PHYSICAL REVIEW D VOLUME 7, NUMBER 2 15 JANUARY 1973

Scaling Behavior in a Four-Dimensional Solvable Field Theory

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(Received 30 March 1972)

We examine the scaling properties of a four-dimensional scale-invariant solvable theory. We find that the scaling laws of the fields and the propagator behave quite differently from the two-dimensional situation. The fermion field scales with a noncanonical dimension while the propagator does not.

I. INTRODUCTION

The recent interest in scaling laws and approximate scale invariance in high-energy physics has produced in the literature a discussion of anomalous dimensions in quantum field theories. The existence of anomalous dimensions has been exhibited in several solvable models[1,2] and in perturbation-theory calculations.[3] Unfortunately the exact solutions have been for two-dimensional field theories with one space dimension. Such theories have been shown not to be the best proving ground for discussions of scaling laws.[4] In this note we examine the scaling properties of a four-dimensional solvable field theory which is the scale-invariant analog of one of the two-dimensional models previously discussed. We find that the scaling laws of the fields and the propagators are quite different from the two-dimensional situation. In particular the fermion field possesses an anomalous dimension which does not however appear in the fermion propagator.

In Sec. II we discuss the model and compare the scaling laws of the fermion propagator with the existing results for field theories in two dimensions. In Sec. III we examine the underlying canonical structure of the model and show how it can give rise to an anomalous dimension which is purely imaginary and hence cancels out in structures which are bilinear in the spinor fields.

II. SCALING OF GREEN'S FUNCTIONS IN TWO AND FOUR DIMENSIONS

We begin by summarizing the results of previous work on the scaling behavior of two solvable field theories.

The self-coupled fermion model described by the Lagrangian[5]

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi + g(\bar{\psi}\gamma^\mu\psi)^2 \tag{2.1}$$

and the Okubo models[6] given by

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - \tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi + \bar{\psi}\gamma^\mu\psi\,\partial_\mu F(\varphi) \tag{2.2}$$

for the special choice $F(\varphi) = g\varphi$, are both formally scale-invariant in a two-dimensional world with one space dimension. The two-point Green's function for both models is of the form

$$\begin{aligned} G_+(x) &\equiv i\langle 0|\,T(\psi(x)\bar{\psi}(0))\,|0\rangle \\ &= G_+^0(x)e^{ib[\Delta_+^0(x)-\Delta_+^0(0)]}, \end{aligned} \tag{2.3}$$

where

$$i\gamma^\mu\partial_\mu G_+^0(x) = \delta^4(x),$$

$$\Box^2\Delta_+^0(x) = \delta^4(x).$$

The constant b depends on the method of solution as follows:

$$b = g^2$$

for the canonical solution of both models,

$$b = (g^2/\pi)(1 - g^2/4\pi^2)^{-1}$$

for the Johnson-type nonlocal solution of the self-coupled model,[17] and

$$b = g^2(1 - g^2/2\pi)^{-1}$$

for the Johnson-type solution of the one-dimensional Okubo model[8] with

$$F(\varphi) = g\varphi.$$

In both models the wave-function renormalization constant is

$$Z = e^{-ib\Delta_+^0(0)}$$

so that the renormalized Green's function is

$$G_{+R}(x) = G_+^0(x)e^{ib\Delta_+^0(x)}. \tag{2.4}$$

The formal scale invariance of the Lagrangians implies the existence of a time-independent dilatation transformation U such that

$$U\psi(x)U^{-1}=s^{-d}\psi\left(\frac{1}{s}x\right).$$

Applying the invariance transformation to the Green's function gives the well-known result

$$d=\tfrac{1}{2}+b/4\pi. \tag{2.5}$$

The anomalous part $b/4\pi$ is nonvanishing both in the local canonical solutions and the Johnson-type point-splitting solutions.

We shall now examine the situation for the choice of $F(\varphi)$ which gives a formally scale-invariant Okubo model in three space dimensions. The particular choice is

$$F(\varphi)=g\ln(\varphi). \tag{2.6}$$

Ignoring the question of the precise meaning of such an expression as $\ln\varphi$, we have from Okubo's formal canonical analysis that the Green's function for this case is given by[9]

$$G_+(x)=G_+^0(x)(\cosh\tfrac{1}{2}\pi g)^{-1}\cosh(g\sin^{-1}[\Delta_+^0(x)/\Delta_+^0(0)]). \tag{2.7}$$

The wave-function renormalization constant is

$$Z=(\cosh\tfrac{1}{2}\pi g)^{-1}$$

so that the renormalized Green's function is given by

$$G_{+R}(x)=G_+^0(x)\cosh(g\sin^{-1}[\Delta_+^0(x)/\Delta_+^0(0)]). \tag{2.8}$$

The Green's function given by Eq. (2.8) can be seen by inspection to scale as $G_+^0(x)$ and hence with no anomalous dimension. In contrast with the one-dimensional choice, $F(\varphi)=g\varphi$, we see that here the quantity $\Delta_+^0(0)$ is not removed by wave-function renormalization. However, it is just the presence of the $\Delta_+^0(0)$ in the renormalized propagator which maintains the scaling to be nonanomalous. In our analysis there has been no nonlocal treatment of the bilinear structures in $\bar\psi$ and ψ; however, in the one-space-dimension models there was an anomalous behavior in both the Johnson-type solutions and the local canonical solutions.

In the next section we consider the canonical structure of the Okubo models to determine if the nonappearance of anomalous dimensions in $G_{+R}(x)$ really implies no anomalous dimensions for the field $\psi(x)$.

III. ANOMALOUS DIMENSIONS OF THE FERMION FIELDS

The Lagrangians of the form

$$\mathcal{L}=i\bar\psi\gamma^\mu\partial_\mu\psi-\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi+\bar\psi\gamma^\mu\psi\,\partial_\mu F(\varphi)$$

are by the transformation

$$\psi'=e^{-iF(\varphi)}\psi \tag{3.1}$$

cast into the form

$$\mathcal{L}'(\psi',\varphi)=i\bar\psi'\gamma^\mu\partial_\mu\psi'-\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi. \tag{3.2}$$

The fields ψ' and φ are easily seen to be canonical free fields and the transformation given by Eq. (3.1) can be generated by a formal unitary mapping.

We shall discuss the scaling properties of the interacting field ψ in terms of the free fields.

In particular, we construct the dilatation transformation $U=e^{iD}$ for the free fields. Where

$$U\psi'U^{-1}=s^{-3/2}\psi'\left(\frac{1}{s}x\right),$$

$$U\varphi U^{-1}=s^{-1}\varphi\left(\frac{1}{s}x\right),$$

writing $s=e^\alpha$, we have for the free-field system that

$$D=\alpha\int d^3x[T^{0\nu}(x)x_\nu-\tfrac{3}{2}\pi_{\psi'}(x)\psi'(x)-\pi_\varphi(x)\varphi(x)]. \tag{3.3}$$

Since there is no need to use an "improved" tensor[10] we have for $T^{\mu\nu}$ the canonical tensor. The dimensionality of ψ will be calculated by considering the commutator

$$[D(y^0),\psi(x)]_{x_0=y_0=0,\ \vec{x}=0}.$$

By means of the transformation Eq. (3.1) the above expression can be written as

$$[D(y^0),\psi(x)]=[D(y^0),e^{iF(x)}\psi'(x)]. \tag{3.4}$$

Evaluating the free-field commutators we obtain

$$[D(y^0),\psi(0)]_{y^0=0}=i\alpha\left(\tfrac{3}{2}+i\varphi\frac{\partial F}{\partial\varphi}\right)\psi(0). \tag{3.5}$$

Equation (3.5) tells us that the interacting field ψ scales as

$$\psi(x)\to s^{-[3/2+i(\partial F/\partial\varphi)\varphi]}\psi\left(\frac{1}{s}x\right) \tag{3.6}$$

when

$$x^\mu\to sx^\mu.$$

The calculation has produced an anomalous dimension which for an arbitrary $F(\varphi)$ is an anti-Hermitian q number. For the scale-invariant choice, $F(\varphi)=g\ln\varphi$ and therefore $(\partial F/\partial\varphi)\varphi=g$. In this case the anomalous part is the c number ig. Thus in the scale-invariant four-dimensional Okubo model the interacting-fermion field scales as

$$\psi(x)\to s^{-(3/2+ig)}\psi\left(\frac{1}{s}x\right). \tag{3.7}$$

The appearance of the purely imaginary anomalous dimension serves only to multiply the field ψ by the phase $e^{-ig\ln s}$. This behavior is consistent with our previous result that the Green's function exhibits no anomalous dimension.

The result obtained above can also be shown to hold to all orders in perturbation theory.

IV. SUMMARY

We have shown within the framework of a local analysis that the fermion field in a scale-invariant four-dimensional solvable model possesses an anomalous dimension which takes the form of a phase transformation on the field. For this reason observables of the fermion field, since they are bilinear in ψ and $\bar{\psi}$, do not exhibit an anomalous scaling in the model. This result is quite different from the behavior of the analogous two-dimensional theory. In the two-dimensional case the local solutions exhibit a nontrivial anomalous dimension which does not cancel in structures bilinear in ψ and $\bar{\psi}$.

ACKNOWLEDGMENT

We would like to thank Professor Paul Roman for a discussion concerning this work.

APPENDIX: NONLOCAL SOLUTION OF THE TWO-DIMENSIONAL SCALE-INVARIANT OKUBO MODEL

In one space and one time dimension we choose for $F(\varphi)$ the function

$$F(\varphi)=g\varphi \tag{A1}$$

which yields a formally scale-invariant Lagrangian. For this choice we have the equations of motion:

$$\gamma^{\mu}\frac{1}{i}\partial_{\mu}\psi(x)=g\varphi_{,\mu}(x)\gamma^{\mu}\psi(x)\ \text{ and }\ \Box^2\varphi=0\,. \tag{A2}$$

The equation of motion for the Green's function is

$$\gamma^{\mu}\frac{1}{i}\partial_{\mu}G_{+}(x,y)=\delta^2(x-y)+g\gamma^{\mu}i\langle 0|\,T(\varphi_{,\mu}(x)\psi(x)\bar{\psi}(x))\,|0\rangle\,, \tag{A3}$$

where

$$G_{+}(x,y)\equiv i\langle 0|\,T(\psi(x)\bar{\psi}(y))\,|0\rangle\,. \tag{A4}$$

In order to solve this equation we define, following Johnson, the vertex functions:

$$\begin{aligned}\Gamma(\xi,x,y)&\equiv i\langle 0|\,T(\varphi(\xi)\psi(x)\bar{\psi}(y))\,|0\rangle\,,\\ \Gamma_{\mu}(\xi,x,y)&\equiv i\langle 0|\,T(\varphi_{,\mu}(\xi)\psi(x)\bar{\psi}(y))\,|0\rangle\,.\end{aligned} \tag{A5}$$

Since φ commutes with ψ and $\bar{\psi}$ at equal times we have

$$\Gamma_{\mu}(\xi,x,y)=\partial^{\xi}_{\mu}\Gamma(\xi,x,y)\,. \tag{A6}$$

In this theory the fields and the conserved fermion current are related by

$$\varphi^{,\mu}=\pi^{\mu}_{\varphi}+gj^{\mu}\,. \tag{A7}$$

Since $\int j^0(x)d^3x$ generates phase transformations we must have

$$\begin{aligned}&[j^0(\xi),\psi(x)]_{\xi^0=x^0}=a\delta(\xi-x)\psi(x)\\ &\text{and}\\ &[\bar{j}^0(\xi),\psi(x)]_{\xi^0=x^0}=\bar{a}\delta(\xi-x)\psi(x)\,,\end{aligned} \tag{A8}$$

where a and $\bar{a}$ are to be determined. Putting this all together we find that

$$\Box_{\xi}{}^2\Gamma(\xi,x,y)=ag[\delta^2(\xi-x)-\delta^2(\xi-y)]G_{+}(x,y) \tag{A9}$$

which integrates immediately to give

$$\Gamma(\xi,x,y)=ag[\Delta^0_{+}(\xi-x)-\Delta^0_{+}(\xi-y)]G_{+}(x,y) \tag{A10}$$

and hence

$$\Gamma_{\mu}(\xi,x,y)=ag\,\partial^{\xi}_{\mu}[\Delta^0_{+}(\xi-x)-\Delta^0_{+}(\xi-y)]G_{+}(x,y)\,. \tag{A11}$$

Substituting Eq. (11) into Eq. (3) and integrating gives the result:

$$G_{+}(x,y)=G^0_{+}(x,y)e^{ag^2[\Delta^0_{+}(x-y)-\Delta^0_{+}(0)]}\,. \tag{A12}$$

In an exactly analogous manner we can compute the four-point function and find

$$\begin{aligned}G(xx'yy')=&\exp\{iag^2[\Delta^0_{+}(x-x')-\Delta^0_{+}(x-y')+\Delta^0_{+}(y-y')-\Delta^0_{+}(y-x')]\}G^0_{+}(x,y)G^0_{+}(x',y')\\ &-\exp\{iag^2[\Delta^0_{+}(x-x')-\Delta^0_{+}(x-y)-\Delta^0_{+}(y-y')-\Delta^0_{+}(y'-x')]\}G^0_{+}(x,y')G^0_{+}(x',y)\,.\end{aligned} \tag{A13}$$

A vertex similar to (5) may be defined with the current $j^{\mu}=\bar{\psi}\gamma^{\mu}\psi$ replacing $\varphi_{,\mu}$. This vertex may likewise be evaluated with the aid of Eq. (8) to get

$$i\langle 0|\,T(j^{\mu}(\xi)\psi(x)\bar{\psi}(y))\,|0\rangle=(ag^{\mu\nu}+\bar{a}\gamma_5\epsilon^{\mu\nu})\partial^{\xi}_{\nu}[\Delta^0_{+}(\xi-x)-\Delta^0_{+}(\xi-y)]\,. \tag{A14}$$

The constants a and $\bar{a}$ are now determined by the requirement that in a well-defined limit we have

$$\gamma^{\mu}_{y'}G(xx'yy') \xrightarrow[x'\to\xi\,;\,y'\to\xi]{} i\langle 0|\,T(j^{\mu}(\xi)\psi(x)\bar{\psi}(y))|0\rangle\,. \quad \text{(A15)}$$

Comparing with Johnson's limiting procedure[7] we see that

$$\gamma^{\mu}_{y'}G(xx'yy') \xrightarrow[x'\to\xi\,;\,y'\to\xi]{} [(1+ag^{2}/2\pi)g^{\mu\nu}+\gamma_5\epsilon^{\mu\nu}] \times\partial^{\xi}_{\nu}[\Delta^{0}_{+}(\xi-x)-\Delta^{0}_{+}(\xi-y)]\,. \quad \text{(A16)}$$

Comparing Eq. (16) with Eq. (14) we get finally

$$a=\frac{1}{1-g^{2}/2\pi} \quad\text{and}\quad \bar{a}=1\,. \quad \text{(A17)}$$

Here we see that the interaction modifies a but not $\bar{a}$; whereas in the Thirring model a and $\bar{a}$ are both affected in a symmetrical way. In both theories the axial-vector current is the dual of the vector current for the fermions. In the Okubo model, however, it is the field $\varphi_{,\mu}$ which appears in the vertex (A5). It is the appearance of the current j^{μ} in the vertex of the Thirring model which brings in a and $\bar{a}$ in this symmetrical way.

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Closed-Loop Corrections to the $SU_3 \times SU_3$ σ Model: One- and Two-Point Functions

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We present the calculation of one-loop corrections to the one- and two-point functions in the renormalizable SU_3 σ model with a symmetry-breaking term $L_{SB}=\epsilon_0\sigma_0+\epsilon_8\sigma_8$. We renormalized at the masses of $\pi, K, \eta, \eta', \sigma$, and f_π. The second-order corrections are found to be small compared to tree-approximation values. The measure of octet breaking, $b=\langle\sigma_8\rangle/\sqrt{2}\,\langle\sigma_0\rangle$, changes less than 5%. The value of $\alpha=\epsilon_8/\sqrt{2}\,\epsilon_0$ is insignificantly changed. The scalar-meson masses are shifted by less than 10%, with the exception of the σ'. The widths are large, with the exception of the π_N. We calculate corrections to f_K/f_π, the wave-function renormalization constants, mixing angles, and the renormalized π and K propagators at $q^2=0$.

I. INTRODUCTION

Numerous Lagrangian models of strong interactions have been constructed with currents that satisfy chiral algebra and with current divergences that are proportional to fields.[1] In addition to providing a convenient method of imposing current-algebra constraints, these models have been used to study low-energy dynamics. The majority of work on the latter has been confined to the tree-graph approximation. Among these models we consider the renormalizable SU_3 σ model to be most attractive. Not only does it give quite a good account of spin-zero mesons,[2] but also provides a framework for calculating higher orders. We present in this paper calculations of the one-closed-loop contributions to the one-point and two-point functions. The details of the renormalization formalism are given in a separate paper[3] referred to as II. We find that the corrections to masses and symmetry-breaking parameters are quite small, generally less than 10%. This is surprising considering that we are dealing with strong interactions and large SU_3 mass splittings. The second order corrections supply two-body analyticity without changing the essential features of the