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Dual $\pi\pi$ Partial-Wave Amplitudes. II. Asymptotic Behavior*

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The integral representations, developed for the partial waves of the dual $\pi\pi$ amplitudes, are used to show that all the partial waves can be expressed as sums involving s waves only, and to prove that the isospin-zero and -two partial waves are bounded by powers in the right half-plane of the energy-squared variable and diverge exponentially along any ray in the left half-plane, thus confirming the behavior conjectured by Tryon. For the isospin-one waves, the derived bounds agree with those of Park and Desai. It is also proved that the discontinuities of the isospin-zero and -two partial waves diverge exponentially along the left-hand cuts and those for the isospin-one case are bounded by a power.

I. INTRODUCTION

In a previous paper,[1] hereafter referred to as I, we presented a formulation of the integral representations of the partial waves of the dual $\pi\pi$ scattering amplitude of Lovelace[2] and Shapiro[3] and studied the threshold behavior in detail. In the present paper, we explore further the representations of I in an investigation of the asymptotic behavior of the partial waves. This subject has already been considered by many authors,[4-12] partly in connection with the K-matrix unitarization scheme and partly in connection with dispersion-theoretic studies.

It has been suggested by Drago and Matsuda[4] and Sivers and Yellin[5,12] that, in this model, partial-wave dispersion relations cannot be used. However, Park and Desai[8] have shown that for the amplitude with isospin one, the partial waves are bounded in the complex energy-squared plane, as is the discontinuity along the cut, so that partial-wave dispersion relations can be obtained for this case.

For the amplitudes with isospin zero and two, Tryon[10] has recently conjectured that the partial waves grow faster than any power along any ray extending into the left half-plane of the energy-squared variable, in which case no dispersion relations can be written down.

We will show in this paper that the integral representations of I furnish a unified treatment of the partial waves for all isospin states and yield, among other results, the explicit exponential divergence of the isospin-zero and -two partial waves in the left half-plane of the energy-squared variable.

In Sec. II, we summarize some of the results obtained in I. We deduce in Sec. III functional relations among the partial waves, which show that all partial waves are essentially finite sums of s waves. In Sec. IV, we study the asymptotic behavior of the partial waves, obtaining the same bounds for the $I=1$ case as found by Park and Desai,[8] and an exponential divergence for the $I=0$ and 2 partial waves in the left half-plane of the energy-squared variable, which confirms Tryon's conjecture[10]; some mathematical details which enter in the proof of this divergence are given in the Appendix. The asymptotic behavior of the discontinuities along the left-hand cut is examined in Sec. V, where it is shown that for $I=1$ the discontinuities of the partial waves are bounded by a power, and for $I=0$ and 2 they are exponentially divergent. Finally, in Sec. VI, we summarize our results, including a comment on the related unitarization problem.

II. PARTIAL-WAVE PROJECTION

The $\pi\pi$ scattering amplitudes in the dual resonance model[2,3] are

$$A^0(s,t,u)=\tfrac{1}{2}gF(t,u)-\tfrac{3}{2}g[F(s,t)+F(s,u)],$$
$$A^1(s,t,u)=g[F(s,u)-F(s,t)], \quad (2.1)$$
$$A^2(s,t,u)=-gF(t,u),$$

where g is the over-all coupling constant, s, t, u are the usual Mandelstam variables, and

$$F(x,y)=\frac{\Gamma(1-\alpha(x))\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))} + \text{secondary terms}, \quad (2.2)$$

with the linear Regge trajectory $\alpha(x)=ax+b$. We shall take the simplest form of (2.2), neglecting the secondary terms, and assume that $a>0$, $0<b<1$.[1]

In I, we have shown that the partial waves $V_l^{(\pm)}(s)$ and $V_l(s)$, defined by

$$F(s,t)\pm F(s,u)=\sum_{l=0}^{\infty}(2l+1)V_l^{(\pm)}(s)P_l(\cos\theta),$$
$$F(t,u)=\sum_{l=0}^{\infty}(2l+1)V_l(s)P_l(\cos\theta), \quad (2.3)$$

are given by

$$V_l^{(\pm)}(s)=(1\pm e^{i\pi l})\frac{\alpha(s)}{\sin\pi\alpha(s)}\tfrac{1}{2}i^{l-1}h_l(\nu),$$
$$V_l(s)=(1+e^{i\pi l})\tfrac{1}{2}i^l(4a\nu-2b+1)g_l(\nu), \quad (2.4)$$

where ν is the square of the c.m. momentum, $s=4(\nu+m_\pi^2)$, and

$$h_l(\nu)=e^{i\pi[\alpha(s)-2a\nu+b]}H_l(\nu;2a\nu-b+1,\alpha(s)+1)-e^{-i\pi[\alpha(s)-2a\nu+b]}\hat{H}_l(\nu;2a\nu-b+1,\alpha(s)+1),$$
$$g_l(\nu)=G_l(\nu;2a\nu-b+1,4a\nu-2b+2). \quad (2.5)$$

Here, the functions $H_l(\nu;\mu,\rho)$, $\hat{H}_l(\nu;\mu,\rho)$, and $G_l(\nu;\mu,\rho)$ are defined by

$$H_l(\nu;\mu,\rho)=\int_{-\infty}^{\infty}dx\,e^{i\delta}\frac{e^{-xe^{i\delta}\mu}}{(1+e^{-xe^{i\delta}})^{\rho}}j_l(-i2a\nu(i\pi+xe^{i\delta})),$$
$$\hat{H}_l(\nu;\mu,\rho)=\int_{-\infty}^{\infty}dx\,e^{i\delta}\frac{e^{-xe^{i\delta}\mu}}{(1+e^{-xe^{i\delta}})^{\rho}}j_l(-i2a\nu(-i\pi+xe^{i\delta})), \quad (2.6)$$
$$G_l(\nu;\mu,\rho)=\int_{-\infty}^{\infty}dx\,e^{i\delta}\frac{e^{-xe^{i\delta}\mu}}{(1+e^{-xe^{i\delta}})^{\rho}}j_l(i2a\nu xe^{i\delta}),$$

where the $j_l(z)$ are the spherical Bessel functions, and θ (the phase of ν) and δ are constrained by

$$|\theta+\delta|\le\tfrac{1}{2}\pi,\quad |\delta|<\tfrac{1}{2}\pi. \quad (2.7)$$

The analyticity domains of $h_l(\nu)$ and $g_l(\nu)$ in the complex ν plane are subject to the above constraints, where we may include also that portion of the real negative axis between $\nu=0$ and $\nu=-\nu_L$ [ν_L is less than the smaller of $(1-b)/4a$ and $(4am_\pi^2+2b)/4a$ for $h_l(\nu)$, and less than $(1-b)/4a$ for $g_l(\nu)$].

III. FUNCTIONAL RELATIONS

From the integral representations of the partial waves as set forth in the preceding section, we can deduce some useful functional relations. After an integration by parts (or by differentiating with respect to δ), we find, for example, that

$$H_l(\nu;\mu,\rho)=-\frac{(-\mu)}{(-i2a\nu)}\int_{-\infty}^{\infty}dx\,e^{i\delta}\frac{e^{-xe^{i\delta}\mu}}{(1+e^{-xe^{i\delta}})^{\rho}}zj_l(z)-\frac{(+\rho)}{(-i2a\nu)}\int_{-\infty}^{\infty}dx\,e^{i\delta}\frac{e^{-xe^{i\delta}(\mu+1)}}{(1+e^{-xe^{i\delta}})^{\rho+1}}zj_l(z)$$
$$-\int_{-\infty}^{\infty}dx\,e^{i\delta}\frac{e^{-xe^{i\delta}\mu}}{(1+e^{-xe^{i\delta}})^{\rho}}zj_l'(z), \quad (3.1)$$

where $z=-i2a\nu(i\pi+xe^{i\delta})$, and we have used

$$0=-\mu\int_{-\infty}^{\infty}dx\,e^{i\delta}\frac{e^{-xe^{i\delta}\mu}}{(1+e^{-xe^{i\delta}})^{\rho}}j_l(z)+\rho\int_{-\infty}^{\infty}dx\,e^{i\delta}\frac{e^{-xe^{i\delta}(\mu+1)}}{(1+e^{-xe^{i\delta}})^{\rho+1}}j_l(z)-i2a\nu\int_{-\infty}^{\infty}dx\,e^{i\delta}\frac{e^{-xe^{i\delta}\mu}}{(1+e^{-xe^{i\delta}})^{\rho}}j_l'(z) \quad (3.2)$$

(the latter expression can also be readily established by an integration by parts).

Substituting the identities

$$zj_l'(z)=-(l+1)j_l(z)+zj_{l-1}(z),$$
$$zj_l(z)=\sum_{n=0}^{m}(-)^n(2l-4n-1)j_{l-2n-1}(z)+(-)^{m+1}zj_{l-2m-2}(z),$$

we obtain, from (3.1),

$$lH_l(\nu;\mu,\rho)=\frac{(-\mu)}{(-i2a\nu)}\sum_{n=0}^{m}(-)^n(2l-4n-1)H_{l-2n-1}(\nu;\mu,\rho)$$
$$+\frac{(+\rho)}{(-i2a\nu)}\sum_{n=0}^{m}(-)^n(2l-4n-1)H_{l-2n-1}(\nu;\mu+1,\rho+1)+\sum_{n=0}^{m}(-)^n(2l-4n-3)H_{l-2n-2}(\nu;\mu,\rho)$$
$$+(-)^{m+1}\frac{(-\mu)}{(-i2a\nu)}\int_{-\infty}^{\infty}dx\,e^{i\delta}\frac{e^{-xe^{i\delta}\mu}}{(1+e^{-xe^{i\delta}})^{\rho}}zj_{l-2m-2}(z)$$
$$+(-)^{m+1}\frac{(+\rho)}{(-i2a\nu)}\int_{-\infty}^{\infty}dx\,e^{i\delta}\frac{e^{-xe^{i\delta}(\mu+1)}}{(1+e^{-xe^{i\delta}})^{\rho+1}}zj_{l-2m-2}(z)+(-)^{m+1}\int_{-\infty}^{\infty}dx\,e^{i\delta}\frac{e^{-xe^{i\delta}\mu}}{(1+e^{-xe^{i\delta}})^{\rho}}zj_{l-2m-3}(z)\,. \tag{3.3}$$

We see therefore that $H_l(\nu;\mu,\rho)$ can be expressed in terms of lower l values. Eventually, it reduces to a finite sum of functions of the type H_0. To clarify the nature of this reduction, let us suppose that l is even, in which case choose m such that $m=\frac{1}{2}(l-2)$. Then the last three terms of (3.3) add up to zero. This can be shown easily by noting that $zj_0(z)=\sin z$ and $zj_{-1}(z)=\cos z$ and integrating by parts. Similarly, suppose now that l is odd. If we set $m=\frac{1}{2}(l-3)$ and substitute $j_0(z)=(\sin z)/z$ and $j_1(z)=(\sin z)/z^2-(\cos z)/z$, the last three terms in (3.3) become a sum of five terms, three of which add up to zero again, and the remaining part is simply

$$(-)^{(l-1)/2}\frac{(-\mu)}{(-i2a\nu)}H_0(\nu;\mu,\rho)+(-)^{(l-1)/2}\frac{(+\rho)}{(-i2a\nu)}H_0(\nu;\mu+1,\rho+1)\,.$$

We find, therefore, from (3.3) that for l even

$$lH_l(\nu;\mu,\rho)=\frac{(-\mu)}{(-i2a\nu)}\sum_{n=0}^{(l-2)/2}(-)^n(2l-4n-1)H_{l-2n-1}(\nu;\mu,\rho)$$
$$+\frac{(+\rho)}{(-i2a\nu)}\sum_{n=0}^{(l-2)/2}(-)^n(2l-4n-1)H_{l-2n-1}(\nu;\mu+1,\rho+1)+\sum_{n=0}^{(l-2)/2}(-)^n(2l-4n-3)H_{l-2n-2}(\nu;\mu,\rho)\,, \tag{3.4}$$

and for l odd

$$lH_l(\nu;\mu,\rho)=\frac{(-\mu)}{(-i2a\nu)}\sum_{n=0}^{(l-3)/2}(-)^n(2l-4n-1)H_{l-2n-1}(\nu;\mu,\rho)$$
$$+\frac{(+\rho)}{(-i2a\nu)}\sum_{n=0}^{(l-3)/2}(-)^n(2l-4n-1)H_{l-2n-1}(\nu;\mu+1,\rho+1)+\sum_{n=0}^{(l-3)/2}(-)^n(2l-4n-3)H_{l-2n-2}(\nu;\mu,\rho)$$
$$+(-)^{(l-1)/2}\frac{(-\mu)}{(-i2a\nu)}H_0(\nu;\mu,\rho)+(-)^{(l-1)/2}\frac{(+\rho)}{(-i2a\nu)}H_0(\nu;\mu+1,\rho+1)\,. \tag{3.5}$$

It is obvious that we will obtain similar relations for $\hat{H}_l(\nu;\mu,\rho)$ and $G_l(\nu;\mu,\rho)$. In summary, we conclude that $H_l(\nu;\mu,\rho)$, $\hat{H}_l(\nu;\mu,\rho)$, and $G_l(\nu;\mu,\rho)$ can be expressed as finite sums, with respect to k, of $H_0(\nu;\mu+k,\rho+k)$, $\hat{H}_0(\nu;\mu+k,\rho+k)$, and $G_0(\nu;\mu+k,\rho+k)$, respectively. In other words, all partial waves are determined by essentially one function, viz., the s wave.

IV. ASYMPTOTIC BEHAVIOR

A. Asymptotic Behavior of the $I=1$ Partial Waves

As we have seen in the preceding section, it is sufficient to study the functions $H_0(\nu;\mu,\rho)$, $\hat{H}_0(\nu;\mu,\rho)$, and $G_0(\nu;\mu,\rho)$, or their combinations, in examining the asymptotic behavior of the partial waves. For the isospin-one partial waves we therefore introduce the function $W(\nu;k)$,

$$W(\nu;k)=\frac{\alpha(s)}{i\sin\pi\alpha(s)}\{e^{i\pi[\alpha(s)-2a\nu+b]}H_0(\nu;\mu+k,\rho+k)-e^{-i\pi[\alpha(s)-2a\nu+b]}\hat{H}_0(\nu;\mu+k,\rho+k)\}\,, \tag{4.1}$$

where $\mu=2a\nu-b+1$, $\rho=4a\nu+\lambda+b+1$, and $\lambda=4am_\pi^{\,2}$. The isospin-one partial waves $V_l^{(-)}(s)$ are then finite sums of $W(\nu;k)$ over k.

Using the relation

$$j_0(z)=\frac{\sin z}{z}=\frac{1}{2}\int_{-1}^{1}dt\,e^{izt}$$

and the formula[13]

$$\int_{-\infty}^{\infty}dx\frac{e^{-\mu x}}{(e^{\beta/\gamma}+e^{-x/\gamma})^{\nu}}=\gamma\exp[\beta(\mu-\nu/\gamma)]B(\gamma\mu,\nu-\gamma\mu)\,, \tag{4.2}$$

with $\mathrm{Re}(\nu/\gamma)>\mathrm{Re}\mu>0$ and $|\mathrm{Im}\beta|<\pi\mathrm{Re}\gamma$, we find

$$W(\nu;k)=(-)^k\alpha(s)\Gamma(-\alpha(s)-k)\int_{-1}^{1}dt\frac{\Gamma(2a\nu(1+t)-b+1+k)}{\Gamma(-2a\nu(1-t)-\lambda-2b+1)}\,. \tag{4.3}$$

Equation (4.3) corresponds to the partial waves as analytically continued by Drago and Matsuda[4] and taken over by Park and Desai[8] in their study of the asymptotic behavior of the $I=1$ amplitude. To determine the bound of $W(\nu;k)$, we employ the technique used by Park and Desai. We then find that as $|\nu|\to\infty$, for $\mathrm{Im}\nu>0$ and $\gamma>0$,

$$\begin{aligned}&\nu^{1-b+k-\gamma}W(\nu;k)\to 0\ \text{ if }\frac{|\mathrm{Im}\nu|}{\ln|\nu|}\to\infty\,,\\ &\left.\begin{aligned}\nu^{\lambda+2b-\gamma}W(\nu;k)\to 0\\ \nu^{1-b+k-\gamma}W(\nu;k)\to 0\end{aligned}\right\}\ \text{ if }\frac{|\mathrm{Im}\nu|}{\ln|\nu|}\to 0\text{ and }\mathrm{Im}\nu\neq 0\,.\end{aligned} \tag{4.4}$$

For $\mathrm{Im}\nu<0$, we obtain the same bounds in view of the relation $[V_l^{(-)}(s)]^*=V_l^{(-)}(s^*)$.[1]

Since $V_l^{(-)}(s)$ is a finite sum of $W(\nu;k)$ over k, we deduce, for $\gamma>0$,

$$\begin{aligned}&s^{1-b-\gamma}V_l^{(-)}(s)\to 0\ \text{ if }\frac{|\mathrm{Im}s|}{\ln|s|}\to\infty\,,\\ &s^{\chi-\gamma}V_l^{(-)}(s)\to 0\ \text{ if }\frac{|\mathrm{Im}s|}{\ln|s|}\to 0\text{ and }|\mathrm{Im}s|\neq 0\,,\end{aligned} \tag{4.5}$$

where χ is the lesser of $\lambda+2b$ and $1-b$. Equation (4.5) is exactly what Park and Desai[8] have obtained.

B. Asymptotic Behavior of the $I=2$ Partial Waves

To investigate the asymptotic behavior of the isospin-two partial waves $V_l(s)$, we need only study

$$G_0(\nu;k)\equiv G_0(\nu;2a\nu-b+1+k,4a\nu-2b+2+k)\,,$$

since $V_l(s)$ is a finite sum of $G_0(\nu;k)$ over k. Once again we use $j_0(z)=\frac{1}{2}\int_{-1}^{1}dt\,e^{izt}$ and the formula (4.2) to find

$$G_0(\nu;k)=\frac{1}{2}\int_{-1}^{1}dt\,f(t,\nu)\,, \tag{4.6}$$

where

$$f(t,\nu)=B(2a\nu(1+t)-b+1+k,2a\nu(1-t)-b+1)\,.$$

Following Park and Desai[8] again, we divide the range of integration into three parts such that

$$G_0(\nu;k)=\frac{1}{2}\left(\int_{-1}^{-1+|\nu|^{-\beta}}+\int_{-1+|\nu|^{-\beta}}^{1-|\nu|^{-\beta}}+\int_{1-|\nu|^{-\beta}}^{1}\right)dt\,f(t,\nu)\,, \tag{4.7}$$

with $0<\beta<1$.

Consider the integral of $f(t,\nu)$ from $t=-1$ to $t=-1+|\nu|^{-\beta}$. Regardless of the magnitude of ν, there is always a region near $t=-1$ for which $\nu(1+t)$ is small or zero. The asymptotic behavior of $f(t,\nu)$ as $|\nu|\to\infty$ is then given by

$$\ln|f|\sim-[1-b+k+2a(1+t)\mathrm{Re}\nu]\ln|2a\nu|-2a(\mathrm{Re}\nu)[2\ln 2-(1-t)\ln(1-t)]+\text{const}\,,\ \text{ if }|\nu(1+t)|\text{ is bounded;}$$

$$\ln|f|\sim-\tfrac{1}{2}\ln|\nu|-2a(\mathrm{Re}\nu)[2\ln 2-(1+t)\ln(1+t)-(1-t)\ln(1-t)]+(-b+\tfrac{1}{2}+k)\ln(1+t)+\text{const}\,,\ \text{ if }|\nu(1+t)|\to\infty\,.$$

If we restrict ourselves to the region $\mathrm{Re}\nu\geq -N$, where N is an arbitrary, finite, positive number, we see from the above asymptotic behavior that

$$\left|\int_{-1}^{-1+|\nu|^{-\beta}} dt\, f(t,\nu)\right| \leq \text{const}\times\left[|\nu|^{-(1-b+k+\beta)}+|\nu|^{-1/2-\beta(3/2-b+k)}\right]$$
$$\leq \text{const}\times|\nu|^{-(2-b+k)+\gamma},$$

where $\gamma\equiv 1-\beta>0$. We therefore find that as $|\nu|\to\infty$, for $\gamma>0$,

$$\nu^{2-b+k-\gamma}\int_{-1}^{-1+|\nu|^{-\beta}} dt\; f(t,\nu)\to 0 \quad \text{if } \mathrm{Re}\nu \not\to -\infty. \tag{4.8}$$

For values of t in the range $[-1+|\nu|^{-\beta}, 1-|\nu|^{-\beta}]$, $|(1\pm t)\nu|\to\infty$ as $|\nu|\to\infty$ and the asymptotic behavior of $f(t,\nu)$ is given by

$$\ln|f| \sim -\tfrac{1}{2}\ln|\nu| - 2a(\mathrm{Re}\nu)[2\ln 2-(1+t)\ln(1+t)-(1-t)\ln(1-t)]$$
$$+(-b+\tfrac{1}{2}+k)\ln(1+t)+(-b+\tfrac{1}{2})\ln(1-t)+\text{const}.$$

Since

$$2\ln 2-(1+t)\ln(1+t)-(1-t)\ln(1-t)>\beta|\nu|^{-\beta}\ln|\nu|,$$

we see, as $|\nu|\to\infty$, that

$$\left|\int_{-1+|\nu|^{-\beta}}^{1-|\nu|^{-\beta}} dt\, f(t,\nu)\right| \leq \text{const}\times\begin{cases}|\nu|^{-1/2} & \text{if } |\mathrm{Re}\nu|\not\to\infty,\\ |\nu|^{-1/2}\exp[-2a\beta(\mathrm{Re}\nu)|\nu|^{-\beta}\ln|\nu|] & \text{if } \mathrm{Re}\nu\to\infty,\end{cases}$$

which in turn gives, as $|\nu|\to\infty$, for $\gamma>0$,

$$\nu^{1/2-\gamma}\int_{-1+|\nu|^{-\beta}}^{1-|\nu|^{-\beta}} dt\, f(t,\nu)\to 0\begin{cases}\text{if } |\mathrm{Re}\nu|\not\to\infty,\\ \text{exponentially if } \mathrm{Re}\nu\to\infty.\end{cases} \tag{4.9}$$

As t varies from $1-|\nu|^{-\beta}$ to 1, we observe again that there is a region of t in which $|\nu(1-t)|$ remains bounded as $|\nu|\to\infty$. The asymptotic behavior of $f(t,\nu)$ is therefore given by

$$\ln|f| \sim -[1-b+2a(1-t)\,\mathrm{Re}\nu]\ln|2a\nu| - 2a(\mathrm{Re}\nu)[2\ln 2-(1+t)\ln(1+t)]+\text{const}, \text{ if } |\nu(1-t)| \text{ is bounded};$$
$$\ln|f| \sim -\tfrac{1}{2}\ln|\nu| - 2a(\mathrm{Re}\nu)[2\ln 2-(1+t)\ln(1+t)-(1-t)\ln(1-t)]+(-b+\tfrac{1}{2})\ln(1-t)+\text{const}, \text{ if } |\nu(1-t)|\to\infty.$$

We therefore deduce that, as $|\nu|\to\infty$,

$$\left|\int_{1-|\nu|^{-\beta}}^{1} dt\, f(t,\nu)\right| \leq \text{const}\times\left[|\nu|^{-(1-b)-\beta}+|\nu|^{-1/2-\beta(3/2-b)}\right],$$

which yields, for $\gamma>0$,

$$\nu^{2-b-\gamma}\int_{1-|\nu|^{-\beta}}^{1} dt\, f(t,\nu)\to 0 \quad \text{if } \mathrm{Re}\nu\not\to-\infty. \tag{4.10}$$

Combining Eqs. (4.7)–(4.10) and (2.4), we conclude finally that, as $|s|\to\infty$,

$$s^{1-b-\gamma}V_l(s)\to 0 \quad \text{if } \mathrm{Re}s\to\infty;$$
$$s^{-1/2-\gamma}V_l(s)\to 0 \quad \text{if } |\mathrm{Re}s|\not\to\infty, \tag{4.11}$$

where $\gamma>0$.

If we use the integral form (2.6) of $G_0(\nu;k)$ to determine the bounds, we may proceed as follows: Consider the region where $-\frac{1}{2}\pi\leq\theta\equiv\arg\nu\leq\frac{1}{2}\pi$. Then we can always choose $\delta=0$ [see (2.7)]; this extra degree of freedom which we have at our disposal in selecting δ is very useful, as we will see later. Then, using $|j_0(z)|\leq\exp|\mathrm{Im}z|$ and the formula (4.2), we find

$$|G_0(\nu;k)|\leq\int_{-\infty}^{\infty}dx\,\frac{e^{-x(2a|\nu|\cos\theta-b+1+k)}}{(1+e^{-x})^{4a|\nu|\cos\theta-2b+2+k}}\,e^{2a|\nu|x\cos\theta}$$
$$\leq B(1-b+k, 4a|\nu|\cos\theta-b+1).$$

As $|\nu|\to\infty$, this yields the bounds

$$|G_0(\nu;k)| \leq \begin{cases} \text{const} & \text{if } |\theta| = \frac{1}{2}\pi, \\ \text{const} \times |\nu|^{-(1-b+k)} & \text{if } |\theta| < \frac{1}{2}\pi. \end{cases}$$

We see therefore that the former method gives better bounds than the latter. However, we cannot use the former method in most of the region of the left half-plane, or when $\mathrm{Re}\nu \to -\infty$; nor can we find a contour as useful as the one employed for the $I=1$ partial waves (see Appendix C of Ref. 8).

Let us observe, however, that if $\frac{1}{2}\pi < \theta < \pi$, we can always choose δ in our integral representations such that $\theta + \delta = \frac{1}{2}\pi$. Then,

$$G_0(\nu;k) = e^{i\delta} \int_{-\infty}^{\infty} dx \frac{e^{-xe^{i\delta}(2a\nu - b + 1 + k)}}{(1+e^{-xe^{i\delta}})^{4a\nu - 2b + 2 + k}} \frac{\sin(2a|\nu|x)}{2a|\nu|x}. \tag{4.12}$$

Now we use $\int_{-\infty}^{\infty} e^{iky} dy = 2\pi\delta(k)$ to derive the formal identity

$$\frac{\sin(2a|\nu|x)}{x} = \pi\delta(x) - \int_{|2a\nu|}^{\infty} dy \cos xy. \tag{4.13}$$

Substituting[14] (4.13) into (4.12) and making use of (4.2), we obtain, for $\frac{1}{2}\pi < \theta < \pi$,

$$G_0(\nu;k) = \frac{\pi i}{2a\nu}\left(\frac{1}{2}\right)^{4a\nu - 2b + 2 + k} - \frac{1}{4a|\nu|} g(\nu;k), \tag{4.14}$$

where

$$g(\nu;k) = \int_0^{\infty} dt [B(1 - b + k - te^{i\theta}, 4a\nu - b + 1 + te^{i\theta}) + B(4a\nu - b + 1 + k + te^{i\theta}, 1 - b - te^{i\theta})]. \tag{4.15}$$

In the Appendix it is established that

$$|g(\nu;k)| < \text{const} \times |\nu|^{b-1} e^{-2a|\nu|\cos\theta}. \tag{4.16}$$

It follows from (4.14) and (4.16) that $\nu \exp(4a\nu \ln 2) G_0(\nu;k) \to \text{const}$, as $|\nu| \to \infty$. We have therefore deduced that, for $\frac{1}{2}\pi < \theta < \pi$,

$$e^{as\ln 2} V_l(s) \to \text{const}, \quad \text{as } |s| \to \infty. \tag{4.17}$$

The same relation holds for $-\pi < \theta < -\frac{1}{2}\pi$ because $[V_l(s)]^* = V_l(s^*)$.[1] This exponential divergence of the $I=2$ partial waves along any ray into the left half s plane confirms the behavior conjectured by Tryon.[10]

C. Asymptotic Behavior of the $I=0$ Partial Waves

Since the isospin-zero partial waves $\frac{1}{2}V_l(s) - \frac{3}{2}V_l^{(+)}(s)$ are finite sums, over k, of the $W(\nu;k)$ [Eq. (4.1)] and $G_0(\nu;k)$ [Eq. (4.6)], it follows from (4.5), (4.11), and (4.17) that as $|s| \to \infty$,

$$\begin{aligned} &e^{as\ln 2}[\tfrac{1}{2}V_l(s) - \tfrac{3}{2}V_l^{(+)}(s)] \to \text{const} \quad \text{if } \mathrm{Re}s \to -\infty; \\ &s^{-1/2-\gamma}[\tfrac{1}{2}V_l(s) - \tfrac{3}{2}V_l^{(+)}(s)] \to 0 \quad \text{if } |\mathrm{Re}s| \not\to \infty; \\ &s^{1-b-\gamma}[\tfrac{1}{2}V_l(s) - \tfrac{3}{2}V_l^{(+)}(s)] \to 0 \quad \text{if } \mathrm{Re}s \to \infty \text{ and } \frac{|\mathrm{Im}s|}{\ln|s|} \to \infty; \\ &s^{\chi-\gamma}[\tfrac{1}{2}V_l(s) - \tfrac{3}{2}V_l^{(+)}(s)] \to 0 \quad \text{if } \mathrm{Re}s \to \infty,\ \frac{|\mathrm{Im}s|}{\ln|s|} \to 0, \text{ and } |\mathrm{Im}s| \neq 0, \end{aligned} \tag{4.18}$$

where $\gamma > 0$ and χ is the lesser of $1-b$ and $\lambda + 2b$.

V. ASYMPTOTIC BEHAVIOR OF THE DISCONTINUITIES

The analyticity domains of $h_l(\nu)$ and $g_l(\nu)$ include only a small part of the negative real axis, as we have seen in Sec. II (see also I). To show that the partial waves have cuts along the rest of the negative real axis in the energy-squared plane, we use an integral representation of the spherical Bessel function,

$$j_l(z) = \frac{1}{2\pi i^{l+1}} \int_C dt\, e^{izt} Q_l(t), \tag{5.1}$$

where $Q_l(t)$ is the Legendre polynomial of the second kind and the closed contour C contains the real axis between −1 and +1. Equation (5.1), which is a special case of the Whittaker loop integral,[15] can be easily proved.

Substituting (5.1) into (2.6), taking the closed contour C to be infinitesimally close to the real axis between −1 and +1, and using the formula (4.2), we find

$$H_l(\nu;\mu,\rho)=\frac{1}{2\pi i^{l+1}}\int_C dt\, e^{i\pi 2a\nu t}Q_l(t)B(\mu-2a\nu t,\rho-\mu+2a\nu t)\,, \tag{5.2}$$

where, in general, $\mu=2a\nu-b+1+k$, $\rho=4a\nu+\lambda+b+1+k$, and $\lambda=4am_\pi^2$. Since

$$Q_l(t)\Gamma(\mu-2a\nu t)\Gamma(\rho-\mu+2a\nu t)\sim M(\nu)t^{\rho-l-2}/\sin\pi(\mu-2a\nu t) \quad \text{as} \quad |t|\to\infty\,,$$

where $M(\nu)$ is some function of ν, we can open up the contour C and use the residue theorem to obtain

$$H_l(\nu;\mu,\rho)=-\frac{1}{2a\nu i^l}\sum_{n=0}^{\infty}\frac{\Gamma(n+\rho)}{n!\,\Gamma(\rho)}\left[-e^{i\pi\mu}Q_l\left(\frac{n+\mu}{2a\nu}\right)+e^{-i\pi(\rho-\mu)}Q_l\left(-\frac{n+\rho-\mu}{2a\nu}\right)\right], \tag{5.3}$$

if $\mathrm{Re}(\rho-l-2)<-1$. Similarly, for $\mathrm{Re}(\rho-l-2)<-1$, we have

$$\hat{H}_l(\nu;\mu,\rho)=-\frac{1}{2a\nu i^l}\sum_{n=0}^{\infty}\frac{\Gamma(n+\rho)}{n!\,\Gamma(\rho)}\left[-e^{-i\pi\mu}Q_l\left(\frac{n+\mu}{2a\nu}\right)+e^{i\pi(\rho-\mu)}Q_l\left(-\frac{n+\rho-\mu}{2a\nu}\right)\right]. \tag{5.4}$$

It therefore follows from (2.4),(2.5) and (5.3),(5.4) that

$$V_l^{(\pm)}(s)=-(1\pm e^{i\pi l})\frac{\alpha(s)}{2a\nu}\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha(s)+1)}{n!\,\Gamma(\alpha(s)+1)}Q_l\left(1+\frac{n-b+1}{2a\nu}\right), \tag{5.5}$$

for $\mathrm{Re}\alpha(s)<l$. This is the representation used by most authors in their studies of the partial waves; it shows manifestly that we have cuts along the negative real axis in the complex s plane, starting at $s=-(1-b)a^{-1}+4m_\pi^2$, if $\lambda+2b<1+l$.

In this section, we are primarily interested in the asymptotic behavior of the discontinuities. Even though the power bound of $\mathrm{disc}V_l^{(\pm)}(s)$ has already been derived by Park and Desai,[8] we show here how simply we can get the same result by making use of the functional relations of Sec. III. Since $V_l^{(\pm)}(s)$ is a finite sum over k of $W(\nu;k)$, where from (4.1) and (5.1)

$$W(\nu;k)=(-)^{1+k}\frac{\alpha(s)}{2a\nu}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha(s)+1+k+n)}{n!\,\Gamma(\alpha(s)+1+k)}Q_0\left(1+\frac{n-b+1+k}{2a\nu}\right) \tag{5.6}$$

for $\mathrm{Re}\alpha(s)<l-k$, we study the discontinuity Δ_k, defined by

$$\Delta_k=\mathrm{disc}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha(s)+1+k+n)}{n!\,\Gamma(\alpha(s)+1+k)}Q_0\left(1+\frac{n-b+1+k}{2a\nu}\right)$$
$$=\tfrac{1}{2}\pi\sum_{n=0}^{n_0-1-k}\frac{\Gamma(\alpha(s)+1+k+n)}{n!\,\Gamma(\alpha(s)+1+k)}\,, \tag{5.7}$$

where the integer n_0 is equal to $|4a\nu|+b-\epsilon$, with $0\leqslant\epsilon<1$.

Since

$$\sum_{n=0}^{\infty}\frac{\Gamma(n+\sigma)}{n!\,\Gamma(\sigma)}=0$$

for $\mathrm{Re}\sigma<0$, we have

$$-\frac{2}{\pi}\Delta_k\Gamma(\alpha(s)+1+k)=\frac{\Gamma(n_0+\alpha(s)+1)}{\Gamma(n_0-k+1)}+\sum_{n=1}^{\infty}\frac{\Gamma(n+n_0+\alpha(s)+1)}{(n+n_0-k)!}\,. \tag{5.8}$$

If we define the integer N such that $N\geqslant\lambda+2b$, then

$$\sum_{n=1}^{\infty}\frac{\Gamma(n+n_0+\alpha(s)+1)}{(n+n_0-k)!}\leqslant\sum_{n=1}^{\infty}\frac{1}{(n+n_0-k)(n+n_0-k-1)\cdots(n+N+1)}\,,$$

where the right-hand summation can be carried out,[16] with the result

$$\sum_{n=1}^{\infty}\frac{\Gamma(n+n_0+\alpha(s)+1)}{(n+n_0-k)!}\leqslant\frac{1}{(n_0-k)!}\,\frac{\Gamma(N+2)}{n_0-k-N-1}\,. \tag{5.9}$$

Therefore, we see from (5.8) and (5.9) that

$$\Delta_k \sim -\tfrac{1}{2}\pi \frac{\Gamma(n_0+\alpha(s)+1)}{\Gamma(n_0-k+1)\Gamma(\alpha(s)+1+k)}$$
$$\sim \tfrac{1}{2}\sin\pi[\alpha(s)+k]\Gamma(\lambda+2b+1-\epsilon)(4a|\nu|)^{-\lambda-2b-1+\epsilon},$$

which in turn leads to

$$|s|^{\lambda+2b}\,\mathrm{disc}\,V_l^{(\pm)}(s)\to 0 \quad \text{as } s\to-\infty. \tag{5.10}$$

So the discontinuities of the $I=1$ partial waves are bounded by at least $|s|^{-\lambda-2b}$ as $s\to-\infty$.

In a similar way, by using (5.1), (4.2), and the residue theorem, we have

$$V_l(s)=(1+e^{i\pi l})\frac{4a\nu-2b+2}{2a\nu}\sum_{n=0}^{\infty}(-)^n\frac{\Gamma(n+4a\nu-2b+2)}{n!\,\Gamma(4a\nu-2b+2)}Q_l\left(1+\frac{n-b+1}{2a\nu}\right) \tag{5.11}$$

for $\mathrm{Re}\,s<4m_\pi^2+(l+2b-1)a^{-1}$. Thus we see that $V_l(s)$ has cuts starting at $s=4m_\pi^2-(1-b)a^{-1}$. To determine the asymptotic form of $\mathrm{disc}V_l(s)$, it is more convenient to note that $V_l(s)$ is a finite sum, over k, of $G_0(\nu;k)$ [Eq. (4.6)], where

$$G_0(\nu;k)=\frac{1}{2a\nu i^l}\sum_{n=0}^{\infty}(-)^n\frac{\Gamma(2\mu+k+n)}{n!\Gamma(2\mu+k)}\left[Q_0\left(\frac{n+\mu+k}{2a\nu}\right)+Q_0\left(-\frac{\mu+n}{2a\nu}\right)\right] \tag{5.12}$$

and $\mu=2a\nu-b+1$. That is to say, we study the discontinuity $\bar{\Delta}_k$ defined by

$$\bar{\Delta}_k=\mathrm{disc}\,2a\nu i^l G_0(\nu;k)$$
$$=\tfrac{1}{2}\pi\left(\sum_{n=0}^{n_0-1-k}+\sum_{n=0}^{n_0-1}\right)(-)^n\frac{\Gamma(2\mu+n+k)}{n!\,\Gamma(2\mu+k)}, \tag{5.13}$$

which can be rewritten, in view of the identity

$$2^{-\sigma}=\sum_{n=0}^{\infty}(-)^n\frac{\Gamma(n+\sigma)}{n!\,\Gamma(\sigma)},$$

as

$$\frac{2}{\pi}\bar{\Delta}_k=\left(\frac{1}{2}\right)^{2\mu+k-1}-\left(\sum_{n=n_0-k}^{\infty}+\sum_{n=n_0}^{\infty}\right)(-)^n\frac{\Gamma(2\mu+k+n)}{n!\,\Gamma(2\mu+k)}. \tag{5.14}$$

Substituting $n_0=4a|\nu|+b-\epsilon$ and $\mu=-2a|\nu|-b+1$, we see that the second sum in (5.14), for example, has a modulus less than

$$\mathrm{const}\times\sum_{n=0}^{\infty}\Gamma(n+k+2-b-\epsilon)e^n\left(\frac{1}{4a|\nu|+n}\right)^{n-b+2+k-\epsilon}\left(\frac{4a|\nu|}{4a|\nu|+n}\right)^{4a|\nu|+2b-3/2-k}$$

as $\nu\to-\infty$. Since $x!=\Gamma(1+x)=(2\pi)^{1/2}x^{x+1/2}\exp(-x+\theta/12x)$ for $0<\theta<1$ and $x>0$, we find

$$\Gamma(n+k+2-b-\epsilon)=\frac{\Gamma(1+n+k+2-b-\epsilon)}{n+k+2-b-\epsilon}\leq \mathrm{const}\times(n+k+2-b-\epsilon)^{n+k+5/2-b-\epsilon}e^{-n}.$$

It follows therefore that, as $\nu\to-\infty$,

$$\left|\sum_{n=n_0}^{\infty}(-)^n\frac{\Gamma(2\mu+k+n)}{n!\,\Gamma(2\mu+k)}\right|\leq \mathrm{const}\times(4a|\nu|)^{5/2}. \tag{5.15}$$

The same bound for the first sum in (5.14) can be obtained by simply replacing ϵ by $\epsilon+k$ in the preceding argument.

From (5.14) and (5.15) we conclude that

$$\bar{\Delta}_k\sim\mathrm{const}\times(\tfrac{1}{2})^{4a\nu},$$

and

$$e^{as\ln 2}\,\mathrm{disc}\,V_l(s)\to\mathrm{const} \tag{5.16}$$

as $s\to-\infty$. In other words, the discontinuities of the isospin-zero and -two partial waves diverge exponentially as $s\to-\infty$.

VI. SUMMARY

Within the framework of the integral representations of I for the dual $\pi\pi$ partial waves, we have shown how we may reduce all partial waves to finite sums of s waves. We have also seen that we can easily reproduce the power bounds of Park and Desai[8] for the $I=1$ partial waves in the entire s plane (where s is the c.m. energy squared) and for the discontinuities along the left-hand cut by using the reduction to s-wave sums and by applying their technique to our s-wave representation.

Insofar as new results are concerned, we would like to emphasize that the aforementioned reduction, together with the freedom in the choice of contour appearing in the integral representations, is essential to establish the asymptotic exponential divergence of the $I=0$ and 2 partial waves in the left half s plane. We have also established the power bounds of the $I=0$ and 2 partial waves in the right half s plane and the asymptotic exponential divergence of their discontinuities along the left-hand cut.

The outstanding problem of the present dual model is its failure to satisfy unitarity. The technique of unitarizing the dual amplitudes by employing the N/D method,[17] where the discontinuity of an amplitude across the left-hand cut is taken as the input discontinuity, is applicable to the $I=1$ amplitude. This procedure cannot be used when $I=0$ or 2 because of the presence of exponential divergences. In the latter instance, the K-matrix method would appear to be more attractive.

APPENDIX: PROOF OF EQ. (4.16)

In this appendix we establish the bound (4.16) as given in the text. Divide $g(\nu;k)$ [Eq. (4.15)] into two parts, defining $I(\nu;k)$ and $I_3(\nu;k)$ as follows:

$$g(\nu;k)=I(\nu;k)+I_3(\nu;k)\,, \tag{A1}$$

where

$$I(\nu;k)=\int_0^R dt[B(1-b+k-te^{i\theta},Re^{i\theta}-b+1+te^{i\theta}) +B(Re^{i\theta}-b+1+k+te^{i\theta},1-b-te^{i\theta})]\,, \tag{A2}$$

and $I_3(\nu;k)$ is the corresponding integral from $t=R$ to $t=\infty$. Here, $4a\nu\equiv Re^{i\theta}$, and $\frac{1}{2}\pi<\theta<\pi$, which is the only region of interest.

Since the asymptotic form of the integrand of $I_3(\nu;k)$ is

$$\text{const}\times\exp\{(Re^{i\theta}-2b+1+k)\ln(t/R) +(Re^{i\theta}-b+\tfrac{1}{2}+te^{i\theta})\ln(1+R/t) -\tfrac{1}{2}\ln Re^{i\theta}+i\pi te^{i\theta}\}\,,$$

we easily obtain

$$I_3(\nu;k)\to 0 \text{ exponentially as } |\nu|\to\infty\,. \tag{A3}$$

Next, we observe that the integrand in (A2) has no poles in the upper half-plane of $te^{i\theta}$. Therefore, we deform the path of integration to the contour along the imaginary axis from the origin to $Re^{i\pi/2}$ and the arc from $Re^{i\pi/2}$ to $Re^{i\theta}$, thus obtaining

$$I(\nu;k)=I_1(\nu;k)+I_2(\nu;k)\,, \tag{A4}$$

where

$$I_1(\nu;k)=ie^{-i\theta}\int_{\pi/2}^{\theta}Re^{i\phi}d\phi[B(1-b+k-Re^{i\phi},Re^{i\theta}-b+1+Re^{i\phi})+B(Re^{i\theta}-b+1+k+Re^{i\phi},1-b-Re^{i\phi})]\,, \tag{A5}$$

$$I_2(\nu;k)=ie^{-i\theta}\int_0^R dy[B(1-b+k-iy,Re^{i\theta}-b+1+iy)+B(Re^{i\theta}-b+1+k+iy,1-b-iy)]\,. \tag{A6}$$

Let us consider (A5) first. It is easy to show that the modulus of the asymptotic form of the integrand in (A5) is less than

$$\text{const}\times R^{1/2}\exp\{R(\cos\theta+\cos\phi)\ln|e^{i\theta}+e^{i\phi}|-R(\sin\theta+\sin\phi)\arg(e^{i\theta}+e^{i\phi})+R\theta\sin\theta+R(\phi-\pi)\sin\phi\}\,. \tag{A7}$$

Since

$$\sqrt{2}\le 2\cos\tfrac{1}{2}(\theta-\tfrac{1}{2}\pi)\le|e^{i\theta}+e^{i\phi}|\le 2\,,$$

and

$$\tfrac{1}{2}(\theta+\tfrac{1}{2}\pi)\le\arg(e^{i\theta}+e^{i\phi})\le\theta\,,$$

we see that (A7) is less than

$$\text{const}\times R^{1/2}\exp(\tfrac{1}{2}R\cos\theta\ln 2-\tfrac{1}{2}\pi R\sin\theta)\,.$$

So we obtain

$$I_1(\nu;k)\to 0 \text{ exponentially as } |\nu|\to\infty\,. \tag{A8}$$

Finally, we consider the integral in (A6). As $|\nu| \to \infty$, the modulus of the asymptotic form of the integrand in (A6) is less than

$$\text{const} \times |\nu|^{b-1} \exp\left[R\cos\theta \ln\left|1+\frac{it}{Re^{i\theta}}\right| - R\psi\sin\theta - t(\theta+\psi)\right]. \tag{A9}$$

Here, $\psi \equiv \arg(1+it/Re^{i\theta})$. Since $0 \le t \le R$ and $\frac{1}{2}\pi < \theta < \pi$, we have

$$\left|1+\frac{it}{Re^{i\theta}}\right| \ge 1 \text{ and } -\tfrac{1}{2}(\theta-\tfrac{1}{2}\pi) \le \psi \le 0. \tag{A10}$$

From (A9) and (A10), we therefore deduce

$$\begin{aligned}|I_2(\nu;k)| &< \text{const} \times |\nu|^{b-1} e^{R(\sin\theta)[(\theta-\pi/2)/2]} \\ &< \text{const} \times |\nu|^{b-1} e^{-2a|\nu|\cos\theta}.\end{aligned} \tag{A11}$$

Here, we have used the fact that $-\frac{1}{2}(\theta-\frac{1}{2}\pi)\tan\theta < \frac{1}{2}$ for $\frac{1}{2}\pi < \theta < \pi$.

Combining (A3), (A8), and (A11), we obtain

$$|g(\nu;k)| < \text{const} \times |\nu|^{b-1} e^{-2a|\nu|\cos\theta}, \tag{A12}$$

which completes the proof.

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[14]This is equivalent to the statement that

$$f(y) = e^{i\delta} \frac{e^{-ye^{i\delta}(2a\nu-b+1+k)}}{(1+e^{-ye^{i\delta}})^{4a\nu-2b+2+k}},$$

together with

$$2\pi f(0) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(y) e^{ixy},$$

the standard identity encountered in the theory of the Fourier transformation.

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[16]See Eq. (0.243-1) in Ref. 13.

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