Goldstone bosons. This symmetry can be further broken either in the boundary conditions or by the additional vector fields.

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Regge Trajectories in the ϕ^3 Multiperipheral Model with Strong Coupling

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The trajectory function $\alpha(t)$ for small t is studied in the ϕ^3 multiperipheral model as the coupling constant becomes very large. Special emphasis is paid to the analogy of the ϕ^3 model to a one-dimensional gas system. We discuss the upper and lower bounds for $\alpha(t)$ when 2,3,... up to N-body potentials in the gas model are taken into account. The power dependence of α on the coupling constant as well as its dependence on N is determined by our bounds. This power dependence changes as N approaches infinity; it also changes as the mass ratio of the produced final particles to the exchanged particles varies. We show that the equal-spacing approximation suggested by T. D. Lee gives the correct power dependence of α on the coupling constant with a coefficient in error by only a few percent.

I. INTRODUCTION

One of the important problems in hadron physics is to understand the regularities and the general features in the production processes. Feynman[1] and Wilson[2] suggested that the distribution properties of the final particles in the high-energy scattering process should be analogous to the behavior of a real gas contained in a finite volume. This can be understood qualitatively by studying the distribution properties in a multiperipheral model. Mueller[3] demonstrated further that many of the inclusive properties suggested by the gas model can be derived from the assumption of a factorizable leading Regge pole, and thus put the gas model on a rather sound theoretical footing.

Based on the properties of ϕ^3 ladder model, one may also postulate a factorization property of the exclusive amplitude for the process[4] $A+B\to A+B+n\pi$. This exclusive factorization property enables one to analyze the exclusive data in terms of a cluster decomposition and to relate the inclusive and the exclusive spectra in an energy-independent way.

Recently, Lee[5] has demonstrated that the ϕ^3 ladder model is in fact equivalent to a particular one-dimensional gas with only repulsive forces, which can be decomposed into multibody forces in a cyclic way.

In the gas model the trajectory function α corresponds to the pressure, and the coupling constant g^2 is proportional to the fugacity.[4,5] An interesting and important question is: "What is the pressure in the high-density limit?" In other words, how does α depend on g^2 as $g^2\to\infty$? This is the question we will study in this paper from the statistical-mechanical point of view.

This question has been studied before by different approaches.[6-11] In the special case of massless final particles ($m^2=0$ as defined in Sec. II), the solution is known analytically and given by Wick, Cutkosky, and Nakanishi.[6,8] Tiktopoulos and Treiman[7] have given upper and lower bounds in this case which both approach the exact result as $g^2\to\infty$. In the more general case $m^2\neq 0$, the upper bound obtained by Tiktopoulos and Treiman[7] is not optimal since their bound is proportional to g^2 while the exact result should be proportional to $(g^2)^{1/4}$. The correct power dependence on g^2 is obtained by Rosner,[9] by Wyld,[10] and by Cheng and Wu[11] through numerically solving the Bethe-Salpeter equation.

The method presented in this paper for obtaining bounds is very elementary and emphasizes the analogy to the gas model. Special attention is paid to the change of power dependence of α on g^2 as the mass ratio of the final particles to the exchanged particles varies and as all multiparticle potentials are included. Our method also treats the nonforward case when the momentum transfer is small.

In Sec. II we briefly review the ϕ^3 ladder model. In Sec. III we examine the nearest-neighbor approximation by keeping only two-particle potentials. The trajectory functions can be obtained analytically in the strong coupling limit.

In Sec. IV we extend the two-particle potential amplitude to amplitudes including N-particle forces. The basic technique of constructing upper and lower bounds is introduced in this section. In Sec. V this technique is applied to the full ladder amplitudes.

In the strong-coupling limit, the ladder amplitude alone violates the Froissart bound. In Sec. VI we discuss two mechanisms by which our ladder calculations can nonetheless remain relevant in this limit. In the first mechanism, we restore the s-channel unitarity by summing over all multiladder exchanges in the s channel.[12] The trajectory function in the strong-coupling limit then determines the input Regge amplitude in the unitarized Regge model. In the second mechanism, we associate the ladder amplitude with the partition function introduced by Bander.[13] Since the s-channel unitarity is not required on the partition function, the strong-coupling limit of the ladder amplitude may give a satisfactory description here.

In Appendix A we study the bounds on α of the one-dimensional ladder mode. In Appendix B we examine the validity of the equal-spacing approximation.[14]

II. THE MODEL

A. ϕ^3 Ladder Amplitudes

The scattering amplitude which we shall study is the t-channel ladder amplitude in a ϕ^3 model shown in Figs. 1(a) and 1(b). This is the simplest multiperipheral amplitude describing the production process

$$a+b\to a+b+n \text{ additional particles,} \tag{2.1}$$

where $n=0$ to ∞. The final particles, represented

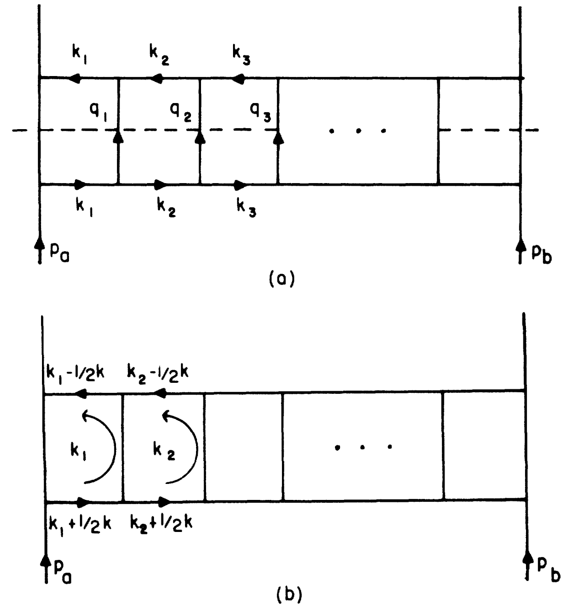

FIG. 1. (a) The forward t-channel ladder amplitude in a ϕ^3 model. This amplitude describes the simplest multiperipheral production process. (b) The nonforward t-channel ladder amplitude in a ϕ^3 model.

by the rungs of the ladder, carry momentum q_i and mass m. The exchanged particles, represented by the sides of the ladder, carry momentum k_i and mass μ. (In general, $\mu\neq m$.)

The ladder amplitudes, properly normalized by the elastic amplitude, have been expressed in a form suitable for our application in Refs. 15 and 16. We shall refer the readers to these two references for notation as well as details. In the following, we write down only the results relevant to our discussion. The properly normalized amplitudes for n additional particles produced in the pionization region ($q_i^0 \ll \sqrt{s}$ = c.m. energy) are given by $(g^2/4\pi\mu^4)^n b^{(n)}$, where g is the coupling constant appearing in $\mathcal{L}_I=\frac{1}{6}g\phi^3$.

For $t=-\vec{k}^2\neq 0$, $b^{(n)}$ is given by [Fig. 1(b)]

$$\begin{aligned}
b^{(n)}_{\vec{k}}(k_1,k_2,\ldots,k_n)={}&\frac{\mu^{4n}}{x_1(x_1-x_2)x_2(x_2-x_3)\cdots(x_{n-1}-x_n)x_n^{\,2}}\left[\frac{(\vec{k}_1+\frac{1}{2}\vec{k})^2+\mu^2}{x_1}\right]^{-1}\left[\frac{(\vec{k}_1-\frac{1}{2}\vec{k})^2+\mu^2}{x_1}\right]^{-1}\\
&\times\left[\frac{(\vec{k}_1-\vec{k}_2)^2+m^2}{x_1-x_2}+\frac{(\vec{k}_2+\frac{1}{2}\vec{k})^2+\mu^2}{x_2}\right]^{-1}\left[\frac{(\vec{k}_1-\vec{k}_2)^2+m^2}{x_1-x_2}+\frac{(\vec{k}_2-\frac{1}{2}\vec{k})^2+\mu^2}{x_2}\right]^{-1}\\
&\times\cdots\left[\frac{(\vec{k}_1-\vec{k}_2)^2+m^2}{x_1-x_2}+\frac{(\vec{k}_2-\vec{k}_3)^2+m^2}{x_2-x_3}+\cdots+\frac{(\vec{k}_{n-1}-\vec{k}_n)^2+m^2}{x_{n-1}-x_n}+\frac{(\vec{k}_n+\frac{1}{2}\vec{k})^2+\mu^2}{x_n}\right]^{-1}\\
&\times\left[\frac{(\vec{k}_1-\vec{k}_2)^2+m^2}{x_1-x_2}+\frac{(\vec{k}_2-\vec{k}_3)^2+m^2}{x_2-x_3}+\cdots+\frac{(\vec{k}_{n-1}-\vec{k}_n)^2+m^2}{x_{n-1}-x_n}+\frac{(\vec{k}_n-\frac{1}{2}\vec{k})^2+\mu^2}{x_n}\right]^{-1},
\end{aligned} \tag{2.2}$$

where $\vec{k}=(k^1,k^2)$ is the momentum transfer, $k_i=(k_i^+,\vec{k}_i)$ is the momentum carried by the ith exchanged particle, and $x_i=k_i^+/p_a^+$ is the fractional longitudinal momentum. In this expression, we assume that the incident particles are moving along the z direction and hence $k_i^+\equiv k_i^0+k_i^3$ is the longitudinal momentum. The variables x_i are ordered according to[15,16]

$$1\geq x_1\geq x_2\geq\cdots\geq x_n>\frac{\mu^2}{s}. \tag{2.3}$$

B. Factorization Properties and the Introduction of Potentials

As is demonstrated in Refs. 15 and 16, the normalized ladder amplitude $b^{(n)}$ satisfies an important factorization property. Whenever $(x_1,\ldots,x_m)\gg(x_{m+1},\ldots,x_n)$, we have

$$b_{\vec{k}}^{(n)}(k_1,\ldots,k_n)=b_{\vec{k}}^{(m)}(k_1,\ldots,k_m)\times b_{\vec{k}}^{(n-m)}(k_{m+1},\ldots,k_n). \tag{2.4}$$

Thus, when sets of the scaled momentum variables x_i differ greatly in magnitude and hence are widely separated in x space, the ladder amplitude breaks into independent amplitudes involving only those x_i which remain "close" to each other. This is a remarkable analogy to the "cluster-decomposition property" of the W function in statistical mechanics.[17] Following the analogy, one can introduce a set of potentials[5] $U^{(n)}$ through

$$b^{(n)}(1,2,\ldots,n)=\exp[-U^{(n)}(1,2,\ldots,n)], \tag{2.5}$$

where $(1,2,\ldots,n)$ stands for the momenta $(k_1,\ldots,k_n)$. The factorization relation (2.4) implies that whenever $(x_1,\ldots,x_m)\gg(x_{m+1},\ldots,x_n)$, we have

$$U^{(n)}(1,2,\ldots,n)=U^{(m)}(1,2,\ldots,m)+U^{(n-m)}(m+1,\ldots,n). \tag{2.6}$$

Thus, the potential becomes the direct sum for widely separated systems.

For later application, it is convenient to introduce an effective potential $E^{(n)}(1,2,\ldots,n)$ through

$$E^{(n)}(1,2,\ldots,n)\equiv U^{(n)}(1,2,\ldots,n)-U^{(n-1)}(1,2,\ldots,n-1). \tag{2.7}$$

$E^{(n)}$ has the physical interpretation as the additional energy required to add the nth particle in the presence of the first $(n-1)$ particles. From (2.7) we obtain

$$\exp(-E^{(n)})=b^{(n)}/b^{(n-1)}\equiv f^{(n)}(1,2,\ldots,n), \tag{2.8}$$

with

$$f^{(n)}(1,2,\ldots,n)=\frac{x_{n-1}\mu^4}{(x_{n-1}-x_n)x_n^2}\left[\frac{(\vec{k}_1-\vec{k}_2)^2+m^2}{x_1-x_2}+\cdots+\frac{(\vec{k}_{n-1}-\vec{k}_n)^2+m^2}{x_{n-1}-x_n}+\frac{(\vec{k}_n+\frac{1}{2}\vec{k})^2+\mu^2}{x_n}\right]^{-1}\times\left[\frac{(\vec{k}_1-\vec{k}_2)^2+m^2}{x_1-x_2}+\cdots+\frac{(\vec{k}_{n-1}-\vec{k}_n)^2+m^2}{x_{n-1}-x_n}+\frac{(\vec{k}_n-\frac{1}{2}\vec{k})^2+\mu^2}{x_n}\right]^{-1}. \tag{2.9}$$

Since $f^{(n)}$ is positive and, for $\mu=m$, less than 1, $E^{(n)}$ is always real and, for $\mu=m$, positive.

It is important to see that $f^{(n)}$ is invariant under the scale transformation $x_i\to cx_i$. This indicates that $f^{(n)}$ depends only on the $n-1$ independent ratios x_i/x_{i+1}, $1\leq i\leq n-1$. In terms of the rapidity variables

$$y_i=-\ln x_i, \tag{2.10}$$

we find that scale invariance implies translational invariance in the rapidity space.[18]

In the special case when $E^{(n)}$ is "two-body separable"—that is, as a combination of two-body nearest-neighbor potentials,

$$E^{(n)}(1,2,\ldots,n)=u(n-1,n)+v(n-2,n-1)+\cdots, \tag{2.11}$$

we have

$$U^{(n)}(1,2,\ldots,n)\equiv\sum_{l=1}^{n}E^{(l)}(1,2,\ldots,l)=\sum_{l=1}^{n}[u(l-1,l)+v(l-1,l)+\cdots]-(v(n-1,n)+\cdots). \tag{2.12}$$

The correction term, $(v(n-1,n)+\cdots)$, is a boundary contribution and will drop out if one imposes periodic boundary conditions. In the language of the gas model, this boundary term corresponds to the surface energy and certainly does not lead to any observable effect in the central region. Hence, it can be ignored. Thus, we can express $U^{(n)}$ as the sum of an effective two-body potential $\bar{E}(l-1,l)$ through

$$U^{(n)}(1,2,\ldots,n)=\sum_l \overline{E}(l-1,l), \tag{2.13}$$

with

$$\overline{E}(l-1,l)=u(l-1,l)+v(l-1,l)+\cdots . \tag{2.14}$$

Since $U^{(n)}$ determines the s dependence and all the distribution properties in the central (pionization) region, it is obvious that we can replace $E^{(n)}(1,2,\ldots,n)$ by the effective two-body potential $\overline{E}(n-1,n)$. This replacement is crucial in our technique because with a two-body potential the problem becomes explicitly solvable.

In the following sections, we shall bound the actual $E^{(n)}(1,2,\ldots,n)$ of the ladder model by certain two-body separable potentials. Since the latter potentials are solvable, we obtain bounds on various physical quantities, including the trajectory functions and the number distributions.

III. THE NEAREST-NEIGHBOR APPROXIMATION

To illustrate the method, we start with a solvable two-body potential. We construct a nearest-neighbor potential $E_2^{(n)}(n-1,n)$ by keeping only the x_{n-1} and x_n dependences in $E^{(n)}(1,2,\ldots,n)$. In other words, $E_2^{(n)}(n-1,n)$ is the two-body potential between x_{n-1} and x_n obtained from $E^{(n)}(1,2,\ldots,n)$ when the remaining $n-2$ particles $(1,2,\ldots,n-2)$ are far away from (x_{n-1},x_n), i.e., when $(x_1,\ldots,x_{n-2})>>(x_{n-1},x_n)$. Using the factorization property (2.6) we find

$$E_2^{(n)}(x_{n-1},x_n)=E^{(2)}(x_{n-1},x_n)\equiv E_2(x_{n-1},x_n). \tag{3.1}$$

A. Forward Amplitude, $t=0$

For simplicity, we concentrate on forward direction, $t=-\vec{k}^2=0$. Then, the two-body nearest-neighbor potential is

$$\begin{aligned}\exp[-E_2(n-1,n)]&\equiv f_2^{(n)}(n-1,n)\\ &=\frac{x_{n-1}\mu^4}{(x_{n-1}-x_n)x_n^2}\left[\frac{(\vec{k}_{n-1}-\vec{k}_n)^2+m^2}{x_n}+\frac{\vec{k}_n^2+\mu^2}{x_n}\right]^{-2}\\ &=\frac{\mu^4}{1-e^{-z_n}}\left\{\vec{k}_n^2+\mu^2+[(\vec{k}_{n-1}-\vec{k}_n)^2+m^2]\frac{e^{-z_n}}{1-e^{-z_n}}\right\}^{-2},\end{aligned} \tag{3.2}$$

where $z_n=\ln x_{n-1}-\ln x_n$ is the rapidity difference of the two nearest-neighbor particles.

For arbitrary values of μ^2 and m^2, the trajectory function α that follows from (3.2) cannot be expressed in a simple form. However, we can study α in various limits. In particular, we shall study the limit of α at large g^2.

Since our potential does not have a hard core, we can show that a large coupling constant always leads to a large density in the rapidity space. Hence, the average rapidity difference $\langle z_n\rangle$ tends to zero as $g\to\infty$. Thus, in the strong coupling limit, we need consider only the small-z_n behavior of (3.2). We then have

$$\begin{aligned}e^{-E_2}&\equiv f_2(z,\vec{k}'s)\\ &=\frac{\mu^4}{z_n}\left\{\vec{k}_n^2+\mu^2+\frac{1}{z_n}[(\vec{k}_{n-1}-\vec{k}_n)^2+m^2]\right\}^{-2}.\end{aligned} \tag{3.3}$$

For $m\neq 0$, the average $\vec{k}_n^2+\mu^2$ is bounded and $[(\vec{k}_{n-1}-\vec{k}_n)^2+m^2]/z_n$ tends to ∞ as $z_n\to 0$. Thus, we can ignore $k_n^2+\mu^2$ in comparison with $[(\vec{k}_{n-1}-\vec{k}_n)+m^2]/z_n$ in the small-z_n region. For $m=0$, μ^2 is the only remaining mass scale and certainly cannot be ignored. However, we can always ignore $\vec{k}_n^2$ in comparison with $(\vec{k}_{n-1}-\vec{k}_n)^2/z_n$ as $z_n\to 0$. This can be seen more transparently after a scale transformation

$$\begin{aligned}\vec{k}_n&=z_n\vec{k}'_n,\\ \vec{k}_{n-1}&=z_{n-1}\vec{k}'_{n-1},\end{aligned} \tag{3.4}$$

which leads to

$$\frac{1}{z_n}(\vec{k}_{n-1}-\vec{k}_n)^2=(\vec{k}'_{n-1}-\vec{k}'_n)^2$$

and (3.5)

$$\vec{k}_n^2=z_n\vec{k}_n'^2\to 0.$$

To anticipate all relevant possibilities, we write

$$f_2(z,\vec{q})=\frac{\mu^4}{z}\left(\mu^2+\frac{\vec{q}^2+m^2}{z}\right)^{-2}, \tag{3.6}$$

with

$$\vec{q}=\vec{k}_{n-1}-\vec{k}_n. \tag{3.7}$$

Note that in the high-density limit f_2 depends only on the rapidity difference z and the transverse momentum difference $\vec{q}$.

Given $f_2(z,\vec{q})$, we can obtain the trajectory function α as a function of the coupling constant $\lambda=g^2/4\pi\mu^4$ and, consequently, obtain all the multiplicity distributions. The explicit form of α can be determined either by the Mellin transform technique[15] or by the statistical-mechanical method,[5] giving

$$\frac{1}{\lambda}=\int_0^\infty dz\int\frac{d^2q}{(2\pi)^2}f_2(z,\vec{q})e^{-z\alpha(0)},\qquad(3.8)$$

where

$$\lambda=\frac{g^2}{4\pi\mu^4}$$

and $\alpha(0)$ stands for $\alpha(t)$ at $t=0$. Combining (3.6) and (3.8) and carrying out the q integration, we obtain

$$\frac{1}{\lambda}=\frac{\mu^2}{4\pi\alpha(0)}\frac{1}{1+m^2\alpha(0)/\mu^2}C(m^2\alpha(0)/\mu^2),\qquad(3.9)$$

where

$$C(x)\equiv(1+x)[1-xe^xE_1(x)]\qquad(3.10)$$

is a slowly varying function satisfying the properties

$$\tfrac{1}{2}<C(x)<1,\quad 0<x<\infty,\qquad(3.11)$$

and

$$C(0)=C(\infty)=1.$$

Thus, as $\alpha(0)\to\infty$, we obtain

$$\frac{1}{\lambda}=\frac{\mu^2}{4\pi\alpha(0)}\frac{1}{1+m^2\alpha(0)/\mu^2}.\qquad(3.12)$$

This equation demonstrates a change of power dependence in $\alpha(0)$ as $m/\mu\to0$: For $m\neq0$, we obtain from (3.12) that, as $\alpha(0)\to\infty$,

$$\alpha(0)=\frac{g}{4\pi m}.\qquad(3.13)$$

For $m=0$, on the other hand, we have

$$\alpha(0)=\frac{g^2}{16\pi^2\mu^2}.\qquad(3.14)$$

This change of power dependences as $m/\mu\to0$ is independent of the nearest-neighbor approximation and also appears when all the multiparticle forces are included. This point will be discussed in Sec. V when we treat the "full" ladder amplitude.

B. Nonforward Amplitude, $t\neq0$

The nonforward nearest-neighbor amplitude is

$$f_2(n-1,n,t)=\frac{x_{n-1}\mu^4}{(x_{n-1}-x_n)x_n}\left[\frac{(\vec{k}_{n-1}-\vec{k}_n)^2+m^2}{x_{n-1}-x_n}+\frac{(\vec{k}_n+\frac{1}{2}\vec{k})^2+\mu^2}{x_n}\right]^{-1}\times\left[\frac{(\vec{k}_{n-1}-\vec{k}_n)^2+m^2}{x_{n-1}-x_n}+\frac{(\vec{k}_n-\frac{1}{2}\vec{k})^2+\mu^2}{x_n}\right]^{-1},\qquad(3.15)$$

where $t=-k^2$ is the momentum transfer. The main purpose of this section is to study the qualitative feature of the t dependence in $\alpha(t)$ near $t=0$. In particular, we wish to make sure that the asymptotic s dependence of the ladder amplitude changes smoothly from $t=0$ to $t\neq0$. Thus, it is sufficient to study $\alpha(t)$ for a finite region of t, say, $0<-t\leq O(\mu^2)$. For $-t=k^2<4\mu^2$, we have particularly simple inequalities, valid for any $A>0$,

$$[A+\vec{k}_n^2+\mu^2+\vec{k}^2/4]^{-2}\leq[A+(\vec{k}_n+\tfrac{1}{2}\vec{k})^2+\mu^2]^{-1}\times[A+(\vec{k}_n-\tfrac{1}{2}\vec{k})^2+\mu^2]^{-1}\leq[A+\mu^2+\vec{k}^2/4]^{-2},\qquad(3.16$$

which imply an upper and a lower bound for f_2. In the high-density limit,

$$z_n=\frac{x_{n-1}-x_n}{x_n}<<1,\qquad(3.17)$$

we can always ignore k_n^2 in the lower bound in comparison with $A=[(\vec{k}_{n-1}-\vec{k}_n)^2+m^2]/z_n$ in the computation of the trajectory function. Then, the upper and the lower bounds on f_2 become the same. Hence, we conclude, at large g^2, that

$$\alpha(t,\mu^2)=\alpha(0,\mu^2+\vec{k}^2/4)\quad\text{for }-t=\vec{k}^2\leq4\mu^2.\qquad(3.18)$$

Since we know the bounds on $\alpha(0)$, we establish easily bounds on $\alpha(t)$ in this region of t.

Equation (3.18) leads to some rather surprising results. For $m\neq0$, we learn from (3.13) that $\alpha(0)$ is independent of μ^2 at large g^2. Hence, (3.18) implies that, for $m\neq0$ and at large g,

$$\alpha(t)=\frac{g}{4\pi m}\left[1+O\left(\frac{1}{g}\right)\right],\quad -t\leq4\mu^2\qquad(3.19)$$

is also independent of t, i.e., all the t dependence in $\alpha(t)$ appears in the nonleading terms at large g. For $m=0$, on the other hand, there is explicit t dependence in $\alpha(t)$; we find

$$\alpha(t)=\frac{g^2}{16\pi^2(\mu^2+\frac{1}{4}\vec{k}^2)}\,,\quad -t=k^2<4\mu^2. \tag{3.20}$$

As we shall see, Eq. (3.18) and the absence of the t dependence in $\alpha(t)$ for $m\neq 0$ are also valid for the "full" ladder amplitude in the high-density limit.

IV. N-BODY POTENTIAL

We shall generalize our considerations to include up to N-body potentials, N fixed. In general, the problem is no longer solvable analytically even in the large-g limit. However, we can find upper and lower bounds on the amplitudes which reduce to solvable potentials in the strong-coupling limit.

The forward $(t=0)$ N-body amplitude in the high-density limit $(1\gg z_i)$ is given by

$$f_N(1,2,\ldots,N)=\frac{\mu^4}{z_N}\left(\mu^2+\frac{\vec{q}_2^{\,2}+m^2}{z_2}+\cdots+\frac{\vec{q}_N^{\,2}+m^2}{z_N}\right)^{-2}, \tag{4.1}$$

where

$$z_i=\ln(x_{i-1}/x_i),\qquad \vec{q}_i=\vec{k}_{i-1}-\vec{k}_i\,. \tag{4.2}$$

We first consider the case $m\neq 0$ so that the μ^2 term in (4.1) can be ignored.

(a) Upper bounds on f_N and $\alpha_N(0)$, $m\neq 0$. Using the fact that the arithmetical mean is larger than the geometrical mean, we obtain an upper bound $f_N^U\equiv\exp(-E_N^U)$ as

$$f_N\leq f_N^U=\frac{\mu^4}{z_N(N-1)^2}\prod_{i=2}^{N}\left(\frac{z_i}{\vec{q}_i^{\,2}+m^2}\right)^{2/(N-1)} \tag{4.3}$$

and

$$-E_N\leq -E_N^U=\ln\frac{\mu^4}{z_N(N-1)^2}+\frac{2}{N-1}\sum_{i=2}^{N}\ln\frac{z_i}{\vec{q}_i^{\,2}+m^2}\,. \tag{4.4}$$

The potential E_N^U is obviously two-body separable. Hence, we can replace E_N^U, and consequently f_N^U, by an equivalent two-body $\bar{E}_N^U$ and $\bar{f}_N^U$, giving

$$-\bar{E}_N^U(z,\vec{q})=\ln\frac{\mu^4}{z(N-1)^2}+2\ln\frac{z}{\vec{q}^{\,2}+m^2}\,, \tag{4.5}$$

$$\bar{f}_N^U(z,\vec{q})=\frac{\mu^4 z}{(N-1)^2(\vec{q}^{\,2}+m^2)^2}\,. \tag{4.6}$$

This latter potential is solvable. Then, the upper bound on the asymptotic trajectory function α_N^U is given by

$$\frac{1}{\lambda}=\int_0^\infty dz\int\frac{d^2q}{(2\pi)^2}\,\frac{\mu^4 z}{(N-1)^2(\vec{q}^{\,2}+m^2)^2}\,e^{-z\alpha_N^U}, \tag{4.7}$$

and consequently

$$\alpha_N^U(0)=\frac{1}{N-1}\,\frac{g}{4\pi m}\,. \tag{4.8}$$

(b) Lower bounds on f_N and $\alpha_N(0)$, $m\neq 0$. We can obtain a useful lower bound on f_N by using the inequality

$$u^{-2}\geq\tfrac{1}{4}e^2e^{-u},\quad u>0. \tag{4.9}$$

Then we find

$$f_N\geq f_N^L=\frac{e^2\mu^4a^2}{4m^4z_N(N-1)^2}\times\exp\left[-\frac{a/m^2}{N-1}\sum_{i=2}^{N}\frac{\vec{q}_i^{\,2}+m^2}{z_i}\right], \tag{4.10}$$

which is also two-body separable. Hence, we can replace f_N^L by an equivalent two-body amplitude

$$\bar{f}_N^L(z,\vec{q})=\frac{e^2\mu^4a^2}{4m^4z(N-1)^2}\exp\left[-\frac{a}{z}\,\frac{\vec{q}^{\,2}+m^2}{m^2}\right]. \tag{4.11}$$

The parameter a appearing in (4.10) and (4.11) is arbitrary and will be chosen to optimize the trajectory function α_N^L. From (4.11) we obtain

$$\frac{1}{\lambda}=\int_0^\infty dz\int\frac{d^2q}{(2\pi)^2}\,\frac{e^2\mu^4a^2}{4m^4z(N-1)^2}\times\exp\left[-\frac{a}{z}\,\frac{\vec{q}^{\,2}+m^2}{m^2}-z\alpha_N^L(0)\right]=\frac{e^2\mu^4a}{8\pi m^2(N-1)^2}\left(\frac{a}{\alpha_N^L}\right)^{1/2}K_1(2(a\alpha_N^L)^{1/2}), \tag{4.12}$$

and, hence,

$$\alpha_N^L(0)=\tfrac{1}{2}e\,[2b^3K_1(2b)]^{1/2}\frac{1}{N-1}\,\frac{g}{4\pi m}\,, \tag{4.13}$$

with

$$b^2=a\alpha_N^L. \tag{4.14}$$

The best lower bound on α_N^L is determined by the maximum value of $b^3K_1(2b)$, which corresponds to

$$b=1.193,$$

and

$$\alpha_N^L(0)=\frac{0.730}{N-1}\,\frac{g}{4\pi m}\,. \tag{4.15}$$

Putting the upper and lower bounds together, we have

$$\frac{0.730}{N-1}\,\frac{g}{4\pi m}<\alpha_N(0)<\frac{1}{N-1}\,\frac{g}{4\pi m}\,. \qquad (4.16)$$

For $m=0$, we find that we can no longer ignore the μ^2 term in f_N. For comparison, we rewrite f_N as

$$f_N(1,2,\ldots,N)=\frac{\mu^4}{z_N}\left[\left(\frac{\vec{q}_2^{\,2}}{z_2}+\frac{\mu^2}{N-1}\right)+\left(\frac{\vec{q}_3^{\,2}}{z_3}+\frac{\mu^2}{N-1}\right)+\cdots+\left(\frac{\vec{q}_N^{\,2}}{z_N}+\frac{\mu^2}{N-1}\right)\right]^{-2}. \qquad (4.17)$$

Then, we can apply the same technique of finding the bounds on f_N by factorizing the individual terms, $\vec{q}_i^{\,2}/z_i+\mu^2/(N-1)$. The two-particle amplitudes derived from the upper and lower bounds are

$$\bar{f}_N^U(z,\vec{q})=\frac{\mu^4}{(N-1)^2 z}\left(\frac{\vec{q}^{\,2}}{z}+\frac{\mu^4}{N-1}\right)^{-2} \qquad (4.18)$$

and

$$\bar{f}_N^L(z,\vec{q})=\frac{e^2a^2}{4z(N-1)^2}\exp\left[-\frac{a}{\mu^2}\left(\frac{\vec{q}^{\,2}}{z}+\frac{\mu^2}{N-1}\right)\right], \qquad (4.19)$$

respectively. The upper and lower bounds on α_N, obtained from (4.18) and (4.19), are

$$\frac{1}{2(N-1)}\,\frac{g^2}{16\pi^2\mu^2}<\alpha_N(0)<\frac{1}{N-1}\,\frac{g^2}{16\pi^2\mu^2}\,. \qquad (4.20)$$

The change of g^2 power dependence in α_N as $m\to 0$ is evident from (4.16) and (4.20).

The result presented here can be generalized easily to nonforward amplitudes. In particular, we find that Eq. (3.18) is also satisfied in the N-body potential approximation, and so are the special features of α for finite t.

Based on the results we have obtained, we can make an estimate on the g^2 and N dependence in α_N when N is comparable with g^2.

Consider first the case $m\neq 0$. Under the N-body potential approximation, we learn that the trajectory function behaves at large coupling constant as

$$\alpha_N\propto\frac{1}{N-1}\left(\frac{g^2}{4\pi m^2}\right)^{1/2}. \qquad (4.21)$$

For any fixed N, (4.21) implies that α_N increases linearly with g at large coupling strength. However, in the full ladder amplitude, we must understand the limit $N\to\infty$. The question then is: "What is the effective N above which the many-body potential can be ignored?" We can make a quick estimate from (2.9) that the N-body potentials remain important if

$$x_N/x_1=\exp\left(-\sum_1^N z_i\right)=O(1), \qquad (4.22a)$$

and can be ignored if

$$x_N/x_1=\exp\left(-\sum_1^N z_i\right)\ll 1. \qquad (4.22b)$$

In other words, the effective N is given by $1/\langle z\rangle$, where $\langle z\rangle$ stands for the average distance between two adjacent particles in the rapidity space,

$$N\approx 1/\langle z\rangle\,. \qquad (4.23)$$

From (3.8), we find that the important z integration region is determined by

$$z\alpha\approx 1. \qquad (4.24)$$

Thus, we obtain

$$N\approx\alpha, \qquad (4.25)$$

and, when combined with (4.21),

$$\alpha\propto\left(\frac{g^2}{4\pi m^2}\right)^{1/4}. \qquad (4.26)$$

Therefore, we anticipate that the trajectory function of the full ladder amplitude behaves like $(g/m)^{1/2}$ for $m\neq 0$. Similar arguments show that α behaves like g/μ for $m=0$. In the next section, we shall establish rigorously the upper and lower bounds for the trajectory functions which confirm the estimated g dependence given here.

V. BOUNDS ON THE FULL LADDER AMPLITUDE

In computing the trajectory functions from the full ladder amplitude, we assume that the energy is high enough so that the average number of final particles $[\bar{n}\sim(g^2d\alpha/dg^2)\ln s]$ greatly exceeds the effective order N $[\sim O(g)]$ above which the many-body potentials may be ignored. For notational simplicity, we reverse our labeling on z such that

$$z_1 = \ln(x_{n-1}/x_n), \quad \vec{q}_1 = \vec{k}_{n-1} - \vec{k}_n,$$
$$z_2 = \ln(x_{n-2}/x_{n-1}), \quad \vec{q}_2 = \vec{k}_{n-2} - \vec{k}_{n-1}, \tag{5.1}$$
$$\cdots\cdot$$

Equation (5.1) implies that

$$\frac{x_n}{x_{n-1} - x_n} = \frac{e^{-z_1}}{1 - e^{-z_1}},$$
$$\frac{x_n}{x_{n-2} - x_{n-1}} = \frac{e^{-(z_1+z_2)}}{1 - e^{-z_2}}, \tag{5.2}$$
$$\cdots\cdot$$

In this notation the high-density limit of the forward ladder amplitude takes the form

$$f^{(n)}(z_i, \vec{q}_i) = \frac{\mu^4}{z_1}\left[\mu^2 + \frac{e^{-z_1}}{z_1}(\vec{q}_1^{\,2} + m^2) + \frac{e^{-(z_1+z_2)}}{z_2}(\vec{q}_2^{\,2} + m^2) + \cdots + \frac{e^{-(z_1+z_2+\cdots+z_{n-1})}}{z_{n-1}}(\vec{q}_{n-1}^{\,2} + m^2)\right]^{-2}, \tag{5.3}$$

where we have ignored $\vec{k}_1^{\,2}$ in comparison with $\vec{q}_1^{\,2}/z_1$.

For $n \gg \langle z \rangle^{-1}$, the dependence of $f^{(n)}$ on n and z_i, $i \gtrsim n$, will be washed out by the damping factor $\exp(-\sum_{i=1}^{n-1} z_i)$. Hence, extending the summation in the square brackets of (5.3) from n to ∞ will introduce only negligible error. This fact is important in establishing the lower bound.

(a) Upper bounds on $f^{(n)}$ and α. To obtain an upper bound, we cut off the sum in $f^{(n)}$ after $(N-1)$ terms, with $N<n$. This leads to

$$f^{(n)}(z_i, q_i) < \frac{\mu^4}{z_1}\left[e^{-z_1}\left(\frac{\vec{q}_1^{\,2} + m^2}{z_1} + \frac{\mu^2}{N-1}\right) + e^{-(z_1+z_2)}\left(\frac{\vec{q}_2^{\,2} + m^2}{z_2} + \frac{\mu^2}{N-1}\right) + \cdots + e^{-(z_1+\cdots+z_{N-1})}\left(\frac{\vec{q}_{N-1}^{\,2} + m^2}{z_{N-1}} + \frac{\mu^2}{N-1}\right)\right]^{-2}. \tag{5.4}$$

Now we can construct a two-body separable potential as an upper bound. The equivalent two-particle amplitude derived from this upper bound is

$$\bar{f}_N^U(z, \vec{q}) = \frac{\mu^4 e^{Nz}}{(N-1)^2 z}\left(\frac{\vec{q}^{\,2} + m^2}{z} + \frac{\mu^2}{N-1}\right)^{-2}, \tag{5.5}$$

which leads to an upper bound $\alpha^U(0)$ on the trajectory function through

$$\begin{aligned}\frac{1}{\lambda} &= \frac{4\pi\mu^4}{g^2} \\ &= \int_0^\infty dz \int \frac{d^2q}{(2\pi)^2}\, \bar{f}_N^U(z,q)\, e^{-z\alpha^U} \\ &= \frac{\mu^4}{4\pi(N-1)^2}\int_0^\infty dz\, e^{-(\alpha^U - N)z}\left(\frac{m^2}{z} + \frac{\mu^2}{N-1}\right)^{-1}.\end{aligned} \tag{5.6}$$

The optimum choice of N is determined by $N-1 = \frac{1}{2}\alpha^U$, giving

$$\begin{aligned}\frac{1}{\lambda} &\equiv \frac{4\pi\mu^4}{g_2} \\ &= \frac{\mu^2}{\pi(\alpha^U)^2}\left[1 - \frac{(\alpha^U)^2 m^2}{4\mu^2}\exp\frac{(\alpha^U)^2 m^2}{4\mu^2}\, E_1\left(\frac{(\alpha^U)^2 m^2}{4\mu^2}\right)\right].\end{aligned} \tag{5.7}$$

Using the inequality

$$\frac{1}{x+1} < e^x E_1(x), \quad x > 0 \tag{5.8}$$

we can replace (5.6) by a much simpler, but slightly worse, constraint,

$$\alpha^2\left(1 + \frac{\alpha^2 m^2}{4\mu^2}\right) < \frac{4g^2}{16\pi^2\mu^2}. \tag{5.9}$$

(b) Lower bounds on $f^{(n)}$ and α. To obtain a lower bound on the amplitude, we put in a lower cutoff in z_i by hand,

$$f^{(n)}(z_i,\vec{q}_i)\geq\frac{\mu^4}{z_1}\left[\mu^2+\frac{e^{-z_1}}{z_1}(\vec{q}_1{}^2+m^2)+\frac{e^{-(z_1+z_2)}}{z_2}(\vec{q}_2{}^2+m^2)+\cdots\right]\prod_{i=1}^{\infty}\theta(z_i-z_0), \tag{5.10}$$

where we have extended the sum in the square brackets to ∞. The cutoff $z_0>0$ will be determined later to give a maximum lower bound α^L. With the presence of the cutoff, we have

$$f^{(n)}(z_i,\vec{q}_i)\geq\frac{\mu^4}{z_1}\left[\mu^2+\frac{e^{-z_0}}{z_1}(\vec{q}_1{}^2+m^2)+\frac{e^{-2z_0}}{z_2}(\vec{q}_2{}^2+m^2)+\cdots\right]^{-2}\prod_{i=1}^{\infty}\theta(z_i-z_0)$$

$$\geq\frac{e^2a^2}{4z_1}\exp\left\{-\frac{a}{\mu^2}\left[\mu^2+\frac{e^{-z_0}}{z_1}(\vec{q}_1{}^2+m^2)+\frac{e^{-2z_0}}{z_2}(\vec{q}_2{}^2+m^2)+\cdots\right]\right\}\prod_{i=1}^{\infty}\theta(z_i-z_0), \tag{5.11}$$

where a is another parameter to be determined. The last expression in (5.11) is clearly two-body factorizable. Thus, we obtain an effective two-body amplitude as a lower bound:

$$\bar{f}^L(z,\vec{q})=\frac{e^2a^2}{4z}\exp\left[-\frac{a}{\mu^2}\left(\mu^2+\frac{\vec{q}^2+m^2}{z_0z}\right)\right]\theta(z-z_0), \tag{5.12}$$

where a and z_0 are parameters chosen to maximize the lower bound. From (5.12), we find a lower bound α^L on the trajectory function through

$$\frac{1}{\lambda}\equiv\frac{4\pi\mu^4}{g^2}$$
$$=\int dz\frac{d^2q}{(2\pi)^2}\bar{f}^L(z,\vec{q})e^{-z\alpha^L}$$
$$=\frac{e^2\mu^2}{16\pi\alpha^2}abe^{-a}\int_b^{\infty}du\,e^{-u-c/u}, \tag{5.13}$$

where

$$b=z_0\alpha,\qquad c=am^2\alpha^2/b\mu^2. \tag{5.14}$$

To give a reasonable lower bound for all m, we choose

$$a=\left(1+\frac{m^2\alpha^2}{4\mu^2}\right)^{-1},\quad b=2, \tag{5.15}$$

and obtain

$$\frac{4\pi\mu^4}{g^2}\geq\frac{\mu^2}{8\pi e\alpha^2(1+m^2\alpha^2/4\mu^2)}. \tag{5.16}$$

Combining (5.9) and (5.16), we have

$$4\frac{g^2}{16\pi^2\mu^2}>\alpha(0)^2\left[1+\frac{\alpha(0)^2m^2}{4\mu^2}\right]$$
$$>\frac{1}{2e}\frac{g^2}{16\pi^2\mu^2}. \tag{5.17}$$

In particular, we have for $m/\mu\neq0$

$$2\left(\frac{g}{4\pi m}\right)^{1/2}>\alpha(0)>0.9262\left(\frac{g}{4\pi m}\right)^{1/2} \tag{5.18}$$

and, for $m=0$,

$$2\left(\frac{g}{4\pi\mu}\right)>\alpha(0)>0.4289\left(\frac{g}{4\pi\mu}\right). \tag{5.19}$$

Let us compare our upper and lower bounds with the results of Rosner,[9] Wyld,[10] and Cheng and Wu,[11] who obtained their results by solving integral equations. Rosner obtained the upper and lower bounds on α, for $m\neq0$,

$$1.23\leq\lim\frac{\alpha(0)}{(g/4\pi m)^{1/2}}\leq1.47 \tag{5.20}$$

which is slightly better than ours. Wyld, Cheng, and Wu obtained for $m\neq0$ the following numerical answer[19]:

$$\alpha(0)=1.4669\left(\frac{g^2}{16\pi^2m^2}\right)^{1/4}+O(1). \tag{5.21}$$

For $m=0$, the trajectory function is known analytically.[6,8] The trajectory function in the forward direction is given by

$$\alpha(0)=\tfrac{1}{2}\left(1+\frac{g^2}{4\pi\mu^2}\right)^{1/2}-\tfrac{1}{2}$$
$$=\left(\frac{g^2}{16\pi^2\mu^2}\right)^{1/2}+O(1). \tag{5.22}$$

Of course, both (5.21) and (5.22) agree with our bounds on $\alpha(0)$.

The advantage of our method lies in the ability of relating the known properties of the real gases to the calculation of the ladder amplitude. For instance, from the property of the "potential," we can anticipate the behavior of the trajectory function which is the analog of the "pressure." We can also understand intuitively how the g-power dependence in the trajectory function may change as the mass ratio m/μ varies. Furthermore, our method has the virtue of being conceptually very simple.

It is important to point out that our analysis can be applied to nonforward direction as well. Using the method outlined in Sec. III B, we find that for $-t=\vec{k}^2<4\mu^2$

$$\alpha(t,\mu^2)=\alpha(0,\mu^2+\tfrac{1}{4}k^2), \tag{5.23}$$

as given in (3.18). Thus, we conclude from the discussion in Sec. III B that for $m\neq 0$ and large g, $\alpha(t,\mu^2)$ is asymptotically independent of both μ and t. However, for $m=0$ and $-t=\vec{k}^2\leq 4\mu^2$, $\alpha(t,\mu^2)$ picks up an explicit t dependence through

$$\alpha(t,\mu^2)=\left[\frac{g^2}{16\pi^2(\mu^2+\frac{1}{4}\vec{k}^2)}\right]^{1/2} \tag{5.24}$$

as $g\to\infty$.

VI. DISCUSSION

Although the main content of this paper is the study of asymptotic bounds on ladder amplitudes at very large coupling constants, we do not pretend that the ladder amplitude alone is necessarily relevant at very large coupling strength. On the contrary, more complicated exchanges may play important roles in this limit. In the following, we wish to mention two possible mechanisms by which our ladder calculations may remain relevant in the strong coupling limit.

In the first case, we consider the strong-coupling limit of a unitarized Regge model. The trajectory function appearing in our calculation is interpreted as the input trajectory function which controls the s dependence of the eikonal function. After imposing the s-channel unitarity, we obtain the full amplitude as the eikonal iteration of the original ladder amplitude.[12] The final-particle distributions can be computed once the distribution in a single ladder is given.

In the second case, we follow Bander[13] by constructing directly from the observed exclusive cross section σ_n a partition function

$$Q(\lambda,s)=\sum_n \lambda^n\sigma_n(s).$$

For multiperipheral processes, we have

$$Q(\lambda,s)\sim s^{\alpha(\lambda)}$$

for all λ. Now, we can study the behavior of $Q(\lambda,s)$ and $\alpha(\lambda)$ at large λ. Physically, a large λ implies that the coupling describing the emission of final particles is artificially enhanced, but the coupling associated with virtual processes, such as the absorptive corrections, remains unchanged. Because these two couplings are different, the amplitude $Q(\lambda,s)$ does not have to satisfy s-channel unitarity and, consequently, $\alpha(\lambda)$ can be larger than one. In particular, if the physical cross section σ_n can be described by the ladder amplitude at $\lambda=1$, then at large λ, $Q(s,\lambda)$ and $\alpha(\lambda)$ behave as the ladder amplitude at the strong-coupling limit. Hence, it is interesting to compare results obtained from $Q(\lambda,s)$ with those predicted by the ladder amplitude.

Recently, Bander made a numerical calculation of $\alpha(\lambda)$ from the p-p data.[13] According to his analysis, $\alpha(\lambda)$ appears to increase as $\sqrt{\lambda}\propto g$ at large λ. It is interesting to note that a linear g dependence emerges naturally in the ϕ^3 model by keeping only the nearest-neighbor or any fixed N-body potential. Analyses using better and higher-energy data to determine phenomenologically the nature of the potentials in the gas model are certainly desirable.

To give an estimate of the effect of multiparticle potentials, we note from our calculation that N-body potential becomes important only if the density in the rapidity space is of the same order of or larger than N. Let us assume that this is a general feature also applicable to the high-energy scattering. The present experimental data seem to support a logarithmic increase of the multiplicity at high energy, $\bar{n}\propto\ln s$. This implies a constant density, $\bar{n}/\ln s\sim 2$, in the rapidity space at high energy. Hence, one might hope that only two- or three-body potentials are important in the actual high-energy hadron-production processes.

In Appendix B we show that if one replaces the full ladder amplitude by an effective two-body potential amplitude through equal-spacing approximation,[14] the derived trajectory function agrees amazingly well with the exact calculation at high density. This indicates that the effect of all multiparticle potentials in our model calculation can be approximated by an effective two-particle potential, in the spirit of the well-known mean-field approximation. The existence of an effective two-body potential may be a general feature of the gas model. We may anticipate that an effective two-body potential can be used to describe phenomenologically the trajectory function and the final-particle distribution.

Added Note: One of us (S.-J.C.) and J. Rosner[20] recently have derived the exact asymptotic result (5.21), (5.22) as well as the $O(1)$ corrections by studying the Bethe-Salpeter equation in coordinate space. A relation is found between the asymptotic Regge trajectory functions and the size of classical orbits described by the coordinate-space wave function.

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APPENDIX A: ONE-DIMENSIONAL LADDER AMPLITUDE

In this appendix we will study the bounds on the trajectory function as the coupling constant approaches infinity for the ladder amplitude in one space dimension. Since the energy dependence of the ladder amplitude is entirely due to longitudinal degrees of freedom, the one-dimensional model possesses all the essential features of the three-dimensional model without the complications of transverse motion.

In the one-dimensional ladder model, the properly normalized amplitude for n additional particles produced in the pionization region is, according to Ref. 15,

$$b^{(n)}(x_1,x_2,\ldots,x_n)=\frac{\mu^{4n}}{x_1(x_1-x_2)x_2(x_2-x_3)\cdots(x_{n-1}-x_n)x_n^2}\left(\frac{\mu^2}{x_1^2}\right)^{-2}\left(\frac{m^2}{x_1-x_2}+\frac{\mu^2}{x_2}\right)^{-2}$$
$$\times\left(\frac{m^2}{x_1-x_2}+\frac{m^2}{x_2-x_3}+\frac{\mu^2}{x_3}\right)^{-2}\cdots\left(\frac{m^2}{x_1-x_2}+\frac{m^2}{x_2-x_3}+\cdots+\frac{m^2}{x_{n-1}-x_n}+\frac{\mu^2}{x_n}\right)^{-2}, \qquad \text{(A1)}$$

where the same notations as in the text are used. The scaled longitudinal momenta x_i are ordered as in (2.3), and the amplitude $b^{(n)}$ also satisfies the factorization property (2.4). The function $f^{(n)}(1,2,\ldots,n)$ introduced in Sec. II for this model is

$$f^{(n)}(1,2,\ldots,n)=\frac{x_{n-1}\mu^4}{(x_{n-1}-x_n)x_n^2}\left(\frac{m^2}{x_1-x_2}+\frac{m^2}{x_2-x_3}+\cdots+\frac{m^2}{x_{n-1}-x_n}+\frac{\mu^2}{x_n}\right)^{-2}. \qquad \text{(A2)}$$

Using the method developed in the text, we can obtain bounds on the amplitude $f^{(n)}$ as well as on the trajectory function α. Since the calculation is straightforward, we shall not reproduce it here. In the following, we list the essential results in the one-dimensional theory for $m\neq 0$ and $g\to\infty$.

1. N-Body Potential

The one-dimensional N-body amplitude at the high-density limit is

$$f_N^{(n)}(z_i)=\frac{\mu^4}{z_N m^4}\left(\frac{1}{z_2}+\frac{1}{z_3}+\cdots+\frac{1}{z_N}\right)^{-2}, \qquad \text{(A3)}$$

where $z_i=\ln(x_{i-1}/x_i)$ measures the rapidity difference as before.

The equivalent two-body amplitudes derived from the upper and lower bounds are

$$\bar{f}_N^U(z)=\frac{\mu^4 z}{m^4(N-1)^2} \qquad \text{(A4)}$$

and

$$\bar{f}_N^L(z)=\frac{e^4\mu^4a^2}{4m^4z(N-1)^2}\,e^{-a/z}, \qquad \text{(A5)}$$

respectively, where a is a parameter to be chosen to give the best lower bound. These two-body amplitudes lead to an upper and lower bound on α_N as

$$\frac{0.823}{N-1}\left(\frac{g^2}{4\pi m^4}\right)^{1/2}<\alpha_N<\frac{1}{N-1}\left(\frac{g^2}{4\pi m^4}\right)^{1/2}. \qquad \text{(A6)}$$

The coefficient 0.823 is obtained by choosing a to optimize the lower bound.

The special case $N=2$ is given explicitly in Ref. 15.

2. Full Ladder Amplitude

The full amplitude at the high-density limit is

$$f^{(n)}(z_i)=\left(\frac{\mu}{m}\right)^4\frac{1}{z_1}\left[\frac{e^{-z_1}}{z_1}+\frac{e^{-(z_1+z_2)}}{z_2}+\cdots+\frac{e^{-(z_1+\cdots+z_{n-1})}}{z_{n-1}}\right]^{-2}, \qquad \text{(A7)}$$

where the labels on z_i are reversed as in (5.4).

For the upper bound, we cut off the sum in $f^{(n)}$ after $(N-1)$ terms, with $N<n$. This leads to an equivalent two-particle amplitude

$$\bar{f}_N^U(z)=\frac{\mu^4 z}{m^4(N-1)^2}\,e^{Nz}, \qquad \text{(A8)}$$

which differs only slightly from (A4). The best upper bound is obtained from (A8) by choosing N to minimize α^U, giving

$$\alpha^U=2\left(\frac{g^2}{4\pi m^4}\right)^{1/4}. \qquad \text{(A9)}$$

To obtain a lower bound on the amplitude, we put in a lower cutoff $\prod_i\theta(z_i-z_0)$ by hand and extend the sum in the square brackets of (A7) to ∞. Following the procedure developed in Sec. V, we obtain an effective two-body amplitude as a lower bound,

$$\bar{f}^L(z)=\frac{e^2\mu^4a^2}{4m^4z}\exp\left(-\frac{a}{z_0z}\right)\theta(z-z_0), \tag{A10}$$

where a and z_0 are two parameters to be determined later. With properly chosen a and z_0, we obtain from (A10) a lower bound

$$\alpha^L=1.13\left(\frac{g^2}{4\pi m^4}\right)^{1/4}. \tag{A11}$$

Combining (A9) and (A11), we have

$$1.13\left(\frac{g^2}{4\pi m^4}\right)^{1/4}<\alpha<2\left(\frac{g^2}{4\pi m^4}\right)^{1/4}. \tag{A12}$$

Note that both the upper and lower bounds on α have the same power dependence in g^2. Only the coefficients in the front are different. Just as in the three-dimensional case, the g^2 power dependence in α changes from g to $\sqrt{g}$ as all the multiparticle potentials are included.

APPENDIX B: EQUAL-SPACING APPROXIMATION

Because of the repulsive nature of the potentials, Lee[14] suggested that it would be a good approximation at high density to assume that the final particles are equally spaced in the rapidity space. Hopefully, we can take the equal spacing as the zeroth approximation, and develop a new approximation scheme by taking various correlations into account. In this appendix we shall study this zero-order approximation and compare our results with the known solutions numerically.

1. One Space Dimension

We substitute

$$z_1=z_2=\cdots=z \tag{B1}$$

into (A7), and obtain, at $n\to\infty$ and $z\to 0$,

$$f(z)=\left(\frac{\mu}{m}\right)^4z^3. \tag{B2}$$

The asymptotic trajectory function is then given by

$$\frac{1}{\lambda}=\frac{4\pi\mu^4}{g^2}=\left(\frac{\mu}{m}\right)^4\int_0^\infty dz\,z^3e^{-z\alpha}=\left(\frac{\mu}{m}\right)^4\frac{6}{\alpha^4}, \tag{B3}$$

which leads to

$$\alpha=1.5651\left(\frac{g^2}{4\pi m^4}\right)^{1/4}. \tag{B4}$$

Now, let us compare (B4) with the exact result. Using the method of Ref. 20, we find that the exact asymptotic expression for α is

$$\alpha=(5.8828)^{1/4}\left(\frac{g^2}{4\pi m^4}\right)^{1/4}=1.5574\left(\frac{g^2}{4\pi m^4}\right)^{1/4}$$

where the coefficient 5.8828 is the maximal value of $2z^4K_0(z)$. Indeed, (B4) gives rise to an α not only with the correct g power dependence, but with a numerical coefficient to within the accuracy of 0.5%. Thus, the correction on α due to the fluctuation is less than one percent.

2. Three Space Dimensions

For the forward three-dimensional ladder amplitude (5.3), we set

$$z_1=z_2=\cdots=z \tag{B5}$$

and

$$\vec{q}_1^{\,2}=\vec{q}_2^{\,2}=\cdots=\vec{q}^{\,2}. \tag{B6}$$

Then we obtain

$$f(z,\vec{q})=\frac{\mu^4}{z}\left(\mu^2+\frac{\vec{q}^{\,2}+m^2}{z^2}\right)^{-2}, \tag{B7}$$

which leads to an equation on the trajectory function,

$$\frac{1}{\lambda}=\frac{4\pi\mu^4}{g^2}=\mu^4\int_0^\infty\frac{dz}{z}\int\frac{d^2q}{(2\pi)^2}\left(\mu^2+\frac{\vec{q}^{\,2}+m^2}{z^2}\right)^{-2}e^{-z\alpha}=\frac{\mu^4}{4\pi}\int_0^\infty\frac{z\,dz}{\mu^2+m^2/z^2}e^{-z\alpha}. \tag{B8}$$

From (B8) we obtain for $m\neq 0$

$$\alpha\approx\left(\frac{3!\,g^2}{16\pi^2m^2}\right)^{1/4}=1.5651\left(\frac{g^2}{16\pi^2m^2}\right)^{1/4}, \tag{B9}$$

and for $m=0$

$$\alpha=\frac{g^2}{16\pi^2\mu^2}. \tag{B10}$$

Note that $\alpha|_{m\neq0}$ obtained in (B9) differs from the numerical result (5.17),

$$\alpha=1.47\left(\frac{g^2}{16\pi^2m^2}\right)^{1/4}, \tag{5.17}$$

by only 7%, while $\alpha|_{m=0}$ obtained in (B10) agrees exactly with the analytic calculation. This indicates that the equal-spacing approximation is indeed a very good approximation at high density.

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Magnetic Dipole and Electric Quadrupole Moments of the $W^\pm$ Meson*

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It is shown that the magnetic dipole and the electric quadrupole moments of the $W^\pm$ meson must be equal to e/m and $-e/m^2$, respectively, if we demand either that the Drell-Hearn sum rule is satisfied up to order α^2 or that the helicity of $W^\pm$ is conserved in the scattering from an arbitrary electromagnetic field at high energies and at small but finite scattering angles.

I. INTRODUCTION

$W^\pm$ vector bosons, which are supposed to mediate the weak interactions,[1] have (in addition to the charge) a magnetic dipole moment and an electric quadrupole moment.[2,3] We assume that the electromagnetic interaction of the $W^\pm$ bosons is invariant under the time-reversal and parity operations, hence the electric dipole moment[4] is zero. The values of these moments greatly affect the total production cross sections, the energy-angle distributions, and the decay correlations in the processes such as $e^+e^- \to W^+W^-$,[5] $\gamma Z \to W^+W^-$ + anything,[6] $\nu_\mu Z \to \mu W$ + anything,[7] etc. Therefore if $W^\pm$ bosons are discovered it is relatively easy to find these moments. It is interesting to speculate what these moments should be. $W^\pm$ bosons are assumed to have no strong interactions,[8] hence the observable moments are expected to be not greatly affected by the radiative corrections, in analogy to the magnetic moment of an electron which[9] is given by

$$\mu_e = \frac{e}{2m_e}\left(1 + \frac{\alpha}{2\pi} - \frac{\alpha^2}{\pi^2}\, 0.328\,479 + \cdots\right)$$
$$\equiv \frac{e}{4m_e}\, g_e \,.$$

As is well known, this is the consequence of the quantum electrodynamics of a spin-$\frac{1}{2}$ particle assuming no anomalous magnetic moment (Pauli term) in the Lagrangian. The absence of the Pauli term in the Lagrangian is commonly believed to be due to the fact that its presence would render