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PHYSICAL REVIEW D VOLUME 7, NUMBER 12 15 JUNE 1973

Dual $\pi\pi$ Partial-Wave Amplitudes. I. Threshold Behavior*

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A formulation of the integral representations of the dual $\pi\pi$ partial-wave amplitudes is presented. One of the many advantages of these representations is that one can extract the threshold coefficients without difficulty. This is demonstrated in a calculation of the scattering lengths and effective ranges for the s, p, d, and f waves.

I. INTRODUCTION

Subsequent to Veneziano's original paper,[1] a dual $\pi\pi$ scattering amplitude which meets the requirements of Regge asymptotic behavior and crossing symmetry and which exhibits zero-width resonance poles was proposed by Lovelace[2] and Shapiro.[3] The partial-wave structure of this amplitude and, in particular, its asymptotic behavior have been studied in the complex angular momentum plane and in the complex energy-squared plane by many authors,[4-11] partly in connection with the K-matrix unitarization scheme, and partly in connection with dispersion-theoretic studies, among other considerations.

In contrast to the partial-fraction expansion of partial waves which has been used by most authors, we present a unified treatment of the corresponding integral representations, which turn out to be more useful in many respects. In this paper, we study, in some detail, the projection of the partial waves in the complex energy-squared plane and their threshold behavior; in a sequel, we will examine their asymptotic behavior and cut structure.

In Sec. II, we perform a partial-wave projection in the s-channel physical region, and, in Sec. III, we carry out the analytic continuation into the entire complex energy-squared plane. We exhibit, in Sec. IV, the normal threshold behavior, with coefficients which can be easily calculated. Here we see, for the first time, the advantage of our partial-wave formulation. By way of application, we calculate the scattering lengths and effective ranges for the s, p, d, and f waves in Sec. V. An appendix is devoted to questions of kinematics and notation.

II. PARTIAL-WAVE PROJECTION IN THE s-CHANNEL PHYSICAL REGION

The dual model for the elastic $\pi\pi$ scattering amplitude was introduced[2,3] by identifying the isospin-two amplitude $A^2(s,t,u)$ of Eq. (A3) with $F(t,u)$, where

$$F(x,y)=\frac{\Gamma(1-\alpha(x))\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))}+\text{secondary terms}, \tag{2.1}$$

and $\alpha(x)=ax+b$ represents a linear Regge trajectory with a and b real.

The other amplitudes are then determined by Eqs. (A2)–(A4), being given by

$$\begin{aligned}A^0(s,t,u)&=\tfrac{1}{2}gF(t,u)-\tfrac{3}{2}g[F(s,t)+F(s,u)],\\ A^1(s,t,u)&=g[F(s,u)-F(s,t)],\\ A^2(s,t,u)&=-gF(t,u),\end{aligned} \tag{2.2}$$

where g is the over-all coupling constant. In our discussion, we take the simplest form of $F(x,y)$, neglecting the secondary terms, and assume that $a>0$ and $0<b<1$ (inasmuch as $b>1$ violates Froissart's bound,[12] and $b=1$ corresponds to the Pomeranchuk trajectory).

The amplitudes given by Eq. (2.2) are analytic functions of complex s, t, and u, having only simple poles on the real axis. In the s-channel physical region, s, t, and u are real; also, $s=4\nu+4m_\pi^2\geq 4m_\pi^2$, $t=-2\nu(1-\cos\theta)\leq 0$, and $u=-2\nu(1+\cos\theta)\leq 0$, where $\nu\equiv q^2$ and q is the c.m. momentum. We therefore find, in view of the identity $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$,

$$F(s,t)=\frac{\alpha(s)}{\sin\pi\alpha(s)}\sin(\pi[\alpha(s)+\alpha(t)])\times\int_0^1 du\,u^{\alpha(s)+\alpha(t)-1}(1-u)^{-\alpha(t)}. \tag{2.3}$$

Changing the variable u to x such that $u=1/(1+e^{-x})$, and substituting the expansion[13]

$$e^{iz\cos\theta}=\sum_{l=0}^{\infty}(2l+1)i^l j_l(z)P_l(\cos\theta),$$

where $j_l(z)$ is the spherical Bessel function $[j_0(z)=(\sin z)/z,\ j_1(z)=(\sin z)/z^2-(\cos z)/z,\ \text{etc.}]$, we obtain, from (2.3),

$$F(s,t)=\frac{1}{2i}\frac{\alpha(s)}{\sin\pi\alpha(s)}\sum_{l=0}^{\infty}(2l+1)i^lP_l(\cos\theta)\bigg[e^{i\pi[\alpha(s)-2a\nu+b]}\int_{-\infty}^{\infty}dx\,\frac{e^{-x(2a\nu-b+1)}}{(1+e^{-x})^{\alpha(s)+1}}j_l(z)-e^{-i\pi[\alpha(s)-2a\nu+b]}\int_{-\infty}^{\infty}dx\,\frac{e^{-x(2a\nu-b+1)}}{(1+e^{-x})^{\alpha(s)+1}}j_l(w)\bigg], \tag{2.4}$$

where $z=-i2a\nu(i\pi+x)$ and $w=-i2a\nu(-i\pi+x)$. In a similar fashion, we can proceed to find a corresponding expansion for $F(s,u)$; the result is that one has simply to substitute $-z$ and $-w$ for z and w in Eq. (2.4).

We define the partial waves $V_l^{(\pm)}(s)$ by

$$F(s,t)\pm F(s,u)=\sum_{l=0}^{\infty}(2l+1)V_l^{(\pm)}(s)P_l(\cos\theta), \tag{2.5}$$

and, using $j_l(-z)=e^{i\pi l}j_l(+z)$, we see easily that

$$V_l^{(\pm)}(s)=(1\pm e^{i\pi l})\frac{\alpha(s)}{\sin\pi\alpha(s)}\tfrac{1}{2}i^{l-1}h_l(\nu), \tag{2.6}$$

where

$$h_l(\nu)=e^{i\pi[\alpha(s)-2a\nu+b]}\int_{-\infty}^{\infty}dx\,\frac{e^{-x(2a\nu-b+1)}}{(1+e^{-x})^{\alpha(s)+1}}j_l(z)-e^{-i\pi[\alpha(s)-2a\nu+b]}\int_{-\infty}^{\infty}dx\,\frac{e^{-x(2a\nu-b+1)}}{(1+e^{-x})^{\alpha(s)+1}}j_l(w). \tag{2.7}$$

Similarly, we define the partial waves $V_l(s)$ by

$$F(t,u)=\sum_{l=0}^{\infty}(2l+1)V_l(s)P_l(\cos\theta), \tag{2.8}$$

and obtain

$$V_l(s)=(1+e^{i\pi l})\tfrac{1}{2}i^l(4a\nu-2b+1)g_l(\nu), \tag{2.9}$$

where

$$g_l(\nu)=\int_{-\infty}^{\infty}dx\,\frac{e^{-x(2a\nu-b+1)}}{(1+e^{-x})^{4a\nu-2b+2}}\,j_l(i2a\nu x)\,. \quad (2.10)$$

Equations (2.6) and (2.9) show the signature factors explicitly.

III. ANALYTIC CONTINUATION

Up to now, we have restricted ourselves to the physical region. Even in this region, we observe that the integrands of $h_l(\nu)$ and $g_l(\nu)$, as functions of complex x, have essential singularities only along the imaginary axis and are analytic elsewhere. Therefore, as long as we can neglect the contribution from the far-distant arc, we can shift the path of integration to any ray in the first and third or the fourth and second quadrants in the complex x plane.

When ν is real and positive (i.e., it assumes a physical value), the integrands of $h_l(\nu)$ are bounded by

$$M(l,\nu)\,|\pm i\pi+x|^l\exp(-(1-b)\,\mathrm{Re}x)\quad\text{as}\quad \mathrm{Re}x\to+\infty\,,$$

and by

$$N(l,\nu)\,|\pm i\pi+x|^l\exp(+(4am_\pi^2+2b)\,\mathrm{Re}x)\quad\text{as}\quad \mathrm{Re}x\to-\infty\,,$$

where $M(l,\nu)$ and $N(l,\nu)$ are functions of l and ν only. Here we have used the inequality

$$\Gamma(\mu+1)\,|J_\mu(z)|\le|\tfrac{1}{2}z|^\mu e^{|\mathrm{Im}z|}\quad\text{for }\mu\ge-\tfrac{1}{2}\,.$$

This shows that the contribution from the far-distant arc indeed vanishes. We see also that the integrand of $g_l(\nu)$ is bounded by $|x|^l\exp(-(1-b)|\mathrm{Re}x|)$ as $|\mathrm{Re}x|\to\infty$, which shows again that the contribution from the far-distant arc vanishes.

We therefore shift the path of integration so that we have, for ν real and positive,

$$h_l(\nu)=e^{i\pi[\alpha(s)-2a\nu+b]}\int_{-\infty}^{\infty}dx\,e^{i\delta}\,\frac{e^{-xe^{i\delta}(2a\nu-b+1)}}{(1+e^{-xe^{i\delta}})^{\alpha(s)+1}}\,j_l(z)$$
$$-e^{-i\pi[\alpha(s)-2a\nu+b]}\int_{-\infty}^{\infty}dx\,e^{i\delta}\,\frac{e^{-xe^{i\delta}(2a\nu-b+1)}}{(1+e^{-xe^{i\delta}})^{\alpha(s)+1}}\,j_l(w)\,, \quad (3.1)$$

where $z=-i2a\nu(i\pi+xe^{i\delta})$ and $w=-i2a\nu(-i\pi+xe^{i\delta})$; similarly,

$$g_l(\nu)=\int_{-\infty}^{\infty}dx\,e^{i\delta}\,\frac{e^{-xe^{i\delta}(2a\nu-b+1)}}{(1+e^{-xe^{i\delta}})^{4a\nu-2b+2}}\,j_l(i2a\nu xe^{i\delta})\,. \quad (3.2)$$

In both cases we have $|\delta|<\frac{1}{2}\pi$.

Now we remove the restriction that ν is real and positive, and extend ν into its complex plane. We see that as long as the phase θ of $\nu=|\nu|e^{i\theta}$ is such that the integrands of $h_l(\nu)$ and $g_l(\nu)$ are bounded at $|x|=\infty$, so as to give the integrals validity, $h_l(\nu)$ and $g_l(\nu)$ are analytic functions in this domain. Since the integrands of $h_l(\nu)$ of (3.1) are bounded by

$$M(\nu,l)\,|\pm i\pi+xe^{i\delta}|^l\exp\{-x[2a|\nu|\cos(\theta+\delta)+(1-b)\cos\delta]+2a|\nu|x|\cos(\theta+\delta)|\}$$

as $x\to+\infty$, and by

$$N(\nu,l)\,|\pm i\pi+xe^{i\delta}|^l\exp\{+x[2a|\nu|\cos(\theta+\delta)+(4am_\pi^2+2b)\cos\delta]-2a|\nu|x|\cos(\theta+\delta)|\}$$

as $x\to-\infty$, we find that the integrals defining $h_l(\nu)$ exist if and only if

$$2a|\nu|\cos(\theta+\delta)+(1-b)\cos\delta-2a|\nu|\,|\cos(\theta+\delta)|>0$$

and

$$2a|\nu|\cos(\theta+\delta)+(4am_\pi^2+2b)\cos\delta-2a|\nu|\,|\cos(\theta+\delta)|>0\,.$$

From these conditions, we deduce that so long as

$$|\theta+\delta|\le\tfrac{1}{2}\pi\quad\text{and}\quad|\delta|<\tfrac{1}{2}\pi\,, \quad (3.3)$$

$h_l(\nu)$ [Eq. (3.1)] is analytic in the complex ν plane. We note also that even if $\theta=\pi$, $h_l(\nu)$ is well defined provided $|\nu|$ is less than the smaller of $(1-b)/4a$ and $(4am_\pi^2+2b)/4a$. Similarly, the integral defining $g_l(\nu)$ [Eq. (3.2)] exists for complex ν if (3.3) holds, because the integrand of $g_l(\nu)$ is bounded by

$$|x|^l\exp\{-2a|\nu x|\cos(\theta+\delta)-|x|(1-b)\cos\delta+2a|\nu x\cos(\theta+\delta)|\}$$

as $|x|\to\infty$. We see also from this bound that the integral still exists for $\theta=\pi$ provided $|\nu|<(1-b)/4a$.

We conclude therefore that $h_l(\nu)$, given by (3.1), and $g_l(\nu)$, given by (3.2), are analytic functions in the entire complex ν plane except along the negative real axis starting at $-\nu_L$ [where ν_L is the lesser of $(1-b)/4a$ and $(4am_\pi^2+2b)/4a$ for $h_l(\nu)$, and $(1-b)/4a$ for $g_l(\nu)$]. They are free of singularities, except possibly at infinity.

From Eq. (3.1), we observe that

$$[h_l(\nu)]^* = e^{-i\pi[\alpha(s^*)-2a\nu^*+b)}\int_{-\infty}^{\infty} dx\, e^{-i\delta}\,\frac{e^{-xe^{-i\delta}(2a\nu^*-b+1)}}{(1+e^{-xe^{-i\delta}})^{\alpha(s^*)+1}}\,j_l(z^*)$$

$$-e^{+i\pi[\alpha(s^*)-2a\nu^*+b]}\int_{-\infty}^{\infty} dx\, e^{-i\delta}\,\frac{e^{-xe^{-i\delta}(2a\nu^*-b+1)}}{(1+e^{-xe^{-i\delta}})^{\alpha(s^*)+1}}\,j_l(w^*),$$

where $z^*=+i2a\nu^*(-i\pi+xe^{-i\delta})$ and $w^*=+i2a\nu^*(+i\pi+xe^{-i\delta})$. We know that the factor $e^{i\delta}$ in $h_l(\nu)$ must appear as $e^{-i\delta}$ in $h_l(\nu^*)$ in virtue of the constraints (3.3). By making use of $j_l(-z)=e^{i\pi l}j_l(+z)$, we find

$$[i^{l-1}h_l(\nu)]^* = i^{l-1}h_l(\nu^*),$$

which in turn gives

$$[V_l^{(\pm)}(s)]^* = V_l^{(\pm)}(s^*). \tag{3.4}$$

Similarly, we see from Eq. (3.2) that

$$[i^l g_l(\nu)]^* = i^l g_l(\nu^*),$$

which yields

$$[V_l(s)]^* = V_l(s^*). \tag{3.5}$$

IV. THRESHOLD BEHAVIOR

Even though the integrals defining $h_l(\nu)$ and $g_l(\nu)$ in Eqs. (3.1) and (3.2) look very complicated, they are very useful and powerful representations of the partial waves. In this section, we show how simply they yield the threshold coefficients.

We substitute the power-series expansion of $j_l(z)$,

$$j_l(z)=\tfrac{1}{2}\sqrt{\pi}\sum_{m=0}^{\infty}(-)^m\frac{(\frac{1}{2}z)^{l+2m}}{m!\,\Gamma(m+l+\frac{3}{2})},$$

into (3.1) and (3.2), obtaining the normal threshold behavior of the partial waves:

$$h_l(\nu)=\nu^l H_l(\nu),\quad g_l(\nu)=\nu^l G_l(\nu), \tag{4.1}$$

where

$$H_l(\nu)=\tfrac{1}{2}\sqrt{\pi}(-ia)^l\sum_{m=0}^{\infty}\frac{(a\nu)^{2m}}{m!\,\Gamma(m+l+\frac{3}{2})}\Bigg[e^{i\pi[\alpha(s)-2a\nu+b]}\int_{-\infty}^{\infty}dx\,e^{i\delta}\,\frac{e^{-xe^{i\delta}(2a\nu-b+1)}}{(1+e^{-xe^{i\delta}})^{\alpha(s)+1}}(i\pi+xe^{i\delta})^{l+2m}$$

$$-e^{-i\pi[\alpha(s)-2a\nu+b]}\int_{-\infty}^{\infty}dx\,e^{i\delta}\,\frac{e^{-xe^{i\delta}(2a\nu-b+1)}}{(1+e^{-xe^{i\delta}})^{\alpha(s)+1}}(-i\pi+xe^{i\delta})^{l+2m}\Bigg],$$

i.e.,

$$H_l(\nu)=2i\tfrac{1}{2}\sqrt{\pi}(-ia)^l\sum_{m=0}^{\infty}\frac{(a\nu)^{2m}}{m!\,\Gamma(m+l+\frac{3}{2})}\left[\left(\frac{d}{dy}\right)^{l+2m}\sin(\pi[\alpha(s)-2a\nu+b+y])\int_{-\infty}^{\infty}dx\,e^{i\delta}\,\frac{e^{-xe^{i\delta}(2a\nu-b+1-y)}}{(1+e^{-xe^{i\delta}})^{\alpha(s)+1}}\right]_{y=0},$$

$$G_l(\nu)=\tfrac{1}{2}\sqrt{\pi}(-ia)^l\sum_{m=0}^{\infty}\frac{(a\nu)^{2m}}{m!\,\Gamma(m+l+\frac{3}{2})}\left[\left(\frac{d}{dy}\right)^{l+2m}\int_{-\infty}^{\infty}dx\,e^{i\delta}\,\frac{e^{-xe^{i\delta}(2a\nu-b+1+y)}}{(1+e^{-xe^{i\delta}})^{4a\nu-2b+2}}\right]_{y=0}. \tag{4.2}$$

The integrals in (4.2) yield[14] simply beta functions, thus giving

$$H_l(\nu)=2\pi i\,\frac{\sqrt{\pi}(-ia)^l}{2\Gamma(\alpha(s)+1)}\sum_{m=0}^{\infty}\frac{(a\nu)^{2m}}{m!\,\Gamma(m+l+\frac{3}{2})}\left[\left(\frac{d}{dy}\right)^{l+2m}\frac{\Gamma(2a\nu-b+1-y)}{\Gamma(1-4am_\pi^2-2b-2a\nu-y)}\right]_{y=0},$$

$$G_l(\nu)=\tfrac{1}{2}\sqrt{\pi}(-ia)^l\sum_{m=0}^{\infty}\frac{(a\nu)^{2m}}{m!\,\Gamma(m+l+\frac{3}{2})}\left[\left(\frac{d}{dy}\right)^{l+2m}B(2a\nu-b+1+y,\,2a\nu-b+1-y)\right]_{y=0}. \tag{4.3}$$

If we expand $H_l(\nu)$ and $G_l(\nu)$ around $\nu=0$, i.e.,

$$H_l(\nu)=C_l+D_l\nu+O(\nu^2),\quad G_l(\nu)=E_l+F_l\nu+O(\nu^2), \tag{4.4}$$

we find that

$$\begin{aligned}
C_l&=\frac{\sqrt{\pi}(-ia)^l}{2\Gamma(l+\frac{3}{2})}\,\frac{2\pi i}{\Gamma(1+\lambda+b)}\left(\frac{d}{dy}\right)^l\frac{\Gamma(1-b-y)}{\Gamma(1-\lambda-2b-y)},\\
D_l&=\frac{\sqrt{\pi}(-ia)^l}{2\Gamma(l+\frac{3}{2})}\,\frac{4\pi a i}{\Gamma(1+\lambda+b)}\left(\frac{d}{dy}\right)^l\frac{\Gamma(1-b-y)}{\Gamma(1-\lambda-2b-y)}\left[\psi(1-b-y)+\psi(1-\lambda-2b-y)-2\psi(1+\lambda+b)\right],\\
E_l&=\frac{\sqrt{\pi}(-ia)^l}{2\Gamma(l+\frac{3}{2})}\left(\frac{d}{dy}\right)^l B(1-b+y,1-b-y),\\
F_l&=\frac{\sqrt{\pi}(-ia)^l}{2\Gamma(l+\frac{3}{2})}\,2a\left(\frac{d}{dy}\right)^l B(1-b+y,1-b-y)\left[\psi(1-b+y)+\psi(1-b-y)-2\psi(2-2b)\right],
\end{aligned} \tag{4.5}$$

where $y=0$, $\lambda=4am_\pi^2$, and $\psi(z)=d\ln\Gamma(z)/dz$.

V. SCATTERING LENGTHS AND EFFECTIVE RANGES

As a simple application of our partial-wave formulation, we compute the scattering lengths and effective ranges for the s, p, d, and f waves.

Since we have only the real part of the amplitude in the dual model, we identify the real part of the right-hand side of Eq. (A10) with its left-hand side, where $A^I(s,\theta)$ is given by (2.2). That is to say, we have

$$\frac{\nu^l[(a_l^I)^{-1}+\frac{1}{2}r_l^I\nu+O(\nu^2)]}{[(a_l^I)^{-1}+\frac{1}{2}r_l^I\nu+O(\nu^2)]^2+\nu^{2l+1}}=\frac{1}{2\sqrt{s}}\int_{-1}^{1}d(\cos\theta)\,A^I(s,\theta)P_l(\cos\theta). \tag{5.1}$$

If we define x_l^I and y_l^I such that

$$\int_{-1}^{1}d(\cos\theta)\,A^I(s,\theta)P_l(\cos\theta)=\nu^l[x_l^I+\nu y_l^I+O(\nu^2)], \tag{5.2}$$

then we can express a_l^I and r_l^I in terms of x_l^I and y_l^I as follows:

$$\begin{aligned}
a_l^I&=\frac{x_l^I}{4m_\pi},\\
r_l^I&=-2a_0^I\delta_{l0}+\frac{1}{m_\pi^2a_l^I}-\frac{y_l^I}{2m_\pi(a_l^I)^2}.
\end{aligned} \tag{5.3}$$

On the other hand, we find from Eqs. (2.2)–(2.9), (4.1), (4.4), and (5.2) that

$$\begin{aligned}
x_l^0&=\tfrac{1}{2}g(1+e^{i\pi l})i^l[(1-2b)E_l+3i\sigma C_l],\\
y_l^0&=\tfrac{1}{2}g(1+e^{i\pi l})i^l[4aE_l+(1-2b)F_l+3i\xi C_l+3i\sigma D_l],\\
x_l^1&=-g(1-e^{i\pi l})i^{l-1}\sigma C_l,\\
y_l^1&=-g(1-e^{i\pi l})i^{l-1}(\xi C_l+\sigma D_l),\\
x_l^2&=-g(1+e^{i\pi l})i^l(1-2b)E_l,\\
y_l^2&=-g(1+e^{i\pi l})i^l[4aE_l+(1-2b)F_l],
\end{aligned} \tag{5.4}$$

where

$$\sigma\sin\pi(\lambda+b)=\lambda+b,$$
$$\xi\sin\pi(\lambda+b)=4a[1-\pi(\lambda+b)\cot\pi(\lambda+b)].$$

Substituting (5.4) and (4.5) into (5.3), we can calculate the scattering lengths and effective ranges. For example, the scattering lengths for the s, p, d, and f waves are

$$a_0^0=\frac{g}{4m_\pi}\left[\frac{\Gamma(1-b)\Gamma(1-b)}{\Gamma(1-2b)}-6\frac{\Gamma(1-b)\Gamma(1-\lambda-b)}{\Gamma(1-\lambda-2b)}\right], \tag{5.5}$$

$$a_0^2=-\frac{g}{2m_\pi}\,\frac{\Gamma(1-b)\Gamma(1-b)}{\Gamma(1-2b)}, \tag{5.6}$$

$$a_1^1=\frac{g}{6m_\pi^3}\,\frac{\lambda\Gamma(1-b)\Gamma(1-\lambda-b)}{\Gamma(1-\lambda-2b)}[\psi(1-b)-\psi(1-\lambda-2b)], \tag{5.7}$$

$$\begin{aligned}
a_2^0=\frac{g}{120m_\pi^5}\lambda^2\Gamma(1-b)\Bigg(&\frac{\Gamma(1-b)}{\Gamma(1-2b)}\psi'(1-b)\\
&-\frac{3\Gamma(1-\lambda-b)}{\Gamma(1-\lambda-2b)}\{[\psi(1-b)-\psi(1-\lambda-2b)]^2+\psi'(1-b)-\psi'(1-\lambda-2b)\}\Bigg),
\end{aligned} \tag{5.8}$$

$$a_2^2 = -\frac{g}{60 m_\pi^5} \frac{\lambda^2 \Gamma(1-b)\Gamma(1-b)}{\Gamma(1-2b)} \psi'(1-b), \tag{5.9}$$

$$a_3^1 = \frac{g}{840 m_\pi^7} \frac{\lambda^3 \Gamma(1-b)\Gamma(1-\lambda-b)}{\Gamma(1-\lambda-2b)}$$
$$\times \{ [\psi(1-b) - \psi(1-\lambda-2b)]^3 + 3[\psi(1-b) - \psi(1-\lambda-2b)][\psi'(1-b) - \psi'(1-\lambda-2b)]$$
$$+ \psi''(1-b) - \psi''(1-\lambda-2b)\}. \tag{5.10}$$

Now, from Eqs. (5.5) and (5.6), we get, for $b \neq \frac{1}{2}$,

$$\frac{a_0^0}{a_0^2} = -\frac{1}{2}\left[1 - 6\frac{\Gamma(1-\lambda-b)\Gamma(1-2b)}{\Gamma(1-b)\Gamma(1-\lambda-2b)}\right], \tag{5.11}$$

from which we infer:

(i) for $\lambda \equiv 4am_\pi^2 = 0$, $b \neq \frac{1}{2}$,

$$\frac{a_0^0}{a_0^2} = +\tfrac{5}{2}; \tag{5.12}$$

(ii) for λ and b such that $0 < \lambda \ll 1$ and $b = \frac{1}{2} - \frac{1}{4}\lambda$ (i.e., b satisfies Lovelace's condition[2] $am_\pi^2 + b = \frac{1}{2}$),

$$\frac{a_0^0}{a_0^2} = -\tfrac{7}{2} - 6\lambda \ln 2 + O(\lambda^2). \tag{5.13}$$

The first term, $-\frac{7}{2}$, is Weinberg's ratio.[15] To obtain (5.13), we have used $\psi(1) - \psi(\frac{1}{2}) = 2\ln 2$.

Similarly, from (5.5)–(5.7), we deduce

$$2a_0^0 - 5a_0^2 = 18 m_\pi^2 a_1^1 \Delta(\lambda, b), \tag{5.14}$$

where

$$\lambda[\psi(1-b) - \psi(1-\lambda-2b)]\Delta(\lambda,b) = \frac{\Gamma(1-b)\Gamma(1-\lambda-2b)}{\Gamma(1-2b)\Gamma(1-\lambda-b)} - 1. \tag{5.15}$$

If $\lambda \ll 1$, we can expand $\Delta(\lambda, b)$ in powers of λ, viz.,

$$\Delta(\lambda, b) = 1 + \lambda\Sigma + O(\lambda^2), \tag{5.16}$$

where

$$2[\psi(1-b) - \psi(1-2b)]\Sigma = [\psi(1-b) - \psi(1-2b)]^2 - \psi'(1-b) - \psi'(1-2b). \tag{5.17}$$

Lovelace's value[2] $b = 0.483$, for example, gives $\Sigma = -1.502$ and

$$2a_0^0 - 5a_0^2 = 18 m_\pi^2 a_1^1 [1 - 1.502\lambda + O(\lambda^2)]. \tag{5.18}$$

To obtain numerical values for the scattering lengths and effective ranges, we take the experimental pion mass (0.139 GeV) and ρ-meson mass (0.765 GeV), and adjust the parameters a and b such that

$$am_\rho^2 + b = 1 \quad \text{and} \quad am_\pi^2 + b = \tfrac{1}{2}.$$

The second condition was taken by Lovelace[2] so that the dual model will satisfy Adler's self-consistency condition.[16] We adjust the over-all coupling constant g so that the p-wave resonance corresponds to the experimental ρ-meson resonance width (0.125 GeV). We then find $g = 1.208$, $a = 0.883$, and $b = 0.483$. The resulting scattering lengths and effective ranges are listed below:

$$a_0^0 = +0.230 m_\pi^{-1}, \quad r_0^0 = -6.43 m_\pi^{-1},$$
$$a_0^2 = -0.062 m_\pi^{-1}, \quad r_0^2 = +44.7 m_\pi^{-1},$$
$$a_1^1 = +0.047 m_\pi^{-3}, \quad r_1^1 = +17.1 m_\pi,$$
$$a_2^0 = +0.12 \times 10^{-2} m_\pi^{-5}, \quad r_2^0 = +8.69 \times 10^2 m_\pi^3,$$
$$a_2^2 = -0.43 \times 10^{-4} m_\pi^{-5}, \quad r_2^2 = +5.99 \times 10^4 m_\pi^3,$$
$$a_3^1 = +0.21 \times 10^{-4} m_\pi^{-7}, \quad r_3^1 = +6.00 \times 10^4 m_\pi^5.$$

VI. CONCLUDING REMARKS

We would like to emphasize that the integral representations we have formulated for the partial waves of the dual $\pi\pi$ scattering amplitudes are very useful and powerful in many respects. As we have seen, one of the advantages of this formalism is that we can extract the threshold coefficients in a straightforward way.

To exploit this advantage, we have calculated the scattering lengths and effective ranges for the s, p, d, and f waves. We can of course extend this procedure to the higher angular momentum waves as well, and may use them in the effective-range-approximation approach to unitarize the dual partial-wave amplitudes in the low-energy region.

The merits of these integral representations will probably be more manifest when we study the asymptotic behavior of the partial waves in the complex energy-squared plane. This subject will be investigated in a subsequent paper.

ACKNOWLEDGMENT

One of us (B.K.C.) would like to express his thanks to Professor K. Kang for helpful discussions.

APPENDIX: KINEMATICS AND NOTATION

We consider the elastic $\pi\pi$ scattering amplitude for the process $a+b\to c+d$. We follow closely the notation used in Ref. 3, defining the S matrix by

$$S=1+i(2\pi)^4\delta^4(\textstyle\sum p)\,T(s,t,u),$$

with

$$M^{dcba}(s,t,u)=T^{dcba}(s,t,u)/16\pi.$$

The quantities a, b, c, and d are Cartesian-basis vectors for the isospins of the particles, and s,t,u are the usual Mandelstam variables $(p_a+p_b)^2$, $(p_a+p_c)^2$, and $(p_a+p_d)^2$, respectively, all p's being physical energy-momentum vectors. The most general amplitude satisfying Bose statistics, isospin conservation, and crossing symmetry[17] is

$$M^{dcba}(s,t,u)=A(s,t,u)\delta_{ab}\delta_{cd}+B(s,t,u)\delta_{ac}\delta_{bd}+C(s,t,u)\delta_{ad}\delta_{bc}, \tag{A1}$$

where

$$A(s,t,u)=A(s,u,t)=B(t,s,u)=C(u,t,s). \tag{A2}$$

The total-isospin s-channel amplitudes are then given by

$$\begin{aligned}A^0(s,t,u)&=3A+B+C,\\ A^1(s,t,u)&=B-C,\\ A^2(s,t,u)&=B+C.\end{aligned} \tag{A3}$$

We observe, in particular, that if $A^2(s,t,u)$ is known throughout the Mandelstam diagram, then the entire amplitude is determined by

$$A(s,t,u)=-\tfrac{1}{2}A^2(s,t,u)+\tfrac{1}{2}A^2(u,s,t)+\tfrac{1}{2}A^2(t,u,s). \tag{A4}$$

We define the volume element of the one-particle contribution to Lorentz-invariant phase space to be $(2\pi)^{-3}d^3p/2E$. The Lorentz-invariant normalization of states is then

$$\langle\vec{p}\,'|\vec{p}\rangle=(2\pi)^3 2E\delta^3(\vec{p}\,'-\vec{p}).$$

In terms of $M(s,t,u)$ and $A(s,\theta)$, we have

$$\begin{aligned}(d\sigma^I/d\Omega)_{\rm el}&=(4/s)|M^I|^2\\ &=(4/s)|A^I(s,\theta)|^2,\end{aligned}$$

and the unitarity condition $SS^\dagger=S^\dagger S=1$ gives

$$\mathrm{Im}A^I(s,\theta)=\frac{q}{4\pi\sqrt{s}}\int d\Omega' A^I(s,\theta'')[A^I(s,\theta')]^* \tag{A5}$$

for s below the inelastic threshold. Here, q is the c.m. momentum, $\cos\theta''=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi'$, and θ is the scattering angle in the c.m. system.

We define the partial waves $A_l^I(s)$ by

$$A^I(s,\theta)=\sum_{l=0}^{\infty}2(2l+1)A_l^I(s)P_l(\cos\theta), \tag{A6}$$

where the extra factor 2 is due to Bose statistics. Elastic unitarity, Eq. (A5), gives

$$\mathrm{Im}A_l^I(s)=\frac{2q}{\sqrt{s}}|A_l^I(s)|^2. \tag{A7}$$

The phase shift $\delta_l^I(s)$ is defined by

$$A_l^I(s)=\frac{\sqrt{s}}{2q}e^{i\delta_l^I(s)}\sin\delta_l^I(s), \tag{A8}$$

so that it satisfies (A7) in the elastic region.

The "scattering lengths" a_l^I and the "effective ranges" r_l^I are defined by

$$q^{2l+1}\cot\delta_l^I(s)=\frac{1}{a_l^I}+\tfrac{1}{2}r_l^I q^2+O(q^4). \tag{A9}$$

It follows from Eqs. (A6), (A8), and (A9) that

$$\frac{1}{2\sqrt{s}}\int_{-1}^{1}d(\cos\theta)\,A^I(s,\theta)P_l(\cos\theta)=\frac{q^{2l}}{(a_l^I)^{-1}+\frac{1}{2}r_l^I q^2+O(q^4)-iq^{2l+1}}. \tag{A10}$$

Near a resonance, we obtain the Breit-Wigner relation

$$e^{i\delta_l^I(s)}\sin\delta_l^I(s)=M_R\Gamma_l^I/(M_R^2-s-iM_R\Gamma_l^I), \tag{A11}$$

where M_R and Γ_l^I are the mass and the width of the resonance, respectively.

*This work was supported in part by the U. S. Atomic Energy Commission.

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Dual $\pi\pi$ Partial-Wave Amplitudes. II. Asymptotic Behavior*

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The integral representations, developed for the partial waves of the dual $\pi\pi$ amplitudes, are used to show that all the partial waves can be expressed as sums involving s waves only, and to prove that the isospin-zero and -two partial waves are bounded by powers in the right half-plane of the energy-squared variable and diverge exponentially along any ray in the left half-plane, thus confirming the behavior conjectured by Tryon. For the isospin-one waves, the derived bounds agree with those of Park and Desai. It is also proved that the discontinuities of the isospin-zero and -two partial waves diverge exponentially along the left-hand cuts and those for the isospin-one case are bounded by a power.

I. INTRODUCTION

In a previous paper,[1] hereafter referred to as I, we presented a formulation of the integral representations of the partial waves of the dual $\pi\pi$ scattering amplitude of Lovelace[2] and Shapiro[3] and studied the threshold behavior in detail. In the present paper, we explore further the representations of I in an investigation of the asymptotic behavior of the partial waves. This subject has already been considered by many authors,[4-12] partly in connection with the K-matrix unitarization scheme and partly in connection with dispersion-theoretic studies.

It has been suggested by Drago and Matsuda[4] and Sivers and Yellin[5,12] that, in this model, partial-wave dispersion relations cannot be used. However, Park and Desai[8] have shown that for the amplitude with isospin one, the partial waves are bounded in the complex energy-squared plane, as is the discontinuity along the cut, so that partial-wave dispersion relations can be obtained for this case.

For the amplitudes with isospin zero and two, Tryon[10] has recently conjectured that the partial waves grow faster than any power along any ray extending into the left half-plane of the energy-squared variable, in which case no dispersion relations can be written down.

We will show in this paper that the integral representations of I furnish a unified treatment of the partial waves for all isospin states and yield, among other results, the explicit exponential divergence of the isospin-zero and -two partial waves in the left half-plane of the energy-squared variable.

In Sec. II, we summarize some of the results obtained in I. We deduce in Sec. III functional relations among the partial waves, which show that all partial waves are essentially finite sums of s waves. In Sec. IV, we study the asymptotic behavior of the partial waves, obtaining the same bounds for the $I=1$ case as found by Park and Desai,[8] and an exponential divergence for the $I=0$ and 2 partial waves in the left half-plane of the energy-squared variable, which confirms Tryon's conjecture[10]; some mathematical details which enter in the proof of this divergence are given in the Appendix. The asymptotic behavior of the discontinuities along the left-hand cut is examined in Sec. V, where it is shown that for $I=1$ the discontinuities of the partial waves are bounded by a power, and for $I=0$ and 2 they are exponentially divergent. Finally, in Sec. VI, we summarize our results, including a comment on the related unitarization problem.

II. PARTIAL-WAVE PROJECTION

The $\pi\pi$ scattering amplitudes in the dual resonance model[2,3] are