The constants a and $\bar{a}$ are now determined by the requirement that in a well-defined limit we have

$$\gamma_{y'}^{\mu} G(xx'yy') \xrightarrow[x'\to\xi\,;\,y'\to\xi]{} i\langle 0|\, T(j^{\mu}(\xi)\psi(x)\bar{\psi}(y))|0\rangle . \tag{A15}$$

Comparing with Johnson's limiting procedure[7] we see that

$$\gamma_{y'}^{\mu} G(xx'yy') \xrightarrow[x'\to\xi\,;\,y'\to\xi]{} [(1+ag^2/2\pi)g^{\mu\nu}+\gamma_5\epsilon^{\mu\nu}] \times \partial_\nu^\xi[\Delta_+^0(\xi-x)-\Delta_+^0(\xi-y)] . \tag{A16}$$

Comparing Eq. (16) with Eq. (14) we get finally

$$a=\frac{1}{1-g^2/2\pi} \quad \text{and} \quad \bar{a}=1 . \tag{A17}$$

Here we see that the interaction modifies a but not $\bar{a}$; whereas in the Thirring model a and $\bar{a}$ are both affected in a symmetrical way. In both theories the axial-vector current is the dual of the vector current for the fermions. In the Okubo model, however, it is the field $\varphi_{,\mu}$ which appears in the vertex (A5). It is the appearance of the current j^{μ} in the vertex of the Thirring model which brings in a and $\bar{a}$ in this symmetrical way.

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Closed-Loop Corrections to the $SU_3 \times SU_3$ σ Model: One- and Two-Point Functions

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We present the calculation of one-loop corrections to the one- and two-point functions in the renormalizable SU_3 σ model with a symmetry-breaking term $L_{SB}=\epsilon_0\sigma_0+\epsilon_8\sigma_8$. We renormalized at the masses of π, K, η, η', σ, and f_π. The second-order corrections are found to be small compared to tree-approximation values. The measure of octet breaking, $b=\langle\sigma_8\rangle/\sqrt{2}\,\langle\sigma_0\rangle$, changes less than 5%. The value of $\alpha=\epsilon_8/\sqrt{2}\,\epsilon_0$ is insignificantly changed. The scalar-meson masses are shifted by less than 10%, with the exception of the σ'. The widths are large, with the exception of the π_N. We calculate corrections to f_K/f_π, the wave-function renormalization constants, mixing angles, and the renormalized π and K propagators at $q^2=0$.

I. INTRODUCTION

Numerous Lagrangian models of strong interactions have been constructed with currents that satisfy chiral algebra and with current divergences that are proportional to fields.[1] In addition to providing a convenient method of imposing current-algebra constraints, these models have been used to study low-energy dynamics. The majority of work on the latter has been confined to the tree-graph approximation. Among these models we consider the renormalizable SU_3 σ model to be most attractive. Not only does it give quite a good account of spin-zero mesons,[2] but also provides a framework for calculating higher orders. We present in this paper calculations of the one-closed-loop contributions to the one-point and two-point functions. The details of the renormalization formalism are given in a separate paper[3] referred to as II. We find that the corrections to masses and symmetry-breaking parameters are quite small, generally less than 10%. This is surprising considering that we are dealing with strong interactions and large SU_3 mass splittings. The second order corrections supply two-body analyticity without changing the essential features of the tree approximation.

The elegant works of Lee[4] and Symanzik[5] have shown that the chiral-symmetric Lagrangians with symmetry breaking that is linear in the fields can be renormalized with all divergent counter-terms absorbed into the coupling constants of the symmetric Lagrangian. No new parameters are needed other than those that are present in the tree level. Further Basdevant and Lee[6] have demonstrated explicitly that the second-order corrections are small in the SU_2 σ model and through the use of Padé approximants are able to provide dynamics beyond the current algebra. Namely they are able to reproduce $\pi\pi$ S-wave phase shifts and generate the ρ and f mesons. Calculations through fourth order for nonchiral Lagrangians have shown that the Padé approximants converge very rapidly.[7]

The SU_3 σ model is a much richer one than the SU_2 counterpart. It contains a nonet of pseudoscalar mesons and a nonet of scalar mesons. It is by no means clear that the inclusion of more channels would alter the $SU_2 \times SU_2$ results for better or worse. However it is tempting to speculate that when SU_3 is included a nonet of vector mesons and tensor mesons can be generated through the Padé and perhaps also the axial-vector nonet in the scalar-pseudoscalar channel giving most of the known mesons. The $\pi\pi$ channel has higher thresholds in this model, i.e., $K\bar{K}$ and $\eta\eta$. This is important for the calculation of $\pi\pi$ S-wave phase shifts up into the region where present experimental ambiguities lie. πK phase shifts are readily available for comparison. The K_{l3} problem can be treated in an unambiguous way. The $\eta' \to \eta\pi\pi$ decay may be reexamined in this model. Numerous additional applications are possible.

Even if this model is not capable of fitting the vast amount of experimental data on mesons it is an important theoretical tool to study the breaking of SU_3 symmetry. The question of singularities in symmetry-breaking parameters can be studied in second order. Carruthers and Haymaker[8] have made the suggestion that mesons as described by the σ model may be analogous to a system near a critical point. If this is the case then model-independent results might be obtained from Wilson's renormalization-group techniques.[9]

To carry out calculations in this model, six physical quantities are needed to fix the six parameters in the Lagrangian. It is for this reason that we studied one- and two-point functions thoroughly. Pseudoscalar meson masses are the best determined experimental quantities. We show that the model can be completely determined at this level and give numerous predictions. In the spirit of this level of calculation we also renormalized at the same masses used to fit the model, hence picking up no second-order corrections for these masses. Further this means that the two-body thresholds composed of these masses will occur at the correct positions, which can be quite important in studying the highly unstable scalar mesons. This also has a distinct computational advantage in that once the model is fitted at the tree level, no further fitting need be done.

However our choice to renormalize at the masses makes it difficult to study symmetry limits.[10] It is nevertheless clear that the model is close to a spontaneously broken $SU_2 \times SU_2$ limit which is a consequence of a small m_π^2. Whether the model produces a spontaneously broken or symmetric $SU_3 \times SU_3$ limit is not so clear. The question has been studied in the tree approximation for this model[9,2] and found that either limit is possible within a reasonable range of fits to the data. Specifically if the tree value of m_σ (the low mass scalar isosinglet) is greater than 525 MeV then the spontaneously broken limit is realized. However with the σ width around 200–300 MeV, the second-order correction to this value can be significant. Even though our choice of σ mass is 450 and 500 MeV, it is not clear whether the spontaneously broken or the normal limit is realized. Nonetheless as long as the σ mass is close to that critical value the validity of any expansion about the $SU_3 \times SU_3$ would be questionable.

The $SU_3 \times SU_3$ algebra is defined by the equal-time current-commutation relations

$$[V_i^0(x), V_j^0(y)] = i f_{ijk} V_k^0(x)\delta(x-y),$$

$$[V_i^0(x), A_j^0(y)] = i f_{ijk} A_k^0(x)\delta(x-y),$$

$$[A_i^0(x), A_j^0(y)] = i f_{ijk} V_k^0(x)\delta(x-y).$$

The most general renormalizable meson chiral Lagrangian with a partial conservation of current (PCC) type of symmetry breaking is

$$\mathcal{L} = -\tfrac{1}{2}\operatorname{tr}(\partial_\mu M \partial_\mu M^\dagger) - \tfrac{1}{2}\mu^2 \operatorname{tr} MM^\dagger + g(\det M + \text{H.c.}) + f_1(\operatorname{tr} MM^\dagger)^2 + f_2 \operatorname{tr} MM^\dagger MM^\dagger - \epsilon_0\sigma_0 - \epsilon_8\sigma_8 . \qquad (1.1)$$

Here M is a 3×3 matrix of fields transforming as $(3, \bar{3})$ defined by

$$M = \frac{1}{\sqrt{2}} \sum_{i=0}^{8} \lambda^i (\sigma_i + i\phi_i),$$

where ϕ_i, σ_i are nonets of pseudoscalar and scalar fields, respectively.[11]

We define perturbation theory as an expansion in powers of λ which is defined through the relation

$$\mathcal{L}(M,\lambda)=\frac{1}{\lambda^2}\mathcal{L}(\lambda M). \tag{1.2}$$

λ is introduced for the purpose of power counting and is set equal to unity in the end. This is in effect an expansion in the number of closed loops. This expansion preserves the symmetry of the Lagrangian order by order.[12]

In Sec. II we show how various parameters can be determined by the known masses and the pion decay constant in the tree approximation. In Sec. III we give explicit calculations of the one-loop correction to the one- and two-point functions and describe the specific renormalization procedure. Finally we summarize the numerical results in Sec. IV. This section is fairly self-contained for those readers only interested in results.

II. TREE APPROXIMATION

The tree approximation is the starting point for calculating in perturbation theory. We review the fitting procedure and numerical results and point out the sensitivity of the solutions for small variations which is important in what follows.

The pseudoscalar masses are by far the best known meson parameters and hence we want to use them to fix the model. The procedure to solve for masses in the tree approximation is given in paper II, which we outline here. We start from the Lagrangian Eq. (1.1) which allows fields to have nonzero vacuum expectation values. The fields are translated such that the new fields have zero vacuum expectation values.

$$\sigma_i=s_i+\xi_i,\quad i=0,8$$
$$\langle s_i\rangle=0,$$
$$\langle\sigma_i\rangle=\xi_i,$$
$$b\equiv\xi_8/\sqrt{2}\,\xi_0.$$

The ξ_i can be found by demanding that the Lagrangian contain no linear terms in the fields, which gives equations for ϵ_0 and ϵ_8. The masses can then be found from the coefficients of the bilinear terms of the new fields s, ϕ. The mass formulas and the equations for the ϵ's are given in Table I. The ϵ equations simplify when expressed in terms of masses

$$\epsilon_0+\frac{\epsilon_8}{\sqrt{2}}=m_\pi^2\xi_0(1+b),$$
$$\epsilon_0-\frac{\epsilon_8}{\sqrt{2}}=m_K^2\xi_0(1-\tfrac{1}{2}b). \tag{2.1}$$

The decay constants f_π and f_K are given by

$$f_\pi=-(\tfrac{2}{3})^{1/2}\xi_0(1+b),$$
$$f_K=-(\tfrac{2}{3})^{1/2}\xi_0(1-\tfrac{1}{2}b), \tag{2.2}$$

or in terms of the ϵ's

$$f_\pi=-(\tfrac{2}{3})^{1/2}(\epsilon_0+\epsilon_8/\sqrt{2}\,),$$
$$f_K=-(\tfrac{2}{3})^{1/2}(\epsilon_0-\epsilon_8/2\sqrt{2}\,).$$

Noting the functional form of the masses (Table I) leads to the following fitting procedure:

$$\left.\begin{matrix}m_\pi^2\\ m_K^2\\ m_\eta^2\\ m_{\eta'}^2\end{matrix}\right\}\text{determines}\left\{\begin{matrix}\xi_0^2f_1-\mu^2/4(1+2b^2)\\ \xi_0^2f_2\\ \xi_0 g\\ b\,.\end{matrix}\right. \tag{2.3}$$

From this we can predict m_{π_N}, m_K, the $\eta\eta'$ mixing angle, and f_K/f_π. We choose the lower isoscalar scalar (σ) to fix μ^2, then the other scalar (σ') and the mixing are determined. Finally using Eq. (2.2) we can determine ξ_0 from f_π. This completes the fit. ϵ_0 and ϵ_8 are given by Eq. (2.1).

The masses for this fit are shown in Figs. 1 and 2. Figure 1 gives the masses of K, π_N, and κ as functions of b for fixed $m_\pi=0.1381$ GeV, $m_\eta=0.5488$ GeV, $m_{\eta'}=0.9571$ GeV. Although b could be fixed by the K mass as indicated in Eq. (2.3) the central mass of the K doublet is not known and the value of b is extremely sensitive to this value. The K mass curve in fact has a very shallow minimum at $b=-0.28$, $m_K^2=0.2446$ GeV^2, which is higher than the charged-K mass^2 (0.2438 GeV^2). For $-0.33\leq b\leq-0.20$ the curve lies below the neutral-K

TABLE I. The ϵ of Eq. (A2) and the mass matrices $(m_{ij}^\phi)^2$ and $(m_{ij}^s)^2$ of Eq. (A2) are tabulated, where $W_R=\mu^2-2\xi_0^2f_1A_R^1-2\xi_0^2f_2A_R^2-(2g/\sqrt{3})\xi_0A_R^3$ and W_R represent the corresponding row of ϵ_i/ξ_i, $(m_{ij}^\phi)^2$, and $(m_{ij}^s)^2$.

	W_R	A_R^1	A_R^2	A_R^3
	ϵ_0/ξ_0	$-2(1+2b^2)$	$-\frac{2}{3}(-2b^3+6b^2+1)$	$(1-b^2)$
	ϵ_8/ξ_8	$-2(1+2b^2)$	$2(b^3-b^2+1)$	$(1+b)$
$(m_{ij}^\phi)^2$:	π	$2(1+2b^2)$	$\frac{2}{3}(1+b)^2$	$(1-2b)$
	K	$2(1+2b^2)$	$\frac{2}{3}(7b^2-b+1)$	$(1+b)$
	η_{00}	$2(1+2b^2)$	$\frac{2}{3}(1+2b^2)$	-2
	η_{88}	$2(1+2b^2)$	$\frac{2}{3}(3b^2-2b+1)$	$(1+2b)$
	η_{80}	0	$\frac{2}{3}\sqrt{2}b(2-b)$	$\sqrt{2}b$
$(m_{ij}^s)^2$:	π_N	$2(1+2b^2)$	$2(b+1)^2$	$-(1-2b)$
	κ	$2(1+2b^2)$	$2(b^2-b+1)$	$-(1+b)$
	σ_{00}	$2(3+2b^2)$	$2(1+2b^2)$	2
	σ_{88}	$2(1+6b^2)$	$2(3b^2-2b+1)$	$-(1+2b)$
	σ_{80}	$4b\sqrt{2}$	$2\sqrt{2}b(2-b)$	$-\sqrt{2}b$

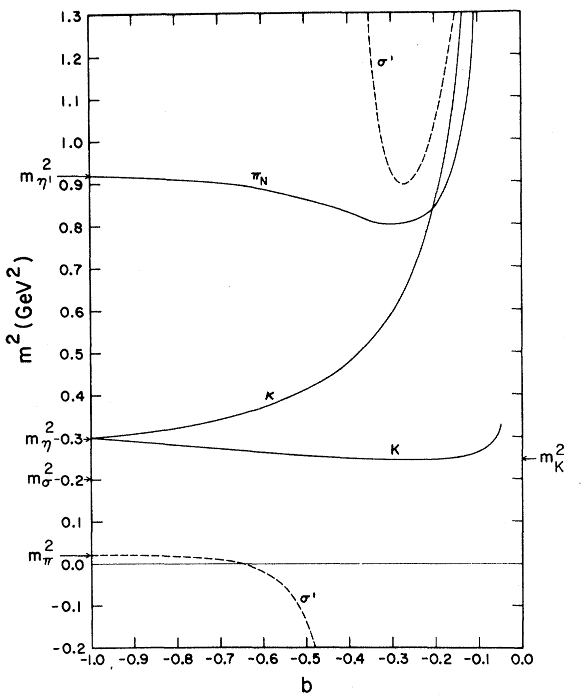

FIG. 1. Tree approximation masses as a function of b for fixed $m_\pi = 138.1$ MeV, $m_\eta = 548.8$ MeV, $m_{\eta'} = 957.1$ MeV. The value of b could be determined in principle from the K mass curve. But since the curve is extremely flat at the experimental K mass, b is very poorly determined. The σ' curve is the only one that is also a function of m_σ, in this example $m_\sigma = 450$ MeV.

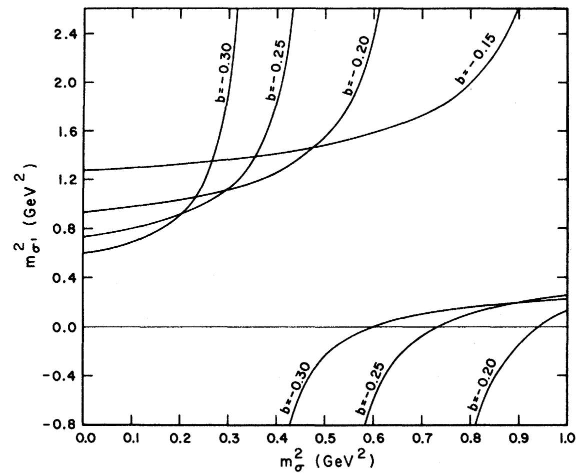

FIG. 2. $m_{\sigma'}^2$ as a function of m_σ^2 for a set of values of b in the tree approximation. The π, η, η' masses were fixed at physical values as in Fig. 1.

mass² (0.2478 GeV²). This means that we have a bonus degree of freedom in the value of b over a fairly large range at our disposal. The precise shape of $m_K^2(b)$ is sensitive to the values of the π, η, and η' masses but the above description is valid over the range of experimental error.

Also shown in Fig. 1 is the σ' mass for fixed $m_\sigma = 450$. (This value was chosen as an illustration and a further discussion is given in Sec. IV.[13]) The σ' as well as the κ have a sizable variation over the above range of b. Figure 2 gives $m_{\sigma'}$ as a function of m_σ for various values of b.

The existence of these scalar mesons is by no means well established. However some structures have been reported in these mass ranges.[14] It appears that two poles are needed in the $I = 0$ channel to fit the experimental $\pi\pi$ phase shift. As in the $\pi\pi$ case we may very well have a κ pole with a large width in πK scattering. Whether the structure as seen at the mass range of 950 MeV referred to as π_N is actually a resonance or a threshold enhancement is still rather controversial. In view of the difficulties of detecting these scalar mesons, we regard this model as a good starting point for doing dynamics.

III. SECOND-ORDER CALCULATION

A. General Formulation

In paper II we formulated the procedure to calculate one-loop corrections in this model. We showed that all divergences can be absorbed into the redefinition of f_1, f_2, g, and μ^2, and outlined a specific renormalization procedure. The notation of paper II, involving an 18 component label, was chosen for simplicity in demonstrating the cancellation of divergences. In this section we attempt to cast the formulas in a more directly usable form. For this purpose we introduce the more conventional nine component label for the scalars and pseudoscalars.

The Lagrangian Eq. (1.2) can be written in the following way[15]:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu s_i)^2 + \tfrac{1}{2}(\partial_\mu \phi_i)^2 - \tfrac{1}{2}(m^2 + \lambda^2\delta m^2)^s_{ij} s_i s_j - \tfrac{1}{2}(m^2 + \lambda^2\delta m^2)^\phi_{ij}\phi_i\phi_j + \lambda(G + \lambda^2\delta G)^s_{ijk} s_i s_j s_k$$
$$- 3\lambda(G + \lambda^2\delta G)^\phi_{ij,k}\phi_i\phi_j s_k + \tfrac{1}{3}\lambda^2(F + \lambda^2\delta F)_{ijkl}(s_i s_j s_k s_l + \phi_i\phi_j\phi_k\phi_l)$$
$$+ 2\lambda^2(\tilde{F} + \lambda^2\delta\tilde{F})_{ij,kl}\phi_i\phi_j s_k s_l - \frac{1}{\lambda}(E + \lambda^2\delta E)_i s_i\,. \qquad (3.1)$$

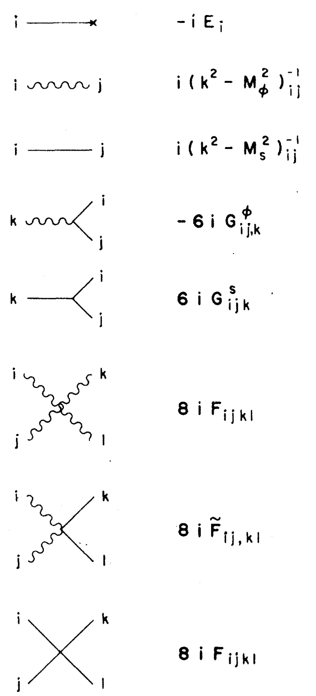

FIG. 3. Feynman rules. Solid lines are scalar, wiggly lines pseudoscalars.

We arrive at this form by using Eq. (1.2), introducing a nine component label, translating the scalar fields σ_0 and σ_8, $\sigma_i = s_i + \xi_i$, and introducing second-order correction terms to all parameters in the Lagrangian. We have not renormalized the fields at this stage. The ξ_i as well as the δ terms are as yet to be determined. We refer the reader to Appendix A for the formulas relating Eq. (3.1) to Eq. (1.2). The Feynman rules are given in Fig. 3.

A word is in order concerning the δ terms in Eq. (3.1). The parameters

$$\{P_i\} = \{m^{2\phi,s}, G^{\phi,s}, F, \tilde{F}, E\} \tag{3.2}$$

are explicit functions of the parameters

$$\{Q_i\} = \{f_1, f_2, g, \mu^2, \epsilon_0, \epsilon_8, \xi_0, \xi_8\} \tag{3.2$'$}$$

as given in Appendix A. Then

$$\delta P_i = \sum_j \frac{\partial P_i}{\partial Q_j} \delta Q_j \ . \tag{3.3}$$

The $\{P_i\}$ are linear functions of the $\{Q_i\}$ except for the dependence on ξ_0 and ξ_8. As Eq. (3.3) indicates, only first-order terms are kept. In order to ensure the proper symmetry all equations must be terminated to second order. The $\{\delta Q_i\}$ are finite except for $\delta f_1, \delta f_2, \delta g, \delta\mu^2$. For these we separate a divergent part denoted by D

$$\delta = D + \Delta .$$

The divergent parts have been calculated and are given in paper II. They are also given in Appendix A of this paper.

The 0, 8 components of the scalars and similarly the pseudoscalars get mixed in this model. It is convenient for calculating second-order corrections to define a new basis such that the mass matrix is diagonal in the tree approximation. That is,

$$U^{\phi}_{\alpha i} m^{\phi\,2}_{ij} \tilde{U}^{\phi}_{j\beta} = m^{\phi 2}_{\alpha} \delta_{\alpha\beta} \ ,$$

$$U^{s}_{\alpha i} m^{s\,2}_{ij} \tilde{U}^{s}_{j\beta} = m^{s\,2}_{\alpha} \delta_{\alpha\beta} \ ,$$

where

$$U^{\phi} = \begin{array}{c} \\ \pi \\ K \\ 8 \\ 0 \end{array} \begin{array}{c} \begin{array}{cccc} \pi & K & \eta & \eta' \end{array} \\ \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & \cos\theta_p & -\sin\theta_p \\ & & \sin\theta_p & \cos\theta_p \end{pmatrix} \end{array} .$$

And similarly for U^s with $\theta_p \to \theta_s$. We conform to the convention that in the new basis, a Greek index will be used. For example if O is any object with an SU_3 index then

$$O_j U_{j\beta} = O_\beta .$$

If we take O to be the pseudoscalar field then

$$\eta = \phi_8 \cos\theta_p - \phi_0 \sin\theta_p \ ,$$
$$\eta' = \phi_8 \sin\theta_p + \phi_0 \cos\theta_p \ . \tag{3.4}$$

Similarly

$$\sigma = s_8 \cos\theta_s - s_0 \sin\theta_s \ ,$$

$$\sigma' = s_8 \sin\theta_s + s_0 \cos\theta_s \ .$$

We emphasize that this diagonalization is accomplished only to the tree order. Also we only go to the new basis in internal loops and we do it there in order to treat all internal lines on the same footing. The external lines will all be in the original basis.

The calculation of the one-point function through second order now follows from the Feynman rules in Fig. 3. It is of the form

$$E_i + \lambda^2 \delta E_i + \lambda^2(\text{loop contributions}) = 0. \tag{3.5}$$

This must be zero so that the fields have vanish-

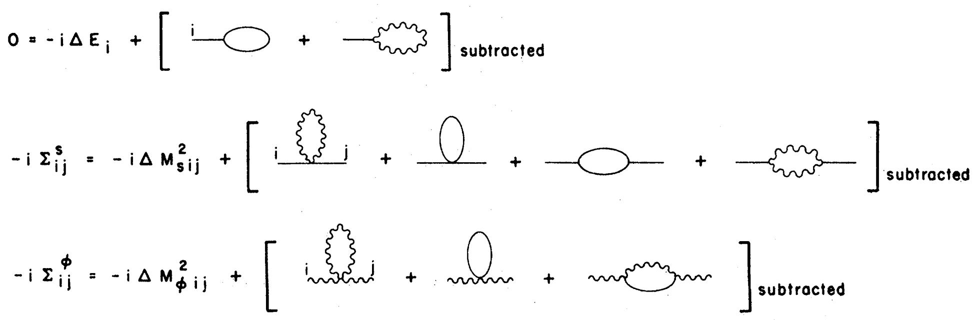

FIG. 4. Feynman graph representations of the second-order contribution to one point function and mass term. The prescription for the subtraction is given in the text.

ing vacuum expectation values. However $E_i=0$ from lowest order which gave the tree solution discussed in Sec. II. Equation (3.5) gives two equations ($i=0, 8$) relating the second-order contributions. The divergent part of δE_i, i.e., DE_i, cancels the divergent loop contributions as was shown in paper I. The second-order part of Eq. (3.5) is explicitly

$$E_i(\Delta f_1,\Delta f_2,\Delta g,\Delta\mu^2)+(m^s_{ij})^2\delta\xi_j-3\sum_\alpha\left[G^s_{\alpha\alpha i}((m^s_\alpha)^2-\mu^2)\bar{B}(0;(m^s_\alpha)^2,\mu^2)-G^\phi_{\alpha\alpha i}((m^\phi_\alpha)^2-\mu^2)\bar{B}(0;(m^\phi_\alpha)^2,\mu^2)\right]=0, \tag{3.6}$$

where

$$\bar{B}(0;(m^s_\alpha)^2,\mu^2)=B(0;(m^s_\alpha)^2,\mu^2)-B,$$

$$B(p^2;(m^s_\alpha)^2,\mu^2)=i\int\frac{d^4k}{(2\pi)^4}\frac{1}{[(p-k)^2-(m^s_\alpha)^2][k^2-\mu^2]},$$

and

$$B=B(0,\mu^2,\mu^2).$$

These integrals are evaluated in Appendix B. The Feynman graphs for this equation are given in Fig. 4. "Subtracted" means that the DE contribution has been combined with the loop integral making it convergent.

In like manner we get equations for the second-order mass terms. For the scalars we have

$$\begin{aligned}\Sigma^s_{ij}(s)=&(m^s_{ij})^2(\Delta f_1,\Delta f_2,\Delta g,\Delta\mu^2)-6G^s_{ijk}\Delta\xi_k\\&-4\sum_\alpha\{\bar{F}_{\alpha\alpha,ij}((m^\phi_\alpha)^2-\mu^2)\bar{B}(0;(m^\phi_\alpha)^2,\mu^2)+F_{\alpha\alpha ij}((m^s_\alpha)^2-\mu^2)\bar{B}(0;(m^s_\alpha)^2,\mu^2)\}\\&+18\sum_{\alpha,\beta}\{G^s_{\alpha\beta i}G^s_{\alpha\beta j}\bar{B}(s,(m^s_\alpha)^2,(m^s_\beta)^2)+G^\phi_{\alpha\beta,i}G^\phi_{\alpha\beta,j}\bar{B}(s,(m^\phi_\alpha)^2,(m^\phi_\beta)^2)\}\ .\end{aligned}\tag{3.7}$$

Similarly for the pseudoscalars:

$$\begin{aligned}\Sigma^\phi_{ij}(s)=&(m^\phi_{ij})^2(\Delta f_1,\Delta f_2,\Delta g,\Delta\mu^2)+6G^\phi_{ij,k}\Delta\xi_k\\&-4\sum_\alpha\{F_{ij\alpha\alpha}((m^\phi_\alpha)^2-\mu^2)\bar{B}(0;(m^\phi_\alpha)^2,\mu^2)+\bar{F}_{ij,\alpha\alpha}((m^s_\alpha)^2-\mu^2)\bar{B}(0;(m^s_\alpha)^2,\mu^2)\}\\&+36\sum_{\alpha\beta}\{G^\phi_{i\alpha,\beta}G^\phi_{j\alpha,\beta}\bar{B}(s;(m^\phi_\alpha)^2,(m^s_\beta)^2)\}\ .\end{aligned}\tag{3.8}$$

The Feynman graphs for these equations are also given in Fig. 4.

We also need formulas for the second-order calculation of the decay constants f_π and f_K. These have been derived in paper I:

$$f_\pi m_\pi^2 = -Z_\pi^{1/2}(\tfrac{2}{3})^{1/2}[\epsilon_0+\lambda^2\Delta\epsilon_0+(\tfrac{1}{2})^{1/2}(\epsilon_8+\lambda^2\Delta\epsilon_8)],$$

$$(f_K+\Delta f_K)m_K^2 = -Z_K^{1/2}(\tfrac{2}{3})^{1/2}[\epsilon_0+\lambda^2\Delta\epsilon_0 - \tfrac{1}{2}(\tfrac{1}{2})^{1/2}(\epsilon_8+\lambda^2\Delta\epsilon_8)]. \quad (3.9)$$

f_π, m_π, and m_K will have no second-order correction as described in Part B.

B. Specific Calculation

We now specialize to the calculation that we carried out. We first describe it in words in hopes that this will save the reader from getting bogged down in formulas. We first adjust the six Lagrangian parameters and ξ_0 and ξ_8, called $\{Q_i\}$ above [Eq. (3.2′)], so that to the tree order π, K, η, and η' have their physical mass[16] and the σ mass and f_π take on acceptable values. This gives six equations to determine six of the Q_i. The remaining two Q_i are determined by Eq. (2.1). In second order there are eight ΔQ_i. We chose to renormalize at the five above masses and f_π giving six equations for the ΔQ_i. The remaining two ΔQ_i are determined from the second-order contribution to the one-point function Eq. (3.6). These five masses and f_π will have no second-order corrections. This is important to keep in mind in sorting out the equations that follow.

We first consider the propagators to get formulas for the second-order corrections to masses and wave function renormalization constants. For the unmixed particles the unrenormalized propagator $D_i(s)$ is

$$D_i^{-1}(s) = s - m_i^2 - \Sigma_i(s), \quad (3.10)$$

where m_i^2 is the appropriate tree mass, i.e., $(m^s)^2$, $(m^\phi)^2$. Since we renormalize at the π and K masses we have

$$D_i^{-1}(m_i^2) = -\Sigma_i(m_i^2) = 0 \quad i=\pi, K.$$

These give two equations for the ΔQ_i. For the π_N and κ cases of Eq. (3.10), the solution of $D_i(s)=0$ occurs for complex s since the particles are unstable. To a good approximation we can get their masses as the zeros of the real part of Eq. (3.10).

We will need the renormalization constants for π and K in our calculation. In addition we use Ward's identities involving these particles as a check on our calculation and hence we need the renormalized π and K propagators. Let us write $D_i^{-1}(s)$, $i=\pi, K$ as follows:

$$D_i^{-1}(s) = (s-m_i^2)[1-\Sigma_i'(m_i^2)] - \Sigma_i(s) + (s-m_i^2)\Sigma_i'(m_i^2)$$

where $\Sigma'(s) = d\Sigma/ds$. To second order this can be written

$$D_i^{-1}(s) = Z_i D_{Ri}^{-1}(s)$$

where

$$Z_i = 1-\Sigma_i'(m_i^2)$$

$$D_{Ri}^{-1}(s) = s - m_i^2 - \Sigma_i(s) + (s-m_i^2)\Sigma_i'(m_i^2)$$

Z_i are the renormalization constants and D_R is the renormalized propagator which now has a pole with unit residue.

For the mixed states we again need the masses. For the pseudoscalars we also need to find the residue matrix at each pole in order to define the mixing angles and the renormalization constants Z_η and $Z_{\eta'}$. For the scalars we only calculate the masses. The question of mixing is discussed in Sec. IV.

We specify to a 2×2 block of D:

$$D = \begin{pmatrix} D_{88} & D_{80} \\ D_{08} & D_{00} \end{pmatrix}.$$

The masses m_1^2 and m_2^2 are determined by the solution of

$$\det D^{-1}(m_a^2) = 0.$$

In the present calculation the η' is stable. Its decay mode $\eta' \to \eta\pi\pi$ has a three-body final state in higher order and is not included in this approximation. Hence Eq. (3.11) has real solutions for both η and η'. Proper renormalization can be carried out to investigate the value of $D^{-1}(0)$. However, in this article we shall restrict ourselves to a procedure that is valid only in the neighborhood of the poles.

The residue at the pole a ($a=\eta, \eta'$) is given by

$$\frac{D_c^{-1}(m_a^2)}{\dfrac{d}{ds}\det D^{-1}(s)\Big|_{s=m_a^2}} \quad \frac{D_c^{-1}(m_a^2)}{\operatorname{tr} D_c^{-1}(m_a^2)\dfrac{d}{ds}D^{-1}(s)\Big|_{s=m_a^2}},$$

where $D_c^{-1}(m_a^2)$, defined as the cofactor of $D^{-1}(m_a^2)$, is factorizable since

$$\det D_c^{-1}(m_a^2) = \det D^{-1}(m_a^2) = 0.$$

We define the angle θ_a by the rotation

$$R(\theta_a) = \begin{pmatrix} \cos\theta_a & -\sin\theta_a \\ \sin\theta_a & \cos\theta_a \end{pmatrix}$$

such that

$$R(\theta_a)D_c^{-1}(m_a^2)\tilde{R}(\theta_a)$$

becomes diagonal. The renormalized propagators for η and η' are

$$D_{R\eta}(s)=Z_\eta^{-1}(\cos\theta_\eta,\ -\sin\theta_\eta)D\begin{pmatrix}\cos\theta_\eta\\ -\sin\theta_\eta\end{pmatrix},$$

$$D_{R\eta'}(s)=Z_{\eta'}^{-1}(\sin\theta_{\eta'},\ \cos\theta_{\eta'})D\begin{pmatrix}\sin\theta_{\eta'}\\ \cos\theta_{\eta'}\end{pmatrix},$$

where

$$Z_a=\frac{\mathrm{tr}\,D_c^{-1}(m_a^2)}{\mathrm{tr}D_c^{-1}(m_a^2)\dfrac{d}{ds}D^{-1}(s)\Big|_{s=m_a^2}}\ .$$

The renormalized fields are given by

$$\eta^R=Z_\eta^{-1/2}(\phi_8\cos\theta_\eta-\phi_0\sin\theta_\eta),$$

$$\eta'^R=Z_{\eta'}^{-1/2}(\phi_8\sin\theta_{\eta'}+\phi_0\cos\theta_{\eta'})\,.$$

Because of the energy dependence of the second-order corrections, the two states are no longer orthogonal and cannot be described by a single mixing angle.

Equations (3.6), (3.8), and (3.9) can be combined to yield

$$F_\pi m_\pi^2 =-(\tfrac{2}{3})^{1/2}Z_\pi^{-1/2}[(\xi_0+\Delta\xi_0)+(\tfrac{1}{2})^{1/2}(\xi_8+\Delta\xi_8)]D_{R\pi}^{-1}(0),$$

$$(F_K+\Delta F_K)m_K^2 =-(\tfrac{2}{3})^{1/2}Z_K^{-1/2}[(\xi_0+\Delta\xi_0) -\tfrac{1}{2}(\tfrac{1}{2})^{1/2}(\xi_8+\Delta\xi_8)]D_{RK}^{-1}(0). \quad (3.12)$$

This is a statement of the Goldstone theorem to second order and can be derived directly from the Ward's identities connecting the one- and two-point functions.

IV. NUMERICAL RESULTS

Our goal in this calculation was to examine the predictions of masses, widths and symmetry-breaking parameters. We not only can make predictions at this level but also gain valuable insight into how best to choose subtraction points in calculating phase shifts and decay processes. However, by limiting our scope we cannot compare our results directly with phase shifts in the scalar meson channels and hence there is some uncertainty in our fit. For this reason we give a few sample values of those free parameters that are left. These examples show that reasonable scalar masses and widths are possible in this model. Qualitatively we find that the one-loop corrections to the tree approximation are generally about 10% or less. We are quite confident that this result will remain in further refinements of the calculation.

Our results are summarized in Table II which we discuss here. The first three entries, f_π, $b=\langle\sigma_8\rangle/\langle\sigma_0\rangle\sqrt{2}$, and m_σ, we took as free parameters to vary. The experimental value of f_π is 0.095 GeV. However, its value is very important in determining the size of corrections and it is valuable to see how results depend on it. The value of b can in principle be determined from the K mass. However, as was pointed out in Sec. II for a wide range in b, i.e., between −0.2 and −0.33, the K mass lies between the neutral and charged experimental values. For the σ mass we took values in a range that can be associated with the maximum derivative of the $\pi\pi$ phase shift, i.e., 450 to 500 MeV. This was partly forced on us if we want to keep the σ' mass from getting too high. Figure 2 shows the relationship between these two masses in the tree approximation, and the corrected values do not differ a great deal from these in Fig. 2.

(i) Expansion parameter. The expansion parameter that determines roughly the size of corrections is proportional to $1/\xi_0^2$. As ξ_0 gets large, all widths and corrections go to zero. This is a consequence of our fixing certain masses. This can be seen by writing the coupling constants f_1, f_2, and g as

$$\frac{(f_i\xi_0^2)}{\xi_0^2}\ ,\quad \frac{(g\xi_0)}{\xi_0}\ .$$

We note that the values in the numerator are fixed by masses. (g always enters as the square or in a combination with the f_i.) f_π and f_K are simple functions of ξ_0:

$$f_\pi=(\tfrac{2}{3})^{1/2}\xi_0(1+b),$$

$$f_K=(\tfrac{2}{3})^{1/2}\xi_0(1-\tfrac{1}{2}b).$$

The parameter $1/f_K^2$ is a better estimate of the strength than $1/f_\pi^2$ since it is less sensitive to b in the range of b considered.

(ii) Symmetry-breaking parameters. A remarkable feature of this calculation is the smallness of corrections to the octet symmetry-breaking parameter b. It means that the symmetry-breaking picture derived from the tree approximation is very good. The parameter $a=\epsilon_8/\sqrt{2}\,\epsilon_0$ has an insignificant correction over the range of b studied. Its value is near the $SU_2\times SU_2$ limit value of −1. A better estimate of the size of the correction to a is perhaps $\delta a/(1+a)$ since δa is zero order by order if the pion mass were zero, i.e., $a=-1$. That is, the symmetry breaking of

the Lagrangian is close to the spontaneous realization of $SU_2 \times SU_2$ symmetry whereas the states are closer to a realization of SU_3 symmetry since b is small.

f_K/f_π has an interesting behavior. For a tree value of 1.5, the corrections lower it to 1.46 to 1.47. However, for the tree value of 1.42 the corrections lowers this less than 0.01. If we assume that the axial-vector Cabibbo angle is equal to the vector angle, the $\pi_{\mu 2}$, $K_{\mu 2}$, π_{e3}, and K_{e3} decay rates give a relation between f_K/f_π and the K_{l3} vector form factor $f_+(0)$ (Ref. 1)

$$\frac{f_K}{f_\pi} \approx 1.22 f_+(0).$$

Our range of f_K/f_π corresponds to $f_+(0) = 1.17 - 1.20$,[17] which is high to compare with current theoretical models. However for a fair comparison we should wait until $f_+(0)$ is properly calculated in this model.

TABLE II. Tree graph, λ^0, and second-order correction, λ^2, results for four sets of parameters. The first three entries f_π, b, m_σ, were taken as variables to show the dependence of results on these parameters. The quantities $m_\pi = 138.1$ MeV, $m_\eta = 548.8$ MeV, and $m_{\eta'} = 957.1$ MeV were fixed. We renormalized at these masses and hence they have no second-order corrections. The quantities denoted by an asterisk have no second-order corrections for the same reason. The quantities denoted by a dagger have a cutoff-dependent correction after renormalization.

	λ^0	λ^2	λ^0	λ^2	λ^0	λ^2	λ^0	λ^2
f_π	0.110	*	0.110	*	0.110	*	0.095	*
b	−0.250	0.012	−0.220	−0.001	−0.250	0.012	−0.250	0.016
m_σ (GeV)	0.450	*	0.450	*	0.500	*	0.450	*
θ_η	5.7°	−0.5°	3.13°	0.05°	5.7°	−0.5°	5.7°	−0.6°
$\theta_{\eta'}$	5.7°	0.9°	3.13°	1.50°	5.7°	1.0°	5.7°	1.2°
m_K (GeV)	0.4949	*	0.4961	*	0.4949	*	0.4949	*
m_{π_N} (GeV)	0.8973	−0.0208	0.9059	−0.0361	0.8973	−0.0228	0.8973	−0.0275
Γ_{π_N} (GeV)	···	0.059	···	0.072	···	0.059	···	0.077
m_κ (GeV)	0.8347	−0.024	0.8847	−0.053	0.8347	−0.024	0.8346	−0.034
Γ_κ (GeV)	···	0.190	···	0.290	···	0.190	···	0.260
Γ_σ (GeV)	···	0.170	···	0.210	···	0.220	···	0.240
$m_{\sigma'}$ (GeV)	0.9533	0.119	0.9851	0.081	0.9958	0.214	0.9534	0.174
$\Gamma_{\sigma'}$ (GeV)	···	0.090	···	0.110	···	0.200	···	0.130
θ_σ	−59.1°	10°	−63.6°	12°	−54.2°	12°	−59.1°	13°
$\theta_{\sigma'}$	−59.1°	7°	−63.6°	9°	−54.2°	8°	−59.1°	10°
f_K/f_π	1.500	−0.031	1.423	−0.006	1.500	−0.036	1.500	−0.040
ξ_0 (GeV)	0.180	−0.010	0.173	−0.008	0.180	−0.011	0.155	−0.012
ϵ_0 (GeV)^3	−0.0339	−0.0010	−0.0323	−0.0021	−0.0339	−0.0010	−0.0292	−0.0012
a	−0.9242	−0.0009	−0.9205	−0.0010	−0.9242	0.0013	−0.9242	0.001
f_1	−1.534	†	−0.1.270	†	−1.982	†	−2.056	†
f_2	−0.8090	†	−1.623	†	−0.809	†	−1.085	†
g (GeV)	1.200	†	1.259	†	1.200	†	1.390	†
μ^2 (GeV)^2	0.1502	†	0.1750	†	0.0851	†	0.1502	†
$Z_\pi^{1/2}$	1.0	−0.0429	1.0	−0.0519	1.0	−0.0477	1.0	−0.0575
$Z_K^{1/2}$	1.0	−0.0511	1.0	−0.0607	1.0	−0.0538	1.0	−0.0686
$Z_\eta^{1/2}$	1.0	−0.0512	1.0	−0.0601	1.0	−0.0538	1.0	−0.0686
$Z_{\eta'}^{1/2}$	1.0	−0.0727	1.0	−0.0578	1.0	−0.0701	1.0	−0.0971
$D_\pi^R(0)^{-1}$ (GeV)^2	−0.019062	0.000024	−0.019062	0.000048	−0.019062	0.000020	−0.019062	0.000031
$D_K^R(0)^{-1}$ (GeV)^2	−0.24494	0.00211	−0.24610	0.00235	−0.24494	0.00215	−0.24494	0.00282

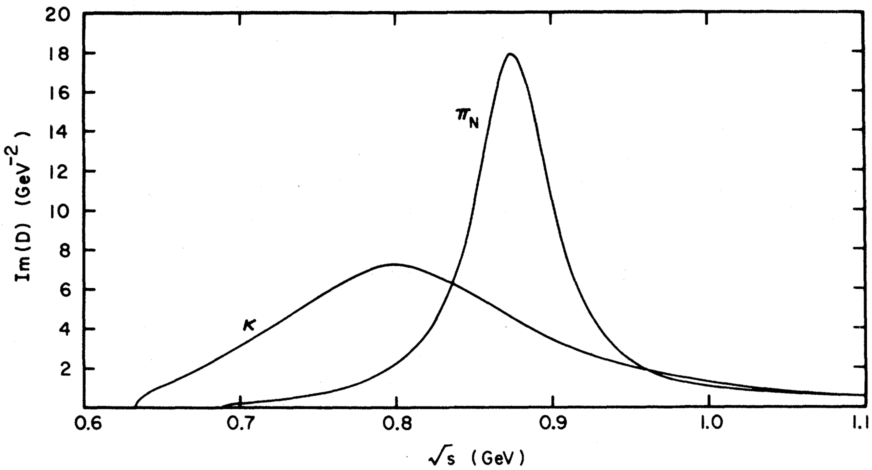

FIG. 5. The imaginary part of the propagator for π_N and κ for $f_\pi = 110$ MeV, $m_\sigma = 450$ MeV, $b = -0.25$.

(iii) Scalar mesons. The masses and widths are given in Table II. Figures 5 and 6 display the imaginary part of the propagator to show the spectrum more clearly. We emphasize that these graphs are not to be compared directly with experiment. Further background and changes in positions and widths is expected in a phase-shift calculation.

The π_N mass is remarkably stable under varying parameters and has a relatively small width. We find its position at about 870 MeV which is possibly too low. The κ on the other hand is quite wide, $\Gamma_\kappa \sim 200$ to 300 MeV, and with an acceptable mass of 800 MeV. This clearly prefers the slowly varying πK phase-shift solution in this mass range over the solution that rises very sharply through 90°.

The $\pi\pi$ S-wave spectrum is shown in Fig. 6. Since there are two states in this channel, the propagator is a 2×2 matrix. To get the $\pi\pi$ projection we take

$$\operatorname{Im} D = \operatorname{Im}\left[(\gamma_{\pi\pi 8}, \gamma_{\pi\pi 0})\begin{pmatrix} D_{88} & D_{08} \\ D_{80} & D_{00}\end{pmatrix}\begin{pmatrix}\gamma_{\pi\pi 8} \\ \gamma_{\pi\pi 0}\end{pmatrix}\right]\frac{1}{\gamma_{\pi\pi 0}{}^2 + \gamma_{\pi\pi 8}{}^2} \ .$$

The γ's are the tree-order coupling constants for $\pi\pi$ coupling to σ_0 and σ_8. The position of the lower σ was fixed as input. Specifically we demanded that

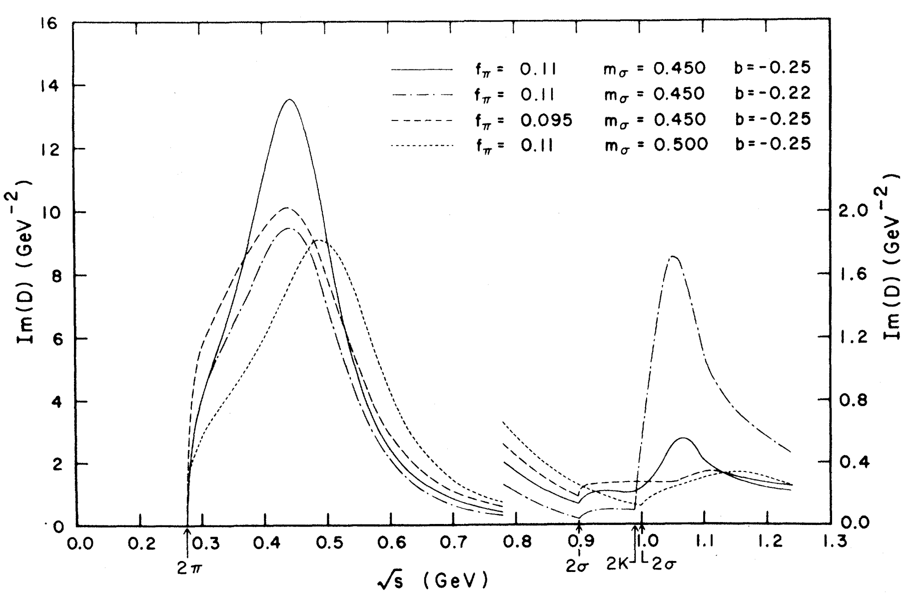

FIG. 6. The imaginary part of the σ, σ' progagator in the $\pi\pi$ channel. The units for f_π and m_σ are given in GeV.

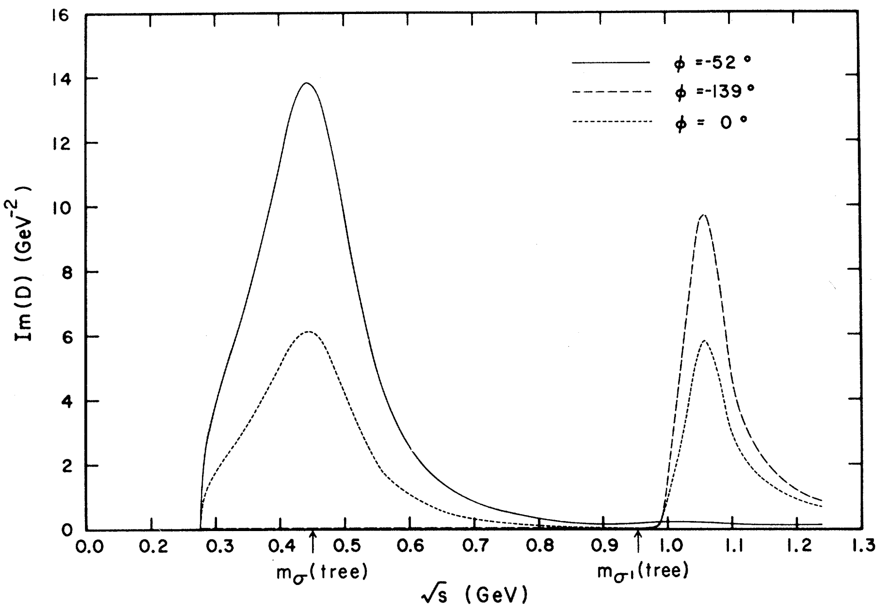

FIG. 7. Imaginary part of the σ, σ' propagator for various linear combinations of singlet and octet. See text for definition of the angle ϕ.

$$\mathrm{Re}\det\begin{pmatrix} D^{-1}{}_{88} & D^{-1}{}_{80} \\ D^{-1}{}_{08} & D^{-1}{}_{00} \end{pmatrix} = 0 \tag{4.1}$$

for s at the tree value of m_σ^2. The σ has a large width consistent with the recent phase-shift determination. A sharp resonance in this region has been ruled out in the phase shift analysis.[14] We note that the resonance shape is distorted by the $\pi\pi$ threshold. We believe this is a fluke due to our choice of renormalization procedure described by Eq. (4.1). The phase shift at this level of approximation would in fact go through 90° at $s = m_\sigma^2$ (tree) which is clearly wrong. What we really want is the phase shift to have its maximum derivative in this region. But that is outside the scope of this calculation. We have in fact found a virtual state to be below threshold in our method of fitting. The question of whether it is really there in this model will have to wait for a proper phase-shift calculation.

The σ' has a somewhat smaller width than the σ. Its partial width into $\pi\pi$ is quite small as shown in Fig. 6. In two cases shown in Fig. 6 the mass lies near the $K\bar{K}$ threshold. Since it couples very weakly to $\pi\pi$ its width is sharply reduced by the $K\bar{K}$ threshold factor.

(iv) Mixing angles. The mixing angles for η and η' are defined in Eq. (3.4). We note from Table II that the η is almost purely octet and η' almost purely singlet. In the second-order calculation we need two mixing angles, one to describe each particle. We notice that the second-order contribution to mixing angles is quite small. The scalar mixing angles are shown in Fig. 7 for one choice of parameters. We plot

$$\mathrm{Im}(\cos\phi,\ -\sin\phi)\begin{pmatrix} D^{-1}{}_{88} & D^{-1}{}_{80} \\ D^{-1}{}_{08} & D^{-1}{}_{00} \end{pmatrix}\begin{pmatrix} \cos\phi \\ -\sin\phi \end{pmatrix}$$

for three angles. The angles were chosen such that one resonance or the other had zero coupling. We also give $\phi = 0$ which shows both states present in approximately equal amounts. The results are quoted in Table II using the conventional mixing angle. Note that the canonical angle in which the upper couples only to $K\bar{K}$ and the lower to $\pi\pi$ is $-54.7°$. The scalars are clearly very near that angle. This corresponds in quark language to the σ having no strange quarks and the σ' only strange quarks.

(v) Renormalization constants, Ward's identities. The wave function renormalization constants $Z^{1/2}$ are given in Table II. They get corrections from 5% to 7%. However, the SU_3 breaking of these between the π, K, and η are less than 1%. We also quote the renormalized propagator at $s = 0$ for π and K. These were calculated directly and used as a check on our calculation. They enter in Ward's identities as described in Sec. III, Eq. (3.12).

Since the renormalized propagation has the same pole position and residue as the tree propagation, $\Delta D_R^{-1}(0)$ measures only the correction of the $D_R^{-1}(s)$ arising from the presence of continuum contributions to the propagator.

ACKNOWLEDGMENT

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APPENDIX A

We give here the formulas relating the Lagrangian used in Sec. III Eq. (3.1) to the more familiar form given by Eq. (1.1). We derive the formulas in three steps. (i) We introduce the nine component notion, (ii) we translate the fields, and (iii) we introduce the counterterms.

(i) Equation (1.1) becomes

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \sigma_i)^2 + \tfrac{1}{2}(\partial_\mu \phi_i)^2 - \tfrac{1}{2}\mu^2(\sigma_i{}^2 + \phi_i{}^2) + G_{ijk}(\sigma_i\sigma_j\sigma_k - 3\phi_i\phi_j\sigma_k) + 2\tilde{F}_{ij,kl}\phi_i\phi_j\sigma_k\sigma_l + \tfrac{1}{3}F_{ijkl}(\sigma_i\sigma_j\sigma_k\sigma_l + \phi_i\phi_j\phi_k\phi_l) - \epsilon_i\sigma_i\ , \tag{A1}$$

where

$$G_{ijk} = g\tfrac{1}{3}\epsilon_{abc}\lambda^i_{aa'}\lambda^j_{bb'}\lambda^k_{cc'}\epsilon_{a'b'c'}\ ,$$

$$F_{ijkl} = f_1(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}) + \tfrac{1}{2}f_2(d_{ijn}d_{nkl} + d_{iln}d_{njk} + d_{ikn}d_{njl})$$

$$\tilde{F}_{ij,kl} = f_1\delta_{ij}\delta_{kl} + \tfrac{1}{2}f_2(d_{ijn}d_{nkl} + f_{ikn}f_{njl} + f_{jkn}f_{nil})\ .$$

We have used the following identities:

$$\mathrm{Tr}\lambda^i\lambda^j = 2\delta_{ij}\ ,$$

$$[\lambda^i, \lambda^j] = 2if_{ijk}\lambda^k\ ,$$

$$\{\lambda^i, \lambda^j\} = 2d_{ijk}\lambda^k\ .$$

(ii) We now translate the 0, 8 scalar fields

$$\sigma_i = s_i + \xi_i\ .$$

Equation (A1) now becomes

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu s_i)^2 + \tfrac{1}{2}(\partial_\mu\phi_i)^2 - \tfrac{1}{2}(m^s_{ij})^2 s_i s_j - \tfrac{1}{2}(m^\phi_{ij})^2\phi_i\phi_j + G^s_{ijk}s_is_js_k - 3G^\phi_{ij,k}\phi_i\phi_js_k + 2\tilde{F}_{ij,kl}s_is_j\phi_k\phi_l + \tfrac{1}{3}F_{ijkl}(s_is_js_ks_l + \phi_i\phi_j\phi_k\phi_l) - E_is_i\ ,$$

where

$$\begin{aligned} G^s_{ijk} &= G_{ijk} + \tfrac{4}{3}F_{ijkl}\xi_l\ ,\\ G^\phi_{ij,k} &= G_{ijk} - \tfrac{4}{3}\tilde{F}_{ij,kl}\xi_l\ ,\\ (m^s_{ij})^2 &= \mu^2\delta_{ij} - 6G_{ijk}\xi_k - 4F_{ijkl}\xi_k\xi_l\ ,\\ (m^\phi_{ij})^2 &= \mu^2\delta_{ij} + 6G_{ijk}\xi_k - 4\tilde{F}_{ij,kl}\xi_k\xi_l\ ,\\ E_i &= \epsilon_i + \mu^2\xi_i - 3G_{ijk}\xi_j\xi_k - \tfrac{4}{3}F_{ijkl}\xi_j\xi_k\xi_l\ . \end{aligned} \tag{A2}$$

(iii) The last step to arrive at the desired form is to make the following replacements:

$$\delta(m^s)^2 = (m^s)^2(\delta f_1, \delta f_2, \delta g, \delta\mu^2) - 6G^s_{ijk}\delta\xi_k\ ,$$

$$\delta(m^\phi)^2 = (m^\phi)^2(\delta f_1, \delta f_2, \delta g, \delta\mu^2) + 6G^\phi_{ij,k}\delta\xi_k\ ,$$

$$\delta F = F(\delta f_1, \delta f_2),$$

$$\delta\tilde{F} = \tilde{F}(\delta f_1, \delta f_2),$$

$$\delta G^s = G^s(\delta f_1, \delta f_2, \delta g) + \tfrac{4}{3}F_{ijkl}\delta\xi_l\ ,$$

$$\delta G^{\phi}=G^{\phi}(\delta f_1,\delta f_2,\delta g)-\tfrac{4}{3}\tilde{F}_{ij,kl}\delta\xi_l\,,$$

$$\delta E_i=E_i(\delta f_1,\delta f_2,\delta g,\delta\mu^2,\delta\epsilon)+(m^s_{ij})^2\delta\xi_j\,.$$

For f_1, f_2, g, and μ^2, we separate off a divergent part of the variation δ, $\delta=D+\Delta$. The divergent parts were calculated in paper I and are given as follows:

$$Df_1=8(13f_1^2+3f_2^2+12f_1f_2)B\,,$$

$$Df_2=48f_2(f_1+f_2)B,$$

$$Dg=24g(f_1-f_2)B,$$

$$D\mu^2=16g^2B-(8f_1+48f_2)A\,,$$

where

$$A=i\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2-\mu^2+i\epsilon}\quad,$$

$$B=i\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2-\mu^2+i\epsilon)^2}\quad.$$

The finite part of the integrals was chosen arbitrarily and any change in the choice can be regrouped into the definition of Δ.

APPENDIX B

The integrals needed in the loop calculation are given here. The basic integral needed is

$$\bar{B}(s,x^2,y^2)=B(s,x^2,y^2)-B(0,\mu^2,\mu^2)$$

as defined in Sec. III. For $s<(x-y)^2$

$$\bar{B}(s,x^2,y^2)=\frac{1}{16\pi^2}\left\{-2+\ln\frac{xy}{\mu^2}+\frac{x^2-y^2}{s}\ln\frac{x}{y}+\frac{[(x-y)^2-s]^{1/2}[(x+y)^2-s]^{1/2}}{s}\ln\left[\frac{[(x-y)^2-s]^{1/2}[(x+y)^2-s]^{1/2}+s-x^2-y^2}{-2xy}\right]\right\};$$

for $(x-y)^2<s<(x+y)^2$

$$\bar{B}(s,x^2,y^2)=\frac{1}{16\pi^2}\left\{-2+\ln\frac{xy}{\mu^2}+\frac{x^2-y^2}{s}\ln\frac{x}{y}+\frac{[s-(x-y)^2]^{1/2}[(x+y)^2-s]^{1/2}}{s}\sin^{-1}\left[\frac{[s-(x-y)^2]^{1/2}[(x+y)^2-s]^{1/2}}{2xy}\right]\right\};$$

for $(x+y)^2<s$

$$\bar{B}(s,x^2,y^2)=\frac{1}{16\pi^2}\left\{-2+\ln\frac{xy}{\mu^2}+\frac{x^2-y^2}{s}\ln\frac{x}{y}+\frac{[s-(x-y)^2]^{1/2}[s-(x+y)^2]^{1/2}}{s}\left[-i\pi+\ln\left(\frac{[s-(x-y)^2]^{1/2}[s-(x+y)^2]^{1/2}+s-x^2-y^2}{2xy}\right)\right]\right\};$$

The following special case of this integral is also needed:

$$(x^2-\mu^2)\bar{B}(0,x^2,\mu^2)=\frac{1}{16\pi^2}\left[x^2\ln\frac{x^2}{\mu^2}-(x^2-\mu^2)\right].$$

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[17]S. Okubo [Phys. Rev. D 3, 2807 (1971)] and L.-F. Li and H. Pagels [Phys. Rev. D 4, 255 (1971)] have derived independently a rigorous bound for $f_+(0)$:

$$|f_+(0)| \le \frac{16}{(m_K - m_\pi)(\sqrt{m_K} + \sqrt{m_\pi})} \left(\frac{\pi\Delta(0)}{3(m_K + m_\pi)}\right)^{1/2}.$$

However, the numerical bound $|f_+(0)| \le 1$ is based on an estimate of $\Delta(0)$ by a model-dependent extrapolation which is at variance with the σ model. We have computed $\Delta(0)$ for various parameters in our model which gives $|f_+(0)| < 1.5$–2.0. Therefore, our value of $f_+(0)$ does not violate the rigorous bound.

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Renormalization of the $SU_3 \times SU_3$ σ Model

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We have carried out the renormalization procedure for the SU_3 σ model through second order in the presence of symmetry-breaking terms. We show explicitly that no new divergent counterterms are needed other than those required in the symmetric theory. The theory can be completely determined in terms of known masses and the decay constant f_π.

I. INTRODUCTION

Most of the results of current algebra[1,2] can be obtained from the tree approximation in the SU_3 σ model.[3,4] That is, the currents in the model satisfy chiral algebra, their divergences are proportional to fields and in the tree-order amplitudes are approximated by poles as is the case of current-algebra extrapolations. If we adopt this model as a starting point for doing dynamics it would be very interesting to see how higher-order corrections affect these results. We propose to do this by calculating corrections in standard perturbation theory. This paper presents the renormalization formalism for this model with symmetry breaking through second order. The explicit calculation of one- and two-point functions is done in a separate paper.[5]

It may very well be doubtful whether perturbation theory is meaningful for strong interactions. However we are encouraged by the recent successes in similar models which indicate that second-order corrections indeed may be sufficiently small.[6,7] Lee and Basdevant[6] have shown in the SU_2 σ model that the perturbation expansion parameter turned out to be about 0.1. They show that the Padé approximant to second order already gives interesting results for $\pi\pi$ scattering. The SU_3 σ model is much richer in predictive power than the SU_2 counterpart. It would be interesting to see if the same program can be carried out in SU_3. This would provide a dynamical model to study further the breaking of SU_3.

We focus most of our attention on the one- and two-point functions where most of the renormalization difficulties occur. This will not only pro-