Z_2 finite through fourth order.

A. Gauge Dependence of Z_2

Beginning with our functional equation for Δ, Eq. (2.10), we replace all internal photon lines representing the full unrenormalized photon propagator $D_{\mu\nu}$ by the full renormalized propagator $\tilde{D}_{\mu\nu}$ according to the substitution $e_c^2 D_{\mu\nu} \to e^2 \tilde{D}_{\mu\nu}$. Graphically this is equivalent to replacing a prior summation of all meson proper self-energy graphs that differ only by their photon self-energy parts by a single graph whose photon lines are the full renormalized propagators. Next, we remove meson self-energy divergences by subtracting Eq. (2.10) at $p^2 = -m^2$ and writing it in terms of the physical mass m^2:

$$\Delta^{-1}(p^2) = p^2 + m^2 + \Sigma(p^2; \Delta, e^2\tilde{D}) - \Sigma(-m^2; \Delta, e^2\tilde{D}). \tag{3.1}$$

We assume that λ is fixed to cancel the ultraviolet (uv) divergent part of all M-part subgraphs in the iteration of Eq. (3.1). The iterative expansion of Eq. (3.1) will in general contain logarithmic uv divergences that are related to Z_2. In order to study the structure of the uv-divergent part of Z_2 we will make the assumption that all uv divergences related to Z_2 can be isolated in Eq. (3.1) by replacing $e^2\tilde{D}_{\mu\nu}$ by its asymptotic limit [Eq. (1.1)]:

$$e^2\tilde{D}_{\mu\nu}(k) \underset{k^2 \gg m^2}{\sim} \left(g_{\mu\nu} + (G-1)\frac{k_\mu k_\nu}{k^2} \right) \frac{e_0^2}{k^2} \equiv e_0^2 D^0_{\mu\nu}(k).$$

This will be proved in Sec. VI (see Ref. 31) provided one makes the additional assumption that h decreases asymptotically with power-law behavior. This assumption then allows us to study Z_2 in a model of scalar electrodynamics without charge renormalization ($Z_3 = 1$) and with coupling constants e_0^2 and $\lambda(e_0^2)$.

The renormalizability of this simplified theory and the existence of the $m=0$ limit of Δ after mass renormalization imply that the structure of Z_2 in an arbitrary covariant gauge G is of the form[17]

$$Z_2 = A(e_0^2, G)(\Lambda^2/m^2)^{g(e_0^2,G)}. \tag{3.2}$$

Here A and g are functions of e_0^2 and G alone. The uv momentum cutoff Λ, necessary to define Z_2 in perturbation theory, is introduced in our analysis by replacing each internal momentum integration $\int d^4s$ by $\int d^4s\,\Lambda^2/(\Lambda^2+s^2)$ with $\Lambda^2 \gg p^2$ and $\Lambda^2 \gg m^2$.[18] Such a cutoff procedure preserves gauge invariance and, in self-energy graphs without closed loops or λ vertices, is equivalent to the Feynman prescription of replacing the free photon propagator $D^0_{\mu\nu}(k)$ by

$$D^0_{\mu\nu}(k,\Lambda) = D^0_{\mu\nu}(k)\frac{\Lambda^2}{k^2+\Lambda^2-i\epsilon}.$$

It is the presence of the $\phi^{\dagger 2}\phi^2$ counterterm in the theory that necessitates this stronger cutoff. Quantities that would be finite in the absence of a cutoff, such as photon-photon scattering skeleton graphs,[19] remain unchanged as $\Lambda\to\infty$.

The behavior of the unrenormalized meson propagator Δ under a gauge transformation induced by

$$D^0_{\mu\nu}(x) \to D^0_{\mu\nu}(x) + (G'-G)\partial_\mu\partial_\nu M(x),$$

where

$$M(x) = -\int \frac{d^4k}{(2\pi)^4}\left(\frac{\Lambda^2}{k^2+\Lambda^2}\right)\frac{e^{ik\cdot x}}{(k^2-i\epsilon)^2},$$

is summarized by the simple relation[20]

$$\Delta(x;G') = \exp\{ie_0^2(G'-G)[M(x)-M(0)]\}\Delta(x;G). \tag{3.3}$$

The unrenormalized and renormalized meson propagators Δ and $\tilde{\Delta}$, expressed in terms of the physical mass m, are related by

$$\Delta(x) = Z_2\tilde{\Delta}(x), \tag{3.4}$$

where $\tilde{\Delta}$ is a finite function independent of Λ in the limit $\Lambda\to\infty$. From Eqs. (3.3) and (3.4) we obtain

$$Z_2(G')\tilde{\Delta}(x;G') = \exp\{ie_0^2(G'-G)[M(x)-M(0)]\} \times Z_2(G)\tilde{\Delta}(x;G). \tag{3.5}$$

The behavior of Z_2 under a gauge transformation can be extracted from Eq. (3.5) by letting $x^2\to\infty$. Before doing this we imagine that the theory has been supplied with an infrared cutoff μ^2 to make the singularity in Δ at $p^2=-m^2$ rigorously a pole. Then in any gauge

$$\tilde{\Delta}(x) \underset{x^2\to\infty}{\sim} \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot x}}{p^2+m^2-i\epsilon}.$$

If we redefine $M(x)$ as

$$M(x) = -\frac{i}{16\pi^2}\int_{\mu^2}^{\infty}\frac{dk^2}{k^2}\left(\frac{\Lambda^2}{k^2+\Lambda^2}\right)\int\frac{d\Omega_k}{2\pi^2}e^{ik\cdot x}, \tag{3.6}$$

then $M(x)\to 0$ as $x^2\to\infty$. Combining these results with the $x^2\to\infty$ limit of Eq. (3.5) gives

$$Z_2(G') = e^{-ie_0^2(G'-G)M(0)}Z_2(G). \tag{3.7}$$

If Eq. (3.2) is now substituted in Eq. (3.7), we obtain an explicit expression for the behavior of the cutoff-dependent part of Z_2 under a gauge transformation:

$$Z_2(G') = A(e_0^2, G)\left(\frac{\Lambda^2}{m^2}\right)^{g(e_0^2,G)}\left(\frac{\Lambda^2}{\mu^2}\right)^{e_0^2(G-G')/16\pi^2}$$

The condition

$$g(e_0^2,G) - \frac{e_0^2(G'-G)}{16\pi^2} = 0$$

fixes the gauge in which Z_2 is finite in the limit $\Lambda\to\infty$. The proof that G can be calculated as a power series in α_0 requires a more detailed study which we will now give.

Suppose the expansion of Σ to any order in e_0^2 is a renormalizable approximation to Δ. By construction, the expansion truncated at any order in e_0^2 preserves the gauge-covariance of Δ. Therefore, if the $m=0$ limit of Δ exists in each order of the expansion, we expect that the functions G and g can be calculated as power series in e_0^2. Explicitly, the gauge

$$G = G_0 + G_2\left(\frac{\alpha_0}{4\pi}\right) + \cdots + G_{2n}\left(\frac{\alpha_0}{4\pi}\right)^n \tag{3.8}$$

that makes Z_2 finite to order α_0^{n+1} is fixed by the condition

$$g_{2n+2}\,\alpha_0^{n+1} - G_{2n}\left(\frac{\alpha_0}{4\pi}\right)^{n+1} = 0. \tag{3.9}$$

Here $g_{2n+2}\alpha_0^{n+1}$ is the coefficient of the part of Z_2 that diverges like a single power of $\ln\Lambda^2$ when calculated to order α_0^{n+1} in the gauge

$$G = G_0 + G_2\left(\frac{\alpha_0}{4\pi}\right) + \cdots + G_{2n-2}\left(\frac{\alpha_0}{4\pi}\right)^{n-1} \tag{3.10}$$

that makes Z_2 finite to order α_0^n. The second term, $-G_{2n}(\alpha_0/4\pi)^{n+1}$, has its origin in the change in Z_2 induced by the change of gauge $G\to G+G_{2n}(\alpha_0/4\pi)^n$, where G is given by Eq. (3.10). It is the coefficient of the term that diverges as $\ln\Lambda^2$ in the expression for Z_2 obtained from the graphs depicted in Fig. 5(a) when calculated with $\Delta(p^2)=1/p^2$ and

$$D^0_{\mu\nu}(k) = G_{2n}\left(\frac{\alpha_0}{4\pi}\right)^n\frac{k_\mu k_\nu}{k^4}.$$

We expect that the renormalizability, gauge covariance, and the existence of the $m=0$ limit of the approximation to Δ obtained by truncating the expansion of Σ at order α_0^{n+1} will result in the cancellation of any higher powers of $\ln\Lambda^2$ present in Z_2 in order α_0^{n+1} provided Z_2 is finite to order α_0^n. We now turn to the proof of these conjectures.

B. Calculation of G

The existence of Z_2 in each order of the expansion of $\Sigma(p^2;\Delta, e^2\tilde{D})$ is most easily studied with the help of the equation for $\Gamma_\mu(p,p)$ $[=\partial\Delta^{-1}(p^2)/\partial p^\mu]$ obtained by setting $q_\mu=0$ in Eq. (2.11) and replacing $e_c^2 D_{\mu\nu}$ by $e^2\tilde{D}_{\mu\nu}$:

$$\Gamma_\mu(p,p)=2p_\mu+\int\frac{d^4s}{(2\pi)^4}K_\mu(p,s)\Delta(s+p)+\int\frac{d^4s}{(2\pi)^4}K(p,s)\Delta(s+p)\Gamma_\mu(s+p,s+p)\Delta(s+p). \tag{3.11}$$

The quantities K and K_μ are now power series in the exact Δ and $\tilde{D}$ functions. The mass-dependent part of Δ is assumed calculated in terms of the physical mass m. Since our method of expanding Σ preserves Ward's identity order by order, $Z_1=Z_2$ in each order.

In this section we will prove that Z_1 can be made finite in each order of the expansion of Σ provided the approximate expression for Δ obtained by truncating Σ at any order in α_0 is renormalizable. The renormalization-group analysis used to derive Eq. (3.2) indicates that the uv-divergent part of Z_2, and hence Z_1, is insensitive to m in a theory of scalar electrodynamics without charge renormalization. We expect this to be true in the full theory provided $h(k^2/m^2,\alpha)$ vanishes with power-law behavior for $k^2\gg m^2$. As we will see below, the $m=0$ limit of scalar electrodynamics results in considerable calculational simplification.

Thus, we begin by taking the $m=0$ limit of Eq. (3.11) and substituting $e_0^2D^0_{\mu\nu}$ for $e^2\tilde{D}_{\mu\nu}$ at each internal photon line. The external momentum p is taken to be spacelike so that the integrals in Eq. (3.11) can be converted to integrals over a four-dimensional Euclidean space. Then in lowest-order perturbation theory, where $\Delta(p^2)=1/(p^2+m^2)$,

$$K(p,s)=-ie_0^2(2p+s)^\mu D^0_{\mu\nu}(s)(2p+s)^\nu,$$

and

$$K_\mu(p,s)=4ie_0^2 g_\mu{}^\alpha D^0_{\alpha\beta}(s)(2p+s)^\beta,$$

Eq. (3.11) gives in the limit $p^2\gg m^2$ and $m\to 0$

$$\Gamma^{(2)}_\mu(p,p)=2p_\mu\left(1-\frac{3e_0^2}{32\pi^2}-\frac{G_0e_0^2}{16\pi^2}+(G_0-3)\frac{e_0^2}{16\pi^2}\int_{p^2}^{\infty}\frac{ds^2}{s^2}\right). \tag{3.12}$$

Hence in the gauge $G_0=3$ (the Yennie gauge) Z_1 is finite through order e_0^2. In this gauge

$$\Gamma^{(2)}_\mu(p,p)=2p_\mu\left(1-\frac{9e_0^2}{32\pi^2}\right), \tag{3.13}$$

and the $m=0$ limit exists through order e_0^2. Also, from the Ward identity $\Gamma_{\mu\nu}(p,p,p)=\partial\Gamma_\mu(p,p)/\partial p^\nu$, we get

$$\Gamma^{(2)}_{\mu\nu}(p,p,p)=2g_{\mu\nu}\left(1-\frac{9e_0^2}{32\pi^2}\right). \tag{3.14}$$

Finally, after mass renormalization, we get

$$\Delta^{(2)}(p^2)=\frac{1+9e_0^2/32\pi^2}{p^2+m^2}, \tag{3.15}$$

thus proving the existence of $\Delta^{(2)}$ at $m=0$. We will now prove by induction that a gauge can be found that makes Z_1 finite to all orders.

Let us assume that a gauge (3.10) has been found in which Z_1 $(=Z_2=Z_4)$ is finite to order α_0^n and that $\Delta^{(2n)}$, $\Gamma^{(2n)}_\mu$, and $\Gamma^{(2n)}_{\mu\nu}$ all exist at $m=0$. The uv-divergent part of the sum of all graphs in each order through order α_0^n with four external charged-meson lines (M parts) without meson self-energy parts and with finite vertex and Compton insertions is assumed to be canceled by the $\phi^{\dagger 2}\phi^2$ counterterm. This is required in order that the factorization (3.4) will continue to hold in higher orders. The special subclass of M parts, the graphs belonging to the perturbation expansion of K, will still diverge logarithmically. It will be shown below that this divergence does not interfere with the

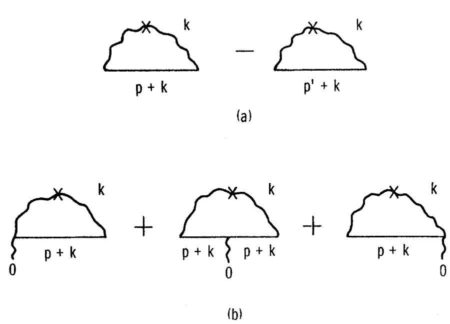

FIG. 5. (a) Diagrams contributing to Z_2 in order α_0^{n+1} obtained from the change of gauge $G\to G+G_{2n}(\alpha_0/4\pi)^n$. The crossed, wavy line represents the gauge term $G_{2n}(\alpha_0/4\pi)^n$ in $D^0_{\mu\nu}$. The momentum p' has the magnitude $p'^2=-m^2$. (b) Diagrams contributed by the gauge term $G_{2n}(\alpha_0/4\pi)^n$ in $D^0_{\mu\nu}$ to the value of Z_1 in order α_0^{n+1}.

removal of divergences in Z_1.

Let the values of K, K_μ, Γ_μ, and $\Gamma_{\mu\nu}$ obtained by setting $m=0$ be denoted by K^a, K^a_μ, Γ^a_μ, and $\Gamma^a_{\mu\nu}$. From the assumption that

$$\Delta^{(2n)}(p^2) \underset{m\to 0}{\sim} C/p^2$$

to order $\alpha_0{}^n$, we obtain with the help of the local Ward identities for $\Gamma_\mu(p,p)$ and $\Gamma_{\mu\nu}(p,p,p)$ the result

$$\Gamma^{a(2n)}_\mu(p,p)=2p_\mu/C\,,$$

$$\Gamma^{a(2n)}_{\mu\nu}(p,p,p)=2g_{\mu\nu}/C\,.$$

Here C is a polynomial of order n in $e_0{}^2$ with finite numerical coefficients: $C=1+C_2e_0{}^2+\cdots+C_{2n}e_0{}^{2n}$. From Eq. (3.15) we get $C=1+9e_0{}^2/32\pi^2$ in second order.

Taking the $m=0$ limit of Eq. (3.11) gives

$$\Gamma^{a(2n+2)}_\mu(p,p)=2p_\mu+\int\frac{d^4s}{(2\pi)^4}\left[K^{a(2n+2)}_\mu(p,s)+(1+C_2e_0{}^2)K^{a(2n)}_\mu(p,s)+\cdots+CK^{a(2)}_\mu(p,s)\right]\frac{1}{(p+s)^2}$$
$$+\int\frac{d^4s}{(2\pi)^4}\left[K^{a(2n+2)}(p,s)+(1+C_2e_0{}^2)K^{a(2n)}(p,s)+\cdots+CK^{a(2)}(p,s)\right]\frac{2(s+p)_\mu}{(p+s)^4}\,. \tag{3.16}$$

Since the finiteness of C to a fixed order in $e_0{}^2$ requires that the $m=0$ limits of $K(p,s)$ and $K_\mu(p,s)$ exist to the same order [see Eq. (3.30)] we have only to study the $m=0$ limits of $K^{(2n+2)}(p,s)$ and $K^{(2n+2)}_\mu(p,s)$ as a preliminary step to establishing the existence of the $m=0$ limit of $\Gamma^{(2n+2)}_\mu(p,p)$. Hence, we require that the kernels $K^{a(2n+2)}$ and $K^{a(2n+2)}_\mu$ defined by

$$\lim_{m\to 0}K^{(2n+2)}(p,s;\Delta(p'^2),e^2\tilde{D})=K^{(2n+2)}(p,s;C/p'^2,e_0{}^2D^0)\equiv K^{a(2n+2)}(p,s)\,, \tag{3.17}$$

$$\lim_{m\to 0}K^{(2n+2)}_\mu(p,s;\Delta(p'^2),e^2\tilde{D})=K^{(2n+2)}_\mu(p,s;C/p'^2,e_0{}^2D^0)\equiv K^{a(2n+2)}_\mu(p,s) \tag{3.18}$$

exist. The proof of this follows almost verbatim the proof of the infrared convergence of $K^{a(2n+2)}(0,s)$ in spinor electrodynamics given by Johnson, Willey, and Baker,[9] and will be omitted here. The only graphs in $K^{a(2n+2)}(0,s)$ that are not infrared-convergent are those with two-photon intermediate states, and these are excluded from the equation for Γ_μ on grounds of C invariance. Since $K^{a(2n+2)}(0,s)$ is infrared-finite in scalar electrodynamics, so are $K^{a(2n+2)}(p,s)$ and $K^{a(2n+2)}_\mu(p,s)$. The latter result follows since the replacement of a lowest-order one-photon vertex in the upper right- or left-hand corner of a graph belonging to $K^{a(2n+2)}(p,s)$ by $-2g_{\mu\nu}$ has no effect on the graph's infrared behavior. Finally, the uv-divergent part of $K^{a(2n+2)}(p,s)$ may be temporarily suppressed with a momentum cutoff Λ, where $\Lambda^2\gg p^2$, $\Lambda^2\gg s^2$.

There remains for consideration those subintegrations in Eq. (3.16) involving the external momentum s of $K^a_\mu(p,s)$ and $K^a(p,s)$. In order to study the small-s behavior of these integrals we make the translation $s+p\to s$, neglecting unimportant finite surface terms. Thus we are led to consider integrals of the form

$$\int\frac{d^4s}{s^4}K^a(p,s)s_\mu \tag{3.19}$$

and

$$\int\frac{d^4s}{s^2}K^a_\mu(p,s)\,. \tag{3.20}$$

For integrals of the first type, $K^a(p,s)$ must behave like $p\cdot s/s^2$ for $s^2\ll p^2$ and for fixed Λ to be infrared-divergent. Since $K^a(0,s)$ is finite in each order of its expansion and since K^a has the symmetry property $K^a(p,s)=K^a(s,p)$, the kernel $K^a(p,0)$, obtained by letting $s_\mu\to\chi s_\mu$ with $\chi\to 0$ in $K^a(p,s)$, is also finite. Hence all integrals of the form (3.19) converge in the infrared region to order $\alpha_0{}^{n+1}$.

The kernel $K^a_\mu(p,s)$ is uv-finite since the overall degree of divergence[21] of the graphs defining K^a_μ is -1 and all insertions are assumed finite. A simple application of Weinberg's theorem[22] to these graphs plus covariance give

$$\int\frac{d\Omega_s}{2\pi^2}K^a_\mu(p,s)\underset{p^2\gg s^2}{\sim}\frac{p_\mu}{p^2}\times(\text{powers of }\ln p^2),$$

which ensures the convergence of integrals of the form (3.20) in the region of small s to order $\alpha_0{}^{n+1}$.

Therefore, the final integration over s in Eq. (3.16) is infrared-convergent, thereby establishing the existence of the $m=0$ limits of $\Gamma^{(2n+2)}_\mu(p,p)$ and $\Delta^{(2n+2)}(p^2)$.

The gauge (3.8) in which Z_1 is finite to order $\alpha_0{}^{n+1}$ is conveniently calculated from Eq. (3.16) by requiring its right-hand side to be uv-convergent. This requires that

$$\int\frac{d^4s}{(2\pi)^4}K^{a(2n+2)}_\mu(p,s)\frac{1}{(p+s)^2}+\int\frac{d^4s}{(2\pi)^4}K^{a(2n+2)}(p,s)\frac{2(p+s)_\mu}{(p+s)^4} \tag{3.21}$$

be finite. The sum of the remaining integrals in Eq. (3.16) is finite since Z_1 is assumed to be finite through order $\alpha_0{}^n$. The requirement that (3.21) be finite fixes the value of G_{2n} in Eq. (3.8) and is equivalent to the condition that

$$\text{divergent part } \Gamma_\mu^{a(2n+2)}(p,p) + 2p_\mu G_{2n}\left(\frac{\alpha_0}{4\pi}\right)^{n+1}\int_{p^2}^{\infty}\frac{ds^2}{s^2}=0\,. \quad (3.22)$$

The first term is the sum of the ultraviolet-divergent parts of all vertex graphs of order $\alpha_0{}^{n+1}$ calculated in the gauge (3.10) that makes Z_1 finite to order $\alpha_0{}^n$. The second term in Eq. (3.22) is contributed by the three vertex graphs depicted in Fig. 5(b) calculated in the gauge $G=G_{2n}(\alpha_0/4\pi)^n$ with $\Delta(p^2)=1/p^2$ and

$$D^0_{\mu\nu}(k)=G_{2n}\left(\frac{\alpha_0}{4\pi}\right)^n\frac{k_\mu k_\nu}{k^4}\,.$$

In lowest order we found from Eq. (3.12) that $G_0=3$. The value of G_2 will be calculated below.

It is clear from Eq. (3.22) that this whole program will fail unless $\Gamma_\mu^{a(2n+2)}$ diverges as a single power of $\ln\Lambda^2$ when calculated in the gauge (3.10). For this to happen it is first necessary that $K^{a(2n+2)}(p,s)$ be rendered finite after one over-all subtraction. In this case if the subtraction is made at the point $p_\mu=0$, then the second integral in (3.21) can be written as

$$\int\frac{d^4s}{(2\pi)^4}\left[K^{a(2n+2)}(p,s)-K^{a(2n+2)}(0,s)\right]\frac{2s_\mu}{s^4} + \int\frac{d^4s}{(2\pi)^4}K^{a(2n+2)}(0,s)\frac{2s_\mu}{s^4}\,,$$

where we have made the translation $s+p\to s$ and neglected a finite surface term. The last integral above is zero on grounds of covariance, while the subtraction in the first integral removes the uv divergence embedded in $K^{a(2n+2)}(p,s)$.

In our case $K^{a(2n+2)}$ contains no meson or photon self-energy insertions by definition. Furthermore, all vertex and Compton insertions are finite to order $\alpha_0{}^n$ in the gauge (3.10), and all M-part divergences to the same order are assumed to be canceled by the $\phi^{\dagger 2}\phi^2$ counterterm. Therefore, all subintegrations in the graphs contributing to $K^{a(2n+2)}$ are finite. On the basis of Weinberg's theorem we expect the uv divergence arising from the over-all integration involving all lines of $K^{a(2n+2)}$ to be removed by a single subtraction.[23]

Second, it is necessary that all subintegrations in Eq. (3.21) involving the external momentum s of $K^{a(2n+2)}(p,s)$ and $K_\mu^{a(2n+2)}(p,s)$ also be uv-convergent. Now the only divergent subintegrations involving the momentum s and not excluded by the definition of K^a and K^a_μ are Compton subgraphs. Since these are all of order $\alpha_0{}^n$ or less and their sum in each order is finite by prior choice of gauge, these subintegrations will likewise converge. Hence all subintegrations in the graphs defining $\Gamma_\mu^{a(2n+2)}$ converge, and by Weinberg's theorem, the over-all integration involving all lines of $\Gamma_\mu^{a(2n+2)}$ will diverge no worse than a single power of $\ln\Lambda^2$ $(\Lambda\to\infty)$ provided we average over angles first.

In conclusion we have established that a gauge exists in each order of the expansion of Σ that renders Z_1 $(=Z_2=Z_4)$ finite and that gives Δ the asymptotic behavior

$$\Delta(p^2)\underset{p^2\gg m^2}{\sim}\frac{C(e_0{}^2)}{p^2}\,,$$

where C is a polynomial in the asymptotic coupling with finite numerical coefficients.

C. Calculation of C

The calculation of C to order $\alpha_0{}^{n+1}$ is most easily accomplished by first rescaling Eq. (2.11) for $\Gamma_\mu(p_+,p_-)$. Let

$$\tilde\lambda=C^2\lambda\,, \quad (3.23)$$

$$\tilde\Delta(p^2)=\Delta(p^2)/C\,, \quad (3.24)$$

$$\tilde\Gamma_\mu(p_+,p_-)=C\,\Gamma_\mu(p_+,p_-)\,, \quad (3.25)$$

$$\tilde\Gamma_{\mu\nu}(p+q+q',p+q,p)=C\,\Gamma_{\mu\nu}(p+q+q',p+q,p)\,. \quad (3.26)$$

Then

$$\lim_{m\to 0}\tilde\Delta(p^2)=1/p^2\,,$$

$$\lim_{m\to 0}\tilde\Gamma_\mu(p,p)=2p_\mu\,,$$

$$\lim_{m\to 0}\tilde\Gamma_{\mu\nu}(p,p,p)=2g_{\mu\nu}$$

in the gauge in which Z_1 is finite. Substituting Eqs. (3.23)–(3.26) in Eq. (2.11), we obtain an equation for $\tilde\Gamma_\mu(p_+,p_-)$ (in symbolic shorthand):

$$\tilde\Gamma_\mu=2Cp_\mu+C\tilde K_\mu\tilde\Delta+\tilde K\tilde\Delta\tilde\Gamma_\mu\tilde\Delta\,, \quad (3.27)$$

where

$$\tilde K=C^2K(\Delta,\Gamma_\mu,\Gamma_{\mu\nu},\lambda)=K(\tilde\Delta,\tilde\Gamma_\mu,\tilde\Gamma_{\mu\nu},\tilde\lambda)\,, \quad (3.28)$$

$$\tilde K_\mu=CK_\mu(\Delta,\Gamma_\mu,\Gamma_{\mu\nu},\lambda)=K_\mu(\tilde\Delta,\tilde\Gamma_\mu,\tilde\Gamma_{\mu\nu},\tilde\lambda)\,, \quad (3.29)$$

that is, $\tilde K$ $(\tilde K_\mu)$ has the same functional dependence on $\tilde\Delta$, $\tilde\Gamma_\mu$, $\tilde\Gamma_{\mu\nu}$, and $\tilde\lambda$ as K (K_μ) has on Δ, Γ_μ, $\Gamma_{\mu\nu}$, and λ.

After setting $q_\mu = 0$ in Eq. (3.27) and taking the $m=0$ limit, we obtain the following equation for C:

$$(1-C)2p_\mu = C\int \frac{d^4s}{(2\pi)^4}\tilde{K}^a_\mu(p,s)\frac{1}{(p+s)^2} + \int\frac{d^4s}{(2\pi)^4}\tilde{K}^a(p,s)\frac{2(p+s)_\mu}{(p+s)^4}\,. \qquad (3.30)$$

Equation (3.30) indicates that $C = 1 + C_2 e_0{}^2 + \cdots + C_{2n+2}e_0{}^{2n+2}$ can be calculated to order $\alpha_0{}^{n+1}$ given $\tilde{\Gamma}_\mu^{a(2n)}$. This can be calculated from Eq. (3.27) given C to order $\alpha_0{}^n$. In lowest order we found $C = 1 + 9e_0{}^2/32\pi^2$.

Let us summarize our results. By construction the expansion of Σ in terms of the exact Δ and $\tilde{D}$ functions truncated at any order in α_0 preserves the gauge covariance of Δ, Γ_μ, and $\Gamma_{\mu\nu}$. Assuming that this expansion is a renormalizable approximation to Δ and that the nonasymptotic part of $\tilde{D}_{\mu\nu}(k)$ falls off with power-law behavior for $k^2 \gg m^2$, then the entire uv-divergent part of Z_2 can be isolated in the $m=0$ limit in each order. To ameliorate the problem of overlapping divergences we chose to study the equation for $\Gamma_\mu(p,p)$ and the divergences associated with Z_1 $(=Z_2)$. Setting $m=0$ and replacing $e^2\tilde{D}_{\mu\nu}$ with $e_0{}^2 D^0_{\mu\nu}$ at all internal photon lines, we proved by induction that a gauge exists in which Z_1 is finite in each order. Equation (3.22) fixes this gauge to order $\alpha_0{}^n$ provided the vertex $\Gamma_\mu^{a(2n+2)}$, calculated in the gauge (3.10) in which Z_1 is finite to order $\alpha_0{}^n$, diverges as a single power of $\ln\Lambda^2$. This required that $K^{a(2n+2)}$, calculated in the same gauge, be rendered finite after one over-all subtraction. It was also shown that the $m=0$ limit of $\Gamma_\mu(p,p)$, and hence of $\Delta(p^2)$ and $\Gamma_{\mu\nu}(p,p,p)$ exists in each order. Then to order $\alpha_0{}^{n+1}$ in the new gauge (3.8) found from Eq. (3.22), $\Delta(p^2) \sim C/p^2$ as $m \to 0$, where the constant C is calculated to order $\alpha_0{}^{n+1}$ from Eq. (3.30). We will now illustrate the above formalism by calculating the gauge that renders Z_1 finite through fourth order.

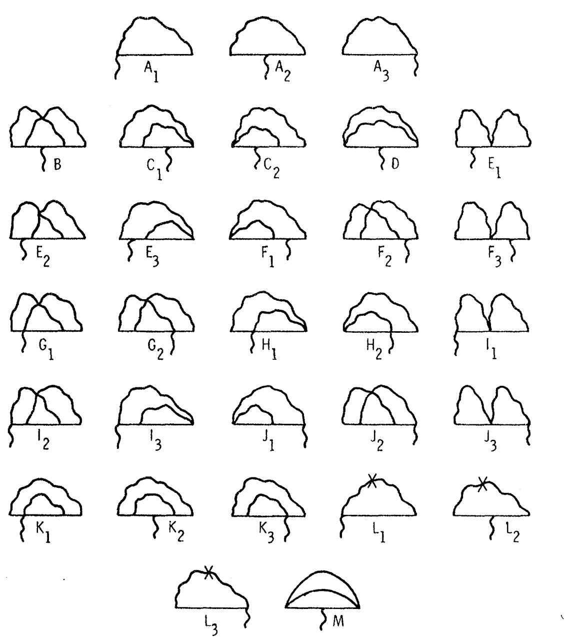

FIG. 6. All diagrams contributing to $\Gamma_\mu(p,p)$ in second and fourth order. The wavy lines in graphs A–K represent $\tilde{D}_{\mu\nu}$ in the Yennie gauge. The crossed, wavy line in graphs L_1–L_3 represent the second-order gauge term in $\tilde{D}_{\mu\nu}$.

D. Calculation of G_2

All graphs that contribute to $\Gamma_\mu(p,p)$ in fourth order are depicted in Fig. 6. Graphs A–K are calculated in the Yennie gauge ($G=3$). Graphs of type L are calculated by replacing $e^2\tilde{D}_{\mu\nu}$ with $G_2(e_0{}^4/16\pi^2)k_\mu k_\nu/k^4$. The final graph M is contributed by the $\phi^{\dagger 2}\phi^2$ counterterm. Table I summarizes the ultraviolet divergences in $\Gamma_\mu^{(4)}(p,p)$ calculated after setting

$$h(k^2/m^2,\alpha)=0$$

and

$$\Delta(p^2) = (1+9e_0{}^2/32\pi^2)/p^2$$

TABLE I. Contribution of graphs in Fig. 6 to the ultraviolet divergences in $\Gamma_\mu^{a\,(4)}(p,p)$.

Graph	Coefficient of $2p_\mu\left(\frac{\alpha_0}{4\pi}\right)^2\ln^2\left(\frac{\Lambda^2}{p^2}\right)$	Coefficient of $2p_\mu\left(\frac{\alpha_0}{4\pi}\right)^2\ln\left(\frac{\Lambda^2}{p^2}\right)$
B	$\frac{9}{4}$	$\frac{27}{2}$
C_1+C_2	$-\frac{9}{4}$	$-\frac{11}{2}$
D	0	12
$E_1+E_2+E_3$	0	$-\frac{27}{2}$
$F_1+F_2+F_3$	0	$-\frac{27}{2}$
G_1+G_2	$-\frac{27}{4}$	-6
H_1+H_2	$\frac{9}{2}$	30
$I_1+I_2+I_3$	$\frac{9}{8}$	$\frac{21}{2}$
$J_1+J_2+J_3$	$\frac{9}{8}$	$\frac{21}{2}$
$L_1+L_2+L_3$	0	G_2
M	$2p_\mu\left(\frac{\lambda}{16\pi^2}\right)^2\ln\left(\frac{\Lambda^2}{p^2}\right)+\text{finite}$	
$A_1+A_2+A_3$ $+K_1+K_2+K_3$	Finite in second and fourth order	

in each graph. The cutoff Λ is introduced by setting

$$\int_E d^4s = \int_0^\Lambda s^3 ds \int d\Omega_s\,,$$

where the subscript E denotes an integral over four-dimensional Euclidean space. Summing, we get

$$\Gamma_\mu^{a(4)}(p,p) = 2p_\mu\left(1 - \frac{9e_0^2}{32\pi^2} - \frac{81e_0^4}{1024\pi^4} + \delta\right) + 2p_\mu\left[(38+G_2)\left(\frac{e_0^2}{16\pi^2}\right)^2 + \left(\frac{\lambda}{16\pi^2}\right)^2\right]\ln\left(\frac{\Lambda^2}{p^2}\right),$$

where δ is a finite constant calculated from graphs B–J and L–M. Therefore, if we choose the gauge

$$G = 3 + G_2\left(\frac{e_0^2}{16\pi^2}\right), \tag{3.31}$$

where

$$G_2 = -\left(\frac{\lambda}{e_0^2}\right)^2 - 38, \tag{3.32}$$

Z_1 will be finite through order e_0^4. Note the separate cancellation of $\ln^2\Lambda^2$ terms among vertices where the external photon attaches directly to an internal photon line. This is a result of the separate cancellation of uv-divergent parts of Compton subintegrations over the external momenta of $K^{a(4)}(p,s)$ and $K_\mu^{a(4)}(p,s)$ in each of the two types of vertex graphs. Whether this separate cancellation of multiple logarithms persists in higher orders is not yet known.

IV. REMOVAL OF M-PART DIVERGENCES

In contrast to spinor electrodynamics, scattering graphs in scalar electrodynamics with four external meson lines (M parts) diverge logarithmically beginning in fourth order. Any discussion of the meson self-mass and Z_3, for example, must necessarily be broadened to include these divergences, as they will eventually appear as subgraphs of Compton vertex insertions in both Σ and the polarization operator $\Pi_{\mu\nu}$ beginning in eighth order. Whether these divergences are suppressed when M parts are summed to all orders is a technical question that must await further developments.[24] Here an alternative approach is given whose virtue is to render all M parts ultraviolet-convergent order by order without an infinite renormalization. We begin this section with an illustration of the ideas involved by studying fourth-order M parts in detail. The generalization of our results to arbitrary order will complete this section.

A. General Considerations

All unrenormalized π^+-π^- scattering graphs can be obtained from the unrenormalized scattering kernel K defined in Sec. II by iterating the Bethe-Salpeter equation

$$A(p,p') = K(p,p') + \int \frac{d^4s}{(2\pi)^4} K(p,s)\Delta^2(s^2)A(s,p') = K + \int K\Delta^2 K + \int K\Delta^2 K\Delta^2 K + \cdots\,. \tag{4.1}$$

The amplitude A is related to the full off-mass-shell π^+-π^- scattering amplitude T by $iT = A$. Δ is the full unrenormalized meson propagator. The mass-dependent part of all meson propagators in (4.1) is assumed calculated in terms of the physical meson mass m.

We begin our study of the uv divergences in (4.1) by replacing $e_c{}^2 D_{\mu\nu}$ with $e^2\tilde{D}_{\mu\nu}$ at all internal photon lines. Assuming that the uv divergences present in each term of the expansion of Eq. (4.1) are insensitive to m, we set $m=0$ and replace K and Δ with K^a and Δ^a to define a new amplitude A^a. The kernel K^a is the same kernel defined by Eq. (3.17). This assumption is based on experience with perturbation theory and is expected to be valid here provided the nonasymptotic part of $\tilde{D}_{\mu\nu}$ vanishes with power-law behavior as $m \to 0$. The relevant coupling constants are now e_0^2 and λ.

Next, we take p and p' to be spacelike in order that the integrals on the right-hand side of (4.1) can be converted to integrals over four-dimensional Euclidean space. The uv divergences in (4.1) may be temporarily suppressed by cutting off the integrals at Λ, where $\Lambda^2 \gg p^2$, $\Lambda^2 \gg p'^2$. The iterated kernels in Eq. (4.1) and the two-photon annihilation graphs in K^a also contain logarithmic infrared divergences which can be rendered finite by cutting off the lower limit of integration at μ, where $\mu^2 \ll p^2$, $\mu^2 \ll p'^2$, and $\mu^2 \ll \Lambda^2$. These infrared singularities will not interfere with the calculations in this section.

In the past the ultraviolet M-part divergences present in (4.1) have been absorbed in an infinite renormalization of λ.[25] In this case it is arbitrary whether λ acts as a purely compensatory term or has a nonzero value after renormalization. Our method of fixing λ will be unconventional since the power-series expansion of

$$\lambda = \sum_{n=1}^{\infty} \lambda_{2n} e_0^{2n}$$

will begin in second[26] instead of fourth order in e_0^2, with each term λ_{2n} in the expansion chosen so that $A^{a(2n+2)}(p,p')$ is uv-finite. The advantage of this is that every term λ_{2n} in the expansion of λ is *finite* and well defined.

The disadvantage of this procedure is that the value of each expansion coefficient λ_{2n} is in general complex, so that the series for λ is of the form

$$\lambda = \sum_{n=1}^{\infty} (a_n \pm i b_n)\alpha_0{}^n ,$$

with each a_n and b_n real and finite. Thus, the theory we are developing here has a Hermitian Lagrangian only for those values of $\alpha_0 > 0$ for which

$$\sum_{n=1}^{\infty} b_n \alpha_0{}^n = 0 .$$

This constraint on α_0 appears unavoidable as long as we insist on expanding the Bethe-Salpeter kernel K in terms of the exact Δ and D functions, while at the same time asking for a finite theory of conventional unrenormalized scalar electrodynamics [as defined by the Lagrangian (2.1)]. Even at this early stage, the self-consistency of this program seems doubtful in view of the probable additional constraint on α_0 required for a finite value of Z_3.

B. Calculation of λ_2

Let us illustrate these remarks by looking at some low orders of perturbation theory. In second order we obtain

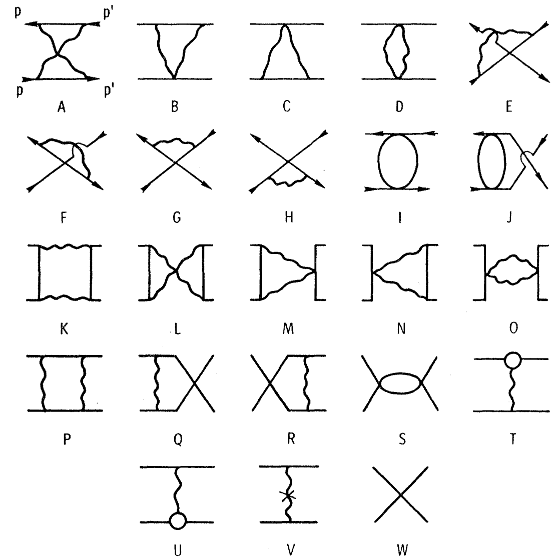

FIG. 7. All fourth-order π^+-π^- scattering graphs contributing to A^a. The blobs in graphs T and U represent the three second-order graphs contributing to Γ^a_μ. The crossed, wavy line in graph V represents the second-order gauge term in $D^0_{\mu\nu}$.

$$A^{a(2)}(p,p') = K^{a(2)}(p,p') = -2i\lambda_2 e_0{}^2 - ie_0{}^2(p+p')^\mu \left(\frac{g_{\mu\nu}}{(p-p')^2} + 2\frac{(p-p')_\mu (p-p')_\nu}{(p-p')^4}\right)(p+p')^\nu , \qquad (4.2)$$

where λ_2 is assumed finite. All fourth-order π^+-π^- scattering graphs contributing to A^a are depicted in Fig. 7. The numerical weight factors associated with graphs of type E, I, J, Q, and S are 2, 4, 2, 2, and 4, respectively. Graph W is simply $-2ie_0{}^4\lambda_4$. All graphs containing photon lines are calculated in the Yennie gauge except graph V which is calculated with

$$D^0_{\mu\nu}(k) = G_2\left(\frac{\alpha_0}{4\pi}\right)\frac{k_\mu k_\nu}{k^4} .$$

Here $-G_2 = \lambda_2{}^2 + 38$ according to Eq. (3.32). Since the second-order vertex insertions are finite in the Yennie gauge, only graphs A–S contribute to the uv-divergent part of $A^{a(4)}$. The calculation of the coefficients of the uv-divergent part of these graphs is straightforward. Table II lists the results. Explicitly, for $\Lambda^2 \gg p^2 > p'^2$,

$$A^{a(4)}(p,p') = -(10\lambda_2{}^2 + 12\lambda_2 + 12)\frac{ie_0{}^4}{16\pi^2}\ln\left(\frac{\Lambda^2}{p^2}\right) + \text{cutoff-independent terms} , \qquad (4.3)$$

so that the values of λ_2 for which

$$10\lambda_2{}^2 + 12\lambda_2 + 12 = 0 \qquad (4.4)$$

or

$$\lambda_2 = \tfrac{1}{5}(-3 \pm i\sqrt{21}) \qquad (4.5)$$

remove the uv divergences present in $A^{a(4)}$.

Suppose we wanted to calculate λ_2 in another gauge. The renormalized π^+-π^- scattering amplitude on the mass shell must be gauge-invariant, including its divergent part. Therefore, the value of λ_2 calculated in the Yennie gauge must render

TABLE II. Contribution of graphs A–S of Fig. 7 to the ultraviolet divergent part of $A^{a(4)}$.

Graph	Coefficient of $\left(\frac{ie_0{}^4}{16\pi^2}\right)\ln\left(\frac{\Lambda^2}{p^2}\right)$
A, K, L, P	-9
B, C, M, N	18
D, O	-24
E, F	$6\lambda_2$
G, H, Q, R	$-6\lambda_2$
I, S	$-4\lambda_2{}^2$
J	$-2\lambda_2{}^2$

the fourth-order renormalized π^+-π^- scattering amplitude finite in any gauge. We may verify this by calculating the divergent part of the renormalized scattering amplitude in the Landau gauge ($G=0$). In this case, account must be taken of the factors of $Z_2^{1/2}$ from the four external meson lines. These were ignored in the Yennie gauge since Z_2 was finite.

The renormalized amplitude $\tilde{A}$ is obtained from A by introducing renormalized propagators, vertex functions, and coupling constants $\tilde{\Delta}$, $\tilde{\Gamma}_\mu$, $\tilde{\Gamma}_{\mu\nu}$, and $\tilde{\lambda}$ defined by the equations $\Delta = Z_2\tilde{\Delta}$, $\Gamma_\mu = Z_1^{-1}\tilde{\Gamma}_\mu$, $\Gamma_{\mu\nu} = Z_4^{-1}\tilde{\Gamma}_{\mu\nu}$, and $\lambda = Z_2^{-2}\tilde{\lambda}$. We may neglect electric charge renormalization in this discussion. By counting internal lines and vertices we get $A = Z_2^{-2}\tilde{A}$. Thus

$$\tilde{A} = -2i\lambda Z_2^{\,2} + \tilde{M},$$

where $\tilde{M}$ consists of all π^+-π^- scattering graphs, excluding the point-interaction graph, with renormalized insertions. In the Landau gauge,

$$Z_2 = 1 + \frac{3e_0^{\,2}}{16\pi^2}\ln\left(\frac{\Lambda^2}{m^2}\right) + \text{cutoff-independent terms},$$

to second order. Only graphs D, I, J, O, and S in Fig. 7 contribute to the divergent part of $\tilde{M}$ in fourth order, since $k^\mu D^0_{\mu\nu}(k) = 0$ in the Landau gauge. We find that

$$\tilde{A}^{(4)} = -(10\lambda_2^{\,2} + 12\lambda_2 + 12)\frac{ie_0^{\,4}}{16\pi^2}\ln\left(\frac{\Lambda^2}{m^2}\right) + \text{cutoff-independent terms},$$

which leads to the same equation for λ_2 as in the Yennie gauge for a finite renormalized scattering amplitude in fourth order. We conjecture that the value of λ calculated in higher orders by our procedure will continue to be gauge-invariant.

Anticipating the results below, the value of λ that will suppress all M-part divergences will be of the form

$$\lambda = \sum_{n=1}^{\infty}(a_n \pm i b_n)\alpha_0^{\,n},$$

with each a_n and b_n real and finite. The plus (minus) sign corresponds to taking the plus (minus) sign in Eq. (4.5) for λ_2 in the calculation of higher-order terms in λ. Whichever value is selected for λ_2, we will require that

$$\sum_{n=1}^{\infty} b_n \alpha_0^{\,n} = 0$$

in order to regain a Hermitian Lagrangian and, it is hoped, a unitary S matrix. We defer until Sec. VII the discussion of this constraint on the asymptotic coupling α_0.

C. Calculation of λ_{2n}

Beginning in sixth order the equation for λ_{2n} ($n \geq 2$) becomes linear. In fact, in order $\alpha_0^{\,n+1}$, the entire coefficient of λ_{2n} is obtained from graphs E–J and graphs Q–S of Fig. 7 when λ_2 is replaced by $\sum_{m=1}^{n}\lambda_{2m}e_0^{\,2m}$. The photon lines of these graphs are in the Yennie gauge. Retaining terms of order $\alpha_0^{\,n+1}$, we get an equation for λ_{2n} given $A^{\alpha(2n+2)}$:

$$\begin{aligned}\text{uv-divergent part } A^{\alpha(2n+2)}(p,p';\lambda_2,\ldots,\lambda_{2n}) &= (-20\lambda_2\lambda_{2n} - 12\lambda_{2n})\frac{ie_0^{\,2n+2}}{16\pi^2}\ln\left(\frac{\Lambda^2}{p^2}\right)\\ &\quad + \text{uv-divergent part } A^{\alpha(2n+2)}(p,p';\lambda_2,\ldots,\lambda_{2n-2})\\ &= 0,\end{aligned}$$

or, from Eq. (4.5),

$$\mp 4i\sqrt{21}\,\lambda_{2n}\frac{ie_0^{\,2n+2}}{16\pi^2}\ln\left(\frac{\Lambda^2}{p^2}\right) + \text{uv-divergent part } A^{\alpha(2n+2)}(p,p';\lambda_2,\ldots,\lambda_{2n-2}) = 0 \tag{4.6}$$

for $\Lambda^2 \gg p^2 > p'^2$ and $n \geq 2$. The minus (plus) sign in (4.6) corresponds to taking the plus (minus) sign in (4.5). The last term in (4.6) is the sum of all uv-divergent graphs contributing to $A^{\alpha(2n+2)}$ that are independent of λ_{2n}. They are calculated in the gauge (3.10) that renders Z_2 finite to order $\alpha_0^{\,n}$ and with the value of λ ($=\sum_{m=1}^{n-1}\lambda_{2m}e_0^{\,2m}$) that makes $A^{\alpha(2n)}$ cutoff-independent.

The success of this calculation requires that the last term in Eq. (4.6) diverge like a single power of $\ln\Lambda^2$ in all orders of perturbation theory. The graphs contributing to

$$A^{\alpha(2n+2)}(p,p';\lambda_2,\ldots,\lambda_{2n-2})$$

naturally divide into two groups: those belonging to $K^{\alpha(2n+2)}$ and those with at least one π^+-π^- intermediate state. The latter class of graphs is obtained from the iterative expansion of A symbolically summarized by Eq. (4.1). In Sec. III it was noted that to find a gauge in which Z_1 is finite to order $\alpha_0^{\,n+1}$, the sum of all vertex graphs calculated from the kernels $K^{\alpha(2n+2)}$ and $K_\mu^{\alpha(2n+2)}$ in the gauge that makes Z_1 finite to order $\alpha_0{}^n$ must diverge no worse than a single power of $\ln\Lambda^2$ [see Eqs. (3.21) and (3.22)]. This in turn required that $K^{a(2n+2)}$ be cutoff-independent after one subtraction. Since

$$K^{a(2n+2)}(p, 0; \lambda_2, \ldots, \lambda_{2n-2})$$

is infrared-convergent (with the exception of its two-photon annihilation graphs), its uv divergence must therefore be of the form $\ln(\Lambda^2/p^2)$. Thus both groups of graphs contributing to

$$A^{a(2n+2)}(p, p'; \lambda_2, \ldots, \lambda_{2n-2})$$

must separately diverge as $\ln\Lambda^2$.

The proof of this is hindered by M-M overlaps and by overlaps between M parts and Compton vertices. We have studied this problem in detail in sixth order using an extension of Ward's[27] method of differentiation with respect to external momenta to deal with the overlap problem and have found that both groups of graphs contributing to $A^{a(6)}$, when separately summed, do indeed diverge like a single power of $\ln\Lambda^2$, provided $A^{a(4)}$ is uv-finite and Z_1 is finite to second order. We know this is true if the calculations are done in the gauge specified by Eqs. (3.31) and (3.32) and λ_2 satisfies condition (4.4). Hence, from Eq. (4.6), a value of λ_4 exists that renders the full amplitude $A^{a(6)}$ uv-finite. The actual value of λ_4 has not been calculated since more than 300 graphs contribute to the uv divergent part of $A^{a(6)}$.

In higher orders we expect that if

$$A^{a(2n)}(p, p'; \lambda_2, \ldots, \lambda_{2n-2})$$

is cutoff-independent for

$$\lambda = \sum_{m=1}^{n-1} \lambda_{2m} e_0{}^{2m}$$

and if Z_1 $(=Z_2=Z_4)$ is finite through order $\alpha_0{}^n$, then the renormalizability of scalar electrodynamics, when combined with the $\phi^{\dagger 2}\phi^2$ counterterm, will guarantee that all subintegrations in

$$A^{a(2n+2)}(p, p'; \lambda_2, \ldots, \lambda_{2n-2})$$

converge. On the basis of Weinberg's theorem the final integration involving all lines of the superficially logarithmically divergent quantity

$$A^{a(2n+2)}(p, p'; \lambda_2, \ldots, \lambda_{2n-2})$$

will be rendered finite after one over-all subtraction. If in addition its divergence is no worse than $\ln\Lambda^2$, then a value of λ_{2n} can be calculated from Eq. (4.6) that will make

$$A^{a(2n+2)}(p, p'; \lambda_2, \ldots, \lambda_{2n})$$

cutoff-independent.

We may give a heuristic proof[28] that the graphs belonging to

$$A^{a(2n+2)}(p, p'; \lambda_2, \ldots, \lambda_{2n-2})$$

with at least one π^+-π^- intermediate state do indeed diverge no worse than $\ln\Lambda^2$. We proceed by induction: (1) We know this to be true in fourth order. Now assume that the amplitudes

$$A^{a(4)}(p, p'; \lambda_2), \ldots, A^{a(2n)}(p, p'; \lambda_2, \ldots, \lambda_{2n-2})$$

are all uv-finite. (2) Take all graphs belonging to

$$A^{a(2n+2)}(p, p'; \lambda_2, \ldots, \lambda_{2n-2})$$

with at least one π^+-π^- intermediate state and route an external momentum q through the bottom meson line pointing from left to right as indicated in Fig. 8(a). Set p and $p'=0$. These graphs are now rendered infrared-convergent by the momentum q.[29] Thus, any uv divergence in these graphs must be of the form $\ln^m(\Lambda^2/q^2)$. (3) Divide each graph into two parts by vertically cutting a π^+-π^- intermediate state. (4) Repeat this procedure on the bisected graphs until no further division is possible. This generates an ordered sequence of scattering graphs whose two ends and middle pieces are the kernels K^a of order $2n$ or less linked together by two charged meson lines as illustrated in Fig. 8(b). (5) Differentiate the bottom meson lines pointing from left to right with respect to q. The kernels K^a may be differentiated in an arbitrary manner except that topologically identical parts must be differentiated in the same way. (6) Regroup the various differentiated pieces and identi-

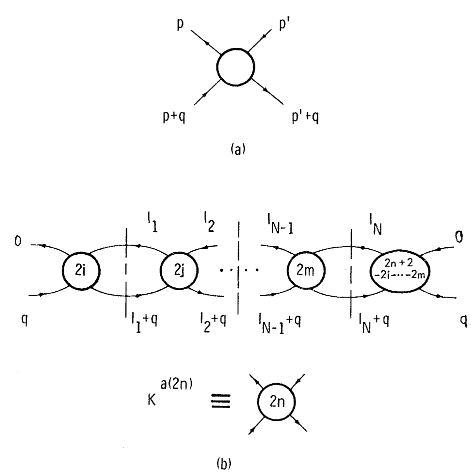

FIG. 8. (a) Typical graph in the series for $A^{a(2n+2)}$ with at least one π^+-π^- intermediate state. (b) Decomposition of the graph in Fig. 8(a) into an ordered sequence of kernels K^a, each linked together by two charged-meson lines.

fy the amplitudes

$$A^{a(4)}(\lambda_2), \ldots, A^{a(2n)}(\lambda_2, \ldots, \lambda_{2n-2}).$$

All subintegrations in the regrouped graphs converge under the assumption that

$$A^{a(4)}(\lambda_2), \ldots, A^{a(2n)}(\lambda_2, \ldots, \lambda_{2n-2})$$

are finite. Finally, the integration over all lines of the differentiated scattering graphs is superficially convergent since differentiation of an M part lowers its superficial degree of divergence from 0 to −1. According to Weinberg's theorem, the differentiated scattering graphs are cutoff-independent, and hence the original undifferentiated graphs diverge no worse than a single power of $\ln\Lambda^2$.

A similar heuristic proof can be given for

$$K^{a(2n+2)}(p, p'; \lambda_2, \ldots, \lambda_{2n-2})$$

by setting $p'=0$ and recalling that $K^{a(2n+2)}(p,0)$ is infrared-convergent.[29] Thus, the uv divergence in

$$K^{a(2n+2)}(p, 0; \lambda_2, \ldots, \lambda_{2n-2})$$

must be of the form $\ln^m(\Lambda^2/p^2)$. Now differentiate with respect to p. The only possible divergent subgraphs in the differentiated graphs are one-photon vertices, Compton vertices, and M parts. These are assumed to be rendered finite by previous choice of gauge and of $\lambda_2, \ldots, \lambda_{2n-2}$. Thus

$$\frac{\partial}{\partial p_\mu} K^{a(2n+2)}(p, 0; \lambda_2, \ldots, \lambda_{2n-2})$$

is cutoff-independent, and hence $m \leq 1$.

V. ASYMPTOTIC MESON PROPAGATOR

Having defined a procedure for making Z_2 and M parts finite we are now in a position to study the behavior of the renormalized solution for the meson propagator $\Delta(p^2)$ in the region $p^2 \gg m^2$. Unlike spinor electrodynamics, the assumption that the nonasymptotic piece h of the renormalized photon propagator $\tilde{D}_{\mu\nu}$ vanishes with power-law behavior is not itself sufficient to enable us to calculate the rate of falloff of the nonasymptotic part of Δ. Only after making the additional assumption that $\tilde{D}_{\mu\nu}$ has Type 1 asymptotic behavior have we been able to obtain quantitative information about the asymptotic behavior of Δ.

A. Asymptotic Part of Δ

We begin our study of the high-p behavior of the renormalized meson propagator by rewriting our functional equation for Δ expressed in terms of the renormalized parameters e^2 and m^2 [Eq. (3.1)]:

$$\Delta^{-1}(p^2) = p^2 + m^2 + \Sigma(p^2; \Delta, e^2\tilde{D}) - \Sigma(-m^2; \Delta, e^2\tilde{D}). \tag{3.1}$$

Each internal photon line represents the full renormalized photon propagator in the gauge in which Z_2 is finite. The coupling constant λ is assumed to be a known power series in the asymptotic coupling e_0^2 whose finite expansion coefficients have been calculated according to the method outlined in Sec. IV. Because of our choice of gauge, only one subtraction is necessary to render Δ finite.

Imagine for the moment that $h=0$ so that $\alpha\tilde{D}_{\mu\nu} = \alpha_0 D^0_{\mu\nu}$. Making this replacement in Eq. (3.1) and iterating, we generate the renormalized perturbation expansion for Δ in the absence of charge renormalization and with coupling constant e_0^2. From this expansion we may define an asymptotic expansion of Δ by keeping m fixed and dropping all terms in each order of the perturbation series that vanish as $p^2 \to \infty$.

For the case of finite h, suppose that $h(k^2/m^2, \alpha)$ behaves like $(m^2/k^2)^\kappa$ for $k^2 \gg m^2$, where $\kappa > 0$. The nonasymptotic part of $\tilde{D}_{\mu\nu}$ will always make a contribution to the asymptotic expansion of Δ, as we have defined it, through the nonasymptotic piece $\Sigma(-m^2)$. Because the difference in Eq. (3.1) is cutoff-independent, a scaling argument indicates that the contribution of graphs having at least one photon line replaced by h [Fig. 9(a)] to the asymptotic expansion of Δ is of the form

$$Ap^2(m^2/p^2)^\kappa + Bm^2, \quad p^2 \gg m^2 \tag{5.1}$$

where A and B are constants. Graphs with multiple h-dependent photon lines [Fig. 9(b)] will modify B and contribute additional p-dependent terms to (5.1) that fall off faster than $p^2(m^2/p^2)^\kappa$ for $p^2 \gg m^2$. If $\kappa \geq 1$, then the only effect of h on the

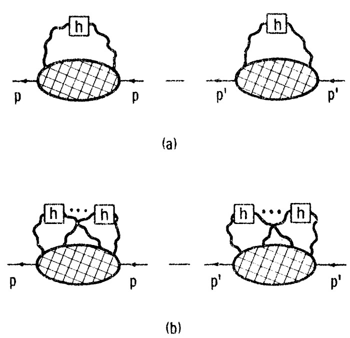

FIG. 9. (a) Contribution to the renormalized meson mass operator from one factor of h. The momentum p' has the magnitude $p'^2 = -m^2$. (b) Contribution to the renormalized meson mass operator from multiple factors of h.

asymptotic expansion of Δ is to contribute an additional constant piece proportional to m^2 in each order of perturbation theory.

Our aim is to sum *all* terms in the asymptotic expansion of Δ and define the resulting function as the "asymptotic part" of Δ. For this procedure to have any connection with reality the nonasymptotic parts of Δ that we neglected in each order of the expansion of (3.1) must not sum to an asymptotically dominant result. This assumption will be made here and is hereafter referred to as the asymptotic hypothesis.

It is unfortunate that the method we use to sum the asymptotic expansion for Δ is not powerful enough to work under the mild assumption that $\kappa > 0$. As we will indicate below, this would require finding the asymptotic solution of a coupled set of integral equations for h and Δ, which we were not able to do. Therefore, to make any progress at all, we have had to make the more restrictive assumption that $\kappa \geq 1$ (Type 1 behavior). Then, as we indicated above, all trace of h disappears from the asymptotic expansion of Δ except for a constant piece proportional to m^2 in each order of perturbation theory. In the subsequent analysis this constant piece will be removed by subtracting in Eq. (3.1) at $p=p_0$, where $p_0{}^2 \gg m^2$, instead of at the nonasymptotic point $p^2=-m^2$.

B. Equation for the Asymptotic Part of Δ

We have shown that in the limit $m\to 0$, $\Delta(p^2) \to C(e_0{}^2)/p^2$. Because C is calculated from superficially linearly divergent integrals (those defining Γ_μ^a), its value is ambiguous and depends on how the external momentum p is routed through the graphs defining Σ and Γ_μ^a. Accordingly we prefer to study instead the rescaled propagator $\tilde{\Delta}(p^2) = \Delta(p^2)/C(e_0{}^2)$. It is also convenient to define a new function $\tilde{m}(p^2)$ by the equation

$$\tilde{\Delta}^{-1}(p^2)=p^2+\tilde{m}(p^2). \tag{5.2}$$

Rewriting Eq. (3.1) in terms of the meson bare mass m_0 and replacing $\Delta(p^2)$ by $C/(p^2+\tilde{m})$ gives

$$\tilde{m}(p^2)=(C-1)p^2+Cm_0{}^2+C\Sigma(p^2;C\tilde{\Delta},e^2\tilde{D}). \tag{5.3}$$

Suppose that (5.3) is continued to spacelike values of p^2, with all integrals defining Σ taken over four-dimensional Euclidean space. If Eq. (5.3) is now subtracted at the point $p=p_0$ with $p_0 \gg m^2$, we get

$$\tilde{m}(p^2)=\tilde{m}(p_0{}^2)+(C-1)(p^2-p_0{}^2) +C\Sigma(p^2;C\tilde{\Delta},e^2\tilde{D})-C\Sigma(p_0{}^2;C\tilde{\Delta},e^2\tilde{D}). \tag{5.4}$$

At this point we make the simplifying assumptions that $\kappa \geq 1$ and that terms contributed by h to the iteration of Eq. (5.4) which vanish for fixed m^2 and for $p^2, p_0{}^2 \to\infty$ do not become asymptotically dominant when summed. With these assumptions we set $e^2\tilde{D}_{\mu\nu}=e_0{}^2D^0_{\mu\nu}$ in Eq. (5.4) and obtain an equation for $\tilde{m}(p^2)$ valid for $p^2\gg m^2$ and $p_0{}^2\gg m^2$:

$$\tilde{m}(p^2)=\tilde{m}(p_0{}^2)+(C-1)(p^2-p_0{}^2) +C\Sigma(p^2;C\tilde{\Delta},e_0{}^2D^0)-C\Sigma(p_0{}^2;C\tilde{\Delta},e_0{}^2D^0). \tag{5.5}$$

Differentiation of (5.5) with respect to $\tilde{m}(p_0{}^2)$ gives[30]

$$\frac{\partial\tilde{m}(p^2)}{\partial\tilde{m}(p_0{}^2)}=1+C^2\int\frac{d^4s}{(2\pi)^4}[K^*(p,s;C\tilde{\Delta})-K^*(p_0,s;C\tilde{\Delta})] \times\frac{1}{[s^2+\tilde{m}(s^2)]^2}\,\frac{\partial\tilde{m}(s^2)}{\partial\tilde{m}(p_0{}^2)}, \tag{5.6}$$

where

$$K^*(p,s;C\tilde{\Delta})=-(2\pi)^4\frac{\delta\Sigma(p^2;C\tilde{\Delta},e_0{}^2D^0)}{\delta(C\tilde{\Delta}(s^2))}. \tag{5.7}$$

The functional K^* is obtained from the Bethe-Salpeter kernel K defined in Sec. II by excluding all graphs with a two-photon intermediate state and replacing $e_c{}^2D_{\mu\nu}$ and Δ with $e_0{}^2D^0_{\mu\nu}$ and $C\tilde{\Delta}$ in the remaining graphs. The former graphs are not included in K^* since Σ contains no photon self-energy insertions by definition. The existence of the $m=0$ limit of $K(p,s)$ implies that the $m=0$ limit of $K^*(p,s)$, defined by replacing $\tilde{\Delta}(p'^2)$ with $1/p'^2$ at all internal meson lines, also exists in each order of $e_0{}^2$. The work in Sec. IV indicates that $K^*|_{m=0}$, with finite vertex and Compton insertions, will diverge as $\ln\Lambda^2$ in each order of $e_0{}^2$ provided λ is properly chosen. This divergence in K^* is removed by the subtraction at $p=p_0$ in Eq. (5.6). For the moment we cut off the upper limits of the integrals defining $K^*(p,s)$ at Λ, where $\Lambda^2\gg(p^2,s^2,m^2)$.

We can eliminate C completely from Eq. (5.6) by rescaling K^* just as K was rescaled in Sec. III. Thus, when K^* is expressed in terms of the full vertices Γ_μ and $\Gamma_{\mu\nu}$ the reader may easily verify that it has the scaling property

$$C^2K^*(\lambda,\Delta,\Gamma_\mu,\Gamma_{\mu\nu})=K^*(\tilde{\lambda},\tilde{\Delta},\tilde{\Gamma}_\mu,\tilde{\Gamma}_{\mu\nu})\equiv\tilde{K}^*, \tag{5.8}$$

where $\tilde{\lambda}$, $\tilde{\Gamma}_\mu$, and $\tilde{\Gamma}_{\mu\nu}$ are related to λ, Γ_μ, and $\Gamma_{\mu\nu}$ by the scaling transformations given by Eqs. (3.23), (3.25), and (3.26).

It is useful to define a new kernel $\mathfrak{K}^*$ from $\tilde{K}^*$ which remains finite as $\Lambda^2\to\infty$. We therefore write

$$\tilde{K}^*(p,s)=-i\gamma(e_0{}^2)\left[\theta(p^2-s^2)\ln\left(\frac{\Lambda^2}{p^2}\right)+\theta(s^2-p^2)\ln\left(\frac{\Lambda^2}{s^2}\right)\right]+\mathcal{K}^*(p,s)\,, \tag{5.9}$$

where we have dropped all terms that vanish in the limit $\Lambda^2\to\infty$. The kernel $\mathcal{K}^*$ depends only on $\tilde{m}$ and the external momenta p and s. The quantity $-i\gamma(e_0{}^2)$ is the coefficient of the logarithmically divergent part of all graphs defining $\tilde{K}^*$.

Using the definitions (5.8) and (5.9) and taking $p^2>p_0{}^2$, we can rewrite Eq. (5.6) as

$$\frac{\partial\tilde{m}(p^2)}{\partial\tilde{m}(p_0{}^2)}=1+\frac{\gamma(e_0{}^2)}{16\pi^2}\int_0^{p_0{}^2}ds^2\frac{s^2}{[s^2+\tilde{m}(s^2)]^2}\ln\left(\frac{p_0{}^2}{p^2}\right)\frac{\partial\tilde{m}(s^2)}{\partial\tilde{m}(p_0{}^2)}+\frac{\gamma(e_0{}^2)}{16\pi^2}\int_{p_0{}^2}^{p^2}ds^2\frac{s^2}{[s^2+\tilde{m}(s^2)]^2}\ln\left(\frac{s^2}{p^2}\right)\frac{\partial\tilde{m}(s^2)}{\partial\tilde{m}(p_0{}^2)}$$
$$+\frac{i}{16\pi^2}\int_0^\infty ds^2\frac{s^2}{[s^2+\tilde{m}(s^2)]^2}\frac{\partial\tilde{m}(s^2)}{\partial\tilde{m}(p_0{}^2)}\int\frac{d\Omega s}{2\pi^2}[\mathcal{K}^*(p,s)-\mathcal{K}^*(p_0,s)]\,. \tag{5.10}$$

We are now in a position to calculate $\tilde{m}(p^2)$ for large spacelike values of p^2.

C. Calculation of $\tilde{m}(p^2)$ for $p^2\gg m^2$

The function $\tilde{m}(p^2)$ has the dimension of mass, and we will assume that its mass scale is fixed by m. Our aim is to calculate $\tilde{m}(p^2)$ for $p^2\gg m^2$. Since $\tilde{m}(p^2)$ must vanish at $m=0$ to be consistent with the result $\Delta(p^2)\sim C(e_0{}^2)/p^2$ for $p^2\gg m^2$, our interest naturally centers on the value of $\partial\tilde{m}(p^2)/\partial\tilde{m}(p_0{}^2)$ at the point $\tilde{m}(p_0{}^2)=0$ or, equivalently, the value of $\partial\tilde{m}(p^2)/\partial\tilde{m}(p_0{}^2)$ at $m=0$. The assumption that $\tilde{m}(p^2)\to 0$ as $m\to 0$ will be verified below. We cannot immediately take the $m=0$ limit of Eq. (5.10) since the kernel of the first integral on its right-hand side would behave like $1/s^2$ at the origin and become a source of infrared divergences when the equation is iterated. For the moment we will keep $\tilde{m}$ finite in the first two integrals in Eq. (5.10).

Let the result of averaging $\mathcal{K}^*$ over spherical angles in (5.10) be summarized as

$$\frac{i}{16\pi^2}\int\frac{d\Omega_s}{2\pi^2}\mathcal{K}^*(p,s)=k(p^2,s^2,m^2)\,, \tag{5.11}$$

where k is a dimensionless function of p^2, s^2, and m^2. The existence of the $m=0$ limit of $\tilde{K}^*(p,s)$ implies that $k(p^2,s^2,m^2)$ becomes a finite, dimensionless function of the ratio p^2/s^2 in the same limit:

$$\lim_{m\to 0}k(p^2,s^2,m^2)=k^a(p^2/s^2)\,. \tag{5.12}$$

The function $k^a(p^2/s^2)$ is calculated by replacing $\tilde{\Delta}(p'^2)$, $\tilde{\Gamma}_\mu$, and $\tilde{\Gamma}_{\mu\nu}$ in the rescaled version of $K^*(p,s)$ by $1/p'^2$, $\tilde{\Gamma}^a_\mu$, and $\tilde{\Gamma}^a_{\mu\nu}$, averaging over s, and subtracting off the logarithmically divergent terms. Furthermore, it is a property of the graphs defining $\tilde{K}^*$ that

$$k^a(p^2/s^2)-k^a(p_0{}^2/s^2)\underset{s^2\to 0}{\sim}s^2\times(\text{powers of }\ln s^2)$$

for fixed p^2 and $p_0{}^2$. Therefore, the $m=0$ limit of the kernel of the last integral in Eq. (5.10) behaves at worst as a power of $\ln s^2$ as $s^2\to 0$, and we shall accordingly take the $m=0$ limit under that integral sign. Because of the subtraction at the point $p=p_0$ in the last integral in Eq. (5.10), all constant terms in $k^a(p^2/s^2)$ drop out of the calculation of $\partial\tilde{m}(p^2)/\partial\tilde{m}(p_0{}^2)$. To emphasize this we define the quantity

$$k_s(p^2/s^2)=k^a(p^2/s^2)-k^a(0)\,.$$

Using the symmetry property $k(p^2/s^2)=k(s^2/p^2)$ we can now rewrite the last integral in (5.10) as

$$\int_0^\infty\frac{ds^2}{s^2}k_s\left(\frac{p_0{}^2}{s^2}\right)\left(\frac{\partial\tilde{m}(p^2p_0{}^2/s^2)}{\partial\tilde{m}(p_0{}^2)}-\frac{\partial\tilde{m}(s^2)}{\partial\tilde{m}(p_0{}^2)}\right).$$

Substituting this result into Eq. (5.10) and differentiating the entire equation with respect to p^2 twice, we obtain

$$\frac{\partial}{\partial p^2}\left[p^2\frac{\partial}{\partial p^2}\left(\frac{\partial\tilde{m}(p^2)}{\partial\tilde{m}(p_0{}^2)}\right)\right]=-\frac{\gamma(e_0{}^2)}{16\pi^2}\frac{p^2}{[p^2+\tilde{m}(p^2)]^2}\frac{\partial\tilde{m}(p^2)}{\partial\tilde{m}(p_0{}^2)}+\int_0^\infty\frac{ds^2}{s^2}k_s\left(\frac{p_0{}^2}{s^2}\right)\frac{\partial}{\partial p^2}\left[p^2\frac{\partial}{\partial p^2}\left(\frac{\tilde{m}(p^2p_0{}^2/s^2)}{\tilde{m}(p_0{}^2)}\right)\right]. \tag{5.13}$$

We are now free to take the $m=0$ limit everywhere in (5.13) and, in particular, to set $\tilde{m}(p^2)=0$ in the first term on its right-hand side. Making the change of integration variable $t^2=p_0^2p^2/s^2$ in (5.13), we obtain in the limit $m=0$ the homogeneous equation

$$p^2\frac{\partial}{\partial p^2}\left[p^2\frac{\partial}{\partial p^2}\left(\frac{\tilde{m}(p^2)}{\tilde{m}(p_0^2)}\right)\right]=-\frac{\gamma(e_0^2)}{16\pi^2}\frac{\partial\tilde{m}(p^2)}{\partial\tilde{m}(p_0^2)}+\int_0^\infty dt^2k_s\left(\frac{t^2}{p^2}\right)\frac{\partial}{\partial t^2}\left[t^2\frac{\partial}{\partial t^2}\left(\frac{\partial\tilde{m}(t^2)}{\partial\tilde{m}(p_0^2)}\right)\right]. \tag{5.14}$$

Letting $p\to\lambda p$ and $p_0\to\lambda p_0$ in Eq. (5.14), we discover that $\partial\tilde{m}(\lambda^2p^2)/\partial\tilde{m}(\lambda^2p_0^2)$ satisfies the same equation as $\partial\tilde{m}(p^2)/\partial\tilde{m}(p_0^2)$. Thus, $\partial\tilde{m}(p^2)/\partial\tilde{m}(p_0^2)$ depends only on the ratio p^2/p_0^2. Assuming a solution of the form

$$\frac{\partial\tilde{m}(p^2)}{\partial\tilde{m}(p_0^2)}=\left(\frac{p^2}{p_0^2}\right)^\epsilon \tag{5.15}$$

and substituting it in Eq. (5.14), we obtain the following equation for ϵ in terms of $\gamma(e_0^2)$ and k_s:

$$\epsilon^2=-\frac{\gamma(e_0^2)}{16\pi^2}+\epsilon^2\int_0^\infty\frac{dt^2}{t^2}k_s\left(\frac{t^2}{p^2}\right)\left(\frac{t^2}{p^2}\right)^\epsilon, \tag{5.16}$$

which is valid for $-1<\epsilon<1$.

From Eq. (5.15) and the condition that $\tilde{m}(p^2)=0$ when $\tilde{m}(p_0^2)=0$, we get

$$\tilde{m}(p^2)=\tilde{m}(p_0^2)\left(\frac{p^2}{p_0^2}\right)^\epsilon,$$

and hence

$$\tilde{m}(p^2)=A(e_0^2)m^2\left(\frac{p^2}{m^2}\right)^\epsilon, \tag{5.17}$$

where $A(e_0^2)$ is a constant to be fixed below and $p^2\gg m^2$. We will now show that $\mathrm{Re}\,\epsilon<1$ in the weak coupling $(e_0^2/4\pi\ll1)$ limit, thereby justifying our assumption that $\tilde{m}(p^2)\to0$ as $m\to0$.

D. Calculation of ϵ

The graphs contributing to $\tilde{K}^*$ in lowest order are depicted in Fig. 10(a). Explicitly, in the limit $m=0$,

$$\tilde{K}^*(p,t)=-2i\lambda_2e_0^2-ie_0^2(p+t)^\mu\left(\frac{g_{\mu\nu}}{(p-t)^2}+2\frac{(p-t)_\mu(p-t)_\nu}{(p-t)^4}\right)(p+t)^\nu\equiv\mathcal{K}^*(p,t).$$

Thus,

$$k^a(t^2/p^2)=\frac{i}{16\pi^2}\int\frac{d\Omega_t}{2\pi^2}\mathcal{K}^*(p,t)=\frac{e_0^2}{16\pi^2}(2\lambda_2+3),$$

so that

$$k_s(t^2/p^2)=k^a(t^2/p^2)-k^a(0)=0.$$

Since $\tilde{K}^*$ is finite in lowest order, $\gamma(e_0^2)=0$, and hence $\epsilon=0$ according to (5.16).

All graphs contributing to $\tilde{K}^*$ in fourth order are depicted in Fig. 10(b). These graphs are calculated in the limit $m=0$ by setting $\Delta(p^2)=1/p^2$ at all meson lines and $G=3$ in all photon lines except in the penultimate graph, where

$$D^0_{\mu\nu}(k)=-\frac{e_0^2}{16\pi^2}(\lambda_2^2+38)\frac{k_\mu k_\nu}{k^4}.$$

We find that

$$\tilde{K}^*(p,t)=-\frac{ie_0^4}{16\pi^2}(6\lambda_2^2-3)\left[\theta(t^2-p^2)\ln\left(\frac{\Lambda^2}{t^2}\right)+\theta(p^2-t^2)\ln\left(\frac{\Lambda^2}{p^2}\right)\right]+\mathcal{K}^*(p,t), \tag{5.18}$$

where Λ is an ultraviolet cutoff and all terms that vanish in the limit $\Lambda^2\to\infty$ have been dropped. The first term in (5.18) is contributed by the first 10 graphs in Fig. 10(b). Comparing Eqs. (5.9) and (5.18) we see that

$$\gamma(e_0^2)=\frac{e_0^4}{16\pi^2}(6\lambda_2^2-3) \tag{5.19}$$

in fourth order. Substituting the value for λ_2 given by (4.5) in (5.19) we get

$$\gamma(e_0^2)=-(\tfrac{147}{25}\pm i\tfrac{36}{25}\sqrt{21})\frac{e_0^4}{16\pi^2}. \tag{5.20}$$

Let us assume that ϵ can be expanded about $e_0^2=0$ and write $\epsilon=\epsilon_2e_0^2+\epsilon_4e_0^4+\cdots$. Inserting this in (5.16) we find

$$\epsilon_2^2e_0^4=(\tfrac{147}{25}\pm i\tfrac{36}{25}\sqrt{21})\frac{e_0^4}{(4\pi)^4}+O(e_0^6). \tag{5.21}$$

Equation (5.21) follows from the fact that the integral in Eq. (5.16) remains finite as $\epsilon\to0$ while k_s, as calculated from $\mathcal{K}^*$ in (5.18), is itself of order e_0^4. Consequently, the second term in (5.16) will not enter until eighth order. Stated differently, ϵ_2 and ϵ_4 are independent of $\mathcal{K}^*$. Thus

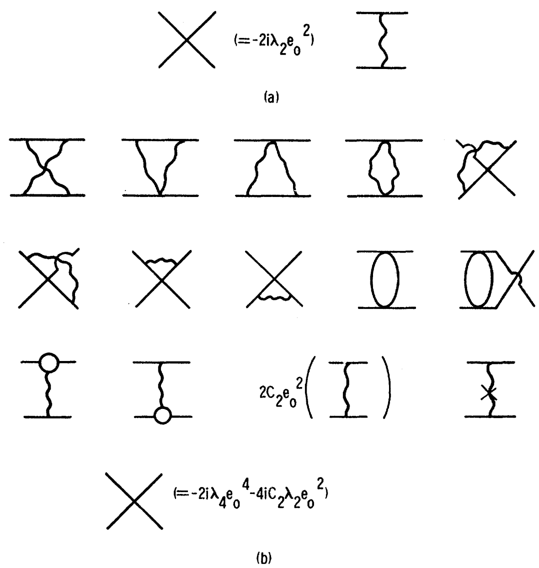

FIG. 10. (a) Lowest-order diagrams in the series for $\tilde{K}^*$. (b) All fourth-order diagrams in the series for $\tilde{K}^*$. The blobs in the first two graphs on the third row represent the second-order contribution to Γ_μ^a. The crossed, wavy line in the penultimate graph represents the second-order gauge term in $D_{\mu\nu}^0$.

$$\epsilon = \pm \frac{e_0^2}{80\pi^2}\left[\left(\frac{5\sqrt{1953}+147}{2}\right)^{1/2} \pm i\left(\frac{5\sqrt{1953}-147}{2}\right)^{1/2}\right] + O(e_0^4), \tag{5.22}$$

where the minus (plus) sign inside the bracket is taken when ϵ is calculated with λ_{2+} (λ_{2-}). This demonstrates the consistency of our assumption that $\tilde{m}(p^2)\to 0$ as $m\to 0$ in the approximation where the first two terms in the power-series expansion of $\tilde{K}^*$ are retained.

E. Final Form of $\tilde{m}(p^2)$

We have found two acceptable solutions of $\tilde{m}(p^2)$ when ϵ is calculated to order e_0^2:

$$\tilde{m}_\pm(p^2) = m^2 A_\pm(e_0^2)(p^2/m^2)^{\epsilon_\pm},$$

where ϵ_+ (ϵ_-) is the value of ϵ obtained by taking the over-all plus (minus) sign in Eq. (5.22). As a boundary condition on $\tilde{m}(p^2)$ we require that it join smoothly with the asymptotic part of Δ calculated from the iterative solution of (3.1) in which $e^2\tilde{D}_{\mu\nu}$ is replaced with $e_0^2 D_{\mu\nu}^0$. Recall that this is defined by keeping m fixed and dropping all terms in each order of the perturbation series that vanish as $p^2\to\infty$. We will now determine what linear combination of $\tilde{m}_+$ and $\tilde{m}_-$ is required to achieve this.

In lowest order the graphs contributing to Σ are the first three graphs depicted in Fig. 1. The first two graphs are canceled by the subtraction at $p^2 = -m^2$ in (3.1), giving

$$\Delta^{-1}(p^2) = p^2 + m^2 + ie_0^2\int \frac{d^4s}{(2\pi)^4} D_{\mu\nu}^0(s)(2p+s)^\mu \Delta(p+s)(2p+s)^\nu - (p\to p',\ p'^2 = -m^2), \tag{5.23}$$

where $D_{\mu\nu}^0$ is in the Yennie gauge. The first iteration of (5.23) is given by (3.15),

$$\Delta^{-1}(p^2) = \left(1 - \frac{9e_0^2}{32\pi^2}\right)(p^2+m^2),$$

so that

$$\tilde{m}(p^2) = m^2 \tag{5.24}$$

through order e_0^2. This requires that $\tilde{m}(p^2)$ be a linear combination of $\tilde{m}_+$ and $\tilde{m}_-$ with equal coefficients:

$$\tilde{m}(p^2) = m^2 A(e_0^2)\left[\left(\frac{p^2}{m^2}\right)^{\epsilon_+} + \left(\frac{p^2}{m^2}\right)^{\epsilon_-}\right]. \tag{5.25}$$

The constant $A(e_0^2)$ may be calculated in renormalized perturbation theory by iterating the rescaled version of Eq. (3.1). Setting $\Delta = C/[p^2 + \tilde{m}(p^2)]$ in (3.1) we expect, in our gauge,

$$\tilde{m}(p^2) = m^2\left[a_0(e_0^2) + a_1(e_0^2)\ln\left(\frac{p^2}{m^2}\right) + a_2(e_0^2)\ln^2\left(\frac{p^2}{m^2}\right) + \cdots\right], \tag{5.26}$$

for m fixed and $p^2\to\infty$. Expanding (5.25),

$$\tilde{m}(p^2) = m^2 A(e_0^2)\left[2 + (\epsilon_+ + \epsilon_-)\ln\left(\frac{p^2}{m^2}\right) + \left(\frac{\epsilon_+^2 + \epsilon_-^2}{2!}\right)\ln^2\left(\frac{p^2}{m^2}\right) + \cdots\right],$$

and comparing with (5.26) we get $A(e_0^2) = \frac{1}{2}a_0(e_0^2)$. It should be emphasized that although $h(k^2/m^2, \alpha)$ makes no contribution to $a_1, a_2, \ldots$ in Eq. (5.26) because of our assumption that $\kappa \geq 1$ (Type 1 behavior) it does contribute to $a_0(e_0^2)$ through the nonasymptotic piece $\Sigma(-m^2; \Delta, e^2\tilde{D})$ in Eq. (3.1).

Finally we cannot conclude that all odd powers of $\ln(p^2/m^2)$ are absent from the power-series expansion of $\tilde{m}(p^2)$ since the integral on the right-hand side of (5.16) could introduce an asymmetry in ϵ_+ and ϵ_- beginning with the eighth-order term in the expansion of ϵ. A consistent calculation of this term requires knowledge of $\tilde{K}^*$ through tenth order.

VI. MESON SELF-MASS

Let us now define the conditions under which the assumption of a finite meson bare mass m_0 is valid. We begin by rewriting our original unsubtracted functional equation for Δ, Eq. (2.10), in terms of the exact renormalized photon propagator $\tilde{D}_{\mu\nu}$ by making the substitution $e_c^2 D_{\mu\nu} \to e^2 \tilde{D}_{\mu\nu}$ in $\Sigma(p^2; \Delta, e_c^2 D)$:

$$\Delta^{-1}(p^2) = p^2 + m_0^2 + \Sigma(p^2; \Delta, e^2 \tilde{D}) . \qquad (6.1)$$

The gauge G is fixed by the condition that Z_2 be finite. We assume that λ is a known power series in e_0^2 whose expansion coefficients are fixed to suppress all M-part divergences according to the procedure outlined in Sec. IV. Finally, we continue (6.1) to spacelike values of p^2 and convert all integrals defining Σ to integrals over four-dimensional Euclidean space. We recall here our convention of always routing p through the Δ functions in self-energy and vertex graphs.

Our plan is to study Eq. (6.1) under the following conditions: (a) The nonasymptotic part of $\tilde{D}_{\mu\nu}(k)$, $h(k^2/m^2, \alpha)$, vanishes asymptotically as a power of k^2/m^2, and hence

$$e^2 \tilde{D}_{\mu\nu} \underset{m\to 0}{\sim} e_0^2 D^0_{\mu\nu} .$$

(b) The full meson propagator Δ, calculated in terms of the physical mass m, has the asymptotic behavior

$$\Delta(p^2) \underset{p^2 \gg m^2}{\sim} \frac{C(e_0^2)}{p^2}\left[1 - \frac{a_0(e_0^2) m^2}{2p^2}\left(\frac{m^2}{p^2}\right)^{\epsilon}\right], \qquad (6.2)$$

where C can be calculated as a power series in e_0^2 following the procedure given in Sec. III. Starting with the above assumption regarding the asymptotic behavior of h, we were able to prove in Sec. III that the $m=0$ limit of Δ exists in each order of the expansion of K. Here we *assume* $\mathrm{Re}\,\epsilon > -1$ when calculated to all orders in e_0^2. If $h(k^2/m^2, \alpha)$ vanishes as $(m^2/k^2)^{\kappa}$ with $\kappa \geq 1$ (Type 1 behavior) and the asymptotic hypothesis is valid, then ϵ and $a_0(e_0^2)$ can be calculated as in Sec. V. (c) The bare mass m_0 is finite.

Before we can test the consistency of (6.1) with these conditions, the integrals defining Σ must be made finite and definite by introducing a uv convergence factor. Aside from the requirements that the convergence factor preserve the gauge-covariance and original singularity structure of the integrals (e.g., no mass singularities should be introduced), its specific form is arbitrary. Here we will regularize Σ by replacing all integrals $\int d^4 s$ over internal momenta with $\int d^4 s[\Lambda^2/(s^2+\Lambda^2)]^2$. All integrals now converge and Σ itself can be expressed in terms of a finite function of Λ^2/m^2 and p^2/m^2 times an over-all scaling factor.

We isolate the uv divergences associated with the meson self-mass by studying Eq. (6.1) in the limit $\Lambda^2 \gg p^2 \gg m^2$. Expanding Σ about $m=0$ and retaining terms of order m^2, we get

$$\begin{aligned} C^{-1}\left[p^2 + \tfrac{1}{2} a_0 m^2 \left(\frac{m^2}{p^2}\right)^{\epsilon}\right] &= p^2 + m_0^2 + \Sigma(\Lambda^2/m^2, p^2/m^2, \Lambda^2)\big|_{m=0} + m^2 \frac{\partial}{\partial m^2}\Sigma(\Lambda^2/m^2, p^2/m^2, \Lambda^2)\bigg|_{m=0} \\ &= p^2 + m_0^2 + \Sigma^a(p^2, \Lambda^2) - m^2 \int \frac{d^4 s}{(2\pi)^4}\left(\frac{\Lambda^2}{s^2+\Lambda^2}\right)^2 K^*(p, s)\big|_{m=0} \frac{\partial \Delta(s^2)}{\partial m^2}\bigg|_{m\sim 0} \\ &\quad - i m^2 \int \frac{d^4 s}{(2\pi)^4}\left(\frac{\Lambda^2}{s^2+\Lambda^2}\right)^2 C_{\mu\nu}(p, s)\big|_{m=0} \frac{1}{s^2}\left(g^{\mu\nu} - \frac{s^\mu s^\nu}{s^2}\right) \frac{\partial[e^2 h(s^2/m^2, \alpha)]}{\partial m^2}\bigg|_{m\sim 0}, \end{aligned} \qquad (6.3)$$

where

$$C_{\mu\nu}(p, s) = i(2\pi)^4 \frac{\delta\Sigma(p^2)}{\delta(e^2 \tilde{D}^{\mu\nu}(s))} . \qquad (6.4)$$

The quantity $\Sigma^a(p^2, \Lambda^2)$ is the $m=0$ limit of the regularized meson mass operator. It is calculated by making the replacements $\Delta(p'^2) \to C/p'^2$ and $e^2\tilde{D}_{\mu\nu} \to e_0^2 D^0_{\mu\nu}$ in all meson and photon lines of the functional Σ. The quantity $K^*(p, s)|_{m=0}$ is obtained from Eq. (5.7) by replacing $\tilde{\Delta}(p'^2)$ with $1/p'^2$.

The functional $C_{\mu\nu}$ consists of both proper and improper uncrossed forward scattering Compton graphs. Looked at from the t channel, $C_{\mu\nu}$ is the sum of all π^+-π^- uncrossed two-photon annihilation graphs irreducible with respect to π^+-π^- vertical cuts. Some typical graphs contributing to $C_{\mu\nu}$ are depicted in Fig. 11. All internal meson and photon lines in $C_{\mu\nu}$ stand for the exact Δ and $\tilde{D}$ functions.

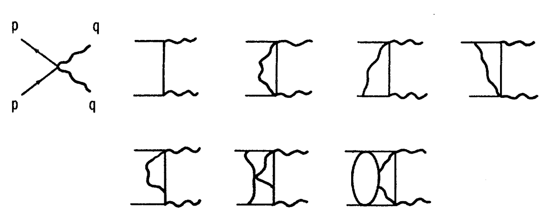

FIG. 11. Some typical graphs in the series for $C_{\mu\nu}$.

We assume that we can study the infrared and ultraviolet behavior of $C_{\mu\nu}$ in each order by making the substitutions $\Delta(p'^2)\to C/p'^2$ and $e^2\tilde{D}_{\mu\nu}\to e_0{}^2 D^0_{\mu\nu}$, with G calculated according to the scheme outlined in Sec. III. The sum of these graphs defines a new quantity $C^a_{\mu\nu}$ which we define to be the quantity $C_{\mu\nu}|_{m=0}$ appearing in Eq. (6.3). Substituting the values of G and λ_2 given by (3.31), (3.32), and (4.5) we find that $C^a_{\mu\nu}$ is uv-finite through fourth order. It is also infrared-convergent. In higher orders, we expect $C^{a(2n)}_{\mu\nu}(p,s)$ to remain finite provided it is calculated in the gauge that renders Z_4 finite through order $2n$ and with a value of λ that renders M parts uv-finite through order $2n-2$. The infrared convergence of $C^{a(2n)}_{\mu\nu}(p,s)|_{m=0}$ is implied by the existence of the $m=0$ limits of $\Gamma^{(2n)}_\mu$ and $\Gamma^{(2n)}_{\mu\nu}$.

There is one point that requires further consideration. It might be thought that all proper Compton graphs of a given order must be summed to obtain a uv-convergent result in the gauge in which Z_4 is finite to the same order. It turns out that, provided one integrates symmetrically, the sum of all Compton graphs of order $2n$ with a π^+-π^- intermediate state (looking down the t channel) converges separately from the remaining Compton graphs in the gauge (3.10) in which Z_4 is finite through order $2n$. Therefore, $C^a_{\mu\nu}(p,s)$ is well defined and remains finite as $\Lambda\to\infty$.[31]

We now return to Eq. (6.3) and the term $\Sigma^a(p^2,\Lambda^2)$. The only possible form this term can have in the limit $\Lambda^2 \gg p^2$ that is consistent with the finiteness of Z_2, the existence of the $m=0$ limit of Δ, and the definition of Σ is

$$\Sigma^a(p^2,\Delta^2)=(C^{-1}-1)p^2+g(\alpha_0)\Lambda^2\,, \tag{6.5}$$

where $g(\alpha_0)$ is a power series in α_0 $(=e_0{}^2/4\pi)$ to be defined below. There are no terms in (6.5) of the form

$$\Lambda^2 \sum_{\substack{m=2\\0<n<m}} \alpha_{mn}\, e_0{}^{2m}\ln^n(\Lambda^2/p^2) \tag{6.6}$$

or

$$\lim_{m\to 0}\Lambda^2 \sum_{\substack{m=2\\0<n<m}} \beta_{mn}\, e_0{}^{2m}\ln^n(p^2/m^2)\,, \tag{6.7}$$

since according to the local Ward identity $\Gamma_\mu(p,p) = \partial\Delta^{-1}(p^2)/\partial p^\mu$, Γ^a_μ would contain terms that diverge like Λ^2. This is inconsistent with the definition of Σ and Γ_μ (no meson self-energy insertions) and the asymptotic behavior of Δ and $\tilde{D}_{\mu\nu}$ for $m\to 0$. Nor are there terms of the form

$$\lim_{m\to 0}\Lambda^2 \sum_{\substack{m=2\\0<n<m}} \gamma_{mn}\, e_0{}^{2m}\ln^n(\Lambda^2/m^2) \tag{6.8}$$

since $\partial\Sigma/\partial m^2$ would likewise contain terms that diverge as Λ^2. This possibility is also excluded by the absence of meson self-energy insertions in K^* and $C_{\mu\nu}$ and by the asymptotic behavior of Δ and $\tilde{D}_{\mu\nu}$.

The function $g(\alpha_0)$ is the sum of the coefficients of Λ^2 calculated from the regularized expression for Σ in which the substitutions $e^2\tilde{D}_{\mu\nu}\to e_0{}^2D^0_{\mu\nu}$ and $\Delta(p'^2)\to C/p'^2$ have been made and the limit $\Lambda^2 \gg p^2$ taken. It is shown in the Appendix that if m_0 is assumed to be gauge-invariant, then $g(\alpha_0)$ is likewise gauge-invariant. The gauge invariance of m_0 would follow, for example, if it is imagined to be the mass of a degenerate isotopic multiplet of spin-0 mesons calculated in the absence of the Maxwell field.

Taking the result (6.5) and substituting it in (6.3), we get an equation for the electromagnetic self-mass $\delta m^2 = m^2 - m_0{}^2$:

$$\delta m^2 = m^2 + g(\alpha_0)\Lambda^2 + \text{less singular} \tag{6.9}$$

for $\Lambda^2 \gg p^2 \gg m^2$. In view of our assumptions (a) and (b) regarding the asymptotic behavior of the mass-dependent parts of Δ and $\tilde{D}_{\mu\nu}$, the last two integrals on the right-hand side of (6.3) will be less singular than Λ^2 for $\Lambda^2 \gg p^2$ and $\Lambda^2 \gg m^2$. Hence, we obtain the result that *no finite solution for the meson self-mass, calculated under assumptions* (a)–(c), *exists unless the value of the asymptotic coupling* α_0 *satisfies the eigenvalue condition*

$$g(\alpha_0)=0\,. \tag{6.10}$$

Stated differently, the quadratic divergences present in perturbation-theory estimates of δm^2 appear to be intrinsic to scalar electrodynamics when combined with a $\phi^{+2}\phi^2$ counterterm unless (6.10) is satisfied. It should be emphasized that the generality of (6.9) and (6.10) is limited by our expansion of Σ in terms of the exact Δ and $\tilde{D}$ functions.

The expression for $g(\alpha_0)$ calculated from the second- and fourth-order graphs contributing to Σ (the first nine graphs in Fig. 1) is found to be

$$g(\alpha_0)=(3-2\lambda_2)\left(\frac{\alpha_0}{4\pi}\right) + \left[2\zeta(2)-\tfrac{28}{5}-\tfrac{33}{5}\lambda_2-32\pi^2\lambda_4\right]\left(\frac{\alpha_0}{4\pi}\right)^2, \tag{6.11}$$

where $\zeta(2)$ is the Riemann zeta function and λ_2 is given by Eq. (4.5). The value of λ_4 can be calculated from Eq. (4.6). Unfortunately, $g(\alpha_0)$ is sensitive to how the regulator mass Λ^2 is introduced in Σ. For example, if instead of a form-factor-type cutoff the upper limits of all Euclidean space integrals over internal momenta in Σ are arbitrarily cut off at Λ, then $g(\alpha_0)$ becomes

$$g(\alpha_0) = (3 - 2\lambda_2)\left(\frac{\alpha_0}{4\pi}\right) - \left(\tfrac{47}{10} + \tfrac{21}{5}\lambda_2 + 32\pi^2\lambda_4\right)\left(\frac{\alpha_0}{4\pi}\right)^2$$

for $\Lambda^2 \gg p^2 \gg m^2$. This circumstance is unsatisfactory since the eigenvalue condition that fixes the value of the asymptotic coupling α_0 for a finite theory of scalar electrodynamics ought to be well defined.

We recall here that the quantities λ_n which enter the expression for $g(\alpha_0)$ are generally complex. Hence, the function $g(\alpha_0)$ is real-valued only at the (conjectured) physical value of α_0 for which $\mathrm{Im}\lambda = 0$.

Even if the equation for $g(\alpha_0)$ can be made definite and a physical value of α_0 $[0 < \alpha < \alpha_0]$ is found to satisfy (6.10), additional, less-singular uv divergences will remain in δm^2. These are partly introduced by K^* which, according to Eqs. (5.8) and (5.9), diverges as a single power of $\ln\Lambda^2$ in each order of α_0:

$$K^*(p, s) \underset{\Lambda\to\infty}{\sim} -i\,\frac{\gamma(\alpha_0)}{C^2}\ln\left(\frac{\Lambda^2}{s^2}\right) + \text{finite}$$

for fixed s^2, p^2 $(<s^2)$, and m. Still more uv divergences in δm^2 will enter from the second integral in Eq. (6.3) if $h(k^2/m^2, \alpha)$ vanishes as $(m^2/k^2)^\kappa$, where $\kappa \leq 1$.

In this section we have merely shown what is possible. Any further comment must await more detailed calculations.

VII. CONCLUSION

We have examined the question of whether a completely finite, closed theory of scalar electrodynamics combined with a $\phi^{\dagger 2}\phi^2$ counterterm is internally consistent. We arrived at the conclusion that it is probably not.

Specifically, it was shown that if the value of the charge renormalization constant Z_3 is assumed finite, then the renormalization constants Z_1 $(=Z_2 = Z_4)$ are rendered finite order by order in a unique gauge. It was then shown that the boson-boson coupling λ is uniquely determined by the requirement that it have a finite power-series expansion in α_0. The expansion coefficients are in general complex. Consequently, the theory can have a Hermitian Lagrangian only if the asymptotic coupling is a finite number independent of α fixed by the condition $\mathrm{Im}\lambda(\alpha_0) = 0$. Our study of the meson self-mass δm^2 led to the discovery that the quadratic divergence, when summed to all orders, is of the form $g(\alpha_0)\Lambda^2$, where $g(\alpha_0)$ can be expressed as a power series in α_0 that is finite term by term. Therefore, the assumption of a finite meson bare mass m_0 is consistent only if α_0 is further restricted by the condition $g(\alpha_0) = 0$. Finally, on the basis of the renormalization group, we conjectured that Z_3 is finite only if α_0 is a zero of the Gell-Mann–Low function $\psi(x)$ calculated in scalar electrodynamics. It appears, then, that all of the divergences in scalar electrodynamics cannot be eliminated without overdetermining α_0.

The generality of our reasoning is limited by the perturbation treatment of the meson mass operator and by the assumption that the nonasymptotic part of the renormalized photon propagator $\tilde{D}_{\mu\nu}$ vanishes with power-law behavior. The ambiguity in the definition of $g(\alpha_0)$ is unsatisfactory and remains to be resolved.

Assuming that $\tilde{D}_{\mu\nu}$ has Type 1 asymptotic behavior (defined in Sec. I), we were able to show that the asymptotic meson propagator has the rather simple form given by Eqs. (1.9)–(1.10).

The basic problem of obtaining a finite value for Z_3 still remains. The speculative remarks in Sec. I as to how this might be achieved have yet to be put on a sound footing in scalar electrodynamics.

Note added in proof. The functional differentiation (2.4) of self-energy tadpole graphs in the series for Δ^{-1} gives apparent C-violating contributions to Γ_μ. The nonvanishing of tadpole graphs with an external photon line is related to the ambiguities of perturbation-theory integrals in scalar electrodynamics. Here, we have adopted the convention of subtracting off these apparent C-violating graphs from Γ_μ. Gauge invariance is maintained by making corresponding subtractions in the equations for Δ^{-1} and $\Gamma_{\mu\nu}$.

ACKNOWLEDGMENTS

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APPENDIX: GAUGE INVARIANCE OF $g(\alpha_0)$

We show here that the coefficient of the quadratic divergence in the meson self-mass defined by Eq. (6.9) is gauge-invariant. Suppose we recalculate $g(\alpha_0)$ in a new gauge labeled by G'. In order to do this we need to know the high-p behavior of the unrenormalized meson propagator $\Delta(p^2)$ in the new gauge G'. This is determined by the small-x behavior of the coordinate-space propagator $\Delta(x; G')$. Since $\Delta(p^2; G)$ is cutoff-independent in the gauge G in which Z_2 is finite and exists at $m=0$, $\Delta(p^2; G) \sim C/p^2$ for $p^2 \gg m^2$. The constant C can be calculated as a power series in α_0 by the procedure described in Sec. III. Hence,

$$\Delta(x;G) \underset{x^2\to 0}{\sim} \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \Delta(p^2;G)\Big|_{p^2 \gg m^2} \underset{x^2\to 0}{\sim} \frac{iC}{4\pi^2}\frac{1}{x^2}. \tag{A1}$$

The small-distance behavior of $\Delta(x;G')$ is related to that of $\Delta(x;G)$ by Eq. (3.3):

$$\Delta(x;G') = \exp\{ie_0^2(G'-G)[M(x)-M(0)]\}\Delta(x;G), \tag{A2}$$

where

$$M(x)-M(0) = -\int \frac{d^4k}{(2\pi)^4}\left(\frac{\Lambda^2}{\Lambda^2+k^2}\right)^2 \frac{e^{ik\cdot x}-1}{(k^2-i\epsilon)^2}. \tag{A3}$$

For consistency, we have introduced the same cutoff in M as we used to regularize the integrals defining Σ in Sec. VI. From Eq. (A3) we find that

$$[M(x)-M(0)] \underset{x^2\to 0;\ \Lambda^2\to\infty}{\sim} \frac{i}{16\pi^2}[\ln(\Lambda^2x^2)+2\gamma-\ln 4-1], \tag{A4}$$

where γ (=0.5772...) is Euler's constant. Substituting (A1) in (A2) and using (A4) we learn that

$$\Delta(x;G') \underset{x^2\to 0;\ \Lambda^2\to\infty}{\sim} \frac{iC}{4\pi^2} e^{\beta\epsilon}\frac{(x^2\Lambda^2)^\epsilon}{x^2}, \tag{A5}$$

with $\beta = 2\gamma - \ln 4 - 1$ and $\epsilon = -(\alpha_0/4\pi)(G'-G)$. Thus, for $\Lambda^2 \gg p^2 \gg m^2$, we obtain

$$\Delta(p^2;G') \sim \int d^4x e^{-ip\cdot x}\Delta(x;G')\Big|_{\Lambda^2 \gg x^2} \sim Ce^{\beta\epsilon}\left(\frac{4^\epsilon\,\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)}\right)\frac{1}{p^2}\left(\frac{\Lambda^2}{p^2}\right)^\epsilon, \tag{A6}$$

which is valid for $-1<\epsilon<\frac{1}{4}$.

Suppose that the full finite-mass expression for $\Delta(p^2;G')$ is now substituted in Eq. (6.1) with the gauge-dependent part of $\tilde{D}_{\mu\nu}$ fixed by the new gauge constant G'. We assume that the integrals defining $\Sigma(p^2;\Delta, e^2\tilde{D})$ are regularized with the same cutoff used to define Σ in the previous gauge G. Then, in the limit $\Lambda^2 \gg p^2 \gg m^2$ we expect on grounds of consistency that the sum of the graphs defining Σ gives the result

$$\Sigma(p^2;\Delta(G'), e^2\tilde{D}(G')) \sim p^2\left[C^{-1}e^{-\beta\epsilon}\frac{\Gamma(1-\epsilon)}{4^\epsilon\,\Gamma(1+\epsilon)}\left(\frac{\Lambda^2}{p^2}\right)^{-\epsilon} - 1\right] + g(\alpha_0, G')\Lambda^2 + \text{less singular}. \tag{A7}$$

The remaining terms in (A7) are contributed by the mass-dependent terms in Δ and $\tilde{D}_{\mu\nu}$. Their contribution to the uv divergences in the meson self-mass is expected to be less singular than Λ^2. Again, because of the absence of meson self-energy insertions in Σ, the quadratic mass divergence has the simple form $g(\alpha_0, G')\Lambda^2$. Then, proceeding as in Sec. VI, the meson self-mass in the gauge G' is

$$\delta m^2(G') = m^2 + g(\alpha_0, G')\Lambda^2 + \text{less singular}. \tag{A8}$$

If $m_0(G) = m_0(G')$, where $m^2 = m_0^2 + \delta m^2$, we obtain from (A8) and (6.9) the result

$$g(\alpha_0, G) = g(\alpha_0, G')$$

in the limit $\Lambda^2 \to \infty$.

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†U. S. National Research Council Postdoctoral Resident Research Associate.

[1]K. Johnson, M. Baker, and R. Willey, Phys. Rev. 136, B1111 (1964); K. Johnson, R. Willey, and M. Baker, Zh. Eksp. Teor. Fiz. 52, 318 (1967) [Sov. Phys. JETP 25, 205 (1967)]; Phys. Rev. 163, 1699 (1967); M. Baker and K. Johnson, *ibid.* 183, 1292 (1969); Phys. Rev. D 3, 2516 (1971); 3, 2541 (1971).

[2]A recent result of S. Adler [Phys. Rev. D 5, 3021 (1972)] indicates that Z_3^{-1} in spinor electrodynamics cannot be finite unless $F^{[1]}(x)$ has an infinite-order positive zero. The reader is referred to Adler's paper for a discussion of the consequences of this essential singularity on the work of Johnson, Baker, and Willey.

[3]A. Salam, Phys. Rev. 86, 731 (1952).

[4]P. T. Matthews and A. Salam, Phys. Rev. 94, 185 (1954).

[5]E. Poggio [Ph.D. thesis, Massachusetts Institute of Technology, 1971 (unpublished)] has emphasized the special care that must be exercised in the order-by-order verification of the renormalizability of the meson propagator Δ in ϕ^4 theory. Thus, for example, defining

$$m(p^2) = -p^4\frac{\partial}{\partial p^2}\left(\frac{1}{p^2\Delta(p^2)}\right),$$

Poggio shows that the formally finite quantity $\partial m(p^2)/\partial p^2 - \partial m(p^2)/\partial p^2|_{p^2=0}$, when calculated without a cutoff and for spacelike values of p^2, diverges beginning in fourth order. Consistency with the renormalization program is restored when $\partial m(p^2)/\partial p^2$ is recalculated

with an ultraviolet cutoff Λ that is held fixed until all integrals defining $\partial m(p^2,\Lambda^2)/\partial p^2 - \partial m(p^2,\Lambda^2)/\partial p^2|_{p^2=0}$ are evaluated. The point we wish to make is that the ambiguities of the perturbation-theory integrals in scalar electrodynamics with a $\phi^{\dagger 2}\phi^2$ counterterm may invalidate formal manipulations and lead to results that are at variance with its renormalizability.

[6]The determination of the shift in the energy levels of pionic atoms due to radiative corrections to the Klein-Gordon equation faces formidable experimental and theoretical difficulties. For example, in the $2p$-$1s$ (10.7 keV) transition in pionic helium, the π-nucleon interaction, vacuum polarization from electron-positron pairs, and nuclear-size effects shift the $1s$ level upward by about 60 eV, whereas the upward displacement from the Lamb shift is only of the order of 0.1 eV. In addition, the $1s$ level is broadened due to pion absorption by the helium nucleus. Present experimental resolution for this transition is only of the order of 60 eV, while the strong-interaction shift (≈ 100 eV) is known only to about 20%. The situation is not much improved for non-s-state transitions. For a review of pionic atoms see G. Backenstoss, Ann. Rev. Nucl. Sci. 20, 467 (1970).

[7]M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

[8]S. Adler, Phys. Rev. D 5, 3021 (1972).

[9]K. Johnson, R. Willey, and M. Baker, Phys. Rev. 163, 1699 (1967).

[10]M. Baker and K. Johnson, Phys. Rev. 183, 1292 (1969); Phys. Rev. D 3, 2541 (1971).

[11]Specifically, Adler's fundamental identity, Eq. (77) of Ref. 8, fails in the presence of the boson-boson counterterm. The reason is that there are graphs in Adler's $\pi_c^{[1]}$ which, when calculated in scalar electrodynamics, cannot be obtained by linking $2n-2$ external vertices of $2n$-point single-loop current correlation functions ($n \geq 2$) with $n-1$ free photon propagators and integrating over the four-momenta carried by these propagators. There is some ambiguity in defining a "single-loop" vacuum polarization graph in scalar electrodynamics with a $\phi^{\dagger 2}\phi^2$ counterterm. Here, a single-loop graph is defined as one free of internal charged boson loops joined to the remainder of the graph *solely* by photon lines.

[12]This statement is qualified by our definition of Adler's "loopwise" summation in scalar electrodynamics. If this is defined to be the sequential summation of graphs in π_c (defined in Ref. 8) containing 0, 1, 2, etc. internal charged boson loops joined to the remainder of the graph solely by photon lines, then M parts of the same order are mixed in the various loopwise sums. As a result, these partial sums contain uncompensated M-part divergences.

[13]K. Johnson, M. Baker, and R. Willey, Phys. Rev. Letters 11, 518 (1963).

[14]D. Flamm and P. G. O. Freund, Nuovo Cimento 32, 486 (1964).

[15]J. Schwinger, Proc. Natl. Acad. Sci. U. S. 37, 452 (1951); Phys. Rev. 91, 713 (1953).

[16]For brevity the term "π^+-π^-" will be used interchangeably with "meson-antimeson."

[17]The general form of Z_2 in the absence of photon self-energy parts may be derived in precise analogy with earlier work on Z_2 in a model of spinor electrodynamics with $Z_3=1$. See Gell-Mann and Low, Ref. 7.

[18]A stronger cutoff will be needed in Sec. VI when we deal with the meson self-mass.

[19]We assume that these graphs are made gauge-invariant by including the contribution from the line-integral definition of the meson current [see Ich-Joh Kim and C. R. Hagen, Phys. Rev. D 2, 1511 (1970)]. This ensures that the photon-photon scattering amplitude

$$A_{\mu_1\cdots\mu_4}(k_1,k_2,k_3,k_4)$$

will have the property

$$A_{\mu_1\cdots\mu_4}(0,k_2,k_3,k_4)=0$$

required by gauge invariance for *fixed* loop momenta. Thus, our convergence factor is not altered by shifts of loop momenta, and hence does not interfere with gauge invariance in this instance.

[20]Equation (3.3) may be derived directly from the functional equation for Δ in the absence of charge renormalization [obtained from (2.2) and (2.3) by replacing $D_{\mu\nu}$ and e_c with $D^0_{\mu\nu}$ and e_0] with the help of the equations

$$\exp\left\{-e_0\partial^{(x)}_\mu\int d^4x' M(x-x')\frac{\delta}{\delta\langle A_\mu(x')\rangle}\right\}\Delta(x,x'|A_\nu(\xi)) = \Delta(x,x'|A_\nu(\xi)-e_0\partial^{(x)}_\nu M(x-\xi))$$

and

$$\Delta(x,x'|A_\mu(\xi)+e_0\partial^{(\xi)}_\mu M(x-\xi)) = \exp\{ie_0^2[M(0)-M(x-x')]\}\,\Delta(x,x'|A_\mu(\xi)).$$

[21]The degree of divergence of a graph is obtained by adding -2 for each internal meson and photon line, $+1$ for each single-photon vertex, and $+4$ for each internal integration. Compton vertices contribute nothing.

[22]S. Weinberg, Phys. Rev. 118, 838 (1960).

[23]Because of the complicated overlapping divergences present in K^a, these remarks are somewhat oversimplified. See Sec. IV for more details.

[24]Some evidence in support of this conjecture was obtained by A. Salam and R. Delbourgo, Phys. Rev. 135, B1398 (1964). The first approximation to their scheme for calculating the basic renormalized Green's functions of scalar electrodynamics consists in writing down the two-particle unitarity equations for Δ and $D_{\mu\nu}$ with $\Gamma_\mu(p',p)$ approximated by the manifestly gauge-covariant expression

$$(p+p')_\mu[\Delta^{-1}(p'^2)-\Delta^{-1}(p^2)]/(p'^2-p^2).$$

The solutions they obtained for Δ and $D_{\mu\nu}$ result in the uv convergence, for example, of graphs A, K, L, and P depicted in Fig. 7 in all gauges *except* the Yennie gauge. Whether their technique continues to render M parts uv-finite when the full vertices Γ_μ and $\Gamma_{\mu\nu}$ are calculated in the two-particle unitarity approximation and beyond remains unanswered.

[25]P. T. Matthews, Phil. Mag. 41, 185 (1950).

[26]The idea of beginning the expansion of λ in second order was first proposed by A. Bhattacharyya, Ph.D. thesis, Massachusetts Institute of Technology, 1969 (unpublished). Bhattacharyya only requires π^+-π^- scattering graphs without π^+-π^- or two-photon intermediate states to be uv-convergent. In particular, we see from Table II that the choice $\lambda_2=\pm(\frac{1}{2})^{1/2}$ renders the

fourth-order graphs $A-J$ in Fig. 7 uv-convergent in the Yennie gauge. However, with this choice of λ_2, it is impossible to find a finite value of λ_4 that will render the same class of graphs uv-convergent in sixth order due to the uv-divergent subgraphs $K-S$ illustrated in Fig. 7. Thus, in his scheme, λ still requires an infinite renormalization in fourth and subsequent orders of its expansion in e_0^2.

[27]J. C. Ward, Proc. Phys. Soc. (London) A64, 54 (1951); Phys. Rev. 84, 897 (1951). See also T. T. Wu, *ibid.* 125, 1436 (1962).

[28]Our proof is based on a procedure originally devised by Yang and Mills for treating overlapping divergences in photon self-energy graphs with multiphoton intermediate states. An outline of their prescription is given by T. T. Wu, Phys. Rev. 125, 1436 (1962).

[29]The two-photon-rung ladder graph with finite insertions continues to diverge in the infrared for finite q with p and $p'=0$. However, since $\Gamma_\mu^a(s,0)=s_\mu/C$ and $s^\mu\Gamma_\mu^a(s+q,q)=(s^2+2s\cdot q)/C$ in the gauge in which Z_1 is finite, it is trivial to show that this graph diverges in the uv region as $\ln\Lambda^2$. The two-photon annihilation graphs belonging to

$$K^{a(2n+2)}(p,0;\lambda_2,\ldots,\lambda_{2n-2})$$

with finite insertions can be made infrared-convergent by giving the photon a small mass. By setting $p=0$ and differentiating these graphs with respect to the photon mass it is easy to show that these graphs likewise diverge no worse than $\ln\Lambda^2$ in the gauge in which Z_1 is finite to order α_0^{n-1}.

[30]Had we not assumed $\kappa\geq 1$ and invoked the asymptotic hypothesis, then Eq. (5.6) would have contained an additional term

$$\int d^4k\,\frac{\delta\Sigma(p^2;\,C\tilde{\Delta},e^2\tilde{D})}{\delta(e^2\tilde{D}_{\mu\nu}(k))}\,\frac{\delta(e^2\tilde{D}_{\mu\nu}(k))}{\delta\tilde{m}(p_0^2)}\ .$$

In order to solve (5.6) we would have needed an additional equation for $\partial\tilde{D}_{\mu\nu}/\partial\tilde{m}$ obtained from the functional differentiation of the polarization operator expanded in terms of the full $\tilde{D}$ and Δ functions. The asymptotic solution of this coupled pair of integral equations goes beyond the scope of this preliminary study.

[31]We are now in a position to prove the assertion in Sec. III that all uv divergences in Z_2 are isolated by neglecting h provided it vanishes with power-law behavior. The contribution to $\Gamma_\mu(p,p)$ when any one of the internal photon lines of $\Gamma_\mu^a(p,p)$ is replaced by the nonasymptotic part of $e^2\tilde{D}_{\mu\nu}$ is

$$-i\int\frac{d^4k}{(2\pi)^4}\,\frac{\partial}{\partial p^\mu}\,C^a_{\alpha\beta}(p,k)\left(g^{\alpha\beta}-\frac{k^\alpha k^\beta}{k^2}\right)\left(\frac{h(k^2/m^2,\alpha)}{k^2}\right)\ .$$

An application of Weinberg's theorem to the graphs defining $C^a_{\alpha\beta}(p,k)$ shows that to any finite order of perturbation theory and for $k^2\gg p^2$ and

$$\left\langle\frac{\partial}{\partial p^\mu}\,C^a_{\alpha\alpha}(p,k)\right\rangle_k\sim\frac{p_\mu}{k^2}\times\text{(powers of }\ln k^2).$$

Therefore, the above integral is uv-convergent provided

$$h(k^2/m^2,\alpha)\underset{k^2\gg m^2}{\sim}(m^2/k^2)^\kappa,$$

where $\kappa>0$.

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Action-at-a-Distance Theories and Dual Models*

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We write the most general classical formulation of Poincaré-invariant action-at-a-distance theories and review their classical applications. We stress their bootstraplike properties. In particular, we try to view dual amplitudes in terms of the radiation reaction of "dual atoms."

I. INTRODUCTION

The study of the strong interactions in the limit of short separations has led to the revival of the conformal group.[1] Yet, the physical applications of this group have been hampered by the noninvariance of the sign of x^2 under its *finite* transformations, thus causing an apparent violation of causality. To circumvent this difficulty, modern "conformists" require only *infinitesimal* conformal invariance and break the full invariance by the specification of physically reasonable boundary conditions (for instance, through an $i\epsilon$ prescription). As it is evident that conformal invariance must be broken in some way, it may prove useful, as well as instructive, to consider alternatives to this procedure. One such alternative is provided by the classical treatment of electrodynamics through action at a distance,[2] as formulated by Feynman and Wheeler.[3] Their formulation, as pointed out by Professor Gürsey,[4] is conformally invariant; in it particles interact by means of a symmetric combination of advanced and retarded signals, instead of the usual retarded interaction. Causality