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[16]Note the remarks about the determination of b through the K mass in Sec. II.
[17]S. Okubo [Phys. Rev. D 3, 2807 (1971)] and L.-F. Li and H. Pagels [Phys. Rev. D 4, 255 (1971)] have derived independently a rigorous bound for $f_+(0)$:

$$|f_+(0)| \le \frac{16}{(m_K - m_\pi)(\sqrt{m_K}+\sqrt{m_\pi})}\left(\frac{\pi\Delta(0)}{3(m_K+m_\pi)}\right)^{1/2}.$$

However, the numerical bound $|f_+(0)| \le 1$ is based on an estimate of $\Delta(0)$ by a model-dependent extrapolation which is at variance with the σ model. We have computed $\Delta(0)$ for various parameters in our model which gives $|f_+(0)| < 1.5$–2.0. Therefore, our value of $f_+(0)$ does not violate the rigorous bound.

PHYSICAL REVIEW D VOLUME 7, NUMBER 2 15 JANUARY 1973

Renormalization of the $SU_3 \times SU_3$ σ Model

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(Received 19 June 1972)

We have carried out the renormalization procedure for the SU_3 σ model through second order in the presence of symmetry-breaking terms. We show explicitly that no new divergent counterterms are needed other than those required in the symmetric theory. The theory can be completely determined in terms of known masses and the decay constant f_π.

I. INTRODUCTION

Most of the results of current algebra[1,2] can be obtained from the tree approximation in the SU_3 σ model.[3,4] That is, the currents in the model satisfy chiral algebra, their divergences are proportional to fields and in the tree-order amplitudes are approximated by poles as is the case of current-algebra extrapolations. If we adopt this model as a starting point for doing dynamics it would be very interesting to see how higher-order corrections affect these results. We propose to do this by calculating corrections in standard perturbation theory. This paper presents the renormalization formalism for this model with symmetry breaking through second order. The explicit calculation of one- and two-point functions is done in a separate paper.[5]

It may very well be doubtful whether perturbation theory is meaningful for strong interactions. However we are encouraged by the recent successes in similar models which indicate that second-order corrections indeed may be sufficiently small.[6,7] Lee and Basdevant[6] have shown in the SU_2 σ model that the perturbation expansion parameter turned out to be about 0.1. They show that the Padé approximant to second order already gives interesting results for $\pi\pi$ scattering. The SU_3 σ model is much richer in predictive power than the SU_2 counterpart. It would be interesting to see if the same program can be carried out in SU_3. This would provide a dynamical model to study further the breaking of SU_3.

We focus most of our attention on the one- and two-point functions where most of the renormalization difficulties occur. This will not only provide the groundwork for doing a Padé calculation of phase shifts but also give rise to predictions at this level that can be compared with experiment.

The $SU_3 \times SU_3$ algebra is defined by the equal-time current commutation relations

$$[V_i^0(x), V_j^0(y)] = if_{ijk} V_k^0(x)\delta(x-y) \ ,$$
$$[V_i^0(x), A_j^0(y)] = if_{ijk} A_k^0(x)\delta(x-y) \ ,$$
$$[A_i^0(x), A_j^0(y)] = if_{ijk} V_k^0(x)\delta(x-y) \ .$$

The most general chiral-invariant renormalizable spin-zero meson Lagrangian is

$$\mathcal{L}_S = -\tfrac{1}{2}\,\mathrm{Tr}(\partial_\mu M \partial_\mu M^\dagger) - \tfrac{1}{2}\mu^2\,\mathrm{Tr}(MM^\dagger) + g(\det M + \mathrm{H.c.}) + f_1(\mathrm{Tr} MM^\dagger)^2 + f_2\,\mathrm{Tr}(MM^\dagger MM^\dagger) \ .$$

Here M is a 3×3 matrix transforming as $(3, \bar{3})$, defined by

$$M = \frac{1}{\sqrt{2}} \sum_{i=0}^{8} \lambda^i(\sigma_i + i\phi_i) \ ,$$

where σ_i, ϕ_i are nonets of scalar and pseudoscalar fields and λ^i have the standard definition.[4] The renormalization procedure for this model has been worked out by Crater.[8]

If we restrict the symmetry breaking to transform like $(3,\bar{3})+(\bar{3},3)$ and further that the divergence of the currents be proportional to fields, then the most general form of symmetry breaking is given by

$$\mathcal{L}_{SB} = -\epsilon_0\sigma_0 - \epsilon_8\sigma_8 \ .$$

We define perturbation theory as an expansion in powers of λ which is defined through the relation[9,4]

$$\mathcal{L}(M,\lambda) = \frac{1}{\lambda^2}\mathcal{L}(\lambda M) \ .$$

λ is introduced for the purpose of power counting and is set equal to 1 in the end. This is in effect an expansion in the number of closed loops. Introducing λ into the Lagrangian gives (M = field)

$$\mathcal{L} \sim \lambda^2 M^4 + \lambda M^3 + M^2 + \frac{1}{\lambda} M \ .$$

Since the maximum power of the fields in the Lagrangian is four, i.e., λ^2, the theory is renormalizable in the usual sense.

We would like to point out that classifying possible symmetry-breaking terms as to their power of λ gives valuable information on the renormalizability of symmetry-breaking parameters. For example if the symmetry-breaking term goes like $1/\lambda$ (as in our case) then no new divergent counterterms other than those in the symmetric theory are needed. Hence it is possible to calculate corrections to ϵ. However, if the breaking term goes like λ^0 then an additional divergent counterterm to the symmetry-breaking Lagrangian would render the model renormalizable. But in this case it would not be possible to calculate corrections to the symmetry-breaking parameters since they would depend on the cutoff. Finally if the breaking went like λ then divergent counterterms would be required that are not already present in the Lagrangian and thereby completely obscure the role of chiral symmetry in the dynamics. The first type, λ^{-1}, is the only one that has operator partial conservation of current (PCC), i.e., the divergence of the currents is proportional to a field when they are not conserved. This means that operator PCC is not only attractive for current algebra calculations but also serves to limit symmetry breaking to the most attractive form.

In Sec. II we introduce a notation which is convenient for dealing with the large number of particles. In Sec. III we examine the tree approximation and show how to fix parameters. Section IV reviews the renormalization procedure in the symmetric limit to second order. In Sec. V. we give the formalism to calculate second-order corrections in the broken theory. We show the cancellation of divergences and outline a specific renormalization procedure.

II. PARTICLE LABELS AND DEFINITIONS

For the purpose of summing over internal lines it is convenient to have one label for all 18 particles. Hence we define

$$\Psi_I = \Psi_{(\alpha,i)} = \begin{pmatrix}\sigma_i \\ \phi_i\end{pmatrix} \ ,$$

where I represents (α, i), where $\alpha = 1$ for scalars, $\alpha = 2$ for pseudoscalars, and $i = 0, \ldots, 8$. Using this label, the Lagrangian can be written,

$$\mathcal{L}_S = \tfrac{1}{2}(\partial_\mu \Psi_I)^2 - \tfrac{1}{2}\mu^2 \Psi_I{}^2 + \tfrac{1}{3} F_{IJKL}\Psi_I\Psi_J\Psi_K\Psi_L + G_{IJK}\Psi_I\Psi_J\Psi_K \ ,$$
$$\mathcal{L}_{SB} = -\epsilon_I \Psi_I \ ,$$

where F, G, and ϵ are defined through the following expressions:

$$D_{IJ}^K = D_{(\alpha,i)(\beta,j)}^{(\gamma,k)} = \begin{cases} d_{ijk} & \alpha,\beta,\gamma = 1 \text{ or } \alpha,\beta = 2, \gamma = 1 \\ f_{ijk} & \alpha = 1, \beta,\gamma = 2 \\ -f_{ijk} & \beta = 1, \alpha,\gamma = 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$F^1_{IJKL} = \delta_{IJ}\delta_{KL} + \delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK} ,$$

$$F^2_{IJKL} = D^M_{IJ}D^M_{KL} + D^M_{IK}D^M_{JL} + D^M_{IL}D^M_{JK} ,$$

$$F_{IJKL} = f_1 F^1_{IJKL} + \tfrac{1}{2} f_2 F^2_{IJKL} , \tag{2.1}$$

$$h_{ijk} = \tfrac{1}{6}\epsilon_{abc}\lambda^i_{aa'}\lambda^j_{bb'}\lambda^k_{cc'}\epsilon_{a'b'c'} ,$$

$$G_{IJK} = G_{(\alpha,i)(\beta,j)(\gamma,k)}$$

$$= 2g \times \begin{cases} h_{ijk} & \alpha,\beta,\gamma = 1 \\ -h_{ijk} & \text{one index} = 1 \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

$$\epsilon_I = \epsilon_{(\alpha,i)}$$

$$= \delta_{\alpha 1}(\epsilon_0\delta_{i0} + \epsilon_8\delta_{i8}) .$$

Note that F_{IJKL} and G_{IJK} are totally symmetric in their indices. The Feynman graphs are given in Fig. 1.

It has been shown that the perturbation expansion in λ preserves the current algebra and partial conservation of current (PCC) order by order. Introducing λ into the Lagrangian gives

$$\mathcal{L}_S = \tfrac{1}{2}(\partial_\mu \Psi_I)^2 - \tfrac{1}{2}\mu^2 \Psi_I{}^2 + \lambda^2 \tfrac{1}{3} F_{IJKL}\Psi_I\Psi_J\Psi_K\Psi_L + \lambda G_{IJK}\Psi_I\Psi_J\Psi_K , \tag{2.3}$$

$$\mathcal{L}_{SB} = -\frac{1}{\lambda}\epsilon_I \Psi_I . \tag{2.4}$$

Tree graphs correspond to lowest order in this expansion.

III. TREE APPROXIMATION

We first examine the one- and two-point functions in the tree approximation as a starting point for doing renormalization. We review the procedure for solving the model in this approximation and show how the parameters can be fixed in terms of masses and the pion decay constant f_π.[10,11]

We start from a Lagrangian which allows fields to have nonzero vacuum expectation values, $\langle\psi\rangle \neq 0$. The physical fields ψ_I are defined in terms of the previously defined ψ_I by a translation

$$\psi_I = \Psi_I - \xi_I ,$$

where

$$\langle\psi_I\rangle = 0 ,$$

$$\langle\Psi_I\rangle = \xi_I .$$

In terms of the fields ψ_I, the Lagrangian becomes

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\psi_I)^2 - \tfrac{1}{2}m^2{}_{IJ}\psi_I\psi_J + \tfrac{1}{3}F_{IJKL}\psi_I\psi_J\psi_K\psi_L + Q_{IJK}\psi_I\psi_J\psi_K - E_I\psi_I , \tag{3.1}$$

where

$$Q_{IJK} = G_{IJK} + \tfrac{4}{3}F_{IJKL}\xi_L , \tag{3.2}$$

$$m^2{}_{IJ} = \mu^2\delta_{IJ} - 6G_{IJK}\xi_K - 4F_{IJKL}\xi_K\xi_L , \tag{3.3}$$

$$E_I = \epsilon_I + \mu^2\xi_I - \tfrac{4}{3}F_{IJKL}\xi_J\xi_K\xi_L - 3G_{IJK}\xi_J\xi_K . \tag{3.4}$$

The vanishing of the vacuum expectation value of ψ_I implies that

$$E_I = 0 , \tag{3.5}$$

hence giving an equation for ξ_I.

The Feynman graphs for the new Lagrangian Eq. (3.1) are given in Fig. 2. Also shown are graphical representations of Eqs. (3.2)–(3.4) re-

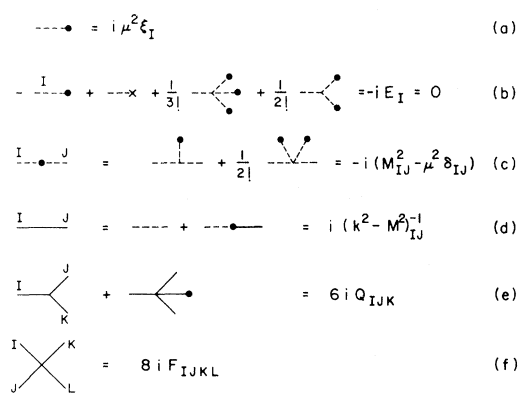

FIG. 2. Feynman graphs in the presence of symmetry breaking, including equations relating these rules to the symmetric rules. The combinatoric factors could have been included in the definition of Feynman rules when particles go to the vacuum, but this is not common practice. For a further discussion see Ref. 14.

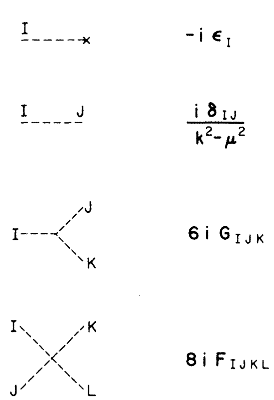

FIG. 1. Feynman graphs in the symmetric limit.

TABLE I. Tree mass matrix. The mass-squared matrix $m^2{}_{IJ}$ Eq. (3.3) is tabulated, where

$$m^2{}_{IJ}=\delta_{IJ}\mu^2-2\xi_0{}^2 f_1 A^1_{IJ}-2\xi_0{}^2 f_2 A^2_{IJ}-\frac{2g}{\sqrt{3}}\xi_0 A^3_{IJ}\ .$$

(I,J)	A^1_{IJ}	A^2_{IJ}	A^3_{IJ}
π	$2(1+2b^2)$	$\frac{2}{3}(1+b)^2$	$(1-2b)$
K	$2(1+2b^2)$	$\frac{2}{3}(7b^2-b+1)$	$(1+b)$
η_{00}	$2(1+2b^2)$	$\frac{2}{3}(1+2b^2)$	-2
η_{88}	$2(1+2b^2)$	$\frac{2}{3}(3b^2-2b+1)$	$(1+2b)$
η_{80}	0	$\frac{2}{3}\sqrt{2}b(2-b)$	$\sqrt{2}b$
π_N	$2(1+2b^2)$	$2(b+1)^2$	$-(1-2b)$
κ	$2(1+2b^2)$	$2(b^2-b+1)$	$-(1+b)$
σ_{00}	$2(3+2b^2)$	$2(1+2b^2)$	2
σ_{88}	$2(1+6b^2)$	$2(3b^2-2b+1)$	$-(1+2b)$
σ_{80}	$4b\sqrt{2}$	$2\sqrt{2}b(2-b)$	$-\sqrt{2}b$

lating the new rules to the rules derived in terms of ψ_I.

For the purpose of strong interactions we take

$$\xi_I=\xi_{(\alpha,i)}=\delta_{\alpha,1}\xi_0(\delta_{i0}+\sqrt{2}\,b\delta_{i8})\ ,$$

where $b=\xi_8/\sqrt{2}\,\xi_0$. The mass matrix, Eq. (3.3) is summarized in Table I. We wish to use the masses to fix as many parameters as possible. There are six parameters in the Lagrangian μ^2, f_1, f_2, g, ϵ_0, and ϵ_8. For the purpose of this discussion let us eliminate ϵ_0, ϵ_8 in terms of ξ_0, ξ_8, Eqs. (3.3)–(3.4) giving a simple relation between them

$$\epsilon_0+\frac{\epsilon_8}{\sqrt{2}}=m_\pi{}^2\xi_0(1+b)\ ,$$
$$\epsilon_0-\frac{\epsilon_8}{2\sqrt{2}}=m_K{}^2\xi_0(1-\tfrac{1}{2}b)\ . \qquad (3.6)$$

The procedure we use to fix parameters is as follows: The π, K, η, and η', masses determine the four quantities:

$$[\mu^2-4(\xi_0{}^2 f_1)(1+2b^2)]\ ,$$
$$\xi_0{}^2 f_2,\quad \xi_0 g,\quad b.$$

From these we can predict η, η' mixing, the π_N and K masses. Finally μ^2 can be determined from the mass of one of the isoscalar σ's. This then determines the other σ and the scalar mixing. Note that the masses are functions of only five parameters in the tree approximation.

The sixth parameter ξ_0 can be fixed by f_π, which is defined through the matrix element

$$\langle 0|A^\mu_i|\phi_j\rangle=\frac{f_{ij}p_\mu}{(2p_0)^{1/2}(2\pi)^{3/2}}\ ,\quad i=1,8\ .$$

It follows that

$$\langle 0|\partial_\mu A^\mu_i|\phi_j\rangle=\frac{f_{ij}m_i{}^2}{(2p_0)^{1/2}(2\pi)^{3/2}}\ . \qquad (3.7)$$

In our Lagrangian we can calculate the operator relation

$$\partial_\mu A^\mu_i=-\epsilon_k d_{ijk}\phi_j\ ,\quad i=1,8.$$

Taking matrix elements in the tree order and comparing with Eq. (3.7) we obtain

$$f_\pi m_\pi{}^2=-(\tfrac{2}{3})^{1/2}\left(\epsilon_0+\frac{\epsilon_8}{\sqrt{2}}\right)\ . \qquad (3.8)$$

Using Eqs. (3.6)–(3.8) we find

$$f_\pi=-(\tfrac{2}{3})^{1/2}\xi_0(1+b)\ .$$

Since b is determined by the masses, ξ_0 is fixed by f_π.

All six parameters are thereby determined. f_K can be predicted and is given by the following expression[12]:

$$f_K m_K{}^2=-(\tfrac{2}{3})^{1/2}\left(\epsilon_0-\frac{\epsilon_8}{2\sqrt{2}}\right)$$
$$f_K=-(\tfrac{2}{3})^{1/2}\xi_0(1-\tfrac{1}{2}b)\ .$$

The ratio f_K/f_π depends only on b:

$$\frac{f_K}{f_\pi}=\frac{(1-\frac{1}{2}b)}{(1+b)}\ .$$

IV. RENORMALIZATION IN THE SYMMETRIC LIMIT

All divergences can be grouped into a redefinition of the Lagrangian parameters f_1, f_2, g, and μ^2. The counter terms required in the symmetric theory, (i.e., $\epsilon_I=0$ and no spontaneous breaking), are sufficient to cancel all divergences in the presence of symmetry breaking. We summarize in this section the renormalization in the symmetric limit.

The Lagrangian, Eq. (2.3) with the counterterms present[13] becomes

$$\mathcal{L}_S=\tfrac{1}{2}(\partial_\mu\Psi_I)^2-\tfrac{1}{2}(\mu^2+\lambda^2\delta\mu^2)\psi_I{}^2+\lambda^2\tfrac{1}{3}(F+\lambda^2\delta F)_{IJKL}\psi_I\psi_J\psi_K\psi_L+\lambda(G+\lambda^2\delta G)_{IJK}\psi_I\psi_J\psi_K\ , \qquad (2.3')$$

where

$$\delta F=F(\delta f_1,\delta f_2)\ ,$$
$$\delta G=G(\delta g)\ .$$

The functional dependences of F and G on the implied variables are given in Eqs. (2.1)–(2.2). Since the wave function renormalization constants are finite we choose not to renormalize the fields at this stage. The counterterms have been calculated by Crater[8] and a manifestly chiral symmetric derivation is given in the Appendix. The results are

$$\begin{aligned}
&\delta f_i = Df_i + \Delta f_i \ , \\
&\delta g = Dg + \Delta g \ , \\
&\delta \mu^2 = D\mu^2 + \Delta \mu^2 \ , \\
&Df_1 = 8(13 f_1^2 + 12 f_1 f_2 + 3 f_2^2)B \ , \qquad (4.1)\\
&Df_2 = 48(f_1 + f_2) f_2 B \ , \\
&Dg = 24(f_1 - f_2) g B \ , \\
&D\mu^2 = 16(5 f_1 + 3 f_2) A - 16 g^2 B \ ,
\end{aligned}$$

where A and B are the divergent integrals

$$A = i \int \frac{d^4k}{(2\pi)^4} (k^2 - \mu^2 + i\epsilon)^{-1} \ , \qquad (4.2)$$

$$B = i \int \frac{d^4k}{(2\pi)^4} (k^2 - \mu^2 + i\epsilon)^{-2} \ . \qquad (4.3)$$

A graphical representation of the divergent part of the counterterms Df_i, Dg, $D\mu^2$ are shown in Figs. 3(a)–3(c). The linearly divergent integral in Fig. 3(d) is in fact zero due to the index summation in the loop. (This is trivial in the formalism used in the Appendix and we comment on it there.) The separation of δ into D and Δ is of course arbitrary. However, by choosing the integrals A and B with specific finite parts, the Δ terms are well defined. The finite parts Δf_i, Δg, and $\Delta\mu^2$ can be determined once the particular renormalization procedure is chosen. The choice of values for these quantities corresponds to choosing certain quantities to be fixed under the perturbation. One particular procedure is to fix the values of the two- and three-point functions, and the F^1 and F^2 types of the four-point functions for all four-momenta zero. Then these four finite parts would be zero.

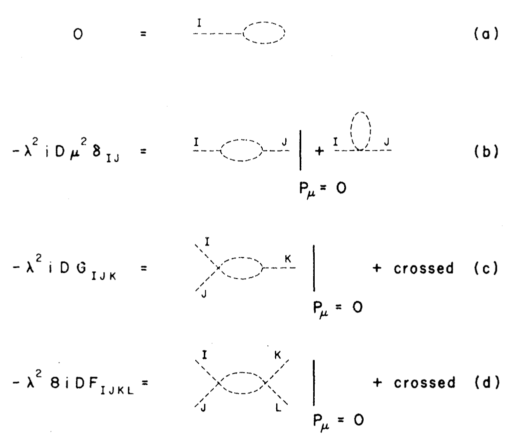

FIG. 3. Graphical representation of divergent counterterms. All graphs are evaluated at all four-momenta equal zero. "Crossed" in both cases refers to two additional graphs obtained by crossing lines.

To recapitulate: Second-order counterterms are included in the Lagrangian. The calculation to second order, including the counterterms, of all physical quantities yields finite results. The particular choice of fixed points under the perturbation determines the four finite contributions to the counterterms Δf_1, Δf_2, Δg, and $\Delta\mu^2$.

V. RENORMALIZATION WITH SYMMETRY BREAKING

We now wish to consider the full Lagrangian with symmetry breaking and with all second-order counterterms present. The Lagrangian before translation is

$$\mathcal{L} = \mathcal{L}_S + \mathcal{L}_{SB} \ ,$$

where $\mathcal{L}_S$ is given by Eq. (2.3′) and $\mathcal{L}_{SB}$ is

$$\mathcal{L}_{SB} = \frac{1}{\lambda} (\epsilon_I + \lambda^2 \delta\epsilon_I) \Psi_I \ .$$

This serves to define $\delta\epsilon_I$. However, it is more convenient to use the translated form of the Lagrangian, which now becomes

$$\begin{aligned}
\mathcal{L} = &\tfrac{1}{2}(\partial_\mu \psi_I)^2 - \tfrac{1}{2}(m^2_{IJ} + \lambda^2 \delta m^2_{IJ}) \psi_I \psi_J \\
&+ \tfrac{1}{3} \lambda^2 (F_{IJKL} + \lambda^2 \delta F_{IJKL}) \psi_I \psi_J \psi_K \psi_L \\
&+ \lambda (Q_{IJK} + \lambda^2 \delta Q_{IJK}) \psi_I \psi_J \psi_K \\
&- \frac{1}{\lambda} (E_I + \lambda^2 \delta E_I) \psi_I \ .
\end{aligned}$$

The δ terms are now given by the following expressions, where the implied functional dependence is given in Eqs. (3.2)–(3.4) and Eqs. (2.1)–(2.2):

$$\delta F_{IJKL} = F_{IJKL}(\delta f_i) \ , \qquad (5.1a)$$

$$\delta Q_{IJK} = Q_{IJK}(\delta f_i, \delta g) + \tfrac{4}{3} F_{IJKL} \delta\xi_L \ , \qquad (5.1b)$$

$$\delta m^2_{IJ} = m^2_{IJ}(\delta f_i, \delta g, \delta\mu^2) - 6 Q_{IJK} \delta\xi_K \ , \qquad (5.1c)$$

$$\delta E_I = E_I(\delta f_i, \delta g, \delta\mu^2, \delta\epsilon) + m^2_{IJ} \delta\xi_J \ . \qquad (5.1d)$$

The δ terms have two parts $\delta = D + \Delta$ where D corresponds to the infinite part and Δ to the finite part in the same way as Eq. (4.1). We take $D\epsilon_I$ to be zero and show that all second-order calculations are finite.

Lee[14] and Symanzik[15] have shown in similar models that no new divergent counterterms are

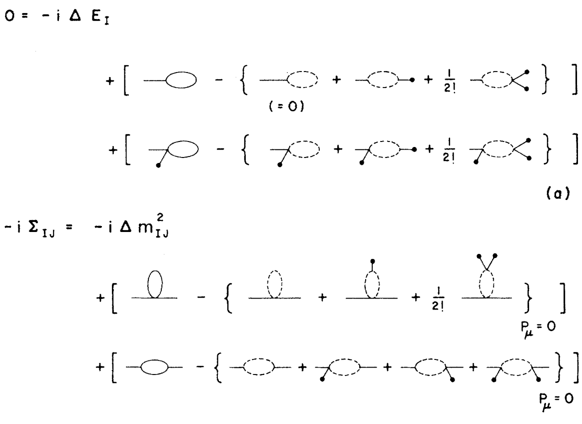

FIG. 4. Cancellation of divergences in the nonsymmetric theory. The graphs in curly brackets arise from the counterterms in the Lagrangian. The expansion of the solid lines in terms of dashed lines shows that the integrals are finite (see text). The one-line graph with a single dashed loop is zero.

needed in the presence of this type of symmetry breaking. The proof is almost parallel for the present model and we shall restrict our discussion to show this explicitly in second order in Part A. In addition we derive formulas for the one- and two-point functions. In Part B we carry out in detail our particular choice of renormalization procedure.

A. Second-Order Corrections and Cancellation of Divergences

We start by examining the one-point function $\langle\psi_I\rangle$. The requirement that this vanish, i.e., that the physical fields have zero vacuum expectation value through second order, is as follows:

$$E_I+\lambda^2\delta E_I-3Q_{IJK}\,\lambda^2 i\int\frac{d^4k}{(2\pi)^4}(k^2-m^2)_{JK}{}^{-1}=0\ . \tag{5.2}$$

However, since $E_I=0$ for the tree result Eq. (3.5), we obtain an equation for the λ^2 part. To show that this equation is finite, we write

$$\delta E_I=DE_I+\Delta E_I\ .$$

DE can be evaluated through Eq. (5.1d) and Eqs. (3.3)–(3.4) to give

$$DE_I=3Q_{IJK}\,i\int\frac{d^4k}{(2\pi)^4}\left[\frac{\delta_{JK}}{k^2-\mu^2}+\frac{(m^2{}_{JK}-\delta_{JK}\mu^2)}{(k^2-\mu^2)^2}\right]\ .$$

Hence Eq. (5.2) can be written

$$\Delta E_I-3Q_{IJK}(m^2{}_{JL}-\mu^2\delta_{JL})(B_{LK}-B\delta_{LK})=0\ ,$$

where

$$B_{IJ}=i\int\frac{d^4k}{(2\pi)^4}\,\frac{(k^2-m^2)_{IJ}{}^{-1}}{(k^2-\mu^2)}\ .$$

The equation for ΔE is manifestly finite and gives a relation between $\Delta\epsilon_I$ and $\Delta\xi_I$. The cancellation of divergences can be seen in Fig. 4(a). The graphs in curly brackets come from the counterterms in the Lagrangian (Fig. 3). That these graphs cancel the divergence can be seen by noting that they are in fact the first few terms in the expansion of the solid line (nonsymmetric propagator) in terms of the dashed line (symmetric propagator), shown in Figs. 2(c) and 2(d). Further terms in the expansion give convergent integrals.

The second-order mass term $\Sigma_{IJ}(p^2)$ can be found in a similar way:

$$\Sigma_{IJ}(p^2)=\delta m^2{}_{IJ}-4F_{IJKL}\,i\int\frac{d^4k}{(2\pi)^4}(k^2-m^2)_{KL}{}^{-1}+18Q_{IKL}\,i\int d^4k\,((p-k)^2-m^2)_{KK'}{}^{-1}(k^2-m^2)_{LL'}{}^{-1}Q_{K'L'J}\ . \tag{5.3}$$

Separating off the divergent part of $\delta m^2{}_{IJ}$ and using Eq. (5.1c) and Eq. (3.3) we find

$$Dm^2{}_{IJ}=4F_{IJKL}\,i\int\frac{d^4k}{(2\pi)^4}\left[\frac{\delta_{KL}}{k^2-\mu^2}+\frac{(m^2{}_{KL}-\mu^2\delta_{KL})}{(k^2-\mu^2)^2}\right]$$
$$-18Q_{IKL}\,i\int d^4k\,\frac{\delta_{KK'}\delta_{LL'}}{(k^2-\mu^2)^2}\,Q_{K'L'J}\ . \tag{5.4}$$

Substituting Eq. (5.4) in Eq. (5.3) gives finally

$$\Sigma_{IJ}(p^2)=\Delta m^2{}_{IJ}$$
$$-4F_{IJKL}(m^2{}_{KJ'}-\mu^2\delta_{KJ'})(B_{J'L}-B\delta_{J'L})$$
$$+18Q_{IKL}[B_{KK',LL'}(p^2)-B\delta_{KK'}\delta_{LL'}]Q_{K'L'J}\ ,$$

where

$$B_{IJ,KL}(p^2)=i\int\frac{d^4k}{(2\pi)^4}((k-p)^2-m^2)_{IJ}{}^{-1}(k^2-m^2)_{KL}{}^{-1}\ .$$

The cancellation of the divergence is illustrated in Fig. 4(b) in the same way as the one-point function.

A similar procedure can be carried out for the three- and four-point functions. They are simpler since the single-particle irreducible graphs are at most logarithmically divergent. One can easily check that counterterms would remove the divergence in exactly the same manner. Single-particle irreducible n-point functions for $n \geq 5$ are convergent.

Now that we have a finite equation for the self-energy, we can find the masses to second order. The unrenormalized propagator D is

$$D^{-1}(s)_{IJ} = s\delta_{IJ} - m^2{}_{IJ} - \Sigma_{IJ}(s) \ .$$

D is almost diagonal; there are two 2×2 blocks corresponding to the 0, 8 components of the scalar and pseudoscalar mesons.

The masses $M_I{}^2$ are given by the location of the poles of D which correspond to

$$s - m_I{}^2 - \Sigma_I(s) = 0 \qquad (5.5a)$$

for the diagonal parts and

$$\det[s - m^2 - \Sigma(s)] = 0 \qquad (5.5b)$$

for the 2×2 submatrices.

If the solution exists for s above the lowest two-body threshold then $M_I{}^2$ is complex and the corresponding particle is unstable. In this case $\mathrm{Re}(M_I)$ is the mass and $-\mathrm{Im}(M_I)$ is the half-width. If the width is small then to a good approximation we can find the masses from the zeros of the real part of Eq. (5.5). For very large widths, the zeros of Eq. (5.5) are not very meaningful, yet Σ_I contributes a continuum contribution to the propagator.

It is convenient to use Eq. (5.6) to rewrite Eq. (5.5) in the form

$$D_{IJ}(s) = \delta_{IJ}[s - M_I{}^2 - \Sigma_I(s) + \Sigma_I(M_I{}^2)]^{-1} \ ,$$

where the once-subtracted mass term becomes explicit.

The properly renormalized second-order propagator can be defined similarly with a twice subtracted mass term.

$$D^R_{IJ}(s) = \delta_{IJ}\{s - M_I{}^2 - [\Sigma_I(s) - \Sigma_I(M_I{}^2) - (s - M_I{}^2)\Sigma'(M_I{}^2)]\}^{-1}$$

and is related to the unrenormalized propagator by the relation (valid up to second order)

$$D_I^R = D_I / Z_I \ ,$$

where

$$Z_I = [1 - \Sigma_I'(M_I{}^2)]^{-1}$$

is the wave-function renormalization constant.

The PCAC (partially conserved axial-vector current) relation in terms of renormalized fields is given by

$$\partial_\mu A_i^\mu = Z_i{}^{1/2} d_{ijk} \phi_k^R(\epsilon_j + \Delta\epsilon_j), \quad i = 1, 8 \ .$$

Taking matrix elements and comparing with Eq. (3.7) we obtain

$$f_\pi m_\pi{}^2 = -Z_\pi{}^{1/2}(\tfrac{2}{3})^{1/2}\left[\epsilon_0 + \lambda^2\Delta\epsilon_0 + \frac{1}{\sqrt{2}}(\epsilon_8 + \lambda^2\Delta\epsilon_8)\right]$$

$$f_K m_K{}^2 = -Z_k{}^{1/2}(\tfrac{2}{3})^{1/2}\left[\epsilon_0 + \lambda^2\Delta\epsilon_0 - \frac{1}{2\sqrt{2}}(\epsilon_8 + \lambda^2\Delta\epsilon_8)\right] \ .$$

B. Renormalization

Apart from the divergence difficulties dealt with above, the renormalization procedure itself refers to choosing a set of variables to be fixed parameters in the model. This means that all higher-order corrections to these variables are zero. The Lagrangian contains six parameters $\{x_i\} = f_1, f_2, g, \mu^2, \epsilon_0, \epsilon_8$. In lowest order, we can determine these six parameters in terms of six physical quantities $\{P_i\}$. In next order we can further determine the $\{\Delta x_i\}$ through the renormalization procedure such that the second-order contributions to six quantities $\{Q_i\}$ vanish. The choice of $\{Q_i\}$ need not be the same as $\{P_i\}$. With the $\{\Delta x_i\}$ determined then the calculation of all other second-order corrections is fixed.

We first describe the procedure that we used in Ref. 5. The $\{P_i\}$ were chosen to be the masses of π, K, η, η' and one isoscalar σ and f_π as described in Sec. III. The experimental situation dictates this choice of $\{P_i\}$. Although f_K is known perhaps better than the σ meson, it can be predicted from the pseudoscalar masses and f_π, and hence is not an independent quantity.

It is natural to choose the $\{Q_i\} = \{P_i\}$, i.e., to hold fixed the experimental parameters that are used to fit the model in lowest order. A consequence of this choice is that once the fitting is done in lowest order no further fitting need be done in higher order. A further advantage is that by renormalizing at the masses of the lowest-lying mesons, the two-particle thresholds involving these particles occur at the correct position. This can be quite important if a resonance occurs in the neighborhood of these thresholds.

One of the many alternative methods is to choose $\{Q_i\}$ to be the masses of π, K, η and ξ_0, ξ_8 yet keeping $\{P_i\}$ the same as before. In this case the value of these variables will be the same as their tree values in second order. There will be finite corrections to f_π and $m_{\eta'}$. However, this means that in order that f_π and $m_{\eta'}$ take on physical values, their tree values will be changed in such a way that their values through second order be physical.

APPENDIX: DIVERGENT RENORMALIZATION CONSTANTS

All divergent integrals occuring in n-point functions in this model have the same $SU_3\times SU_3$ transformation properties as tree-graph n-point functions constructed from the symmetric Lagrangian. In other words the internal loop summations are invariant contractions under the chiral group. It is therefore advantageous to use chiral tensor indices to calculate the divergent part of the renormalization constants.

The non-Hermitian fields M as defined in Sec. I are appropriate for this purpose. They are not fields of definite parity, but this is of no consequence in doing internal summations. We define

$$M=\sum_{i=0}^{8}\lambda^i\chi_i \ ,$$

$$M^\dagger=\sum_{i=0}^{8}\lambda^i\chi_i^\dagger \ .$$

The Feynman rules for these fields are given in Fig. 5. We have used the shorthand notation:

$$\{ijkl\}=\mathrm{Tr}(\lambda^i\lambda^j\lambda^k\lambda^l)\ ,$$

$$[ijk]=\epsilon_{abc}\lambda^i_{aa'}\lambda^j_{bb'}\lambda^k_{cc'}\epsilon_{a'b'c'}\ .$$

Although these expressions can be written in terms

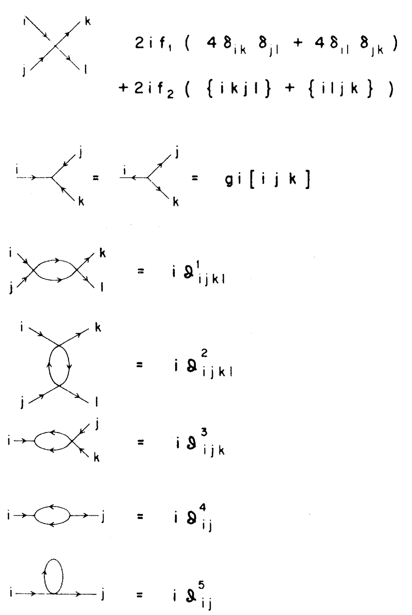

FIG. 5. Feynman rules in the symmetric theory for the non-Hermitian fields M. Also shown are integrals needed to calculate counterterms. The arrows are needed because we use non-Hermitian fields. The in arrows correspond to χ and the out arrows to $\chi^\dagger$.

of f_{ijk} and d_{ijk} it is advantageous to leave them in terms of the λ's.

We need to calculate the integrals $\mathcal{G}^1-\mathcal{G}^5$ shown in Fig. 5, for all four-momenta zero since they will give the counterterms in the Lagrangian. [It is clear from the arrows that one cannot construct the graph in Fig. 3(d).] The calculation of these integrals is straightforward. The following identities involving the λ matrices are useful

$$\mathrm{Tr}(\lambda^i\lambda^j)=2\delta_{ij}\ ,$$

$$\lambda^i_{aa'}\lambda^i_{bb'}=2\delta_{ab'}\delta_{a'b}\ .$$

The results are as follows:

$$\mathcal{G}^1_{ijkl}=-16iB[4(f_1^2+f_2^2)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+2f_1f_2(\{ikjl\}+\{iljk\})]\ ,$$

$$\mathcal{G}^2_{ijkl}=-16iB[4(11f_1^2+12f_1f_2+2f_2^2)\delta_{ik}\delta_{jl}+4f_1^2\delta_{il}\delta_{jk}+(2f_1f_2+3f_2^2)(\{ikjl\}+\{iljk\})]\ ,$$

$$\mathcal{G}^3_{ijk}=-8iBg(f_1-f_2)[ijk]\ ,$$

$$\mathcal{G}^4_{ij}=-16iBg^2\delta_{ij}\ ,$$

$$\mathcal{G}^5_{ij}=i(80f_1+48f_2)\delta_{ij}A\ .$$

The integrals A and B are given in Sec. IV, Eqs. (4.2)–(4.3).

The one-loop corrections to the single-particle irreducible four-point function are

$$\mathcal{G}^1_{ijkl}+\mathcal{G}^2_{ijkl}+\mathcal{G}^2_{ijlk}\ .$$

This is equal to

$$-16iB(13f_1^2+3f_2^2+12f_1f_2)(4\delta_{ik}\delta_{jl}+4\delta_{il}\delta_{jk})-96iBf_2(f_2+f_1)(\{ikjl\}+\{iljk\})\ . \quad \text{(A1)}$$

By comparing the chiral tensors in Eq. (A1) with those in the Feynman rules we can identify Df_1, Df_2 as the negative of the coefficients

$$Df_1=8(13f_1^2+3f_2^2+12f_1f_2)B\ ,$$

$$Df_2=48f_2(f_1+f_2)B\ .$$

For the three-point function we need

$$\mathcal{G}^3_{ijk}+\mathcal{G}^3_{jki}+\mathcal{G}^3_{kij}\ ,$$

which is equal to

$$-24iBg(f_1-f_2)[ijk]\ .$$

Hence the counterterm Dg is

$$Dg=24Bg(f_1-f_2)\ .$$

Finally in the same way $D\mu^2$ is

$$D\mu^2 = 16Bg^2 - (8f_1 + 48f_2)A \ .$$

We give the coupling-constant renormalization constants to second order:

$$Z_{f_1} = 1 + \lambda^2 Df_1/f_1 \ ,$$

$$Z_{f_2} = 1 + \lambda^2 Df_2/f_2 \ ,$$

$$Z_g = 1 + \lambda^2 Dg/g \ .$$

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PHYSICAL REVIEW D VOLUME 7, NUMBER 2 15 JANUARY 1973

Electrodynamics of Spin-0 Mesons at Small Distances*

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It is shown that if Z_3 (where Z_3 is the photon wave-function renormalization constant) is assumed finite and the nonasymptotic part h of the renormalized photon propagator vanishes with power-law behavior, then all the remaining renormalization constants in scalar electrodynamics can be made finite order by order, except the charged-meson self-mass δm^2. The condition that δm^2 be finite forces the asymptotic coupling α_0 to satisfy at least one eigenvalue equation. A second eigenvalue condition for α_0 emerges from the requirement that the theory have a Hermitian Lagrangian. Finally, on the basis of the renormalization group, we expect that the initial assumption of a finite value of Z_3 is self-consistent only if α_0 satisfies a third eigenvalue condition. Hence, we conjecture that a completely finite, closed theory of scalar electrodynamics is probably internally inconsistent. Assuming that h falls off sufficiently rapidly, we are able to show that the meson propagator has a very simple asymptotic form for momenta much greater than its physical mass.

I. INTRODUCTION AND SUMMARY OF RESULTS

The development of relativistic quantum field dynamics during the past quarter-century has been largely dominated by the recurrent question of whether a completely finite, pathology-free local field theory, with some claim of describing physical reality, exists. Attention in this regard has naturally focused on the one theory which has had the most quantitative success – quantum electrodynamics. One of the most systematic attempts to answer this question in quantum electrodynamics, considered as a closed theory, has been the series of papers by Johnson, Baker, and Willey[1] published over the past eight years. Their main conclusion is that all of the renormalization constants of quantum electrodynamics are finite provided (a) the electron bare mass m_0 is zero and (b) the