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K Degeneracy in Nonadditive Dual Resonance Models

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The problem of K degeneracy, which has hitherto prevented calculations with nonadditive dual resonance models, is studied. It is shown that the second gauge condition selects out a unique member from each K-degenerate family. The vector so selected is not, in general, normalizable, but finite results can be obtained after renormalization. Calculation of norms and vertex functions is examined and is shown to be perfectly feasible.

I. INTRODUCTION

While the construction of mathematical dual resonance models of the S matrix has been almost spectacularly successful, the models constructed thus far all possess certain properties which prevent their application to the real world. In particular, no one has yet constructed a general ghost-free model with fermions. Also, the squares of the masses of particles without an additive or multiplicative conserved quantum number are integral, and there is always a zero-mass vector particle.

A class of models which do not necessarily possess the above-mentioned properties has been constructed by Bardakci and Halpern.[1] In such "nonadditive" models, the Virasoro operators $L^{(n)}$, and, in particular, the Hamiltonian $L^{(0)}$, are not simply the sum of operators involving spin and orbital degrees of freedom. The models possess spin-orbit coupling, and, even though resonances are narrow, the trajectories are not necessarily linear.[2] The MacDowell symmetry does not, therefore, imply that physical fermions are parity-doubled.

Nonadditive Bardakci-Halpern models are characterized by the presence of two complete, independent, mutually commuting Virasoro algebras $L^{(n)}$ and $K^{(n)}$. The algebra $L^{(n)}$ is analogous to the Virasoro algebra which exists in all ghost-free dual resonance models, but the second Virasoro algebra has no analog in the simpler models. Associated with the L algebra, one has gauge conditions which remove the ghosts that arise from the orbital degrees of freedom. In models with spin degrees of freedom there exist further ghosts which are not removed by the gauge conditions involving the L's. It is natural to hope that the K algebra may provide gauge conditions which remove these additional ghosts. Bardakci and Halpern showed that such gauge conditions do exist and do eliminate certain states. Whether they suffice to eliminate the ghosts is not yet known, and we do not propose to examine that question here.

Thus far, not much attention has been devoted to nonadditive models, probably because calculation of even the simplest quantities has been hindered by a problem known as K degeneracy. Since all representations of the Virasoro algebra are infinite-dimensional, and since the K algebra commutes with the Hamiltonian, it follows that the eigenstates of the Hamiltonian are all infinitely degenerate. As we have already mentioned, Bardakci and Halpern proposed applying a gauge condition, and they hoped that this condition would remove the degeneracy. They did not study the question, however, and all they were able to calculate was the spectrum of eigenstates of the Hamiltonian. As they were unable to calculate the signs of the norms, they could not determine whether the eigenstates corresponded to particles or ghosts.

Our aim here is to study the question of K degeneracy further. We shall show that the gauge condition does indeed select out one and only one member of each K-degenerate family. The state vector so selected will not in general be normalizable but, with an infinite renormalization, all quantities calculated in this paper will be finite. We shall show that calculation of norms and of vertex functions is perfectly feasible. It is accordingly possible to determine whether any state is a particle or a ghost, and to calculate coupling constants associated with particles or resonances. Thus, while we do not provide a practical method for the complete calculation of scattering amplitudes, we do have a large number of quantities which can be compared with experiment.

In Sec. II we construct the K-degenerate families and state some of their general properties. In Sec. III we show that the gauge condition does select a unique member of each K-degenerate family. The real states so selected will not be normalizable, but will be dominated by their components with very high sector number. We discuss the form of the solution of the gauge condition for such high-sector-number components, we shall

have to make use of plausibility arguments rather than rigorous proofs. In Sec. IV we shall study the equations provided by the gauge condition in more detail and shall give rough numerical calculations for the parameters left undetermined in Sec. III. The calculations of that section should suffice for the determination of the signs of norms, provided that certain parameters ν and γ are not too large. More accurate calculations can, of course, be done if necessary. Finally, in Sec. V, we shall discuss the calculation of vertices.

II. K-DEGENERATE FAMILIES

Nonadditive models have two mutually commuting Virasoro algebras $L^{(n)}$ and $K^{(n)}$, with

$$L^{(n)}+K^{(n)}=S^{(n)}. \tag{2.1}$$

Each set $L^{(n)}$ and $K^{(n)}$ satisfies the commutation relations

$$[L^{(n)}, L^{(m)}]=(n-m)L^{(n+m)}+\tfrac{1}{12}\beta n(n^2-1)\delta_{n,-m}, \tag{2.2a}$$

$$[K^{(n)}, K^{(m)}]=(n-m)K^{(n+m)}+\tfrac{1}{12}\gamma n(n^2-1)\delta_{n,-m}. \tag{2.2b}$$

The operators $L^{(n)}$ are the usual Virasoro operators; $L^{(0)}$, $L^{(1)}$, and $L^{(-1)}$ are associated with the duality projective group; and $L^{(0)}+p^2$ is the Hamiltonian. The operators $K^{(n)}$ form the new Virasoro algebra which is responsible for the K degeneracy. $S^{(0)}+p^2$ is the sector number; a state

$$\prod_{n_r} a^{\dagger(n_r)}\prod_{m_r} b^{\dagger(m_r)}|0\rangle$$

has sector number $\sum n_r+\sum m_r$. Any operator $a^{\dagger(n)}$, $b^{\dagger(n)}$, $L^{\dagger(n)}$, $K^{\dagger(n)}$, etc., increases the sector number by n.

The constants β and γ in (2.2) are the "dimension numbers." For the simple dual resonance model the corresponding constant is the number of dimensions of the space involved, but for nonadditive models β and γ may be fractional or negative.

The vacuum is the only state with sector number zero. From it we can form the K-degenerate family

$$\prod_{r=1}^{N} K^{(n_r)\dagger}|0\rangle, \quad n_r\geq n_s, \ r>s. \tag{2.3}$$

The vacuum will be an eigenstate of $K^{(0)}$:

$$K^{(0)}|0\rangle=\nu_0|0\rangle,$$

so that

$$K^{(0)}\prod_{r=1}^{N} K^{(n_r)\dagger}|0\rangle=\Big(\sum_{r=1}^{N} n_r+\nu_0\Big)\prod_{r=1}^{N} K^{(n_r)\dagger}|0\rangle. \tag{2.4}$$

Now let us consider states in the first sector. One of these will be $K^{(1)\dagger}|0\rangle$. The states in the first sector orthogonal to $K^{(1)\dagger}|0\rangle$ will be eigenstates of $K^{(0)}$, since $K^{(0)}$ and $S^{(0)}$ commute. Let us call them $|1a\rangle$, $|1b\rangle$, etc., with

$$K^{(0)}|1a\rangle=\nu_{1a}|1a\rangle, \tag{2.5}$$

etc. We can form new K-degenerate families

$$\prod_{r=1}^{N} K^{(n_r)\dagger}|1a\rangle, \qquad \prod_{r=1}^{N} K^{(n_r)\dagger}|2a\rangle, \tag{2.6}$$

etc. Each member of one of the K-degenerate families associated with the states $|0\rangle$, $|1a\rangle$, $|1b\rangle,\ldots$ will be orthogonal to any member of the other families, as may easily be shown from the Virasoro algebra.

In a similar way we can examine the second sector. The states $(K^{(1)\dagger})^2|0\rangle$, $K^{(2)\dagger}|0\rangle$, $K^{(1)\dagger}|1a\rangle$, $K^{(1)\dagger}|1b\rangle,\ldots$ will be members of the second sector. From the states of the second sector orthogonal to these states we can construct new K-degenerate families as before.

In this way we can decompose the whole Hilbert space into mutually orthogonal K-degenerate families. Each family will consist of states of the form

$$\prod_{r=1}^{N} K^{(n_r)\dagger}|a\rangle, \quad n_r\geq n_s, \quad r>s. \tag{2.7}$$

The state $|a\rangle$ will be called the basic state associated with the family. It is characterized by two numbers, the eigenvalues of $K^{(0)}$ and $S^{(0)}$:

$$K^{(0)}|a\rangle=\nu_a|a\rangle, \tag{2.8a}$$

$$S^{(0)}|a\rangle=(n_a-p^2)|a\rangle, \quad n_a \text{ integral}. \tag{2.8b}$$

From (2.1) and the eigenvalue condition

$$L^{(0)}|a\rangle=|a\rangle,$$

it follows that

$$p=n_a-\nu_a-1. \tag{2.9}$$

Equations (2.8) may be generalized to arbitrary states:

$$K^{(0)}\prod_{r=1}^{N} K^{(n_r)\dagger}|a\rangle=\Big(\sum_{r=1}^{N} n_r+\nu_a\Big)\prod_{r=1}^{N} K^{(n_r)\dagger}|a\rangle, \tag{2.10a}$$

$$S^{(0)}\prod_{r=1}^{N}K^{(n_r)}|a\rangle=\left(\sum_{r=1}^{N}n_r+n_a-p^2\right)\prod_{r=1}^{N}K^{(n_r)}|a\rangle\,. \tag{2.10b}$$

Equation (2.9) remains true for all members of the K-degenerate family.

III. THE GAUGE CONDITION

Bardakci and Halpern suggest that one remove the K degeneracy by demanding that real states satisfy the equation

$$(K^{(n)}-K^{(0)})|R\rangle=0\quad(n>0)\,. \tag{3.1}$$

This restriction is to be imposed in addition to the usual gauge restriction involving the L operators.

Let us first demonstrate that this restriction isolates one and only one member from each K-degenerate family. We do so by finding the scalar product between the vector $|R\rangle$ and the vectors $\langle 0|\prod_{r=1}^{N}K^{(n_r)}$ $(N\geq 0)$, which span the subspace associated with the K-degenerate family. Thus

$$\left\langle 0\left|\prod_{r=1}^{N}K^{(n_r)}\right|R\right\rangle=\left\langle 0\left|\prod_{r=1}^{n-1}K^{(n_r)}K^{(0)}\right|R\right\rangle\quad[\text{by (3.1)}]$$
$$=\left(\sum_{s=1}^{N-1}n_s+\nu\right)\left\langle 0\left|\prod_{r=1}^{N-1}K^{(n_r)}\right|R\right\rangle,$$

by (2.4) applied to the bra vector $\langle 0|\prod_{r=1}^{N-1}K^{(n_r)}$. Repeating this procedure, we obtain the result

$$\left\langle 0\left|\prod_{r=1}^{N}K^{(n_r)}\right|R\right\rangle=\prod_{r=1}^{N}\left(\sum_{s=1}^{n_r-1}n_s+\nu\right)\,. \tag{3.2}$$

Since the scalar product of the vector $|R\rangle$ with a complete set of vectors is determined, the vector $|R\rangle$ is unique. In fact, the vectors $\langle 0|\prod_{r=1}^{N}K^{(n_r)}$ form an extremely overcomplete set, because one obtains a complete set if one takes only those n_r's satisfying $n_r\leq n_s$, $r>s$. However, it is not difficult to show that the Eqs. (3.2), with $n_r\leq n_s$ if $r>s$, imply similar equations for all combinations of n_r, so that the Eqs. (3.2) are consistent.

On the other hand, if $|R\rangle$ does satisfy (3.2), the matrix element

$$\left\langle a\left|\prod_{r=1}^{N}K^{(n_r)}K^{(l)}\right|R\right\rangle$$

is independent of l. It follows that the vector $(K^{(l)}-K^{(0)})|R\rangle$ has zero scalar product with all members of a complete set, and it must be identically zero. Thus the vector $|R\rangle$ satisfies the required condition if and only if (3.2) is true, and the existence and uniqueness of a vector satisfying (3.1) is proved.

We have not shown, however, that the vector $|R\rangle$ is normalizable, and we shall now give arguments which indicate that $|R\rangle$, in general, diverges logarithmically as the sector number is increased. Let

$$|R\rangle=\sum_{n=0}^{\infty}|R\rangle_n\,, \tag{3.3}$$

where $|R\rangle_n$ is the component of $|R\rangle$ with sector number n. We first show that

$$|R\rangle_n=(n+\nu)^{-1}K^{(1)\dagger}|R\rangle_{n-1}\,,\quad n\to\infty\,. \tag{3.4}$$

Our demonstration of (3.4) will be a plausibility rather than a rigorous argument. We shall further verify that, in certain cases where $|R\rangle$ can be calculated exactly, (3.4) is true, and, in Sec. IV, we shall show that some of the consequences of (3.4) are at least consistent with our calculations.

Let us determine the scalar product of the right-hand side of (3.4) with a general vector. Thus, if $\sum n_r=n$,

$$\left\langle a\left|\prod_{r=1}^{N}K^{(n_r)}K^{(1)\dagger}\right|R\right\rangle_{n-1}=\sum_{s=1}^{N}(n_s+1)\left\langle a\left|\prod_{r=1}^{s-1}K^{(n_r)}K^{(n_s-1)}\prod_{r=s+1}^{N}K^{(n_r)}\right|R\right\rangle_{n-1}$$
$$=\sum_{s=1}^{N}(n_s+1)\prod_{r=1}^{s}\left(\sum_{t=1}^{r-1}n_t+\nu\right)\prod_{r=s+1}^{N}\left(\sum_{t=1}^{r-1}n_t+\nu-1\right)\quad[\text{by (3.2)}]$$
$$=\sum_{s=1}^{N}\Bigg[\left(\sum_{t=1}^{s}n_t+\nu\right)\prod_{r=1}^{s}\left(\sum_{t=1}^{r-1}n_t+\nu\right)\prod_{r=s+1}^{N}\left(\sum_{t=1}^{r-1}n_t+\nu-1\right)$$
$$-\left(\sum_{t=1}^{s-1}n_t+\nu\right)\prod_{r=1}^{s-1}\left(\sum_{t=1}^{r-1}n_t+\nu\right)\prod_{r=s}^{N}\left(\sum_{t=1}^{r-1}n_t+\nu-1\right)\Bigg]$$
$$=\left(\sum_{t=1}^{N}n_t+\nu\right)\prod_{r=1}^{N}\left(\sum_{s=1}^{r-1}n_s+\nu\right)-\nu\prod_{r=1}^{N}\left(\sum_{s=1}^{r-1}n_s+\nu-1\right)$$
$$=(n+\nu)\prod_{r=1}^{N}\left(\sum_{s=1}^{r-1}n_s+\nu\right)-\nu\prod_{r=1}^{N}\left(\sum_{s=1}^{r-1}n_s+\nu-1\right). \tag{3.5}$$

If n is large, the factor $n+\nu$ in the first term will be much larger than the factor ν in the second term. Also, if there are a large number of factors $\sum n_s+\nu$, the product in the first term will be much larger

than that in the second; if all the n_r's are equal to 1, the terms will differ by an additional factor n. On neglecting the second term of (3.5) we notice that the scalar product of the vector $(n+\nu)K^{(1)\dagger}|R\rangle_{n-1}$ with any other vector is equal to the scalar product of $|R\rangle_n$ with that vector. Equation (3.4) is thus proved.

We shall not attempt to justify further the dropping of the second term of (3.5), but shall assume that we can do so and study the consequences.

First let us determine how the norm of the state $|R\rangle_n$ varies with n when n is large. Using (3.6), we may write

$$\begin{aligned}\langle R|_{n+1}|R\rangle_{n+1} &= (n+1+\nu)^{-2}\langle R|_n K^{(1)}K^{(1)\dagger}|R\rangle_n \\ &= (n+1+\nu)^{-2}(2\langle R|_n K^{(0)}|R\rangle_n + \langle R|_n K^{(1)\dagger}K^{(1)}|R\rangle_n) \\ &= (n+1+\nu)^{-2}(2\langle R|_n K^{(0)}|R\rangle_n + \langle R|_{n-1}K^{(0)}K^{(0)}|R\rangle_{n-1}) \quad [\text{from (3.1)}] \\ &= (n+1+\nu)^{-2}[2(n+\nu)\langle R|_n|R\rangle_n + (n-1+\nu)^2\langle R|_{n-1}|R\rangle_{n-1}]. \end{aligned} \tag{3.6}$$

We assume that the norm of $|R\rangle_n$ varies smoothly with n and write

$$\langle R|_{n+1}|R\rangle_{n+1} = \langle R|_n|R\rangle_n + \frac{d}{dn}\langle R|_n|R\rangle_n + O(n^{-2}),$$

$$\langle R|_{n-1}|R\rangle_{n-1} = \langle R|_n|R\rangle_n - \frac{d}{dn}\langle R|_n|R\rangle_n + O(n^{-2}).$$

From (3.6), we then find that

$$\frac{d}{dn}\langle R|_n|R\rangle_n = -\frac{1}{n}\langle R|_n|R\rangle_n,$$

or

$$\langle R|_n|R\rangle_n = cn^{-1}. \tag{3.7}$$

It follows that the norm of $\sum_{n=0}^{N}|R\rangle_n$ increases logarithmically with N, so that the state $|R\rangle$ is not normalizable.

Let us write the vector $|R\rangle_n$ in the form

$$\begin{aligned}|R\rangle_n = n^{-\nu}\{n!\}^{-1}\ [&c_1(K^{(1)\dagger})^n + c_2(K^{(1)\dagger})^{n-2}K^{(2)\dagger} + c_3(K^{(1)\dagger})^{n-3}K^{(3)\dagger} \\ &+ c_4(K^{(1)\dagger})^{n-4}K^{(2)\dagger}K^{(2)\dagger} + c_5(K^{(1)\dagger})^{n-4}K^{(4)\dagger} + \cdots]\,|a\rangle. \end{aligned} \tag{3.8}$$

If Eq. (3.8) is accurate for a large value of n, it follows from (3.4) that it will be accurate for all larger values of n. Thus, by taking a sufficiently large number of terms in (3.8), we can obtain an accurate representation of $|R\rangle$ for all sufficiently large values of n. When n is large, (3.8) will be substantially equal to the vector

$$\begin{aligned}|R\rangle_n = \{&c_1 n^{-\nu}(n!)^{-1}(K^{(1)\dagger})^n + c_2(n-2)^{-\nu-2}[(n-2)!]^{-1}(K^{(1)\dagger})^{n+2}K^{(2)\dagger} \\ &+ c_3(n-3)^{-\nu-3}[(n-3)!]^{-1}(K^{(1)\dagger})^{n-3}K^{(3)\dagger} + \cdots\}|a\rangle, \end{aligned}$$

or

$$|R\rangle \simeq \sum_{n=0}^{\infty}\frac{(K^{(1)\dagger})^n}{n!}(c_1 n^{-\nu} + c_2 n^{-\nu-2}K^{(2)\dagger} + c_3 n^{-\nu-3}K^{(3)\dagger} + \cdots)|a\rangle, \tag{3.9}$$

Eq. (3.9) being interpreted in the sense that it is valid for the large-n components $|R\rangle_n$. As the vector $|R\rangle$ is, in general, not normalizable, it will be dominated by its large-n components, so that (3.9) may be regarded as an essentially correct equation.

If ν is equal to 0 or 1, the second term of Eq. (3.5) is strictly zero, and our treatment is exact for all values of n. We may therefore write

$$|R\rangle = \sum_{n=0}^{\infty}\frac{(K^{(1)\dagger})^n}{n!}|a\rangle, \quad \nu = 0 \tag{3.10a}$$

$$|R\rangle = \sum_{n=0}^{\infty}\frac{(K^{(1)\dagger})^n}{(n+1)!}|a\rangle, \quad \nu = 1. \tag{3.10b}$$

The value of the norm of $(K^{(1)\dagger})^n|a\rangle$ may be found by commuting a factor $K^{(1)}$ through the $K^{(1)}$'s:

$$\langle 0|(K^{(1)})^n(K^{(1)\dagger})^n|a\rangle = 2\sum_{r=1}^{N}\langle 0|(K^{(1)})^{n-1}(K^{(1)\dagger})^{r-1}K^{(0)}(K^{(1)\dagger})^{n-r}|a\rangle$$
$$= 2\sum_{r=1}^{N}(n-r+\nu)\langle 0|(K^{(1)})^{n-1}(K^{(1)\dagger})^{n-1}|a\rangle \quad [\text{from (2.4)}]$$
$$= n(n-1+2\nu)\langle 0|(K^{(1)})^{n-1}(K^{(1)\dagger})^{n-1}|a\rangle,$$

so that

$$\langle 0|(K^{(1)})^n(K^{(1)\dagger})^n|a\rangle = n!\Gamma(n+2\nu)[\Gamma(2\nu)]^{-1}. \tag{3.11}$$

If $\nu=0$, the factor $[\Gamma(2\nu)]^{-1}$ in (3.11) implies that all components of $|R\rangle$ other than $|R\rangle_0$ have zero norm. However, if ν is small but not precisely zero, Eqs. (3.10a) and (3.11) show that the square of the norm of $|R\rangle_n$ is equal to $n^{-1}[\Gamma(2\nu)]^{-1}$, in agreement with (3.7). If ν is equal to 1, it follows from Eqs. (3.10) and (3.11) that the square of the norm of $|R\rangle_n$ is $(n+1)^{-1}$, again in agreement with (3.7).

It is not difficult to verify directly that the vectors on the right-hand sides of Eqs. (3.10) satisfy (3.1) when ν is equal to 0 or 1, respectively. We also notice that Eqs. (3.10) are special cases of Eqs. (3.9).

If ν is equal to zero, the basic state $|a\rangle$ itself satisfies the equation

$$K^{(n)}|a\rangle=0, \quad n\geq 0$$

so that it satisfies the gauge conditions (3.1). In that particular case the state $|R\rangle$ is normalizable. It may happen in other special cases that the constant c in (3.7) is zero, i.e., that the state $|R\rangle_n$ approaches a zero-norm state as n approaches infinity. We would then require higher negative powers of n in (3.7), and the state $|R\rangle$ would again be normalizable. We shall show in the following paper that such is the case for the simplest nonadditive model without any internal degrees of freedom.

If a model possesses both normalizable and non-normalizable states, we would expect the normalizable states to disappear after renormalization. In particular, the Pomeranchuk trajectory, which is always present, even in nonadditive models,[3] has $\nu=0$. Thus, if the model possesses any non-normalizable trajectories, the Pomeranchuk trajectory will be renormalized away.

IV. DETERMINATION OF THE REAL STATES

In this section we shall calculate explicitly the vectors $|R\rangle_n$ for $n\leq 4$. We shall show that the norms do appear to be approaching the limits inferred in Sec. III, and we shall also conclude that the calculations are sufficient to determine the signs of the square of the norms if ν is not too large.

We write

$$|R\rangle = |a\rangle + bK^{(1)\dagger}|a\rangle + c(K^{(1)\dagger})^2|a\rangle + dK^{(2)\dagger}|a\rangle + e(K^{(1)\dagger})^3|a\rangle + fK^{(1)\dagger}K^{(2)\dagger}|a\rangle + gK^{(3)\dagger}|a\rangle$$
$$+ h(K^{(1)\dagger})^4|a\rangle + k(K^{(1)\dagger})^2K^{(2)\dagger}|a\rangle + lK^{(1)\dagger}K^{(3)\dagger}|a\rangle + mK^{(2)\dagger}K^{(2)\dagger}|a\rangle + nK^{(4)\dagger}|a\rangle + \cdots. \tag{4.1}$$

In order to illustrate the method of determining the constants, we apply the gauge condition (3.1) to a few of the lowest sectors. The simplest equation is

$$K^{(1)}|R\rangle_1 = K^{(0)}|R\rangle_0$$

or

$$bK^{(1)}K^{(1)}|a\rangle = K^{(0)}|a\rangle.$$

We commute $K^{(1)}$ with $K^{(1)\dagger}$, since $K^{(1)}|a\rangle=0$:

$$2bK^{(0)}|a\rangle = K^{(0)}|a\rangle$$

or

$$b=\tfrac{1}{2}. \tag{4.2}$$

For the second sector, we require two equations

$$K^{(1)}|R\rangle_2 = K^{(0)}|R\rangle_1, \tag{4.3a}$$

$$K^{(2)}|R\rangle_2 = K^{(0)}|R\rangle_0. \tag{4.3b}$$

Equation (4.3a) gives

$$cK^{(1)}(K^{(1)\dagger})^2|a\rangle + dK^{(1)}K^{(2)\dagger}|a\rangle = bK^{(0)}K^{(1)\dagger}|a\rangle,$$

or, after commuting $K^{(1)}$ through to the right-hand side,

$$2c(K^{(0)}K^{(1)\dagger}|a\rangle + K^{(1)\dagger}K^{(0)}|a\rangle) + 3dK^{(1)\dagger}|a\rangle = bK^{(0)}K^{(1)\dagger}|a\rangle,$$

i.e.,

$$2c(2\nu+1)K^{(1)\dagger}|a\rangle + 3dK^{(1)\dagger}|a\rangle = b(\nu+1)K^{(1)\dagger}|a\rangle,$$

whence

$$2(2\nu+1)c+3d=3(\nu+1)b. \tag{4.4a}$$

From Eq. (4.3b),

$$cK^{(2)}(K^{(1)\dagger})^2|a\rangle + dK^{(2)}K^{(2)\dagger}|a\rangle = K^{(0)}|a\rangle,$$

or

$$3cK^{(1)}K^{(1)\dagger}|a\rangle + d(4K^{(0)} + \tfrac{1}{2}\gamma)|a\rangle = K^{(0)}|a\rangle,$$

i.e.,

$$6\nu cK^{(0)}|a\rangle + d(4K^{(0)} + \tfrac{1}{2}\gamma)|a\rangle = K^{(0)}|a\rangle,$$

from which

$$6\nu c + (4 + \tfrac{1}{2}\gamma)d = \nu. \tag{4.4b}$$

Equations (4.4) enable us to determine the constants c and d.

The constants e, f, and g associated with the third sector, and the subsequent constants associated with the higher sectors, may be determined in a similar manner. In general, the complete series of conditions (3.1) will give us a redundant set of equations, but the equations will always be consistent, since we know that there exists a vector which satisfies all the gauge conditions.

For stating the results of the solution of the equations, it is convenient to define

$$\alpha = \tfrac{1}{2}\gamma + 4\nu, \tag{4.5a}$$

$$D_1 = \alpha(2\nu+1) - 9\nu, \tag{4.5b}$$

$$D_2 = 4(2\alpha-1)[-5\alpha(2\nu+3) + 32\nu^2 + 101\nu - 33]. \tag{4.5c}$$

Then

$$b = \tfrac{1}{2}, \tag{4.6a}$$

$$c = (4D_1)^{-1}[\alpha(\nu+1) - 6\nu], \tag{4.6b}$$

$$d = (2D_1)^{-1}\nu(\nu-1), \tag{4.6c}$$

$$e = (24D_1)^{-1}[\alpha(\nu+2) - 9\nu], \tag{4.6d}$$

$$f = (4D_1)^{-1}\nu(\nu-1), \tag{4.6e}$$

$$g = 0, \tag{4.6f}$$

$$h = (24D_1D_2)^{-1}[-10\alpha^3(\nu+2)(\nu+3) + \alpha^2(32\nu^3 + 353\nu^2 + 617\nu - 102) + \alpha(-448\nu^3 - 1749\nu^2 + 271\nu + 66) + 18\nu(3\nu+5)(8\nu-3)], \tag{4.6g}$$

$$k = (2D_1D_2)^{-1}[-10\alpha^2(\nu+2) + \alpha(32\nu^2 + 137\nu + 1) - 64\nu^2 - 135\nu + 9], \tag{4.6h}$$

$$l = (2D_1D_2)^{-1}[5\alpha^2 + \alpha(-30\nu - 61) + 6(6\nu^2 + 32\nu + 3)], \tag{4.6i}$$

$$m = (2D_1D_2)^{-1}[2\alpha(-5\nu^2 - 7\nu - 6) + \nu(37\nu + 101)], \tag{4.6j}$$

$$n = (2D_1D_2)^{-1}[-2\alpha^2(\nu+3) + 2(3\nu^2 + 26\nu + 33) - 75\nu^2 - 237\nu - 45]. \tag{4.6k}$$

By comparing the calculations of the present section, insofar as they apply to the fourth sector, with the fourth-sector components of (3.9), we can obtain rough values of the constants $c_1, \ldots, c_5$. More accurate results would necessitate taking the calculations of the present section to higher sector numbers.

Having found the states $|R\rangle_n$ ($n \leq 4$), we can calculate their norms. This may be done fairly simply by making use of (3.1). Thus, to take $|R\rangle_2$ as an example,

$$\langle R|_2|R\rangle_2 = \langle R|_2[c(K^{(1)\dagger})^2|a\rangle + dK^{(2)\dagger}|a\rangle].$$

Now, by (3.1),

$$\langle R|_2K^{(1)\dagger} = \langle R|_1K^{(0)} = (1+\nu)\langle R|_1,$$

$$\langle R|_2(K^{(1)\dagger})^2 = (1+\nu)\langle R|_1K^{(1)\dagger} = \nu(1+\nu)\langle a|,$$

Hence

$$\langle R|_2|R\rangle_2 = c\nu(1+\nu) + d\nu.$$

By using this equation and a similar equation for $|R\rangle_3$, we find

$$\langle R|_1|R\rangle_1 = \tfrac{1}{2}\nu, \tag{4.7a}$$

$$\langle R|_2|R\rangle_2 = (4D_1)^{-1}\nu[\alpha(\nu+1)^2 - 4\nu(\nu+2)], \tag{4.7b}$$

$$\langle R|_3|R\rangle_3 = (24D_1)^{-1}\nu(\nu+2)[\alpha(\nu+1)(\nu+2) - 3\nu(\nu+5)]. \tag{4.7c}$$

The norm of $|R\rangle_4$ becomes somewhat unwieldy if written down in terms of ν and α. In terms of the coefficients in (4.1), it is

$$\langle R|_4|R\rangle_4 = \nu[h(\nu+1)(\nu+2)(\nu+3) + k(\nu+2)(\nu+3) + l(\nu+3) + m(\nu+2) + n]. \tag{4.7d}$$

The algebraic results for the expansion parameters of the vector $|R\rangle$, and for the norms, become rather involved as the order n increases, but one can always perform the calculations for particular values of ν and α.

We shall now use the above calculations to verify the behavior of the norms calculated in Sec. III. For $\nu = 0$ or 1, we have seen that the results of Sec. III are exact. Let us therefore calculate the norms for $\nu = -1$. Negative values of ν are of interest in that the norms are not necessarily positive, so that the K degeneracy may convert ghosts into particles. We shall take $\gamma = 4$, though the be-

havior of the norms is not very sensitive to the value of γ. We then find the following results for $\nu=-1$, $\gamma=4$:

$$\langle R|_2|R\rangle_2=-0.091, \quad 2\langle R|_2|R\rangle_2=-0.182,$$
$$\langle R|_3|R\rangle_3=-0.045, \quad 3\langle R|_3|R\rangle_3=-0.135,$$
$$\langle R|_4|R\rangle_4=-0.029, \quad 4\langle R|_4|R\rangle_4=-0.116.$$

While one cannot, of course, infer the asymptotic behavior of $\langle R|_n|R\rangle_n$ from values of $n\leq 4$, the results are at least consistent with the behavior $\langle R|_n|R\rangle_n\simeq cn^{-1}$. They could in fact be better represented by the formula $\langle R|_n|R\rangle_n\simeq c(n-1)^{-1}$, which is asymptotically equivalent to the formula $\langle R|_n|R\rangle_n\simeq cn^{-1}$.

It is also of interest to calculate the positions of the zeros in the norms, since the sign of the square of the norm determines whether or not the K degeneracy converts ghosts to particles and vice versa. The norms have zeros at $\nu=0$, so that, for small negative values of ν, the K degeneracy does interchange ghosts and particles. For $\gamma=4$, we find the next zeros at the following positions:

$$\langle R|_2|R\rangle_2=0, \quad \text{if } \nu=-1.7$$
$$\langle R|_3|R\rangle_3=0, \quad \text{if } \nu=-2$$
$$\langle R|_4|R\rangle_4=0, \quad \text{if } \nu=-2.2.$$

Since the norm of $|R\rangle$ diverges logarithmically as n increases, the sign of its square will be determined by that of $|R\rangle_n$ for large n. Thus, for $\gamma=4$, we can say that ghosts and particles will be interchanged for values of ν between 0 and about -2.2. The squares of the norms have further zeros at larger negative values of ν, but one would have to calculate the norms for values of n larger than 4 to determine their position. The number of zeros on the negative real axis increases with n, and it is likely that the square of the norm $\langle R|R\rangle$ has an infinite number of zeros.

We may mention that the square of the norm $\langle R|_4|R\rangle_4$ has a pole at $\nu=-\frac{3}{8}$ for $\gamma=4$, due to the vanishing of the factor $2\alpha-1$ in D_2. However, the residue at the pole is approximately 2×10^{-6}, and the square of the norm, after changing sign at $\nu=-\frac{3}{8}$, changes sign again within a distance of 10^{-4} to the left. Thus, unless ν is exactly equal to $-\frac{3}{8}$ or extremely close to this value, the pole is of no significance.

Another remark we may make concerns the combination of values $\nu=-\frac{1}{2}$, $\gamma=-\frac{25}{2}$, values of ν and γ which occur in the simple nonadditive model to be studied in the following paper. For $\gamma=-\frac{25}{2}$, we find zeros of $\langle R|_n|R\rangle_n$ at the following values of ν:

$$\langle R|_2|R\rangle_2=0, \quad \text{for } \nu=-0.42$$
$$\langle R|_3|R\rangle_3=0, \quad \text{for } \nu=-0.478$$
$$\langle R|_4|R\rangle_4=0, \quad \text{for } \nu=-0.492.$$

The position of the zero appears to be converging to $\nu=-\frac{1}{2}$ as n becomes large. The vector $|R\rangle$ then has finite norm. As the only values of ν which occur in the model in question are $\nu=0$ and $\nu=-\frac{1}{2}$, the norms are all finite.

V. VERTICES

In Sec. III we examined the states which satisfy the gauge condition (3.1). It is now necessary to construct vertex operators which represent the coupling of such a state with two other states.

We shall treat the simplest state satisfying the gauge condition, i.e., the state (3.9) where $|a\rangle$ is taken to be the vacuum state. The treatment may be extended without essential modification to any state $|a\rangle$ on the leading trajectory. To begin, we consider the first term of (3.9), and we neglect the factor $n^{-\nu}$. The vertex corresponding to the state

$$\frac{(K^{(1)})^n}{n!}|0\rangle \tag{5.1}$$

is simply the canonical vertex

$$V\equiv\exp\left[\sqrt{2}\,k\sum_n\left(\frac{a^{(n)}}{\sqrt{n}}-\frac{a^{(n)\dagger}}{\sqrt{n}}\right)\right],$$

where normal ordering of the a's is implied. In order to justify this statement, we have to prove[4]

$$\lim_{z\to0} z^{L^{(0)}}\exp\left[\sqrt{2}\,k\sum_n\left(\frac{a^{(n)}}{\sqrt{n}}-\frac{a^{(n)\dagger}}{\sqrt{n}}\right)\right]|0\rangle = \sum\frac{(K^{(1)\dagger})^n}{n!}|0\rangle. \tag{5.2}$$

Now it is not difficult to prove the identity

$$\exp\left[\sqrt{2}\,k\sum_n\left(\frac{a^{(n)}}{\sqrt{n}}-\frac{a^{(n)\dagger}}{\sqrt{n}}\right)\right]|0\rangle=\sum\frac{(S^{(1)\dagger})^n}{n!}|0\rangle. \tag{5.3}$$

On applying the factor $z^{L^{(0)}}$ $(z\to0)$ to the right-hand side of (5.3), commuting it through the factors $S^{(1)\dagger}$, and using (2.1), the terms $L^{(1)\dagger}$ drop out, and we are left with the state (5.1). Our assertion is thus proved.

To express the vertex corresponding to the state

$$\sum_n n^{-\nu}\frac{(K^{(1)\dagger})^n}{n!}|0\rangle, \tag{5.4}$$

we define a quantity

$$[K^{(0)}\cdots V]_\mu,$$

which is a generalization of the multiple commutators of $K^{(0)}$ with V. In a representation where $K^{(0)}$ is diagonal,

$$\langle k_1^{(0)}|[K^{(0)}\cdots V]_\mu|k_2^{(0)}\rangle=(k_1^{(0)}-k_2^{(0)})^\mu \times\langle k_1^{(0)}|V|k_2^{(0)}\rangle . \tag{5.5}$$

If μ is a positive integer, the definition (5.5) just gives us the $[K^{(0)}\cdots V]_\mu$ commutator. The vertex corresponding to the state (5.4) is

$$[K^{(0)}\cdots V]_{-\nu} . \tag{5.6}$$

When such a vertex is applied to the vacuum as in (5.2), $k_1^{(0)}$ will be equal to $n+\nu$ and $k_2^{(0)}$ to ν, so that the factor $(k_1^{(0)}-k_2^{(0)})^{-\nu}$ gives us the required factor $n^{-\nu}$.

We also require an equation corresponding to (5.2) when our vertex is applied to the vacuum from the right. In that case $k_1^{(0)}$ is equal to 0 and $k_2^{(0)}$ to n, and we pick up an extra phase factor $e^{-i\pi\nu}$. We can adjust the phase factors correctly by adding a factor $e^{i\pi\nu/2}$ to (3.9). The corresponding formula for the bra vectors would contain a phase factor $e^{-i\pi\nu/2}$, and (5.6) would be replaced by the vertex

$$e^{i\pi\nu/2}[K^{(0)}\cdots V]_{-\nu} . \tag{5.7a}$$

In order for an amplitude constructed from the vertex (5.7) to be dual, it is necessary that the z-dependent vertex

$$e^{i\pi\nu/2}[K^{(0)}\cdots V(z)]_{-\nu} , \tag{5.7b}$$

where

$$V(z)=z^{L^{(0)}}Vz^{-L^{(0)}} ,$$

be local in z space. In Appendix A we shall argue that the vertex is in fact local. The argument will be based on the fact that when our vertex is sandwiched between non-normalizable states $|R\rangle$, only those matrix elements (5.5) where $k_1^{(0)}-k_2^{(0)}$ is large are significant.[5] We shall have a consistency check of the duality properties of our vertices when we calculate the matrix element of the vertex operator between two real states. We shall find that the vertex is symmetrical in the three particles, a property which is necessary for duality, but which we would not expect in a nondual amplitude.

It is not difficult to construct vertices corresponding to the higher terms of (3.9). To take the second term, for instance, we use the identity

$$\sum_{n=0}^{\infty}\frac{(K^{(1)\dagger})^n}{n!}K^{(2)\dagger}|0\rangle=\sum_{m=2}^{\infty}\sum_{n=0}^{\infty}K^{(m)\dagger}\frac{(K^{(1)\dagger})^n}{n!}|0\rangle . \tag{5.8}$$

It follows that the vertex

$$\sum_{m=2}^{\infty}(K^{(m)\dagger}V+VK^{(m)})$$

corresponds to the state (5.8). This is not quite a local vertex, but we can replace it with the vertex

$$V_2\equiv\sum_{m=1}^{\infty}(K^{(m)\dagger}V+VK^{(m)})+\tfrac{1}{2}(K^{(0)}V+VK^{(0)}) , \tag{5.9}$$

which is local. The extra terms added to (5.9) make no difference when matrix elements such as (5.5) are evaluated between states where $k_1^{(0)}-k_2^{(0)}$ is large, as may easily be verified from the calculations of the matrix elements which will be performed further on in the present section. The vertex corresponding to the second term of (3.9), including the factor $n^{-\nu-2}$, may be constructed as before; it is

$$-e^{i\pi\nu/2}[K^{(0)}\cdots V_2]_{-\nu-2} . \tag{5.10}$$

One can construct vertices corresponding to the subsequent terms of (3.9) by a straightforward extension of the above reasoning.

We shall now calculate the matrix elements of the vertices between two states of the form (3.9). To begin, let us calculate the matrix element of the vertex (5.6) between two states represented by the first term of (3.9).[6] In other words, we require the matrix element

$$e^{i\pi(-\nu_a+\nu_c+\nu_b)/2}\sum_{n,m}\left\langle a\left|\frac{(K^{(1)})^n}{n!}n^{-\nu_a}[K^{(0)}\cdots V]_{-\nu_c}m^{-\nu_b}\frac{(K^{(1)\dagger})^m}{m!}\right|b\right\rangle . \tag{5.11}$$

The states $\langle a|$ and $|b\rangle$ are the basic states of the K-degenerate family in question. The two families, as well as the family corresponding to the vertex, have different ν's which have been denoted by ν_a, ν_b, and ν_c, respectively.

The commutator between $K^{(n)}-K^{(0)}$ and V will first be required. We note that $L^{(n)}$ contains a term $-2aL_0^{(n)}$, and $K^{(n)}$ a term $(1+2a)L_0^{(n)}$, where the constant a depends on the particular model, and $L_0^{(n)}$ is the operator $L^{(n)}$ for the simple dual resonance model with no spin degrees of freedom. Hence it is easy to verify that

$$[K^{(n)}-K^{(0)},\,V]=-nk^2(1+2a) ,$$

where k is the momentum of the vertex. For the K-degenerate family of the vacuum

$$2ak^2 = 1\,,$$

so that

$$[K^{(n)} - K^{(0)}, V] = -n(1+k^2)\,,$$

i.e.,

$$[K^{(n)} - K^{(0)}, V] = n\nu_c V\,, \tag{5.12}$$

from (2.9), as the sector number of the vacuum is zero.

We now calculate the matrix element

$$\langle a | V(K^{(1)\dagger})^m | b\rangle\,,$$

which may be written as

$$\langle a | V(K^{(1)\dagger} - K^{(0)})(K^{(1)\dagger})^{m-1} | b\rangle + \langle a | VK^{(0)}(K^{(1)\dagger})^{m-1} | b\rangle\,.$$

Using (5.12), and remembering that $K^{(1)\dagger} = K^{(-1)}$, we may set this expression equal to

$$\nu_c \langle a | (K^{(1)\dagger})^{m-1} | b\rangle + \langle a | (K^{(1)\dagger} - K^{(0)}) V(K^{(1)\dagger})^{m-1} | b\rangle + \langle a | VK^{(0)}(K^{(1)\dagger})^{m-1} | b\rangle\,.$$

Thus, from (2.10a), we may write

$$\langle a | V(K^{(1)\dagger})^m | b\rangle = (m - 1 - \nu_a + \nu_b + \nu_c)\langle a | (K^{(1)\dagger})^{m-1} | b\rangle\,, \tag{5.13}$$

since $\langle a | K^{(1)\dagger} = 0$. By using (5.13) repeatedly, we obtain the equation

$$\langle a | V(K^{(1)\dagger})^m | b\rangle = [\Gamma(m - \nu_a + \nu_b + \nu_c)/\Gamma(-\nu_a + \nu_b + \nu_c)]\langle a | V | b\rangle\,. \tag{5.14}$$

The more general matrix element

$$\langle a | (K^{(1)})^n V(K^{(1)\dagger})^m | b\rangle$$

can be calculated by similar reasoning. Writing the last factor $K^{(1)}$ as $K^{(1)} - K^{(0)} + K^{(0)}$, and commuting $K^{(1)} - K^{(0)}$ through V, we find

$$\begin{aligned}\langle a | (K^{(1)})^n V(K^{(1)\dagger})^m | b\rangle = {} & \langle a | (K^{(1)})^{n-1} K^{(0)} V(K^{(1)\dagger})^m | b\rangle + \nu_c \langle a | (K^{(1)})^{n-1} V(K^{(1)\dagger})^m | b\rangle \\ & - \langle a | (K^{(1)})^{n-1} VK^{(1)}(K^{(1)\dagger})^m | b\rangle + \langle a | (K^{(1)})^{n-1} VK^{(1)}(K^{(1)\dagger})^m | b\rangle\,.\end{aligned} \tag{5.15}$$

The last term of (5.15) can be found by commuting $K^{(1)}$ through the factors $(K^{(1)\dagger})^m$ until it reaches $|b\rangle$, where it gives zero:

$$\begin{aligned}K^{(1)}(K^{(1)\dagger})^m | b\rangle &= 2\sum_{l=1}^{m} (K^{(1)\dagger})^{l-1} K^{(0)}(K^{(1)\dagger})^{m-l} | b\rangle \\ &= m(m - 1 + 2\nu_b)(K^{(1)\dagger})^{m-1} | b\rangle \quad [\text{from (2.4)}]\,.\end{aligned} \tag{5.16}$$

On calculating the first and third terms on the right of (5.15) from (2.4), we obtain the result

$$\begin{aligned}\langle a | (K^{(1)})^n V(K^{(1)\dagger})^m | b\rangle = {} & (n - m + \nu_a + \nu_c - \nu_b - 1)\langle a | (K^{(1)})^{n-1} V(K^{(1)\dagger})^m | b\rangle \\ & + m(m - 1 + 2\nu_b)\langle a | (K^{(1)})^{n-1} V(K^{(1)\dagger})^{m-1} | b\rangle\,.\end{aligned} \tag{5.17}$$

If $n \leq m$, we may apply (5.16) repeatedly, and reduce the matrix element to a sum of terms of the form (5.14). Thus, the difference equation (5.17) together with the boundary condition (5.14) for $n = 0$ enable us to calculate the matrix element for $n \leq m$. The details are given in Appendix B; the result is

$$\begin{aligned}\langle a | (K^{(1)})^n V(K^{(1)\dagger})^m | b\rangle = {} & \frac{m!\,\Gamma(2\nu_a + n)\Gamma(m - n - \nu_a + \nu_b + \nu_c)}{(m-n)!\,\Gamma(2\nu_a)\Gamma(-\nu_a + \nu_c + \nu_b)} \\ & \times {}_3F_2(\nu_a + \nu_c - \nu_b,\, 1 + \nu_a - \nu_c - \nu_b,\, -n;\, 2\nu_a,\, m - n + 1;\, 1)\langle a | V | b\rangle \quad (n \leq m)\,.\end{aligned} \tag{5.18}$$

When m, n, and $m - n$ are large, the generalized hypergeometric function in (5.18) may be replaced by an ordinary hypergeometric function, so that

$$\left\langle a \left| \frac{(K^{(1)})^n}{n!}\, V\, \frac{(K^{(1)\dagger})^m}{m!} \right| b \right\rangle \simeq \frac{n^{2\nu_a - 1}(m-n)^{-\nu_a + \nu_c + \nu_b - 1}}{\Gamma(2\nu_a)\Gamma(-\nu_a + \nu_c + \nu_b)}\, F\left(\nu_a + \nu_c - \nu_b,\, 1 + \nu_a - \nu_c - \nu_b;\, 2\nu_a;\, \frac{n}{n-m}\right) \quad (n \leq m)\,. \tag{5.19}$$

Equation (5.19) is an obvious consequence of (5.18) only if $n < m-n$, but, by expressing (5.18) as a sum of two other generalized hypergeometric series and converting them to ordinary hypergeometric series in the asymptotic limit, one can prove (5.19) subject only to the restriction $n \leq m$.

If $m \leq n$, we can obtain the matrix element by interchanging m and n, and ν_a and ν_b, on the right-hand side of (5.19).

We now carry out the summation over m, for a particular large value of n, in (5.11). The summation can be replaced by an integration, and there will be two terms, one for $m \geq n$ and one for $m \leq n$. Thus[7]

$$
\begin{aligned}
& e^{i\pi(-\nu_a+\nu_c+\nu_b)/2} \sum_m \left\langle a \left| \frac{(K^{(1)})^n}{n!} n^{-\nu_a} [K^{(0)} \cdots V]_{-\nu_c} m^{-\nu_b} \frac{(K^{(1)\dagger})^m}{m!} \right| b \right\rangle \\
& \quad = \int_n^\infty dm\, e^{i\pi(-\nu_a-\nu_c+\nu_b)/2} \frac{n^{\nu_a-1}(m-n)^{-\nu_a+\nu_b-1} m^{-\nu_b}}{\Gamma(2\nu_a)\Gamma(-\nu_a+\nu_c+\nu_b)} F\left(\nu_a+\nu_c+\nu_b, 1+\nu_a-\nu_c-\nu_b; 2\nu_a, \frac{n}{n-m}\right) \langle a \mid V \mid b \rangle \\
& \quad + \int_0^n dm\, e^{i\pi(-\nu_a+\nu_c+\nu_b)/2} \frac{n^{-\nu_a}(n-m)^{\nu_a-\nu_b-1} m^{\nu_b-1}}{\Gamma(2\nu_b)\Gamma(\nu_a+\nu_c-\nu_b)} F\left(-\nu_a+\nu_c+\nu_b, 1-\nu_a-\nu_c+\nu_b; 2\nu_b, \frac{m}{m-n}\right) \langle a \mid V \mid b \rangle \\
& \quad = \frac{2^{-\nu_a-\nu_c-\nu_b+2}\pi^{3/2}}{n\Gamma(\frac{1}{2}(\nu_a+\nu_c+\nu_b))\Gamma(\frac{1}{2}(1-\nu_a+\nu_c+\nu_b))\Gamma(\frac{1}{2}(1+\nu_a-\nu_c+\nu_b))\Gamma(\frac{1}{2}(1+\nu_a+\nu_c-\nu_b))} \langle a \mid V \mid b \rangle \\
& \quad \equiv n^{-1} H(\nu_a, \nu_c, \nu_b) \langle a \mid V \mid b \rangle . \qquad (5.20)
\end{aligned}
$$

The details of the integration are given in Appendix B.

On integrating (5.20) over n, we obtain a divergent factor $\int dn\, n^{-1}$, multiplied by the same combination of Γ functions. The divergent factor is just the factor which appears in the square of the norms of the states, so that the matrix element of the vertex between normalized states is finite.

We confirm that the right-hand side of (5.20) is symmetric in ν_a, ν_b, and ν_c, as is necessary in a dual theory.

Next we calculate some of the vertex functions involving higher terms of (3.9). Let us begin with the matrix element

$$
e^{i\pi(-\nu_a+\nu_c+\nu_b+2)/2} \sum_m \left\langle a \left| \frac{(K^{(1)})^n}{n!} n^{-\nu_a} [K^{(0)} \cdots V]_{-\nu_c} m^{-\nu_b-2} \frac{(K^{(1)\dagger})^m}{m!} K^{(2)\dagger} \right| b \right\rangle . \qquad (5.21)
$$

The calculation will be based on the fact that the only properties of the vector $(K^{(1)\dagger})^m |b\rangle$ which we used in the derivation of (5.20) were the fact that it was an eigenstate of $K^{(0)}$ with eigenvalue $m+\nu_a$, and Eq. (5.16). We therefore attempt to find a vector involving $K^{(2)\dagger}|b\rangle$ with similar properties. In fact,

$$
K^{(1)}[(K^{(1)\dagger})^m K^{(2)\dagger} - \tfrac{3}{2}(1+2\nu_b)^{-1}(K^{(1)\dagger})^{m+2}]|b\rangle = m(m-1+2\nu_b+4)[(K^{(1)\dagger})^{m-1}K^{(2)\dagger} - \tfrac{3}{2}(1+2\nu_b)^{-1}(K^{(1)\dagger})^{m+1}]|b\rangle , \qquad (5.22)
$$

and the vector $[(K^{(1)\dagger})^n K^{(2)\dagger} - \frac{3}{2}(1+2\nu_b)^{-1}(K^{(1)\dagger})^{n+2}]|b\rangle$ is an eigenvector of $K^{(0)}$ with eigenvalue $m+2+\nu_b$. Hence,

$$
\begin{aligned}
& e^{i\pi(-\nu_a+\nu_c+\nu_b+2)/2} \sum_m \left\langle a \left| \frac{(K^{(1)})^n}{n!} n^{-\nu_a} [K^{(0)} \cdots V]_{-\nu_c} m^{-\nu_b-2} \frac{1}{m!} [(K^{(1)\dagger})^m K^{(2)\dagger} - \tfrac{3}{2}(1+2\nu_b)^{-1}(K^{(1)\dagger})^{m+2}] \right| b \right\rangle \\
& \quad = \frac{1}{n} H(\nu_a, \nu_c, \nu_b+2) \langle a \mid V[K^{(2)\dagger} - \tfrac{3}{2}(1+2\nu_b)^{-1}(K^{(1)\dagger})^2]|b\rangle . \qquad (5.23)
\end{aligned}
$$

The matrix element in (5.23) can be found by manipulations similar to those leading to (5.13):

$$
\langle a \mid V[K^{(2)\dagger} - \tfrac{3}{2}(1+2\nu_b)^{-1}(K^{(1)\dagger})^2]|b\rangle = [-\nu_a+2\nu_c+\nu_b - \tfrac{3}{2}(1+2\nu_b)^{-1}(-\nu_a+\nu_c+\nu_b+1)(-\nu_a+\nu_c+\nu_b)]\langle a \mid V \mid b \rangle . \qquad (5.24)
$$

To obtain (5.21) from (5.23), we have to subtract the matrix element involving the second term on the left-hand side within the square brackets. This may immediately be obtained from (5.20), since, for m large, we may write $m^{-2}(m!)^{-1} \simeq [(m+2)!]^{-1}$. Hence

$$e^{i\pi(-\nu_a+\nu_c+\nu_b+2)/2}\sum_m \left\langle a\left|\frac{(K^{(1)})^n}{n!}n^{-\nu_a}[K^{(0)}\cdots V]_{-\nu_c}m^{-\nu_b-2}\frac{1}{m!}(-\tfrac{3}{2})(1+2\nu_b)^{-1}(K^{(1)\dagger})^{m+2}\right|b\right\rangle$$
$$=\frac{3}{2}\frac{1}{n}(1+2\nu_b)^{-1}H(\nu_a,\nu_c,\nu_b)\,. \tag{5.25}$$

From (5.23), (5.24), and (5.25), we obtain the final result:

$$e^{i\pi(-\nu_a+\nu_c+\nu_b+2)/2}\sum_m \left\langle a\left|\frac{(K^{(1)})^n}{n!}n^{-\nu_a}[K^{(0)}\cdots V]_{-\nu_c}m^{-\nu_b-2}\frac{(K^{(1)\dagger})^m}{m!}K^{(2)\dagger}\right|b\right\rangle$$
$$=\frac{-\nu_b^2-\nu_b(1+\nu_c+\nu_a)+2(\nu_c-\nu_a)^2-2\nu_c\nu_a}{(\nu_a+\nu_c+\nu_b)(1-\nu_a+\nu_c+\nu_b)(1+\nu_a-\nu_c+\nu_b)}\,n^{-1}H(\nu_a,\nu_c,\nu_b)\langle a\,|\,V\,|\,b\rangle\,. \tag{5.26}$$

We notice that the result is symmetric in ν_a and ν_c, as it should be for duality.

As another illustration of the method of calculation, we shall find the vertex function where the state vectors correspond to the lowest term of (3.9), but where the vertex operator itself corresponds to a higher term; in particular, we shall work with the vertex (5.10), V_2 being defined by (5.9). It is not strictly necessary to perform such a calculation, since duality enables us to obtain the result simply by interchanging ν_b and ν_c in (5.26). However, we shall give the calculation, first to provide a further verification of duality, and second to illustrate the method of working with more complicated vertices.

The principles of the calculation will be similar to those of the calculation leading to (5.26). In deriving (5.20) the only property of the vertex V which we used was (5.12) with $n=\pm1$, and we attempt to find a vertex resembling (5.10) with a similar property. Now, the vertex

$$U_2=V_2-\tfrac{3}{2}(1+2\nu_c)^{-1}[K^{(0)},[K^{(0)},V]]-\tfrac{1}{2}\nu_c(1-\nu_c)(1+2\nu_c)^{-1}V \tag{5.27}$$

satisfies the commutation relation

$$[K^{(1)}-K^{(0)},U_2]=(\nu_c+2)U_2\,. \tag{5.28}$$

Hence,

$$e^{i\pi(-\nu_a+\nu_c+\nu_b+2)/2}\sum_m \left\langle a\left|\frac{(K^{(1)})^n}{n!}n^{-\nu_a}[K^{(0)}\cdots U_2]_{-\nu_c-2}m^{-\nu_b}\frac{(K^{(1)\dagger})^m}{m!}\right|b\right\rangle=n^{-1}H(\nu_a,\nu_c+2,\nu_b)\langle a\,|U_2|\,b\rangle\,. \tag{5.29}$$

The matrix element on the right-hand side of (5.29) may be calculated directly from (5.27) and (5.9):

$$\langle a\,|U_2|\,b\rangle=[\tfrac{1}{2}(\nu_a+\nu_b)-\tfrac{3}{2}(\nu_a-\nu_b)^2(1+2\nu_c)^{-1}-\tfrac{1}{2}\nu_c(1-\nu_c)(1+2\nu_c)^{-1}]\langle a\,|\,V\,|\,b\rangle\,. \tag{5.30}$$

To obtain the required result, we must subtract matrix elements involving the last two terms of (5.27) from (5.29). Now

$$[K^{(0)}\cdots[K^{(0)},[K^{(0)},V]]]_{-\nu_c-2}=[K^{(0)}\cdots V]_{-\nu_c}\,,$$

so that

$$e^{i\pi(-\nu_a+\nu_c+\nu_b+2)/2}\sum_m \left\langle a\left|\frac{(K^{(1)})^n}{n!}n^{-\nu_a}[-\tfrac{3}{2}(1+2\nu_c)^{-1}][K^{(0)}\cdots[K^{(0)},[K^{(0)},V]]]_{-\nu_c-2}m^{-\nu_b}\frac{(K^{(1)\dagger})^m}{m!}\right|b\right\rangle$$
$$=\tfrac{3}{2}(1+2\nu_c)^{-1}n^{-1}H(\nu_a,\nu_c,\nu_b)\langle a\,|\,V\,|\,b\rangle\,. \tag{5.31}$$

Examining the third term of (5.27), we notice that the operator $[K^{(0)}\cdots V]_{-\nu_c-2}$ involves two negative powers of $k_0^{(1)}-k_0^{(2)}$, in addition to those contained in $[K^{(0)}\cdots V]_{-\nu_c}$. Hence, since we are interested in large values of $k_0^{(1)}-k_0^{(2)}$, we may write

$$\sum_m \left\langle a\left|\frac{(K^{(1)})^n}{n!}n^{-\nu_a}[K^{(0)}\cdots V]_{-\nu_c-2}m^{-\nu_b}\frac{(K^{(1)\dagger})^m}{m!}\right|b\right\rangle=0\,. \tag{5.32}$$

Combining (5.27), (5.29), (5.30), (5.31), and (5.32), we may write

$$e^{i\pi(-\nu_a+\nu_c+\nu_b+2)/2}\sum_m \left\langle a \left| \frac{(K^{(1)})^n}{n!} n^{-\nu_a}[K^{(0)}\cdots V_2]_{-\nu_c-2} m^{-\nu_b}\frac{(K^{(1)\dagger})^m}{m!} \right| b \right\rangle$$

$$= \frac{-\nu_c^2-\nu_c(1+\nu_b+\nu_a)+2(\nu_b-\nu_a)^2-2\nu_b\nu_a}{(\nu_a+\nu_c+\nu_b)(1-\nu_a+\nu_c+\nu_b)(1+\nu_a+\nu_c-\nu_b)} n^{-1}H(\nu_a,\nu_c,\nu_b)\langle a|V|b\rangle . \tag{5.33}$$

We have also calculated the matrix element of the vertex $[K^{(0)}\cdots V]_{-\nu_c}$ between two states corresponding to the second term of (3.9). The calculation is similar to that leading to (5.26); though the algebra is somewhat more complicated, no new principles are involved. We quote the result:

$$e^{i\pi(-\nu_a+\nu_b+\nu_c)/2}\left\langle a \left| K^{(2)}\frac{(K^{(1)})^n}{n!} n^{-\nu_a-2}[K^{(0)}\cdots V]_{-\nu_c} m^{-\nu_b-2}\frac{(K^{(1)\dagger})^m}{m!} K^{(2)\dagger} \right| b \right\rangle$$

$$\begin{aligned}
&= \{4\nu_c^4-2\nu_c^3(3\nu_a+3\nu_b+10)+\nu_c^2[4(\nu_a-\nu_b)^2+9\nu_a\nu_b+11(\nu_a+\nu_b)-4+\tfrac{1}{2}\gamma] \\
&\quad +\nu_c[2(\nu_a-\nu_b)^2+18\nu_a\nu_b+28(\nu_a+\nu_b)+20+\gamma] \\
&\quad +[-(\nu_a-\nu_b)^2+1][2(\nu_a-\nu_b)^2+9\nu_a\nu_b+11(\nu_a+\nu_b)+\tfrac{1}{2}\gamma]\} \\
&\quad \times[(\nu_a+\nu_c+\nu_b)(2+\nu_a+\nu_c+\nu_b)(1-\nu_a+\nu_c+\nu_b)(1+\nu_a+\nu_c-\nu_b)(1+\nu_a-\nu_c+\nu_b) \\
&\quad \times(3+\nu_a-\nu_c+\nu_b)]^{-1} n^{-1}H(\nu_a,\nu_b,\nu_c)\langle a|V|b\rangle .
\end{aligned} \tag{5.34}$$

By proceeding in a similar way, one can calculate vertex functions involving any term in (3.9), or any vertex corresponding to such a term. The calculation becomes unwieldy when expressed as an algebraic function of ν_a, ν_b, and ν_c; if we were interested in calculating higher terms, it would probably be easier to do so for the particular ν's in question. One can calculate the coefficients c in (3.9) by the methods of Sec. III, and the vertex functions involving three real states can thereby be calculated. They will always be in the form of the vertex $\langle a|V|b\rangle$ between basic states, multiplied by a factor which depends only on the K algebra.

VI. CONCLUDING REMARKS

While the work of the preceding sections shows that calculations with nonadditive models are probably feasible, it leaves several questions unanswered. As we do not possess analytic expressions for scattering amplitudes, we have no proof that the models are mathematically consistent or physically acceptable. We have no guarantee, for instance, that there are only a finite number of resonances below any given energy. Most important, we do not know whether ghosts can be eliminated. We are now able to determine whether any particular state is a ghost or a particle, but to prove that all real states are particles is another matter.

In the following paper we shall examine a particular model without internal symmetry. The model question only exists in ten or more dimensions and, in ten dimensions, we shall show that we do have a consistent, ghost-free set of amplitudes. The model has certain simplifying features which are not present in the general nonadditive model, and it is not at present clear whether the results obtained are consequences of these particular features or whether they represent general properties of nonadditive models.

APPENDIX A: LOCALITY PROPERTIES OF THE VERTICES

In this appendix we shall give arguments to support the fact that the vertex (5.7) is local. As we mentioned in the main text, the arguments will be based on the fact that only those elements of (5.5) with large values of $k_1^{(0)}-k_2^{(0)}$ are important.

We first notice that the function $x^{-\nu}$ may be approximated by a function with the following properties:

(i) For sufficiently large real x, it approaches $x^{-\nu}$.

(ii) It is an entire function.

(iii) When expanded in powers of x, the coefficient of the nth term is bounded by $M^{-n}/n!$, where M can be made arbitrarily large. To prove that this can be done, we begin by replacing the function $x^{-\nu}$ by $(x+iN^2)^{-\nu}$. We can move all singularities arbitrarily far from the real axes by making N sufficiently large but, if x is sufficiently large (e.g., larger than N^3) we do not appreciably change the value of the function. Next we show that a function whose singularities are sufficiently far from the real axis (of order M^2) may be approximated by a function with the above properties. It is sufficient to carry out the proof for a function

whose only singularity is a pole, since the theorem may easily be generalized by integration if the singularity is a cut. In fact, the function

$$(z+iM^2)^{-1}$$

may be approximated by the function

$$(z+iM^2)^{-1}\{1-\exp[iM^{-1}(z+iM^2)]\}\,, \tag{A1}$$

which has all of the above properties, as may easily be verified.

We now approximate the function $(k_1^{(0)}-k_2^{(0)})^{-\nu}$ in (5.5) in the above manner. Our vertex has thereby been replaced by a new vertex whose matrix elements between states diagonal in $K^{(0)}$ are

$$F(k_1^{(0)}-k_2^{(0)})\langle k_1^{(0)}|\,V\,|k_2^{(0)}\rangle\,, \tag{A2}$$

where F has the second and third of the properties enumerated above. Expanding F, and remembering that a power $(k_1^{(0)}-k_2^{(0)})^n$ in the matrix element simply gives the n-tuple commutator, we obtain for our vertex

$$\sum_{n=0}^{\infty}\frac{c_n}{n!}[K^{(0)}\cdots[K^{(0)},V]]_n\,, \tag{A3a}$$

where

$$c_n<M^{-n}\,. \tag{A3b}$$

The vertex (A3) is to be commuted with another vertex where V is replaced by $V(z)$. If the vertices are expanded as in (A3a), the resulting terms in the commutator may be evaluated as in the paper of Bardakci and Halpern,[1] Appendix B. Each term will be local in z, but will contain derivatives of the δ function whose order is, at most, equal to the number of commutations of $K^{(0)}$. We therefore obtain sums of the form

$$\sum\frac{g_n}{n!}\,\delta^{(n)}(z)\,,\quad |g_n|<(\tfrac{1}{2}M)^{-n}$$

and the commutator is zero for $|z|<2/M$. Since M can be made arbitrarily large without altering the matrix elements of the vertex function for large values of $k_1^{(0)}-k_2^{(0)}$, the nonlocality can be made as small as we please. The vertex operators would therefore be expected to provide us with a dual model.

APPENDIX B: FORMULAS INVOLVING HYPERGEOMETRIC FUNCTIONS

We shall prove some of the formulas quoted in Sec. V. We begin by proving (5.18) by induction. Let us assume it to be true for any m and a given n; we then prove it for $n+1$.

Applying our induction assumption to (5.16), we find that

$$\begin{aligned}\langle a|(K^{(1)})^{n+1}V(K^{(1)\dagger})^m|b\rangle=&\frac{(m-1)!\,\Gamma(2\nu_a+n)}{(m-n-1)!\,\Gamma(2\nu_a)}\,\frac{\Gamma(m-n-1-\nu_a+\nu_c+\nu_b)}{\Gamma(-\nu_a+\nu_c+\nu_b)}\\&\times[(n-m+\nu_a+\nu_c-\nu_b)(m-n-1-\nu_a+\nu_c+\nu_b)m(m-n)^{-1}\\&\times{}_3F_2(\nu_a+\nu_c-\nu_b,1+\nu_a-\nu_c-\nu_b,-n;2\nu_a,m-n+1;1)\\&+m(m-1+2\nu_b)\,{}_3F_2(\nu_a+\nu_c-\nu_b,1+\nu_a-\nu_c-\nu_b,-n;2\nu_a,m-n;1)]\langle a|V|b\rangle\,.\end{aligned} \tag{B1}$$

If we combine the two terms on the right-hand side using the recursion formula

$$\begin{aligned}&(a-e+1)(e-b)(e-1)^{-1}F(a,b,c;d,e;1)+(-a-b-c+d+e-1)F(a,b,c;d,e-1;1)\\&\qquad=(d-c)F(a,b,c-1;d,e-1;1)\,,\end{aligned}$$

we obtain precisely the expression on the right-hand side of (5.17), with n replaced by $n+1$. According to (5.14), our induction assumption is correct for $n=0$, so that (5.18) is proved.

Next we carry out the integration in (5.20). Let us define

$$x=\frac{n}{n-m}\,.$$

The first integral is then

$$\frac{e^{i\pi(-\nu_a-\nu_b+\nu_c)/2}}{n\Gamma(2\nu_a)\Gamma(-\nu_a+\nu_c+\nu_b)}\int_{-\infty}^{0}dx(-x)^{\nu_a-1}(1-x)^{-\nu_b}F(\nu_a+\nu_c-\nu_b,1+\nu_a-\nu_c-\nu_b;2\nu_a,x)\,. \tag{B2}$$

The integral is a generalized hypergeometric function, so that (B2) is equal to

$$\frac{e^{i\pi(-\nu_a-\nu_c+\nu_b)/2}}{n\Gamma(2\nu_a)\Gamma(-\nu_a+\nu_c+\nu_b)}\,\frac{\Gamma(\nu_a)\Gamma(\nu_c)}{\Gamma(\nu_a+\nu_c)}\,{}_3F_2(\nu_a+\nu_c-\nu_b,-1+\nu_a+\nu_c+\nu_b,\nu_a;2\nu_a,\nu_a+\nu_c;1)\,. \tag{B3}$$

With the particular combination of parameters which occurs in the hypergeometric function of (B3), it is

possible to express the function in terms of Γ functions according to the formula[8]

$$ {}_3F_2(a, b, c; 2c, \tfrac{1}{2}(a+b+1); 1) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+c)\Gamma(\frac{1}{2}(1+a+b))\Gamma(\frac{1}{2}(1-a-b+2c))}{\Gamma(\frac{1}{2}(1+a))\Gamma(\frac{1}{2}(1+b))\Gamma(\frac{1}{2}(1-a+2c))\Gamma(\frac{1}{2}(1-b+2c))} . \tag{B4} $$

Equation (B4) is inserted into (B3), and the result is simplified slightly using the formulas

$$ \Gamma(\nu_a)\Gamma(\nu_a+\tfrac{1}{2}) = 2^{1-2\nu_a}\pi^{1/2}\Gamma(2\nu_a), $$

$$ \Gamma(-\nu_a+\nu_c+\nu_b) = 2^{-1-\nu_a+\nu_c+\nu_b}\pi^{-1/2}\Gamma(\tfrac{1}{2}(-\nu_a+\nu_c+\nu_b))\Gamma(\tfrac{1}{2}(1-\nu_a+\nu_c+\nu_b)), $$

$$ \Gamma(\nu_c)\Gamma(1-\nu_c) = \pi/\sin\pi\nu_c, $$

$$ \Gamma(\tfrac{1}{2}(-\nu_a+\nu_b+\nu_c))\Gamma(\tfrac{1}{2}(2+\nu_a-\nu_b-\nu_c)) = \pi/\sin[\tfrac{1}{2}\pi(-\nu_a+\nu_b+\nu_c)] . $$

The expression is then equal to

$$ \frac{2^{2-\nu_a-\nu_c-\nu_b}\pi\,\Gamma(\frac{1}{2})\sin[\frac{1}{2}\pi(-\nu_a+\nu_b+\nu_c)]e^{i\pi(-\nu_a-\nu_c+\nu_b)/2}}{n\sin\pi\nu_c\,\Gamma(\frac{1}{2}(\nu_a+\nu_c+\nu_b))\Gamma(\frac{1}{2}(1-\nu_a+\nu_c+\nu_b))\Gamma(\frac{1}{2}(1+\nu_a-\nu_c+\nu_b))\Gamma(\frac{1}{2}(1+\nu_a+\nu_b-\nu_c))} . \tag{B5} $$

The second integral in (5.9) will be similar to (B4), except that

(i) ν_a and ν_b are interchanged in all factors except the phase factor.

(ii) The phase factor is $e^{i\pi(-\nu_a+\nu_c+\nu_b)/2}$.

Upon adding the two terms, we obtain (5.20).

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[1]K. Bardakci and M. B. Halpern, Phys. Rev. D 3, 2493 (1971).

[2]It is possible to have nonlinear trajectories in a narrow-resonance model, provided the trajectories intersect one another.

[3]M. B. Halpern, Phys. Rev. D 3, 3068 (1971).

[4]S. Fubini and G. Veneziano, Nuovo Cimento 67A, 29 (1970).

[5]We shall verify this fact explicitly further on in this section.

[6]The first term of (3.9) is understood to be multiplied by a phase factor $e^{(1/2)i\pi\nu}$, as we have explained above.

[7]Note that the phase factor multiplying the first integral is different from that on the left-hand side of the equation. This is because the factor $(n-m)^{-\nu_c}$ in the vertex $[K^{(0)}\cdots V]_{-\nu_c}$ has been canceled against a factor $(m-n)^{\nu_c}$ in (5.19).

[8]Y. L. Luke, *Special Functions and Their Approximations* (Academic, New York, 1969), p. 104.