Electromagnetic Radiation from Charges in Weak Gravitational Fields

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(Received 31 May 1972)

The electromagnetic potential of a point charge in arbitrary motion in a weak gravitational field of a mass M is found by a Green's-function approach. This solution is then applied to the study of the geometrical effects on the generation and propagation of electromagnetic waves to first order in the Riemann tensor. The electromagnetic field is considered as a small perturbation in the sense that its gravitational field is negligible. Generally the received radiation is generated by the charge at two retarded times: that for the direct path of a null ray and that representing geometrical scattering off of the central mass. Explicit expressions for the power radiated are found in the small-$\omega r'$ and large-$\omega r'$ limits, where r' is the distance of the charge from the central mass, for both nonrelativistic and relativistic motion. The analysis shows under what conditions one may suitably make an approximation based on ray optics or photon emission. In the small-$\omega r'$ limit the power radiated by the charge is found to be smaller than calculated by a red-shift argument, and the total power depends on the orientation of the accelerating system to the mass M. For the case of nonrelativistic free-fall acceleration of the charge, geometrical corrections to the power radiated are of order v^2/c^2 smaller than the dominant contributions calculated from flat-space electromagnetism. Although the concept of constrained uniform motion is not precisely defined in a gravitational field, any reasonable definition gives electromagnetic radiation for a uniformly moving charge of a calculable amount in a gravitational field, in contrast to the nonexistence of such radiation in empty flat space. In the extreme relativistic limit ($v \simeq c$) the radiation from a uniformly moving charge is orders of magnitude larger than that from a freely falling charge; in either case large amounts of radiation are received by the observer at a time such that the charge, at the retarded time, is far from the mass and unambiguously moving uniformly. The geometrical origin and implications of this radiation are discussed, and related to the corresponding situation in gravitational radiation.

I. INTRODUCTION

The properties of electromagnetic radiation emitted by accelerated charges in flat space-time are well known.[1] One solves Maxwell's equations, or equivalently solves the electromagnetic wave equations, to find the rate at which energy is lost from the system. Because of the linearity of the field equations, it suffices to describe the radiation potentials of a single charge, so that one can obtain any general solution by superposition of solutions for each charge comprising the system. One can also approach the radiation problem by considering the radiation-reaction forces on the charges; however, this gives information about the energy outflow from the system only over a time average.[1]

It has been long recognized that the presence of gravitational fields modifies the character of electromagnetic radiation. The most striking examples of such effects are the deflection of light by the sun, the gravitational red shift, and the time delay of signals passing the sun.[2] However, derivations of these effects do not start from Maxwell's equations, but rather use ray optics applied to curved space, the equivalence principle, and properties of null geodesics. It is of interest to study solutions of the curved-space electromagnetic equations in order to place limits on the applicability of ray optics. In addition such a study may lead to effects not anticipated from the ray-optics picture.

A general formal solution of the electromagnetic field equations in curved space has been developed by DeWitt and Brehme,[3] following earlier studies of wave equations by Hadamard.[4] The formal solution exhibits scattering of the electromagnetic signals off of the Riemann tensor, which effects are lumped together in the so-called "tail function." DeWitt and Brehme[3] then give a method of solution for the tail function in a power series in the square of the proper-time interval. Such a series is inappropriate for the study of questions such as electromagnetic radiation, however. A method of solution for the radiation reaction for freely falling charges in almost flat space was given later by DeWitt and DeWitt.[5] Here an expansion was made about the flat-space metric $\eta_{\mu\nu}$, and first-order effects in the gravitational potential were calculated. This method was recently applied by Roe,[6] who showed that electromagnetic radiation from a pulsed source in the vicinity of a massive body should be received, to first order, as two distinct pulses, one arriving along the direct route and the other effectively being scattered off of the central body. Such a double-pulse effect has also been shown to occur for gravitational radiation from sources in the vicinity of a massive body.[7]

An alternative approach to electromagnetic radiation in the Schwarzschild metric has been developed along the lines of the Regge-Wheeler treatment of metric perturbations.[8,9] This involves the use of an expansion into generalized spherical harmonics and is most useful when the radiation arises from a given multipole moment of the source. For a point charge such a technique involves an infinite sum, although some advantage is gained in that the solution obtained in this manner is valid for the strong-field region of the Schwarzschild geometry. Recent applications of this approach[10-12] to the problem of synchrotron radiation from particles in relativistic circular orbits have given rise to some unexpected examples of the breakdown of ray optics due to scattering off the Riemann tensor, or equivalently, off the gravitational field.

In this paper we consider the solution, to first order in GM/rc^2, of the electromagnetic field equations for a point charge in the vicinity of a mass M via a Green's-function technique. Our choice of gauge, which differs somewhat from the covariant Lorentz gauge, allows relatively simple explicit solutions for the potentials to be generated. The potential solutions then allow one to compute the flux of electromagnetic waves for a number of significant cases. Although many of the expected effects are generated in this manner, some require reexamination and reinterpretation. Moreover, we have found effects, such as electromagnetic radiation from uniformly moving charges, which had not been anticipated.

In Sec. II we consider the curved-space electromagnetic field equations expanded to first order in the gravitational potential, and give the solution for the radiation fields in terms of a previously defined coordinate-dependent Green's function. This solution is then reduced to simpler expressions in the small-$\omega r'$ and large-$\omega r'$ limits. An alternate semicovariant form of the solution is also given, which is particularly useful in the large-$\omega r'$ limit. In Sec. III we consider a number of applications of the coordinate-dependent solution to problem of the generation of electromagnetic radiation by charges in gravitational fields. In the small-$\omega r'$ limit we consider examples in which nongravitational forces dominate, in which the charge is freely falling, and in which the motion is, in some sense, uniform. In the large-$\omega r'$ limit, we consider examples of extreme relativistic motion ($v \simeq c$). In particular "uniform" motion and free fall (in an extreme hypberbolic orbit) are examined in detail. In Sec. IV we consider applications of the semicovariant solution to similar problems, all in the large-$\omega r'$ limit. In the nonrelativistic limit we consider radiation from both the direct, dominant part of the solution and from the part representing scattering off the Riemann tensor. In the relativistic limit we consider again the problems of "uniform" motion and of free-fall motion. Section V contains a discussion of the results and their implications. An outline of the derivation of the coordinate-dependent solution is given in Appendix A, and an outline of the derivation of the semicovariant solution is given in Appendix B. The relation between these two solutions is given in Appendix C.

II. GREEN'S-FUNCTION SOLUTIONS

A. Electromagnetic Field Equations

The covariant electromagnetic field equations[13]

$$F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\,(F^{\mu\nu}\sqrt{-g})_{,\nu} = 4\pi J^{\mu} \tag{2.1}$$

with

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \tag{2.2}$$

are linear and therefore admit solutions in terms of a Green's function. It suffices to consider, in any given geometry, the solution for a point charge in arbitrary motion, the solution for more complex systems being obtained by superposition. In this calculation we assume that a charge q is moving in the vicinity of a massive body M, and we investigate the dominant effect of the curved space-time on the electromagnetic fields produced by such a charge. We assume that the mass m of the charge q is small compared to M, that the mass M is static and spherically symmetric, and that the gravitational potential of M is sufficiently weak that the metric can be expanded in powers of the potential, with the dominant geometrical influence being given by terms linear in the potential. The mass M is assumed to be electromagnetically inert for simplicity, i.e., its effect on electromagnetism is entirely through its gravitational influence. One could take into account electromagnetically active central masses by finding the contribution to the electromagnetic field from all the charges in the central body, using the Green's-function approach. Within the limits of the above approximations, we cannot, of course, treat the case of charges in the strong-field regions of gravitationally collapsed bodies or black holes. Throughout our discussion the electromagnetic field is considered as a small perturbation in the sense that its gravitational field is negligible.

The geometry of the exterior of the massive body is described by the Schwarzschild metric. Since we are interested in the dominant geometrical effects on electromagnetic fields, it suffices to approximate the true Schwarzschild metric in isotropic coordinates by its weak-field form

$$g_{00}=(1+2\phi), \quad g_{0i}=0, \quad g_{ij}=-\delta_{ij}(1-2\phi), \quad (2.3)$$

where $\phi=-GM/r$ is the gravitational potential of the central mass. Because this approximation neglects terms of order ϕ^2, our electromagnetic field equations will be strictly valid only to order ϕ. Higher-order terms could, if desired, be taken into account by iteration using the Green's function derived here. To order ϕ, the contravariant form of the metric is

$$g^{00}=(1-2\phi), \quad g^{0i}=0, \quad g^{ij}=-\delta_{ij}(1+2\phi), \quad (2.4)$$

and $\sqrt{-g}$, with $g\equiv\det g_{\mu\nu}$, is given by

$$\sqrt{-g}=(1-2\phi). \quad (2.5)$$

Substituting (2.3)–(2.5) into (2.1) and (2.2) yields the electromagnetic field equations,[13] to first order in ϕ,

$$\Box A_0+2\phi_{,k}(A_{0,k}-A_{k,0})-f_{,0}=4\pi J^0, \quad (2.6)$$

$$\Box A_i-4\phi(A_{i,kk}-A_{k,ik})-2\phi_{,k}(A_{i,k}-A_{k,i})-f_{,i}=-4\pi J^i, \quad (2.7)$$

where f is defined to be

$$f=A_{0,0}-A_{k,k}$$

and is determined by the choice of gauge.

Two methods of solution of the electromagnetic field equations will be used in this paper. The first, which we call a coordinate-dependent method, is based on a Green's function which has been used previously in the analogous study of gravitational perturbations in weak-field regions of the Schwarzschild metric.[14] The basic starting point in this approach is the electromagnetic field equations in the coordinate-dependent form, (2.6) and (2.7). A suitable gauge is chosen that allows the Green's function to be applied to this problem, and explicit solutions are then given in the coordinate system in which the metric has the form (2.3). Although valid for near fields as well as radiation fields, we will be concerned here only with the latter. The main disadvantage of this formalism is that it involves an expansion in the parameter $|\omega r'\phi|$, which limits its applicability in the large-$\omega r'$ limit.

An alternate approach, which we call a semicovariant method, starts directly from the covariant equations (2.1). Basically the flat-space solution of (2.1) is first written in a covariant form, and used as a zeroth-order (in the Riemann tensor) approximation to the solution of the curved-space equations (2.1). Since the zeroth-order solution is not an exact solution of (2.1), one must introduce a first-order correction whose effective source is known explicitly in terms of the zeroth-order solution. Thus this method generates a covariant solution to first order in the Riemann tensor. Applied to radiation potentials this solution assumes a semicovariant form. In applications this form is most useful for cases in which $\omega r'$ is large, since in many cases the zeroth-order solution will contain the dominant geometrical effects.

In the following two sections we will give the results for the radiation potentials and fields by these two methods. We will also give the reductions of these potentials and fields in various limits, which give rise to simpler expressions and which are also useful for later applications. The derivations of these potentials and fields are somewhat involved, and thus we will defer to Appendix A the derivation using the coordinate-dependent method, and to Appendix B the derivation using the semicovariant method. In Appendix C we show the equivalence of these two methods in regions in which they are both valid.

B. Coordinate-Dependent Solution

In order to find the electromagnetic radiation generated by the charge q, it suffices to know only the time derivative of the spatial component of the potential A_μ for large r, where r is the distance from the mass M to the point at which the field is observed. In addition we can ignore any terms in the potential proportional to the unit vector $\vec{n}$, where $\vec{n}=\vec{r}/r$, since only transverse components give rise to an energy flux. In Appendix A the radiation fields are found to be propagated from the charge q to the observer at r along two different null paths, with the result that the field at r is generated by the charge q at r' at two different retarded times, t_1' and t_2', where

$$t_1'=t-r+\vec{n}\cdot\vec{r}'(t_1'), \qquad t_2'=t-r-r'(t_2'). \quad (2.8)$$

The time t_1' represents the time retarded by the time it takes a signal to propagate with the speed of light directly from the source to the field point. The time t_2' represents the time retarded by the time it takes a signal to propagate from the source to the mass M and then from M to the field point. In a sense the second signal can be thought of as having been scattered off the central body, although this scattering is purely geometrical, since we have assumed that the central body was electromagnetically inert.

The time derivative of the spatial component of A_i in the radiation zone is then found to be (see Appendix A)

$$A_{i,0}=\frac{-q}{r}F^i(t_1')+\frac{2GM}{r}\,q\left\{\left[\frac{G^i(t')}{(1-\vec{n}\cdot\vec{v})^2}\right]_{t_1'}-\left[\frac{G^i(t')}{(1+\vec{r}'\cdot\vec{v}/r')^2}\right]_{t_2'}+\left[\frac{H^i(t')}{1-\vec{n}\cdot\vec{v}}\right]_{t_1'}-\left[\frac{H^i(t')}{1+\vec{r}'\cdot\vec{v}/r'}\right]_{t_2'}+\left[\frac{K^i(t')}{1+\vec{r}'\cdot\vec{v}/r'}\right]_{t_2'}\right\}, \qquad (2.9)$$

where

$$F^i(t')=\frac{1}{(1-\vec{n}\cdot\vec{v})^2}\left[\left(\dot{v}^i+\frac{\vec{n}\cdot\dot{\vec{v}}v^i}{1-\vec{n}\cdot\vec{v}}\right)\left(1+\frac{GM}{r'}\right)-\frac{GM}{r'^3}\vec{r}'\cdot\vec{v}v^i\right],$$

$$G^i(t')=\frac{v^i}{r'(r'+\vec{n}\cdot\vec{r}')(1-\vec{n}\cdot\vec{v})}\left[\frac{(\vec{r}'\cdot\vec{v}/r'+\vec{n}\cdot\vec{v})^2}{1+\vec{n}\cdot\vec{r}'/r'}+\frac{(\vec{r}'\cdot\vec{v})^2}{r'^2}-v^2+(1-\vec{n}\cdot\vec{v})\right]-\frac{x'^i}{r'(r'+\vec{n}\cdot\vec{r}')^2}\left[\frac{\vec{r}'\cdot\vec{v}}{r'}\left(2+\frac{\vec{n}\cdot\vec{r}'}{r'}\right)+\vec{n}\cdot\vec{v}\right],$$

$$H^i(t')=\frac{1}{2r'(r'+\vec{n}\cdot\vec{r}')}\left\{-v^i+\frac{x'^i}{r'+\vec{n}\cdot\vec{r}'}\left[\frac{\vec{r}'\cdot\vec{v}}{r'}\left(2+\frac{\vec{n}\cdot\vec{r}'}{r'}\right)+\vec{n}\cdot\vec{v}\right]\right\},$$

and

$$K^i(t')=\frac{v^i}{r'(1+\vec{r}'\cdot\vec{v}/r')^2(r'+\vec{n}\cdot\vec{r}')}\left[\frac{(\vec{r}'\cdot\vec{v}/r')^2-v^2}{1-\vec{n}\cdot\vec{v}}-\frac{1}{2}\left(\frac{\vec{r}'\cdot\vec{v}}{r'}+\vec{n}\cdot\vec{v}\right)\left(1+\frac{\vec{r}'\cdot\vec{v}}{r'}\right)-\frac{1}{2}\left(\frac{\vec{r}'\cdot\vec{v}}{r'}+v^2\right)\left(1+\frac{\vec{n}\cdot\vec{r}'}{r'}\right)\right]+\frac{1}{2}\frac{x'^i}{(r'+\vec{n}\cdot\vec{r}')(r'+\vec{r}'\cdot\vec{v})^2}\left[v^2+\left(\frac{\vec{r}'\cdot\vec{v}}{r'}\right)^3+\vec{n}\cdot\vec{v}\left(\frac{\vec{r}'\cdot\vec{v}}{r'}+v^2\right)+\frac{(\vec{r}'\cdot\vec{v}/r'+\vec{n}\cdot\vec{v})^2(r'+\vec{r}'\cdot\vec{v})}{r'+\vec{n}\cdot\vec{r}'}\right],$$

and where v^i is the velocity of the charge q; $\dot{v}^i$ is its acceleration; and the terms in (2.9) are to be evaluated at one or the other of the retarded times (2.8). The above expressions are rather complex, and one can obtain analytic expressions for the power radiated using them only in the case of extreme limits. Thus in general one must resort to numerical integration to find the power radiated.

The expression (2.9) greatly simplifies in the small-$\omega r'$ limit, which limit implies that the motion is nonrelativistic and that the distance r' of q from M is much smaller than one wavelength of the radiation. This simplification arises because the two retarded-time contributions coalesce to form, to lowest order in $\omega r'$, a single retarded-time contribution. As shown in Appendix A, this reduces (2.9) in the small-$\omega r'$ limit to

$$A_{i,0}=\frac{-q}{r}\frac{\partial}{\partial t}\left[v^i-\frac{GM}{r'}\left(v^i-\frac{x'^i}{r'^2}(\vec{r}'\cdot\vec{v})\right)\right]_{\rm ret}, \qquad \omega r'\ll 1 \qquad (2.10)$$

where ret means all terms are evaluated at the time $t'=t-r$. Note that this is the same form as the time derivative of the radiation potential in the absence of the gravitational field, if we replace the velocity v^i by an effective velocity $v^i_{\rm eff}$, given by

$$v^i_{\rm eff}=v^i-\frac{GM}{r'}\left(v^i-\frac{x'^i}{r'^2}\vec{r}'\cdot\vec{v}\right). \qquad (2.11)$$

The expression (2.9) also simplifies in the large-$\omega r'$ limit, though not to the extent that it simplified in the small-$\omega r'$ limit. However, one cannot take an arbitrarily large $\omega r'$ since we must keep $|\omega r'\phi|$ small. Also, in deriving (2.9) we neglected terms in G^i, H^i, and K^i that contained acceleration. Effectively this means that this limit is one of extreme relativistic motion, but of small acceleration, i.e., unbound trajectories. More specifically, our assumptions are that $\gamma\gg 1$ where $\gamma=(1-v^2)^{-1/2}$, and $\vec{v}$ is such that $\gamma\dot{v}_\perp r'\ll 1$ and $\gamma^2\dot{v}_\parallel r'\ll 1$, where $\dot{v}_\perp$ and $\dot{v}_\parallel$ are respectively the components of the acceleration perpendicular and parallel to the velocity. Since these imply extreme hyperbolic motion, the restriction also implies that the deflection angle of the charge, $\Delta\theta$, must satisfy $\gamma\Delta\theta\ll 1$. In particular, if the force on the charge is electric or magnetic, then the approximation holds if $qEr'/m \ll 1$ or $qBr'm \ll 1$. If the acceleration is due to the gravitational attraction of M, then $\gamma^2 GM/r' \ll 1$.

In the extreme relativistic limit the dominant contributions to the potentials arise from the first-retarded-time contributions, as those exhibit the same kind of peaking in the forward direction as is found in flat-space electromagnetism. To find an approximation to the potential (2.9) which is valid for large γ, it is useful to define a quantity ϵ,

$$\epsilon = (\vec{r}' \cdot \vec{v} - \vec{n} \cdot \vec{r}')/r' , \tag{2.12}$$

which is seen to be small (of order γ^{-1}) in the forward peak in the extreme relativistic limit. As shown in Appendix A, the expression (2.9) becomes, for large γ, the sum of the direct contribution to $A_{i,0}$ from the F^i terms and the geometrical contribution to $A_{i,0}$ from only the G^i and H^i terms. The F^i terms can be evaluated for any given acceleration, and the final form would, of course, depend on the particular acceleration used. Explicitly the geometrical contribution becomes, in extreme relativistic limit,

$$A_{i,0}^{(\text{geo})} = \frac{2GMq}{r}\left\{\frac{v^i}{(1-\vec{n}\cdot\vec{v})^3}\left[\left(\frac{2\vec{r}'\cdot\vec{v}}{r'^3} - \frac{\vec{n}\cdot\vec{r}'}{r'^3}\right) + \frac{\vec{n}\cdot\vec{v} - v^2}{r'(r'+\vec{n}\cdot\vec{r}')} + \frac{\epsilon^2(2+\vec{n}\cdot\vec{r}'/r')}{(r'+\vec{n}\cdot\vec{r}')^2}\right] - \frac{x'^i}{r'^3(1-\vec{n}\cdot\vec{v})^2}\left[1 + \frac{\epsilon(2+\vec{n}\cdot\vec{r}'/r')}{(1+\vec{n}\cdot\vec{r}'/r')^2}\right]\right\}_{t_1'} , \quad \omega r' \gg 1, \quad |\omega r'\phi| \ll 1 . \tag{2.13}$$

In deriving (2.13) we have kept only the leading terms that will be needed later.

C. Semicovariant Solution

The structure of the solution of the electromagnetic field equations in a curved space-time has been examined by DeWitt and Brehme.[3] The general solution includes a direct propagation of electromagnetic signals along the light cone, as is the case in flat space-time, together with a "tail," which represents a propagation or smearing out of the signal inside the light cone. This latter contribution can also be interpreted as a scattering of the signal off the Riemann tensor. The solution (2.9) contains both a direct retarded-time contribution, propagated directly from the source to the observer along a null path, as well as a scattered retarded-time contribution, which contributes to the "tail" part of the solution. Although our general solution (see Appendix A) has this latter contribution being indeed smeared out inside the light cone, the specific restriction of radiation fields implies, to first order in the Riemann tensor, that the field will be propagated sharply from the source to the observer. The retarded time for this contribution is indicative of a signal which propagates along a null path directly from the source at r' to the mass M, and then propagates along a null path directly from the mass M to the observer at r.

We can now see why, from a physical point of view, the expression (2.9) has a limited range of validity. It should be noted that the propagation of signals in (2.9) is assumed to be along straight coordinate lines, which are null only in terms of the particular coordinate system used. One would expect physical signals to be propagated along null geodesics, which experience deflection when expressed in our coordinate system, and which do not have zero coordinate interval. Since these two paths are the same in flat space-time, differences in the paths are expected to occur to first order in the gravitational potential. Although some of these differences can be rectified by additional terms of order ϕ (see Appendix C), the effect of the proper time difference between the two paths cannot be rectified for all ω by addition of terms of order ϕ. The reason for this is that the important quantity to evaluate in the latter case is the phase difference between the paths, which depends on frequency and will be small only if $|\omega r'\phi| \ll 1$. This qualitative picture of the limitation on (2.9) is made more quantitative in Appendix C.

In Appendix B a solution of the field equations is given, to first order in the Riemann tensor, in a covariant form. This derivation explicitly uses the fact that direct signals from the source to observer propagate along true null geodesics. Of course we also find scattered contributions in the formalism as well. In the wave zone the direct contribution to the radiation field (see Appendix B for derivation) is given in semicovariant form as

$$A_{\mu,0} = \frac{q}{r}\frac{\partial}{\partial t}\left(\frac{g_{\mu\alpha'}v^{\alpha'}}{n^\mu g_{\mu\beta'}v^{\beta'}}\right)_{\text{ret}} , \tag{2.14}$$

where $n^\mu = (1, \vec{n})$, $g_{\mu\alpha'}$ is the parallel propagator defined by Synge[15] and DeWitt and Brehme,[3] $v^{\alpha'}$ is the velocity $(1, \vec{v})$, and all quantities are to be evaluated at the true retarded time along the direct null geodesic. The potential in (2.14) reduces to the Liénard-Wiechert potential in the limit of flat space-time. It should be remembered that the complete potential, to order ϕ, also contains the terms with the second-retarded-time contribution, which represent scattering off the Riemann tensor.

In order to evaluate (2.14) explicitly to order ϕ for our calculations, we list the components of the parallel propagator, to order ϕ, for the metric (2.3):

$$g^0{}_{0'} = (1+\phi')$$
$$g^0{}_{k'} = g^k{}_{0'} = \psi_{,k'} \qquad (2.15)$$
$$g^k{}_l = \delta^k{}_{l'}(1-\phi) + n^l\psi_{,k'} - n^k\psi_{,l'}$$

where $r \gg r'$, $\phi' = \phi(\vec{r}\,')$, and

$$\psi = -GM\ln[(r' + \vec{n}\cdot\vec{r}\,')/2r]\,. \qquad (2.16)$$

Note that $n^k\psi_{,k'} = \phi'$ and $n^\mu g_{\mu\alpha'}v^{\alpha'} = 1 - \vec{n}\cdot\vec{v} + 2\dot{\psi}$. The true retarded time, to order ϕ, is given by the expression

$$t' = t - r + \vec{n}\cdot\vec{r}\,' - 2\psi\,. \qquad (2.17)$$

The solution (2.14) is not complete without the scattered contributions. We can, however, consider the direct and scattered contributions as independent because the A_μ defined in (2.14) satisfies the Lorentz gauge $n^\mu A_{\mu,0} = A_{0,0} - A_{k,k} = 0$. Since the full potential satisfies the Lorentz gauge for large r, this implies that the scattered contributions also satisfy the Lorentz gauge. Further, the scattered contributions are found to be of order $[\omega(r' + \vec{n}\cdot\vec{r}\,')]^{-1}$ times the direct contributions. For all angles except a narrow cone centered on $\vec{n}\cdot\vec{r}\,'/r' = -1$, this will be of order $(\omega r')^{-1}$, and the scattered terms can be neglected in the large-$\omega r'$ limit. Since we will be applying the semicovariant formalism only in the large-$\omega r'$ limit, we can often ignore the scattered contributions. However, since scattered contributions have a retarded time that is different than that of the direct contribution, they may still produce observable effects, even though small compared with the direct contribution to $A_{i,0}$. Further, we will need the scattered contributions only in the nonrelativistic limit, as they are found to be orders of magnitude in $(1-v^2)$ smaller than the direct contributions in the relativistic limit. From Appendix B, Eq. (B10), we have the scattered contribution

$$A_{i,0}^{(s)} = -\frac{GMq}{r}\left[\frac{\dot{v}^i(r' - \vec{n}\cdot\vec{r}\,') + x'^i(\vec{n}\cdot\dot{\vec{v}} - \vec{r}\,'\cdot\dot{\vec{v}}/r')}{r'(r' + \vec{n}\cdot\vec{r}\,')}\right],$$

$$\omega r' \gg 1,\; v \ll c \qquad (2.18)$$

where quantities are to be evaluated at the second retarded time t_2', given in (2.8).

D. Electromagnetic Radiation

We now turn to the method of evaluation of electromagnetic radiation of a point charge moving in the vicinity of the mass M. The energy radiated by electromagnetic waves can be found by integrating the Poynting flux over a sphere surrounding the system. Although normally expressed in terms of the electric and magnetic fields, an alternate expression for the power radiated into solid angle $d\Omega$, useful for our purposes, is given by

$$\frac{dP}{d\Omega} = -\frac{r^2}{4\pi}[A_{\alpha,0}A^\alpha{}_{,0}] = \frac{r^2}{4\pi}[A_{k,0}A_{k,0} - (n^kA_{k,0})^2]\,. \qquad (2.19)$$

The latter form illustrates the fact that only the spacelike transverse part of $A_{\mu,0}$ contributes to the radiation. The total power radiated is the integral over solid angle of (2.19)

$$\frac{dE}{dt} = P = \int\!\!\int \frac{dP}{d\Omega}\,d\Omega\,. \qquad (2.20)$$

For relativistic problems it is often desirable to evaluate $dE/dt' = P'$, where t' is the retarded time,

$$\frac{dE}{dt'} = P' = \int\!\!\int \frac{dP}{d\Omega}\left(\frac{dt}{dt'}\right)d\Omega\,, \qquad (2.21)$$

and where dt/dt' depends on angle.[1] The interpretation is that P' represents the radiation emitted in time dt' along the particle's path. However, for the potentials we derived in the previous sections, we found that signals from the source are received at two different retarded times. Thus, unless one can ignore interference between these two, one cannot characterize the energy received as having been emitted by the charge at any unique time on its path.

III. RADIATION DERIVED FROM THE COORDINATE-DEPENDENT SOLUTION

In this section we will study applications of the coordinate-dependent formalism to the problem of the generation of electromagnetic waves by the charge q in the vicinity of the mass M. In effect the problem is already solved since the potentials are known explicitly from (2.9) and the power radiated can then be found from (2.20). However, the relatively complicated form of (2.9) involving two different retarded times makes it difficult to visualize what physical effects one might expect to find as a result of the gravitational field of M. Thus we will restrict ourselves in this section only to applications of the two limiting cases, $\omega r' \ll 1$

[for which (2.10) applies] and $\omega r' \gg 1$, $|\omega r'\phi| \ll 1$ [for which (2.13) applies]. Since these limits do not uniquely specify the system, we will find it useful to consider subcases of each limit. More specifically, for the limit of $\omega r' \ll 1$, which implies both nonrelativistic motion as well as the distance r' of q from M being smaller than a wavelength of the radiation, we will consider the three subcases in which the acceleration of q is dominated by nongravitational forces, the acceleration is dominated by the gravitational force of M, and the acceleration is constrained to vanish, i.e., "uniform" motion. For the limit of $\omega r' \gg 1$, $|\omega r'\phi| \ll 1$, which implies both extreme relativistic motion ($v \simeq c$) and only a small deflection of q in an unbound trajectory past M, we will consider the two subcases of constrained "uniform" motion and of acceleration dominated by the free-fall acceleration of q towards M in its trajectory past M.

A. Small $\omega r'$, Nongravitational Forces Dominant

As a first example of the use of (2.10), consider the case in which $\dot{v} \gg v^2/r'$ (still keeping $\dot{v} \ll v/r'$, as is required from the assumption of small $\omega r'$). This could occur for a charge moving in a circle of radius a, where $a \ll r'$, or a harmonically oscillating charge of amplitude a, where a again is $\ll r'$. In particular for periodic or bounded motion this also implies that the acceleration is much larger than the acceleration of gravity due to the mass M, because if the acceleration were comparable to that of gravity, the virial theorem would imply that $\dot{v} \simeq v^2/r'$, which violates our assumption. Also, the restriction of small $\omega r'$ implies that the motion is nonrelativistic and that the mass M is in the near zone of the electromagnetic field of q. Our assumption $\dot{v} \gg v^2/r'$ implies that the time derivative on the right-hand side of (2.10) acts only on the velocity v^i and not the distance r', giving

$$A_{i,0} = \frac{q}{r}\,\dot{v}^i_{\rm eff}\big|_{\rm ret}\,, \tag{3.1}$$

where

$$\dot{v}^i_{\rm eff} = \dot{v}^i - \frac{GM}{r'}\left(\dot{v}^i - \frac{x'^i}{r'^2}\,\vec{r}\,'\cdot\dot{\vec{v}}\right). \tag{3.2}$$

Since the effective acceleration (3.2) does not involve the unit vector n^i, the power radiated has the same form as in nonrelativistic electrodynamics; i.e., from (2.19),

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi}\,\dot{v}_{\rm eff}^{\,2}\sin^2\theta\,, \tag{3.3}$$

where θ is the angle between the direction of $\dot{\vec{v}}_{\rm eff}$ and the observation direction $\vec{n}$. The total power radiated is then found from (2.20) as

$$P = \tfrac{2}{3}q^2\dot{v}_{\rm eff}^{\,2}\,, \tag{3.4}$$

with $\dot{v}_{\rm eff}$ given by (3.2). Note that if the acceleration $\dot{\vec{v}} \parallel \vec{r}\,'$ then the effective acceleration is just the acceleration $\dot{\vec{v}}$, but if $\dot{\vec{v}} \perp \vec{r}\,'$ then the effective acceleration is the acceleration $\dot{\vec{v}}$ multiplied by the factor $(1 - GM/r')$. In other words the power radiated by an accelerating charge in this limit depends on the orientation of the accelerating charge to the central body. In general, if we let ψ be the angle between $\vec{r}\,'$ and $\dot{\vec{v}}$, then the power radiated is, to first order in GM/r',

$$P = \tfrac{2}{3}\,q^2\dot{v}^2\left(1 - \frac{2GM}{r'}\,\sin^2\psi\right). \tag{3.5}$$

One should be somewhat careful in applying (3.5), however, as $\dot{\vec{v}}$ is the coordinate acceleration and not the proper acceleration that could be measured locally in terms of proper rulers and clocks. One advantage of using the isotropic form of the coordinate is that the relation between the proper acceleration $\dot{\vec{V}}$ and $\dot{\vec{v}}$ is only one of scale in the case examined here, and does not depend on the orientation of $\dot{\vec{v}}$ to $\vec{r}\,'$. In particular, $\dot{\vec{V}}$ is seen to be related to $\dot{v}$ by

$$\dot{\vec{V}} = \dot{\vec{v}}\left(\frac{1-\phi}{(1+\phi)^2}\right) \approx \dot{\vec{v}}\left(1 + \frac{3GM}{r'}\right), \tag{3.6}$$

giving the power radiated in terms of the proper acceleration $\dot{\vec{V}}$, to first order in GM/r', as

$$P = \tfrac{2}{3}\,q^2\dot{V}^2\left[1 - \frac{6GM}{r'}\,(1 + \tfrac{1}{3}\sin^2\psi)\right]. \tag{3.7}$$

One might have predicted a different result from an analysis based on a photon picture of radiation, in which one argues that the power radiated from the system immersed in the gravitational potential should be lowered by two red-shift factors $(1 - GM/r')$, one factor from the fact that the photons are received with a lower energy, and the other factor from the fact that the number of photons per second received is less than the number per second emitted. Of course, such an argument assumes that the mass M is in the radiation zone of the electromagnetic field of q, which violates one of the assumptions made in deriving (3.7). Our analysis confirms the fact that in the small-$\omega r'$ limit such an argument does not work, and shows that one generally receives somewhat less radiation than would be given by the photon argument. In fact one finds a power radiated which depends on the orientation of the emitting system to the central mass M.

B. Small $\omega r'$, Gravitational Force Dominant

A second example of interest in the small-$\omega r'$ limit is that of a charge q freely falling, in a bound or unbound orbit, in the gravitational field of the mass M. Here small $\omega r'$ means simply nonrelativistic motion, since the restriction on the acceleration is already taken into account by the assumption of small potential ϕ. Now the radiation fields are given by (3.1) but with the effective acceleration

$$\dot{v}^i_{\rm eff} = -\frac{GM}{r'^3}\left\{x'^i\left[1+3\left(\frac{\vec{r}'\cdot\vec{v}}{r'}\right)^2 - v^2\right] - 2v^i\vec{r}'\cdot\vec{v}\right\} \tag{3.8}$$

so that the power radiated is

$$P = \frac{2}{3}\frac{G^2M^2q^2}{r'^4}\left\{1-2\left[v^2-\left(\frac{\vec{r}'\cdot\vec{v}}{r'}\right)^2\right]\right\}. \tag{3.9}$$

We see that the effect of the extra terms added by geometrical considerations is to modify the acceleration and power by terms which are of order v^2/c^2 smaller. These terms disappear when $\vec{r}'\parallel\vec{v}$, and are largest when $\vec{r}'\perp\vec{v}$. Of course, the expression (3.14) does not contain all of the v^2/c^2 corrections to the power radiated, as there are also the v^2/c^2 terms arising from special-relativistic corrections to the dominant, flat-space, part of the potentials, which were ignored due to the assumption $\omega r' \ll 1$, as well as general-relativistic corrections to the acceleration of a body in a gravitational field. The extra terms found are, of course, small in the nonrelativistic limit, and our expressions are not valid as $v\to c$; however, the fact that the correction terms approach the dominant terms as $v\to c$ leads one to suspect that the geometrical effect may become important for the free-fall case in the relativistic limit.

C. Small $\omega r'$, Constrained "Uniform" Motion

The last example in the small-$\omega r'$ limit is that of the case of vanishing acceleration, i.e., $\dot{v}^i=0$. Although perhaps somewhat artificial, the example has considerable interest, as the result is not zero, and the analysis which follows helps one to understand the relativistic case better. It should be noted that $\dot{v}^i=0$ means that the charge is constrained to follow a straight-line path at uniform velocity by the application of forces which themselves do not contribute to the radiation potentials. Unlike the analogous case of gravitational radiation, where this is not possible, there are many conceivable physical mechanisms by which this can be accomplished for electromagnetic radiation. An examination of the time rate of change of the effective velocity (2.11) leads one to the conclusion that even though $\dot{v}^i=0$, $\dot{v}^i_{\rm eff}\neq 0$ because of the r' dependence of the last term in (2.11). Explicitly, for the "uniform"-motion case,

$$\dot{v}^i_{\rm eff} = \frac{GM}{r'^3}\left[2v^i\vec{r}'\cdot\vec{v}+x'^i\left(v^2-\frac{3(\vec{r}'\cdot\vec{v})^2}{r'^2}\right)\right], \tag{3.10}$$

and the power radiated is then

$$P = \frac{2}{3}\frac{G^2M^2q^2}{r'^4}\left[v^2-\left(\frac{\vec{r}'\cdot\vec{v}}{r'}\right)^2\right]\left[v^2+3\left(\frac{\vec{r}'\cdot\vec{v}}{r'}\right)^2\right]. \tag{3.11}$$

It should be noted that since $\dot{\vec{v}}=0$, there are no relativistic corrections that would contribute in the same order as (3.10) and (3.11). The power in (3.11) is seen to vanish only when $\vec{r}'\parallel\vec{v}$.

The meaning of the expressions (3.10) and (3.11) is not clear at this point, however, because the concept of uniform motion in a curved space-time is not well defined. In fact we have assumed uniform coordinate velocity in the isotropic system of coordinates. In another system of coordinates that motion may not be uniform. We therefore ask if there is a path, slightly deviating from uniform coordinate velocity, such that the total radiation vanishes to this order. This path would have $\dot{v}^i\neq 0$ in order that $\dot{v}^i_{\rm eff}=0$, but $\dot{v}^i$ would clearly be small compared to free-fall accelerations. Setting $\dot{\vec{v}}^i_{\rm eff}=0$ implies that

$$\dot{v}^i = GM\frac{d}{dt}\left[\frac{v^i}{r'}-\frac{x'^i(\vec{r}'\cdot\vec{v})}{r'^3}\right] = GM\frac{d^2}{dt^2}\left(\frac{x'^i}{r'}\right). \tag{3.12}$$

This implies that if we let the path defined by (3.12) be called $\vec{r}''(t')$, where $\vec{r}''(t')=\vec{r}'(t')+\vec{\eta}(t')$ and $\vec{r}'(t')$ is the path defined by $\dot{v}^i=0$, then the effective acceleration vanishes if $\vec{\eta}$ is given by

$$\vec{\eta}(t') = \vec{\eta}(\vec{r}(t')) = GM\vec{r}'/r'. \tag{3.13}$$

Since this path does not depend on velocity or time explicitly, it defines a unique path of no effective acceleration, and no radiation to the order calculated, which one may choose to designate as the path of uniform motion. This designation, however, has numerous difficulties.

If we change coordinates from the system in which the metric is (2.3) to a system in which the path defined in (3.13) is one of uniform coordinate velocity in the new system, we find that the metric in the new (barred) coordinate system becomes (to order ϕ)

$$\bar{g}_{00} = (1+2\bar{\phi}), \quad \bar{g}_{0k} = 0, \qquad \bar{g}_{ij} = -\delta_{ij} - 2\bar{\phi}\bar{x}^i\bar{x}^j, \tag{3.14}$$

This transformation, to first order in ϕ, takes us to the standard Schwarzschild coordinate system (with $\bar{r}$ being an area coordinate) reexpressed in rectangular coordinates by the standard polar transformation. We should note the fact that the path (3.13) is not the path of a light ray, since a light ray undergoes a net coordinate deflection in passing the mass M, where the path we have found undergoes no net coordinate deflection in passing the mass M. One may also object to the path in (3.13) being defined as uniform motion, since that does not correspond to uniform motion as measured by local rulers and clocks. Specifically, we can imagine a clock moving nonrelativistically along the path defined in (3.13), each tick marking out spatial intervals which one would expect to be at constant separation as measured by physical rulers, if the motion is truly uniform. However, one can calculate the measured space intervals along such a path with the result that the proper length ΔL is related to the proper time ΔT and asymptotic velocity v_0 by

$$\Delta L = \frac{\Delta T}{v_0}\left\{1 + \frac{3GM}{r'}\left[1 - \tfrac{1}{3}\left(\frac{\vec{r}'\cdot\vec{v}}{r'v}\right)^2\right]\right\},$$

which does not define constant spatial intervals. Further objections to the path (3.13) being defined as uniform motion are found when the motion along the path is relativistic, as will be seen next.

D. Relativistic, Constrained "Uniform" Motion

We now consider cases in which $\omega r' \gg 1$, $|\omega r'\phi| \ll 1$, for which the expression (2.13) serves as the dominant geometrical contribution to the radiation field (in the forward beam). These restrictions imply that the motion is relativistic ($v \simeq c$) and that the trajectory undergoes only a small deflection. One example of such a situation is the artificial one of a charge being constrained to move uniformly at relativistic velocities, where uniform may be taken to be uniform coordinate velocity with respect to either of our two coordinate systems, which both imply no net deflection in the transit past M. Of course, the requirement that ϕ be small implies that GM/bc^2 is small, where b is the impact parameter of the trajectory. Thus we cannot consider the case in which the particle is aimed directly at M.

Consider first the case in which $\dot{v}^i = 0$ in the coordinate system in which the metric is (2.3). Then the direct contribution to the radiation fields, arising from the F^i term of (2.9), is

$$A_{i,0}^{(\text{direct})} = + \frac{GMqv^i(\vec{r}'\cdot\vec{v}/r'^3)}{r(1-\vec{n}\cdot\vec{v})^2}. \tag{3.15}$$

The geometrical contribution to the radiation field is given by (2.13), and the leading contribution in the extreme relativistic limit ($\gamma \gg 1$) gives

$$A_{i,0} \approx \frac{2GMq}{r}\,\frac{v^i(\vec{r}'\cdot\vec{v}/r'^3)}{(1-\vec{n}\cdot\vec{v})^3}. \tag{3.16}$$

The total radiation field is the sum of (3.15) and (3.16). Because there is only one retarded time we can compute the power radiated in the form (2.21), which then gives an analytic expression for the power radiated from the uniformly moving charge, by integrals which are familiar from standard electromagnetism:

$$P' = \frac{dE}{dt'} = \frac{8}{3}\,\frac{G^2M^2q^2}{(1-v^2)^3}\left(\frac{\vec{r}'\cdot\vec{v}}{r'^3}\right)^2. \tag{3.17}$$

The energy radiated in one transit is the time integral of (3.17), i.e.,

$$\Delta E = \frac{\pi G^2M^2q^2}{3(1-v^2)^3b^3}, \tag{3.18}$$

where b is the impact parameter of the path.

In the nonrelativistic limit it was possible to eliminate the dominant term in the radiation by assuming the charge was moving along the path $\vec{r}' + \vec{\eta}$, where $\vec{\eta}$ was given in (3.13). We can assume the same path for the relativistic case to see if we eliminate the radiation (3.18) at least to that order of magnitude. This introduces the acceleration-dependent terms with a well-defined acceleration. The result is that the dominant contribution of the radiation field for this motion is given by

$$A_{i,0} = \frac{3GMqv^i(\vec{r}'\cdot\vec{v}/r')}{r(1-\vec{n}\cdot\vec{v})^3}\left[1 - \left(\frac{\vec{r}'\cdot\vec{v}}{r'}\right)^2\right],$$

which, when compared with (3.16), gives radiation fields of the same order of magnitude. Thus it appears that a uniformly moving charge in a gravitational field radiates electromagnetic waves.

E. Relativistic Free-Fall Motion; Large Impact Parameter

The other case to which we wish to apply the relativistic limit is the case of free fall in the gravitational field M. Because of the limitation of small ϕ, the impact parameter b must be large enough so that $GM/bc^2 \ll 1$. Since the deflection angle for a relativistic particle passing M is $4GM/bc^2$, this implies that we also only consider trajectories which undergo small deflection in passing M.

Before applying our formalism to this problem, it is interesting to see what radiation one would expect using flat-space electromagnetic formulas together with the acceleration of a body in the gravitational field of M (the geodesic equation to order ϕ). With $\dot{v}^i$ given by

$$\dot{v}^i = -\frac{GMx'^i}{r'^3}(1+v^2) + \frac{4GMv^i\vec{r}'\cdot\vec{v}}{r'^3}, \tag{3.19}$$

the expected radiation can be calculated using the standard formulas.[1] This gives an expression, in the large velocity limit, which is identical with (3.17). As we will see, this is not a coincidence, as the free-fall acceleration terms and uniform-motion terms are of the same order of magnitude, and consideration of both leads to a cancellation, giving power radiated which is orders of magnitude in γ^{-1} smaller than (3.17) in the extreme relativistic limit.

With the free-fall acceleration (3.19) the direct contribution to $A_{i,0}$ is (to order GM/r')

$$A_{i,0}^{(\text{direct})} = \frac{-GMq}{r(1-\vec{n}\cdot\vec{v})^2}\left[\frac{-x'^i}{r'^3}(1+v^2) + \frac{v^i}{1-\vec{n}\cdot\vec{v}}\left[\vec{r}'\cdot\vec{v}(3+\vec{n}\cdot\vec{v}) - \vec{n}\cdot\vec{r}'(1+v^2)\right]\right]. \tag{3.20}$$

The sum of (2.13) and (3.20) then gives, to the dominant order in the radiation fields $[(1-v^2)^{1/2} \ll 1]$,

$$A_{i,0} = \frac{GMq}{r(1-\vec{n}\cdot\vec{v})^3}\left[\frac{2+\vec{n}\cdot\vec{r}'/r'}{(r'+\vec{n}\cdot\vec{r}')^2}\right]\left(v^i\left\{2\epsilon^2 + (\vec{n}\cdot\vec{v}-v^2)\left[1-\left(\frac{\vec{n}\cdot\vec{r}'}{r'}\right)^2\right]\right\} - \frac{2\epsilon x'^i(1-\vec{n}\cdot\vec{v})}{r'}\right). \tag{3.21}$$

It should be noted that the radiation fields are of order $(1-v^2)$ smaller than that found in (3.16), showing the effect of the cancellation of the direct and geometrical terms in the free-fall case. Because of the complex nature of the expression (3.21), analytic expressions for the power radiated, analogous to (3.17), cannot be found for all times.

However, an explicit expression for the power radiated can be obtained at the position of closest approach, and an order-of-magnitude estimate can be given to the power radiated at other times. The position of closest approach occurs when $\vec{r}'\cdot\vec{v}=0$, which we define to occur at the time $t'=0$. At this time $\vec{n}\cdot\vec{r}'$ is small over the forward peak. The power radiated is then found to be given by

$$P' = \frac{dE}{dt'} = \frac{64\pi G^2M^2q^2}{b^4(1-v^2)}, \tag{3.22}$$

which is seen to be of order $(1-v^2)^2$ smaller than the typical power radiated from (3.17), taking into account the fact that the maximum power from (3.17) occurs for $t'\neq 0$. To estimate the total power radiated over one transit we first note that for $t'>0$, $\vec{n}\cdot\vec{r}'/r' \to +1$ for large t' and the radiation drops off for times t' a few times b. However, for $t'<0$, $\vec{n}\cdot\vec{r}'/r' \to -1$ for large negative times, and the factors of $(1+\vec{n}\cdot\vec{r}'/r')^{-1}$ cause the radiation to keep the same order of magnitude as (3.22) for much larger times. Specifically, the power radiated keeps the same order of magnitude for negative times t' out to approximately $t'\approx -b/(1-v^2)^{1/2}$. The net power radiated over one transit for free fall is then found to be of order of magnitude

$$\Delta E \sim \frac{64\pi G^2M^2q^2}{b^3(1-v^2)^{3/2}}. \tag{3.23}$$

For negative times much larger than $b/(1-v^2)^{1/2}$, the power using (3.21) becomes unimportant; in addition, because the mass M is now in the forward peak of the radiation pattern, the second-retarded-time contributions are equally important, and the radiation cannot then be ascribed to having been emitted at a unique retarded time.

The long duration of the radiation is not a consequence of the direct, acceleration-dependent term, but rather results from the geometrical terms in the potential. Thus in the extreme relativistic limit most of the radiation is received, in the free-fall case with large impact parameter, at times at which the charge at the retarded time is moving uniformly an appreciable distance away from the mass M. In fact there is no ambiguity about the motion being uniform in that region, as the gravitational potential is small [of order $(GM/b)(1-v^2)^{1/2}$] in that region. The physical origin of the radiation will be clearer when the problem is reexamined from the point of view of Eq. (2.14).

IV. RADIATION DERIVED FROM THE SEMICOVARIANT SOLUTION

In this section we will consider applications of the semicovariant solution in the large-$\omega r'$ limit. This limit allows us to reproduce effects expected on a ray-optics picture as well as to reconsider the radiation for uniform motion or free-fall motion in the extreme relativistic limits. In the large-

$\omega r'$ limit the expression (2.14) serves as the initial approximation, with the scattered contributions (2.18) considered as corrections. Writing

$$u_\mu \equiv g_{\mu\alpha'} v^{\alpha'} \tag{4.1}$$

and

$$1 - \vec{n}\cdot\vec{v} + 2\dot{\psi} = n^\mu g_{\mu\alpha'} v^{\alpha'} = n^\mu u_\mu \equiv s\,, \tag{4.2}$$

we obtain the radiation fields from (2.14) in the form

$$\begin{aligned} A_{\mu,0} &= \frac{q}{r}\,\frac{1}{s}\,\frac{\partial}{\partial t'}\left(\frac{u_\mu}{s}\right) \\ &= \frac{q}{r}\left[\frac{\dot{u}_\mu}{s^2} - \frac{u_\mu \dot{s}}{s^3}\right], \end{aligned} \tag{4.3}$$

where the derivative with respect to t has been reexpressed in terms of the derivative with respect to t'. The fields (4.3) are the same as are found from the Liénard-Wiechert potentials in flat-space electromagnetism with the replacement of the velocity v^μ by the parallel propagated or apparent velocity u^μ. Unlike the flat-space case, however, the time rate of change of the apparent velocity is not directly proportional to the acceleration, since

$$\dot{u}^\mu = \frac{\partial}{\partial t'}\,(g^\mu{}_{\alpha'} v^{\alpha'}) = \frac{\partial}{\partial t'}\,(g^\mu{}_{\alpha'})\, v^{\alpha'} + g^\mu{}_{\alpha'} \dot{v}^{\alpha'}\,. \tag{4.4}$$

The last term on the right-hand side of (4.4), representing the parallel propagated acceleration, is what one might have expected as a generalization of the flat-space situation. The first term on the right-hand side does not involve the acceleration $\dot{v}^{\alpha'}$, but only the velocity $v^{\alpha'}$; it represents the apparent acceleration caused by the fact that the geodesic, along which the parallel propagation takes place, is moving if the charge or mass is moving. In fact one can easily imagine cases in which the acceleration $\dot{v}^{\alpha'} = 0$, as in uniform motion, but where $\dot{u}^\mu \neq 0$. For such cases one would obtain radiation fields, and thus radiation, even though the charge is moving uniformly.

A. Large $\omega r'$, Nonrelativistic Motion (Dominant Terms)

We examine first the case of large $\omega r'$ and nonrelativistic velocities, which implies that the acceleration term $\dot{v}^{\alpha'}$ dominates (4.4). This limit means that the mass M is in the radiation zone of the charge q. This also implies that the acceleration is much larger than the free-fall acceleration of q towards M, for periodic or bounded motion. The spatial components of (4.3) can then be explicitly evaluated in terms of the parallel propagator (2.15) and the acceleration. Keeping terms only to order ϕ, this results in the power radiated per unit solid angle, from (2.19),

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi}\,[\dot{v}^2(1-2\phi) + 4\psi_{,i}\dot{v}^i\,\vec{n}\cdot\dot{\vec{v}} - (\vec{n}\cdot\dot{\vec{v}})^2(1+2\phi)]\,. \tag{4.5}$$

To reduce the expression (4.5) we define the unit vector

$$\vec{n}' = \vec{n}(1+2\phi) - 2\vec{\nabla}\psi\,. \tag{4.6}$$

It can be recognized from (4.6) that the component of $2\vec{\nabla}\psi$ perpendicular to $\vec{n}$ is just the angle of deflection of the null geodesic from the point $\vec{r}'$ to the point $\vec{r}$. Thus $\vec{n}'$ represents the unit vector at $\vec{r}'$ that is deflected to $\vec{n}$ at $\vec{r}$. Using (4.6) to eliminate the unit vector $\vec{n}$ from (4.5) and expressing the coordinate components of acceleration in terms of the physical components [as in (3.7)], we find the power radiated per unit solid angle to be given by

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi}\,\dot{V}^2 \sin^2\theta'\left(1 - \frac{4GM}{r'}\right), \tag{4.7}$$

where θ' is the angle between $\vec{n}'$ and $\dot{\vec{v}}$. We see that the observed radiation pattern is deflected from the emitted pattern and that the power radiated per unit solid angle is $(1-4GM/r') \approx (1-GM/r')^4$ smaller than that emitted. Two factors of $1-GM/r'$ can be understood on the basis that photons are observed red-shifted by a factor of $1+z \approx 1+GM/r'$ and that the rate at which photons are emitted is also "red-shifted" by the same factor. The remainder can be understood in terms of the transformation of a given solid angle $d\Omega'$ at the source into the solid angle $d\Omega$ at the observer's point, because of the deflection of light rays defining the boundaries of the solid angle. A geometrical calculation then shows that, to order ϕ, the solid angle $d\Omega$ at r that is formed by null geodesics having a solid angle $d\Omega'$ at r' is related to $d\Omega'$ by

$$d\Omega = d\Omega'(1+2GM/r')\,, \tag{4.8}$$

which accounts for the other factors of $(1-GM/r')$ in (4.7)

The total power radiated is presumably the integral over solid angle of (4.7); however, this integral requires some thought. Because of the deflection of null geodesics, a total angle of 4π at r corresponds only to a solid angle of $4\pi(1-2GM/r')$ at the distance r'. The remaining solid angle at r', $8\pi GM/r'$, represents radiation that is beamed directly at the mass M by the charge q. Assuming that the radius of the mass M is small enough, any null geodesics emitted within this solid angle will be found at the same angular position at r as some other null geodesic emitted outside this solid angle.

This feature arises in any system which allows multiple null geodesics between two points. Generally the times for signals to travel on each of the null geodesics will be different. If the optical pathlengths of the null geodesics differ by many wavelengths, the contributions from the multiple paths will be incoherent. However, for the radiation at exactly the backward angle, there are an infinite number of null geodesics that contribute in phase. This gives rise to bright-spot effects which have been studied extensively within the geometrical optics approximation.[16,17] It should also be noted that null geodesics which are deflected by a finite angle are those whose impact parameters are close to the Schwarzschild radius, to which the present analysis does not apply.

Our solution for the direct contribution, (2.14), is only a first approximation to these geometrical features. In particular (2.14) assumes that the signal propagates only along the principal null geodesic. This deficiency is not as severe as one might first guess, since the scattered contributions (2.18) are chosen in such a way that the sum of (2.14) and (2.18) yields a solution of the electromagnetic field equations. For example, if we were to include in (2.14) a sum over multiple null geodesics, then the scattered contributions would no longer be given by (2.18), but would rather involve sums over multiple null geodesics. In our formalism effects due to expected multiple null geodesics at backward angles will be found in an examination of the scattered contribution (2.18). Although one might expect such effects to invalidate (4.7) at backward angles, and prevent an angular integration from being performed, one finds under the assumption of no energy losses associated with M that the integrated energy flux can be found by integrating over $d\Omega'$, using the transformation (4.8). Specifically this gives the power radiated

$$P = \tfrac{2}{3} q^2 \dot{V}^2 \left(1 - \frac{2GM}{r'}\right) \tag{4.9}$$

in agreement with the red-shift argument. As we have seen, this power is radiated in a pattern that gives a flux proportional to $1 - 4GM/r'$ at other than backward angles, and an enhanced flux within an angle from the backward direction of order $(GM/r')^{1/2}$.

If there are a number of charges radiating coherently, it is possible to produce high-order multipole radiation patterns, e.g., patterns for which the radiation is emitted in one or more lobes. For a lobe not aimed at the mass M, we can give a description of the flux and radiation pattern at the point r. First, the lobe will be centered at the deflected angle for a null geodesic emitted at that angle. Second, the energy flux will be decreased by $1 - 4GM/r'$ from that emitted. Third, the angular size of the lobe will be stretched by the factor

$$\left[1 + \frac{2GM}{r' + \vec{n}\cdot\vec{r}'}\right]$$

in the plane of $\vec{r}'$ and $\vec{r}$, and the factor

$$\left[1 + \frac{2GM\vec{n}\cdot\vec{r}'}{r'(r' + \vec{n}\cdot\vec{r}')}\right]$$

perpendicular to the plane of $\vec{r}'$ and $\vec{r}$. This gives a solid angle enhancement of $(1 + 2GM/r')$ as in (4.8). Fourth, the total power radiated into this lobe will be the product of the energy flux and solid-angle size, or a factor of $1 - 2GM/r'$ smaller than emitted, again in agreement with red-shift calculations.

B. Large $\omega r'$, Nonrelativistic Motion (Scattered Terms)

We now examine the scattered signal, given by (2.18), in the large-$\omega r'$, small-velocity limit. Under the assumption that the acceleration terms dominate, we find radiation fields which are of the same order as the gravitational corrections to the dominant terms of (4.3). However, these will generally be incoherent, as the pathlengths differ by many wavelengths. Thus we can, with suitable averaging, compute the flux of radiation from the scattered terms independent of the direct flux (4.5). This yields

$$\frac{dP^{(s)}}{d\Omega} = \frac{G^2M^2q^2}{4\pi r'^2}\left(\frac{r' - \vec{n}\cdot\vec{r}'}{r' + \vec{n}\cdot\vec{r}'}\right)^2 \left[\dot{V}^2 - \left(\frac{\vec{r}\cdot\dot{\vec{V}}}{r'}\right)^2\right]. \tag{4.10}$$

We note that the power radiated in (4.10) is, for most angles, of second order in the gravitational potential, and thus gives an energy flux which is negligible compared to the first-order corrections in (4.7). One may of viewing (4.10) is to write $dP^{(s)}/d\Omega$ as the product of

$$\left[\frac{GM}{r'}\left(\frac{r' - \vec{n}\cdot\vec{r}'}{r' + \vec{n}\cdot\vec{r}'}\right)\right]^2$$

and $dP/d\Omega|_M$, where $dP/d\Omega|_M$ is the power radiated per unit solid angle at the position of the mass M, as evaluated from (4.7). Although small for most angles, the expression (4.10) is not small when $\vec{n}\cdot\vec{r}'$ is too close to $-r'$. In particular, when the angle between $\vec{n}$ and $-\vec{r}'$ is of order $(GM/r')^{1/2}$, the contribution of (4.10) becomes as large as that of (4.7). Note that this angle is the typical angle at which one would expect significant radiation from null geodesics which pass M on the side opposite the path of the direct null geodesic.

The scattered contribution gives us this radiation, but with a different physical interpretation. It might be thought that (4.10) gives an infinite flux at $\vec{n}\cdot\vec{r}' = -r'$; however, it should be remembered that the potentials A_i and $A_i^{(s)}$ are incoherent only so long as $\omega(r'+\vec{n}\cdot\vec{r}') \gg 1$. When the angle between $\vec{n}$ and $-\vec{r}'$ is of order $1/\sqrt{\omega r'}$ they become coherent, and the expression (4.10) no longer applies.

One other consideration that should be made in the interpretation of (4.7) and (4.10) is that of the radius of the mass M. Clearly this should be appreciably larger than the Schwarzschild radius in order that the weak-field approximation be valid. On the other hand if the radius is sufficiently large, the null geodesics from $\vec{r}'$ to $\vec{r}$ will pass through the mass M, and the formulas derived for a $1/r$ potential will no longer be valid. This is not a serious matter, however, because of the fact that the gravitational potential is linear to this order and obeys the principle of superposition. Therefore we can break up the mass M into a number of mass elements, calculate the gravitational correction to the electromagnetic field from each element, and sum over all such contributions. In general there will be one direct contribution from the charge to the observer, deflected by an angle that depends on all the contributions of the mass elements, plus a sum of scattered terms from each mass element. If $\omega R \gg 1$, where R is the radius of M, then different paths of the scattered signals will have different optical lengths, and one will have destructive interference between paths that differ by a half wavelength. In effect this implies that the power radiated by the scattered contributions will be reduced to $(\omega R)^{-3}$ of that given by (4.10) because of random interference effects.

C. Relativistic Motion, Acceleration-Dominated Fields

We next consider the case of acceleration-dominated fields in the extreme relativistic limit $(v \approx c)$. It is convenient to define

$$\beta^i = v^i(1-2\phi)$$

so that $\beta \to 1$ in the extreme relativistic limit. Then in terms of $\vec{n}'$, defined by (4.6), the parameter s in (4.2) becomes, to order ϕ,

$$s = 1 - \vec{n}'\cdot\vec{\beta}\,.$$

The direct contribution (4.3) can then be evaluated and the power can be calculated using (2.19), assuming the acceleration terms dominate the velocity terms in (4.4). This yields

$$P' = \frac{q^2}{4\pi}(1+2\phi)\int d\Omega\left[\dot{\beta}^2 + \frac{2\dot{\vec{\beta}}\cdot\vec{\beta}\,\vec{n}'\cdot\dot{\vec{\beta}}}{s} - \frac{(1-\beta^2)(\vec{n}'\cdot\dot{\vec{\beta}})^2}{s^2}\right]. \quad (4.11)$$

Because of the directional dependence of ψ on $\vec{n}$, $\vec{n}'\cdot\vec{\beta}$ will in general be a complicated function of the angles involved. However, we can use the solid-angle transformation (4.9) to convert (4.14) into an integration over $d\Omega'$, which allows us to carry out the integration explicitly. This results in the power radiated

$$P' = \tfrac{2}{3}q^2\left(1-\frac{2GM}{r'}\right)\left[\frac{\dot{V}_\parallel^{\,2}}{(1-\beta^2)^3} + \frac{\dot{V}_\perp^{\,2}}{(1-\beta^2)^2}\right], \quad (4.12)$$

where $\dot{V}_\parallel$ and $\dot{V}_\perp$ are the components of $\dot{\vec{V}}$ parallel and perpendicular to the velocity $\vec{\beta}$, and $\dot{\vec{V}} = \dot{\vec{v}}(1-3\phi) = \dot{\vec{\beta}}(1-\phi)$ is the physical acceleration as defined in (3.6). We see that (4.12) represents the power radiated by a relativistic charge in flat space decreased by two red-shift factors, as would be expected on a photon picture.

In the extreme relativistic limit the radiation, from (4.12), is found to be peaked about the forward direction ($\vec{n}' \parallel \vec{\beta}$), which at r is peaked about the direction of the deflected $\vec{\beta}$. The size in solid angle of the radiation cone, which is the standard special-relativistic value near r', is increased by the factor $(1+2GM/r')$ at r, with the same anisotropic shape change that was found for the high multipole radiation pattern in the nonrelativistic limit. If the forward cone of the radiation does not include the mass M, then the scattered contributions, analogous to (4.10), will be significantly smaller than (4.12), both because of the dependence on $(GM/r')^2$ as well as because of the fact that the scattered term δ function does not exhibit the peaking and enhancement found for the direct terms. Hence, the scattered power will be orders of magnitude smaller in $(1-\beta^2)$ than (4.12) in general.

D. Relativistic, "Uniform" Motion

For the case of relativistic uniform motion only the first term on the right-hand side of (4.4) contributes to the radiation. As in Sec. III, uniform motion is taken to be uniform with respect to the coordinate system defined by (2.3). Only the direct part of the radiation field (4.3) is significant, the scattered terms being orders of magnitude in $(1-\beta^2)$ smaller. The radiation field in (4.3) is seen to depend on the effective acceleration $\dot{u}_\mu$, which from (4.4) depends only on the time derivative of the parallel propagator (2.15). For the uniform-motion case, the effective acceleration is found to have the components

$$\dot{u}_0 = \dot{\phi} + \dot{\psi}_{,k} v^k , \qquad (4.13a)$$

$$\dot{u}_i = \dot{\psi}_{,i}(1 + \vec{n}\cdot\vec{v}) + n^i \dot{\psi}_{,k} v^k + v^i \dot{\phi} , \qquad (4.13b)$$

with $\dot{s}$ in (4.3) given by

$$\dot{s} = \dot{u}_0 + n^i \dot{u}_i .$$

In order to interpret (4.13) it is useful to write $\psi_{,i}$ in the form

$$\psi_{,i} = \phi n^i + \tfrac{1}{2}\theta_i , \qquad (4.14)$$

where ϕ is the gravitational potential and θ_i is the angular deflection (perpendicular to n^i) of the null geodesic from $\vec{r}'$ to $\vec{r}$. We see that the effective acceleration in (4.13) arises from two effects: the changing potential $\dot{\phi}$, which contributes only when the charge is near M, and the changing deflection angle $\dot{\theta}_i$, which can be appreciable even when the charge is far from M, as viewed along a null geodesic which passes close to M. For the uniform-motion case the $\dot{\phi}$ terms give rise to radiation fields which are orders of magnitude larger in γ than the terms containing $\dot{\theta}_i$. Thus we expect most of the radiation to be emitted when the charge, at the retarded time, is near the mass M. Specifically for r' comparable with b, where b is the impact parameter of the uniform motion path, the radiation field becomes

$$A_{i,0} = \frac{2qv^i \vec{n}\cdot\vec{v}\dot{\phi}}{rs^3} , \qquad (4.15)$$

which reproduces, to the dominant order in γ, (3.17). Similar results are obtained if the second path of uniform acceleration is used to compute the radiation. In either case, however, one receives some radiation, through the $\dot{\theta}_i$ terms, even when the charge is far removed from the mass M and moving uniformly in an unambiguous manner. Although negligible in terms of the total power radiated in the uniform-motion case, those terms will play a dominant role in the case of a path of small-deflection free fall in the gravitational field of M.

E. Relativistic, Small-Deflection Free-Fall Motion

We now assume that the charge is moving with an extreme relativistic velocity on a path with an impact parameter b which is large enough so the deflection angle is small. This will be the case if the potential ϕ along the path is small. Now both terms on the right-hand side of (4.4) are important in computing the effective acceleration. The components of the acceleration of free fall are known from (3.19), and the effective acceleration $\dot{u}_\mu$ can be calculated. The result is the same as that of uniform motion for $\dot{u}_0$, i.e. (4.13a), but the spatial components $\dot{u}_i$ are now given by

$$\dot{u}_i = -\dot{\psi}_{,i}(1 + \vec{n}\cdot\vec{v}) - 3\dot{\phi}v^i + n^i \dot{\psi}_{,k} v^k + \phi_{,i}(1 + v^2) . \qquad (4.16)$$

Substituting (4.16) in (4.3) and taking the dominant terms in the extreme relativistic limit then yields a field which contributes to the radiation in order $(1-\beta^2)$ smaller than (4.15), in agreement with the discussion leading to (3.21). Thus in the relativistic limit the apparent acceleration for the free-fall case is significantly smaller than the apparent acceleration for the uniform-motion case, at least in the region r' near b.

We now examine the power radiated when $r' \gg b$ (and $\vec{n}\cdot\vec{r}' \approx -r'$) so that $\dot{\phi} \approx 0$. This corresponds to the charge being far behind M, as seen at r. For this limit we do not need to distinguish between constrained uniform motion and free-fall motion, since these are the same, i.e., unambiguous uniform motion. Using either (4.13) or (4.16) in (4.3) and keeping only the $\dot{\theta}_i$ terms yields the power radiated per unit solid angle

$$\frac{dP'}{d\Omega} = \frac{q^2}{16\pi s^5}\left[s^2\dot{\theta}^2(1 + \vec{n}\cdot\vec{v})^2 - 4(1 - v^2)(\dot{\theta}_i v^i)^2\right] . \qquad (4.17)$$

In general $\dot{\theta}_i$ varies greatly over the forward radiation cone. If $r' \ll b^2/GM$, $\dot{\theta}_i$ reduces to the relatively simple form

$$\dot{\theta}_i = -\frac{4GM}{b'^2} v_{\perp i} , \qquad (4.18)$$

where b' is the impact parameter of the null geodesic from the source to the observer and $v_{\perp i}$ is the component of velocity perpendicular to $\vec{n}$. Now if $r' \ll b^2/GM$, then b' does not vary significantly over the forward radiation cone, and we can integrate (4.17) over angles, giving

$$P' = \frac{16}{3}\,\frac{G^2M^2q^2}{b^4(1-\beta^2)} , \qquad (4.19)$$

in agreement with the order-of-magnitude estimate made in Sec. III. Note that (4.19) is independent of r', i.e., the power radiated is constant over the allowed range of r', $b \ll r' \ll b^2/GM$. However, there is one other restriction on the applicability of (4.19). If the forward radiation cone includes the mass M, i.e., if $(1-\beta^2)^{1/2} \geq b/r'$, then one must include the scattered contributions as well, which will tend to cancel the direct contributions. Thus (4.19) is valid only for r' smaller than $\sim b(1-\beta^2)^{-1/2}$. In Sec. III we had made a further assumption that $GM/b(1-v^2)^{1/2} \ll 1$, so that (4.19) would in that case be valid for r' out to $\sim b(1-v^2)^{-1/2}$, giving a total energy radiated in agreement with (3.23). In the formulation we have developed in this section, this restriction may be dropped, as the same assumption was not made in deriving (4.19).

We next ask what happens if $r' \gtrsim b^2/GM$. First, the expression for $\dot{\theta}_i$ is changed from (4.18) to

$$\dot{\theta}_i \cong -v_{\perp i}/r'$$

for $r' \gg b^2/GM$, so that the radiation will be cut off at that typical distance. Second, the null geodesics subtend an angle $\lesssim (GM/r')^{1/2}$ to the central mass, which implies that the impact parameter is close to the Schwarzschild radius or the radius of M, which will bring in either higher-order geometrical effects or potential corrections that effectively cut off the radiation. Third, the scattered contribution, or the contribution from multiple null geodesics, becomes important at this angle. What this implies is that if $(1-\beta^2)^{1/2} \ll GM/b$, then the power radiated (4.19) will continue to be emitted out to a distance $r' \sim b^2/GM$, giving a total energy radiated of order

$$\Delta E \sim \frac{GMq^2}{b^2(1-\beta^2)} . \tag{4.20}$$

The striking feature of the expression (4.20) is the linear dependence of ΔE on the mass M, whereas up to this point the dependence has been quadratic. Note that the radiation (4.20) comes entirely from the region in which the charge is moving uniformly. Thus the free-fall radiation would be expected to be of order (4.20) if $(1-\beta^2)^{1/2} \ll GM/b$, since (4.19) is of the same order of magnitude as the power expected near M, the enhancement coming from the fact that the radiation is emitted over a much longer time interval.

V. DISCUSSION OF THE RESULTS

We have seen that many of the traditional views of classical electromagnetism must be modified in curved space. Specifically, for a "point" mass as a source of the gravitational field, electromagnetic signals, to first order in ϕ, can scatter geometrically off the mass even though the mass is electromagnetically inert. This gives rise to interference effects that are not found in flat space. For a complicated system, which is the superposition of many mass elements, one receives signals to first order in the Riemann tensor both along the principal null geodesic (deflected by each mass element) as well as scattered signals due to each mass element. For this situation one has a double kind of superposition – the electromagnetic fields for a complicated system of charges can be obtained by superimposing the fields due to each charge in the system, and the gravitational effect on the field of each charge can be obtained by superimposing the effects due to each point mass in the system. Thus the most general first-order problem is solved by assuming a single arbitrarily moving charge in the gravitational field of a "point" mass, where "point" mass implies the body is spherically symmetric with radius small, but still not so small as to be close to the Schwarzschild radius, where the strong field effects would become important.

The solution of the electromagnetic field equations to second or higher order in the Riemann tensor are, in principle, obtainable by iteration using the lowest-order Green's function. For example, if we were solving a problem in which there were sufficient deviations from a $1/r$ potential due to strong field effects to make a second-order solution desirable, we would need to write the electromagnetic field equations to second order in the gravitational potential. The second-order terms would then be considered as a perturbation in the source, and that part of the solution arising from these terms could be found using the first-order solution as a Green's function. Such an iteration procedure implies that to second or higher order, the electromagnetic signals will be multiply scattered off of the Riemann tensor. Further, the remarkable feature of the first-order radiation fields that signals are sharply propagated only along two different null paths will not be found in the second- or higher-order solutions. This arises because of the multiple scattering as well as because the effective source, from the second- or higher-order terms, is smeared throughout the region in which the potential deviates from $1/r$, and cannot be considered a point source.

Throughout our calculations there was a natural breakup into three cases of particular interest: dominance of nongravitational acceleration, dominance of gravitational free-fall acceleration, and constrained uniform motion, by whatever definition one wishes to impose concerning uniform motion. For the case of dominance of nongravitational acceleration, there were three limits: small-$\omega r'$, large-$\omega r'$ (nonrelativistic), and large-$\omega r'$ (relativistic). In the small-$\omega r'$ limit we found that the two signals, direct and scattered, coalesced to form an effective single signal. This gave rise to a dependence of the total energy radiated on the orientation of the accelerating charge and the central mass. This result, averaged over angles, gave a smaller power radiated than would have been predicted by red-shift arguments. The large-$\omega r'$ limit, in both the nonrelativistic and relativistic limits, allowed comparison of the wave results with expected results from ray optics and the photon picture of light. The deflection of electromagnetic waves was explicitly seen in both the nonrelativistic and relativistic cases. The red shift of signals was implicitly contained in the results as a relation between proper time intervals at the source and observer point. However, the distortion of the signals gave a lower energy flux generally than would be attributed to a simple photon model. The wave picture also produced an enhancement (or bright spot) within a small angular cone on the opposite side of the mass from the charge, which had been found using the ray-optics picture. The scattering off the central mass would not, of course, have been anticipated in either the photon or ray-optics picture. The time delay of signals in gravitational fields was also seen in the form of the direct retarded time in the large-$\omega r'$ limit, in agreement with the photon derivation of the same effect.

The case of constrained uniform motion was treated in both the nonrelativistic and relativistic limits, or, equivalently, in the small-$\omega r'$ and large-$\omega r'$ limits. At first uniform motion was taken to be uniform coordinate velocity, which gave rise to nonvanishing electromagnetic radiation in the nonrelativistic limit. A consideration of other paths which deviated only slightly (to order ϕ) from this path showed that there was a unique path which gave no electromagnetic radiation to the order in v/c that was considered. However, in the extreme relativistic limit, we showed that large amounts of electromagnetic radiation would be found from either of these two paths, and in fact from any path that is, in effect, one of almost uniform motion. This result is, of course, in contrast to what one would expect from a consideration of standard flat-space electromagnetism. However, it has been argued that one might well expect that there should be no "radiationless" trajectory for near uniform motion if the motion is through a region of nonvanishing gravitational fields. Consider the case of charge which "falls" into a Schwarzschild black hole in a yet unspecified trajectory. Israel[18] has shown that the resultant black hole must asymptotically approach the Reissner-Nordström solution. This implies that the electric dipole moment (with respect to the center of the black hole) must be radiated away, no matter how the charge falls into the black hole. Looking at the problem far away from the charge, we would necessarily see the fields (at infinity) change in a nontrivial (i.e., physical, not coordinate) way if the charge moves in any manner with respect to the gravitating body. This follows from the fact that the Newman-Penrose constants[19] change with a change in the static dipole moment. This would suggest that radiationless motion in a gravitational field is impossible.

The case of free fall in the gravitational field of the mass M was examined in both the nonrelativistic (small-$\omega r'$) limit and the relativistic (large-$\omega r'$) limit. In the nonrelativistic limit the geometrical effects were seen to produce v^2/c^2 corrections to the dominant radiation, which were of the same order as relativistic corrections but were of somewhat different form. This suggested that the geometrical effects would be most important in the relativistic limit. In the relativistic limit, free-fall acceleration implied an extreme hyperbolic trajectory with large impact parameter, since the potential ϕ was assumed to be small. One of the striking features about the relativistic free-fall case was the fact that the energy radiated was significantly smaller than that radiated in the uniform-motion case, for the same initial velocity. Another feature was the fact that most of the energy is radiated during the time at which the charge, at the retarded time, is far removed from the mass M, and thus not directly interacting with its gravitational field. This effect was seen to be due to the apparent acceleration at the observation point being nonzero, as a result of the time rate of change of the null geodesics connecting the charge and the observer's point. The requirement of relativistic motion was necessary to obtain this result, so that only the direct signal was important. In the nonrelativistic limit the scattered and direct signals were seen to cancel for the charge being far from M. The power radiated, when the charge is far from M, is the same for the free-fall relativistic case as for the uniform-motion relativistic case. In both cases the charge at the retarded time is undergoing unambiguous uniform motion. The reason that the uniform-motion case gives more total energy radiated in one transit of the trajectory is because there is a much larger power radiated when the charge is near the mass M, and that power dominates the total energy radiated.

Although the calculations described here are specifically for the case of electromagnetic radiation, there are some analogies to previous calculations of the generation of gravitational radiation. Consider the situation of a small mass m moving in a relativistic trajectory past M with large impact parameter. For the case of gravitational radiation we cannot postulate uniform motion without also considering the role of the stresses that constrain the particle to move uniformly. Thus we assume that the trajectory is one of free fall, which also implies that the net deflection past M is small. The gravitational radiation expected from such a system has been calculated[7] using a first-order solution to the perturbed gravitational field equations.[14] This solution is quite similar to the coordinate-dependent solution described in this paper. In particular, gravitational signals are propagated, to first order in ϕ, along the same two null paths found here. In the extreme relativistic limit only the contribution propagated directly from the source to the observer is important, and that gives rise to a gravitational power radiated which is significant even when the mass m is far behind M as seen by the observer.[7] In fact, although the power radiated is of the same order as that emitted when the mass m is near M, the fact that the mass m spends more time away from M than near it [out to a distance of order $b/(1-v^2)^{1/2}$] implies that most of the energy is radiated when the mass m is far from M and essentially moving uniformly.

It may appear that this result is in contradiction with other calculations of gravitational radiation, specifically for particles which fall radially into a Schwarzschild black hole,[20] in which case most of the energy is radiated when the mass m is in the strong field region of M, and thus near M. Our calculation differs from the radial free-fall calculation in two respects. First, the statement that most of the energy is radiated when the mass m is far from M is valid only in the extreme relativistic limit. In the nonrelativistic limit most of the energy is in fact radiated when the mass m is near M. The calculations for radially falling particle assumed a nonrelativistic initial velocity, and those results are thus consistent with our results. Second, it is important that the impact parameter be large in order to derive our results, since we have assumed a small potential over the trajectory. If the impact parameter were small (or zero as in the case of radial motion) then throughout the motion the central mass would be in the forward beam of the particle, and both retarded times would contribute. As in the nonrelativistic case, this leads to a cancellation as a result of the fact that the geometrical contribution to the potentials depends on differences in the same quantity evaluated at two different retarded times. If the forward cone is not beamed at M, which implies both relativistic motion and large impact parameter, then only one retarded-time contribution is important and our results follow. Thus our conclusion is that for free-fall motion of a charge q or a mass m in the gravitational field of M, most of the energy radiated, whether electromagnetic or gravitational, is received by the observer at a time such that the charge or mass, at the retarded time, is far from M, provided that the motion is one of extreme relativistic motion and that the impact parameter is large.

The results obtained here have a wide range of applicability in that most regions of space are in fact suitably weak field regions (perhaps even too weak). Near a condensed body, such as a neutron star, the potential is $\sim\frac{1}{10}$, and dominant effects should be obtainable via this formalism. However, there are two points that should be repeated here. First, it may not be suitable to consider an extended body as a single point mass – one may have to imagine it broken up into its various mass elements, and then superimpose the geometrical effects from each element. Second, physical bodies are not generally electromagnetically inert, but are composed of charged particles. Thus a polarization may result from the influence of an external charge. Since the technique described here uses a Green's function, one can find the polarization contribution to the radiation by integrating over the induced charge and current density in the body and superimposing that on the geometrical solution for the external charge. Some of the results obtained here, e.g., radiation from uniformly moving charges, are of more academic interest, however, since the charge must be constrained by a force which opposes the gravitational force. The importance of such examples is that they illustrate the manner in which the geometry of space-time influences the generation of electromagnetic radiation. As we have seen, if the gravitational field is expected to play some role in a process of generation of electromagnetic waves, then we cannot anticipate its effect based on flat-space electromagnetism. If the gravitational field is sufficiently weak, we can determine its effects by the method given here.

APPENDIX A: DERIVATION OF THE COORDINATE-DEPENDENT SOLUTION

Our starting point in the derivation is the electromagnetic field equations written in coordinate-dependent form, (2.6) and (2.7). In order to apply the Green's-function technique previously developed[7,14] to this problem, it is convenient to rewrite (2.6) and (2.7) in a form in which there are no spatial derivatives acting on A_μ in the mixed ϕA terms. This can be accomplished if we choose the gauge

$$f=4\phi A_{k,k}+2\phi_{,k}A_k\,, \qquad \text{(A1)}$$

where f is defined below (2.7). Note that this gauge is the Lorentz gauge to zeroth order in ϕ, but is neither the flat-space Lorentz gauge nor the curved-space Lorentz gauge to first order in ϕ. With this choice of gauge, and some rearrangement, (2.6) and (2.7) become, to order ϕ,

$$\Box(1-\phi)A_0 = 4\pi J^0(1-\phi) + 4\phi A_{0,00} + 4\phi_{,k}A_{k,0} + \phi_{,kk}A_0\,, \tag{A2}$$

$$\Box(1+\phi)A_i = -4\pi J^i(1-3\phi) + 4\phi A_{i,00} + 4\phi_{,i}A_{0,0} + 2\phi_{,ki}A_k - \phi_{,kk}A_i\,. \tag{A3}$$

The solutions to Eqs. (A2) and (A3) are most easily obtained if we first Fourier-decompose A_μ into its frequency components, i.e.,

$$A_\mu(\vec{r},t) = \int A_\mu(\vec{r},\omega)e^{-i\omega t}d\omega\,, \tag{A4}$$

with a similar decomposition for $J_\mu(\vec{r},t)$. With this substitution the field equations assume the same form as (A2) and (A3), except that time derivatives are replaced by $-i\omega$, and $\Box$ is replaced by $-(\omega^2+\nabla^2)$. In the ϕA terms on the right-hand sides of (A2) and (A3) we can use the zeroth-order solution for A_μ

$$A_0(\vec{r},\omega) = \int J^0(\vec{r}\,',\omega)\frac{e^{i\omega R}}{R}\,d\vec{r}\,'\,, \tag{A5}$$

$$A_i(\vec{r},\omega) = -\int J^i(\vec{r}\,',\omega)\frac{e^{i\omega R}}{R}\,d\vec{r}\,'\,, \tag{A6}$$

since errors made by this substitution will be of order ϕ^2. R is the distance $|\vec{r}-\vec{r}\,'|$. For a point charge q in arbitrary motion,

$$J^\mu(\vec{r},\omega) = \frac{1}{2\pi}\int e^{i\omega t'}J^\mu(\vec{r},t')dt'\,, \tag{A7}$$

where to order ϕ

$$J^\mu(\vec{r},t') = q\int dt' v^\mu(t')[1+2\phi(\vec{z}(t'))]\delta^3(\vec{r}-\vec{z}(t'))\,, \tag{A8}$$

where $v^\mu = dz^\mu/dt' = (c,\vec{v})$, $z^0 = t'$, $z^\mu(t')$ is the equation of the path of the charge in space-time.

The solution of the pair of equations (A2) and (A3) is then obtained using the function $G(\vec{r},\vec{r}\,',\omega)$, defined previously,[7,14] which satisfies

$$-(\nabla^2+\omega^2)G(\vec{r},\vec{r}\,',\omega) = \phi(r)\frac{e^{i\omega R}}{R}\,. \tag{A9}$$

This results in the general first-order expressions for $A_\mu(\vec{r},t)$

$$\begin{aligned}A_0(\vec{r},t) = {} & q(1+\phi)\int dt'\,\frac{\delta(t'-t+R(t'))}{R(t')}\,[1+\phi(t')] \\ & + \frac{q}{2\pi}\int d\omega e^{-i\omega t}\int dt'\int d\vec{r}\,'e^{i\omega t'}\delta^3(\vec{r}\,'-\vec{z}(t'))[-4\omega^2+4i\omega\tilde{\nabla}_k v^k(t')+\tilde{\nabla}^2]G(\vec{r},\vec{r}\,',\omega)\,,\end{aligned} \tag{A10}$$

$$\begin{aligned}A_i(\vec{r}\,'t) = {} & -q(1-\phi)\int dt' v^i(t')\,\frac{\delta(t'-t+R(t'))[1-\phi(t')]}{R(t')} \\ & + \frac{q}{2\pi}\int d\omega e^{-i\omega t}\int dt'\int d\vec{r}\,'e^{i\omega t'}\delta^3(\vec{r}\,'-\vec{z}(t'))[4\omega^2 v^i(t') - 4i\omega\tilde{\nabla}_i - 2v^k(t')\tilde{\nabla}_k\tilde{\nabla}_i + v^i(t')\tilde{\nabla}^2]G(\vec{r},\vec{r}\,',\omega)\,,\end{aligned} \tag{A11}$$

with

$$\tilde{\nabla}_k = \frac{\partial}{\partial x^k} + \frac{\partial}{\partial x^{k'}}\,,\quad R(t') = |\vec{r}-\vec{z}(t')|,\ \phi(t') = \phi(z(t')),\quad v^i(t') = \frac{dz^i}{dt'}\,.$$

We will be interested in the time derivatives of the potentials for $r \gg r'$, as these will be the ones which enter into a discussion of electromagnetic radiation. For large r, the time derivatives of (A10) and (A11) become, explicitly,

$$A_{0,0}(\vec{r},t)=\frac{q}{r}\frac{\partial}{\partial t}\int dt'\delta(t'-t+r-\vec{n}\cdot\vec{z}(t'))[1+\phi(t')]$$

$$+\frac{2GMq}{r}\int dt'\int d\vec{r}'\delta^3(\vec{r}'-\vec{z}(t'))\left[\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial t}\tilde{\nabla}_k v^k(t')+\tfrac{1}{4}\tilde{\nabla}^2\right]$$

$$\times\left[\delta(t'-t+r-\vec{n}\cdot\vec{r}')\ln((r'+\vec{n}\cdot\vec{r}')/2r)+\int_{r'+\vec{n}\cdot\vec{r}'}^{2r}\delta(t'-t+r-\vec{n}\cdot\vec{r}'+u)\frac{du}{u}\right],\quad \text{(A12)}$$

$$A_{i,0}(\vec{r},t)=-\frac{q\partial}{r\partial t}\int dt' v^i(t')\delta(t'-t+r-\vec{n}\cdot\vec{z}(t'))[1-\phi(t')]$$

$$+\frac{2GMq}{r}\int dt'\int d\vec{r}'\delta^3(\vec{r}'-\vec{z}(t'))\left[-v^i(t')\frac{\partial^2}{\partial t^2}+\frac{\partial}{\partial t}\tilde{\nabla}_i-\tfrac{1}{2}v^k(t')\tilde{\nabla}_k\tilde{\nabla}_i+\tfrac{1}{4}v^i(t')\tilde{\nabla}^2\right]$$

$$\times\left[\delta(t'-t+r-\vec{n}\cdot\vec{r}')\ln((r'+\vec{n}\cdot\vec{r}')/2r)+\int_{r'+\vec{n}\cdot\vec{r}'}^{2r}\delta(t'-t+r-\vec{n}\cdot\vec{r}'+u)\frac{du}{u}\right].\quad \text{(A13)}$$

Here $\vec{n}$ is the radial unit vector $\vec{r}/r$.

For applications to radiation problems we need only be concerned with the time derivative of the spatial potential, (A13). In further reducing (A13) we must also recognize that the Green's-function approach used here involves an expansion in powers of $|\omega r'\phi|$, where we have kept terms linear in ϕ. Higher-order terms in ϕ will then be negligible only if $|\omega r'\phi|\ll 1$. In Appendix B we derive an alternate expression for the radiation fields, in semicovariant form, that will allow us to consider the large-$\omega r'$ limit with no such restriction on $\omega r'\phi$. One implication of small $|\omega r'\phi|$ in (A13) is that the velocity in the second, long term on the right-hand side of (A13) can be assumed to be a constant; i.e., terms in the reduced form of (A13) which involve acceleration are neglected. The reduction of (A13) to a form suitable for numerical or analytic calculations then parallels the reduction that was outlined for the analogous case of gravitational potentials.[7]

In the first term on the right-hand side of (A13) we convert the derivative with respect to t, which acts only on the δ function, to a derivative with respect to t' acting only on the δ function:

$$\frac{\partial}{\partial t}\delta(t'-t+r-\vec{n}\cdot\vec{z}(t'))$$

$$=-\frac{1}{1-\vec{n}\cdot\vec{v}(t')}\frac{\partial}{\partial t'}\delta(t'-t+r-\vec{n}\cdot\vec{z}(t')).\quad \text{(A14)}$$

We then integrate by parts with respect to t', leaving the δ function undifferentiated. This allows the integral over t' to be performed, giving for a general integrand $f(t')$

$$\int dt' f(t')\delta(t'-t+r-\vec{n}\cdot\vec{z}(t'))=\left(\frac{f(t')}{1-\vec{n}\cdot\vec{v}(t')}\right)_{t'=t_1'},\quad \text{(A15)}$$

where t_1' is the first retarded time in (2.8), with $\vec{r}'=\vec{z}(t')$. In the second term on the right-hand side of (A13) we first reduce the expressions by performing the differentiations indicated by the $\tilde{\nabla}_i$ operators. Simplification arises because any terms proportional to n^i can be ignored, as those would not contribute to the power radiated. Also, terms which give rise to fields which fall off faster than $1/r$ are ignored, as those also do not contribute to the power radiated. Because the $\tilde{\nabla}_i$ operators involve symmetrized gradients with respect to x^i and x'^i, they do not affect the δ function with argument $\delta(t'-t+r-\vec{n}\cdot\vec{r}'+u)$, and thus they act only on the limits of integration on the second-retarded-time term, giving rise to terms which no longer involve integration. Thus such terms produce δ functions with argument $\delta(t'-t+r+r')$ coming from differentiating the lower limit on the integral. The derivatives with respect to t are then carried out, giving derivatives of the δ functions with respect to their arguments. The integration over $\vec{r}'$ is then performed, with the result that all positions $\vec{r}'$ are now functions of t'. The derivative of the δ functions with respect to their arguments can then be converted into derivatives with respect to t', and then integrated by parts with respect to t' so that the δ functions are no longer differentiated. In doing this one makes use of the assumption that no acceleration terms are produced. The net result of this reduction is to produce some terms with a δ-function argument as in (A15), for which the integration over t' can then be performed as in (A15). The remainder of the terms have a different argument of the δ function, i.e.,

$\delta(t'-t+r+r')$, where $r'=|\vec{z}(t')|$, for which the integral over time t' gives, for a general integrand $f(t')$,

$$\int dt' f(t')\delta(t'-t+r+r') = \left(\frac{f(t')}{1+\vec{r}\cdot\vec{v}/r'}\right)_{t'=t_2'}, \tag{A16}$$

where t_2' is the second retarded time in (2.8). The radiation fields are thus seen to be sharply propagated with two different retarded times, the first representing direct propagation from r' to r, and the second representing scattering off the mass M. The net result of these operations is to produce the general, rather complicated expression for the radiation fields (2.9).

The expression (2.9) is cast into a form which would be suitable for numerical calculations of the power radiated, but not for analytical results. The primary difficulty is in the fact that the fields depend on properties of the source at two different retarded times, generally giving rise to interference effects. However, in two limits the expression (2.9) simplifies enough so that some analytic expressions can be derived. In particular there are the limits of small $\omega r'$ and of large $\omega r'$. We first examine the small-$\omega r'$ limit, starting with the radiation fields in the form of (A12) and (A13).

In the small-$\omega r'$ limit the potentials (A12) and (A13) greatly simplify, since the two retarded-time δ functions coalesce to form, to lowest order in $\omega r'$, a single retarded-time δ function. Also, the time derivative operators can be ignored compared to the spatial derivative operators $\partial/\partial t \ll \tilde{\nabla}_k$, in the small-$\omega r'$ limit. The $\tilde{\nabla}_k$ acting on the last bracket in (A12) or (A13) gives, discarding all terms which fall off faster than $1/r$,

$$\tilde{\nabla}_k\left[\delta(t'-t+r-\vec{n}\cdot\vec{r}')\ln((r'+\vec{n}\cdot\vec{r}')/2r)+\int_{r'+\vec{n}\cdot\vec{r}'}^{2r}\delta(t'-t+r-\vec{n}\cdot\vec{r}'+u)\frac{du}{u}\right]$$
$$=[\delta(t'-t+r-\vec{n}\cdot\vec{r}')-\delta(t'-t+r+r')]\frac{x'^k/r'+n^k}{r'+\vec{n}\cdot\vec{r}'}. \tag{A17}$$

In the small-$\omega r'$ limit retardation effects across the size of the system are small. In effect this means that each of the δ functions can be expanded about the argument $t'-t+r$, i.e.,

$$\begin{aligned}\delta(t'-t+r-\vec{n}\cdot\vec{r}') &\approx \delta(t'-t+r)-\vec{n}\cdot\vec{r}'\delta'(t'-t+r),\\ \delta(t'-t+r+r') &\approx \delta(t'-t+r)+r'\delta'(t'-t+r),\end{aligned} \tag{A18}$$

where δ' indicates differentiation with respect to the argument. That such an expansion is reasonable is most easily seen if one first Fourier-transforms (A17) with respect to time, and then expands terms in powers of $\omega r'$. Using (A18) in (A17) then results in the reduced expression for (A17)

$$\tilde{\nabla}_k[\ ]\cong\frac{\partial}{\partial t}[\delta(t'-t+r)(x'^k/r'+n^k)]. \tag{A19}$$

The second gradient of the above bracket gives, to lowest order,

$$\tilde{\nabla}_k\tilde{\nabla}_j[\ ]=\frac{\partial}{\partial t}\left[\delta(t'-t+r)\left(\frac{\delta_{kj}}{r'}-\frac{x'^k x'^j}{r'^3}\right)\right]. \tag{A20}$$

Since $\partial/\partial t\cong v^k\tilde{\nabla}_k$ in the nonrelativistic limit, we then evaluate the potential (A13) in the small-$\omega r'$ limit as

$$A_{i,0}=\frac{-q}{r}\frac{\partial}{\partial t}\left[v^i-\frac{GM}{r'}\left(v^i-\frac{x'^i}{r'^2}(\vec{r}'\cdot\vec{v})\right)\right]_{\text{ret}}, \tag{A21}$$

where ret means all terms are evaluated at the time $t'=t-r$. This produces the reduced expression given by (2.10), valid in the limit $\omega r'\ll 1$.

Although (A21) is sufficient to calculate the power radiated, it is somewhat instructive to evaluate $A_{0,0}$ and show that gauge invariance still holds. To evaluate $A_{0,0}$ to the desired order it is necessary, in addition to the above steps, to expand the first δ function in (A12) to first order in r' as in (A18). Grouping terms all of the same order then gives

$$A_{0,0}=\frac{q}{r}\frac{\partial}{\partial t}\left[\vec{n}\cdot\vec{v}-\frac{GM}{r'}(\vec{n}\cdot\vec{v}-\vec{n}\cdot\vec{r}'\,\vec{r}'\cdot\vec{v}/r'^2)\right],$$

so that the gauge condition for large r,

$$A_{0,0}-A_{k,k}=A_{0,0}+n^kA_{k,0}=0,$$

is still satisfied for large r.

In the large-$\omega r'$, small-$|\omega r'\phi|$ limit, applied to extreme relativistic motion $[(1-v^2)^{1/2}\ll 1]$, the solution (2.9) also simplifies. This results from the fact that only the first-retarded-time terms give rise to the enhancement and peaking in the forward direction that is characteristic of the corresponding flat-space electromagnetic result in the relativistic limit. This arises because of the $1-\vec{n}\cdot\vec{v}$ factors in the denominator of the terms in (2.9) which are evaluated at the time t_1'. The corresponding factors in the t_2' terms are $1+\vec{r}'\cdot\vec{v}/r'$, which will in general not give such an enhancement. In fact the t_2' terms will become important only if $1+\vec{r}'\cdot\vec{v}/r'$ becomes small, which implies that the mass M is in the forward beam [of width $(1-v^2)^{1/2}$] of the charge. If that is the case, there will tend to be cancellation between the t_1' terms and t_2' terms, reducing the power radiated below what would be expected on the basis of the t_1' terms alone. As is seen in Sec. III, the t_1' terms fall off at a distance such that the mass M is the forward beam, so the only effect of this cancellation is to cut off the radiation at large distances faster than expected. Thus in the large-$\omega r'$, extreme relativistic, limit we keep only the first-retarded-time terms, and also keep only terms which are important over the forward beam of the charge. Since the fields are to be used in computing the radiated power through (2.21) we can estimate the dependence in $(1-v^2)^{1/2}$ of the various terms in (2.9), considering only the angular dependence of each term over the forward beam. This allows us to group terms according to their importance, in the extreme relativistic limit, so far as a calculation of the power is concerned. To this end we define the parameter ϵ by (2.12), which, in the power radiated, gives rise to terms of order $(1-v^2)^{1/2}$. Similarly $1-\vec{n}\cdot\vec{v}$ gives rise to terms of order $(1-v^2)$.

In computing the leading terms in (2.9) one must also take into account whether the term is proportional to v^i or to x'^i. A term proportional to v^i is, in the forward beam, almost entirely longitudinal. Its transverse components, which are the only ones that contribute to the power radiated, are of order $(1-v^2)^{1/2}$ smaller than its longitudinal component. For a term proportional to x'^i the transverse and longitudinal components are of the same order of magnitude (unless r' is very large). Thus, to group terms to find the dominant contribution, those terms proportional to v^i will have to be $(1-v^2)^{1/2}$ larger than those proportional to x'^i in order to contribute to the power radiated in the same order in $(1-v^2)^{1/2}$.

In reducing (2.9) we have two kind of terms to treat. The F^i terms, which in part depend on the acceleration, must be considered separately from the rest of the terms, since an order-of-magnitude estimate to the power radiated cannot be given until the acceleration is specified. The G^i, H^i, and K^i terms contain no acceleration and can be approximated directly. Applying the arguments given to these terms gives the dominant geometrical contribution to the power radiated indicated in (2.13). Actually in deriving (2.13) we have kept terms of one extra power in $(1-v^2)$ than would appear necessary at first glance. This is done because in the case of the acceleration being that of the gravitational acceleration of the charge towards M, the F^i terms cancel out the dominant contribution of (2.13). Thus in order to find the leading contribution to the total field in the case of free-fall acceleration, it is necessary to keep terms of order $(1-v^2)$ smaller than the dominant terms in (2.13).

APPENDIX B: DERIVATION OF THE SEMICOVARIANT SOLUTION

Our starting point for this derivation is the generally covariant field equations for the potential A_μ, obtained by substitution of (2.2) in (2.1):

$$A_{\mu;\nu}{}^{;\nu} - A_{\nu;\mu}{}^{;\nu} = 4\pi J_\mu \,. \tag{B1}$$

In flat space-time the solution to (B1) can be written in generally covariant form as

$$A_\mu(x) = \int d^4x' \sqrt{-g}\, g_{\mu\alpha'}\delta_R(\Omega) J^{\alpha'}(x') \,, \tag{B2}$$

where $g_{\mu\alpha'}$ is the parallel propagator[15] between the two space-time points x and x', Ω is $\frac{1}{2}$ the square of the proper time along the geodesic between x and x', called the world function by Synge,[15] and δ_R indicates a δ function which gives a contribution only when signals are propagated along the forward null cone centered on x'. The expression (B2) reduces to the standard Liénard-Wiechert potentials in a system of rectangular coordinates. Further, the potential defined in (B2) satisfies the covariant Lorentz gauge $A_\mu{}^{;\mu}=0$ in flat space-time.

In a curved space-time the covariant potential (B2) is not an exact solution of (B1). Further, the potential (B2) is not precisely defined, since there will, in general, be more than one geodesic between the two points x and x'. However, since we will be considering the case of an "almost flat" space-time, i.e., the linearized Schwarzschild geometry, (B2) taken along the principal null geodesic between x and x' should be a suitable first approximation, with corrections which are at least of first order in the Riemann tensor (since there are no corrections in flat space-time). Thus we take (B2) to the zeroth-order solution $A_\mu^{(0)}$ to the field equations (B1), and since we are interested here in the solution of the field equations only to first order in ϕ (and thus first order in the Riemann tensor), we need only find the first-order correction $A_\mu^{(1)}$ such that $A_\mu^{(0)}+A_\mu^{(1)}$ satisfies (B1) to second order in the Riemann tensor. It should be remarked that (B2) is not a unique zeroth-order solution – other trial solutions which differ from (B2) by terms of order the Riemann tensor would be equally valid. However, since we will be considering the solution only to first order in the Riemann tensor, differences will be accounted for by our first-order correction.

If we substitute $A_\mu^{(0)}$, given by (B2), into the curved space-time equation (B1), we find that $A_\mu^{(0)}$ satisfies the differential equation

$$A^{(0)}_{\mu;\nu}{}^{;\nu} - A^{(0)}_{\nu;\mu}{}^{;\nu} = 4\pi J_\mu + \int d^4x'\sqrt{-g}\, J^{\alpha'}(x')\{\delta_R(\Omega)[g_{\mu\alpha';\nu}{}^{;\nu} - R^\sigma_\mu g_{\sigma\alpha'}] + \delta_R'(\Omega)[g_{\mu\alpha'}(\Omega_{;\nu}{}^{;\nu} - 4)] - [\delta_R(\Omega)g_{\nu\alpha'}{}^{;\nu}]_{;\mu}\}\,, \tag{B3}$$

where we have used the facts that[15]

$$\Omega^{;\nu}g_{\mu\alpha';\nu} = 0\,,\quad \Omega_{;\nu}\Omega^{;\nu} = 2\Omega\,,$$

$$\delta_R''(\Omega)\Omega = -2\delta_R'(\Omega)\,,\quad g_{\nu\alpha'}\Omega^{;\nu} = -\Omega_{;\alpha'}\,,$$

together with current conservation $J^{\alpha'}{}_{;\alpha'} = 0$, to reduce the rather complicated expression that results from substitution into (B1). The derivatives of the parallel propagator and of the world function can be in principle evaluated to any order in the Riemann tensor. Synge[15] explicitly gives these derivatives to first order in the Riemann tensor, which are all that will be needed here.

Since the second, long term on the right-hand side of (B3) vanishes in flat space-time, it must be at least of first order in the Riemann tensor. Define the second, long term to be $4\pi J^{(1)}_\mu$, an apparent "induced" current density. Of course, there is no actual current density points where $J^{(1)}_\mu \neq 0$, but for mathematical purpo es we can treat it as a perturbed current density and solve for the perturbed potential $A^{(1)}_\mu$ using the zeroth-order Green's function (B2). Since we want $A^{(1)}_\mu$ to be such that substitution of $A^{(0)}_\mu + A^{(1)}_\mu$ into (B1) cancels out $4\pi J^{(1)}_\mu$, we choose the first-order solution $A^{(1)}_\mu$ to be given by

$$A^{(1)}_\mu(x) = -\int d^4x''\sqrt{-g}\, g_{\mu\alpha''}\delta_R(\Omega)J^{(1)\alpha''}(x'')\,. \tag{B4}$$

Thus $A^{(0)}_\mu + A^{(1)}_\mu$ is a solution of (B1) to first order in the Riemann tensor. We see from (B2) that $A^{(0)}_\mu$ is sharply propagated from the source to the observer along a light cone. On the other hand $A^{(1)}_\mu$ from (B4) and (B3) is propagated from x' to an intermediate point x'' along a light cone, and then propagated from x'' to x along another light cone. The term $A^{(1)}_\mu$ thus represents the scattering off the Riemann tensor (or smearing of the signal inside the light cone centered on x').

At this point one may raise a question about our choice of gauge. The field equations (B1) are unchanged if we replace A_μ by $A_\mu + \chi_{;\mu}$. Usually one uses this degree of freedom to impose the Lorentz gauge $A_\mu{}^{;\mu} = 0$. In our solution the gauge is determined by our choice of the zeroth-order solution. In particular if we evaluate $A^{(0);\mu}_\mu$, we find

$$A^{(0);\mu}_\mu = \int d^4x'\sqrt{-g}\, g_{\mu\alpha'}{}^{;\mu}\delta_R(\Omega)J^{\alpha'}(x')\,, \tag{B5}$$

using current conservation again. The right-hand side of (B5) is of the order of the Riemann tensor, and so we have not used the curved-space Lorentz gauge. Similarly one can compute $A^{(1);\mu}_\mu$. However, since $J^{(1)}_\mu$ from (B3) is also conserved, and also of the order of the Riemann tensor, $A^{(1);\mu}_\mu$ will be of second order in the Riemann tensor. Thus the total field A_μ to first order in the Riemann tensor satisfies the gauge condition (B5).

The contribution of $A^{(0)}_\mu$ to the radiation field is easily expressed in semicovariant form by evaluating $\delta_R(\Omega)$ for the linearized Schwarzschild metric and taking the field coordinate r large. This yields the direct contribution to the radiation field

$$A^{(0)}_{\mu,0} = \frac{q}{r}\frac{\partial}{\partial t}\left(\frac{g_{\mu\alpha'}v^{\alpha'}}{n^\mu g_{\mu\beta'}v^{\beta'}}\right)_{\rm ret} \tag{B6}$$

for a point charge q. This is the same as (2.14), with the same definitions that follow (2.14).

We next focus our attention on the scattered contribution $A^{(1)}_\mu$, with particular attention to the form of the geometrical quantities in (B3) that need to be evaluated to first order in the Riemann tensor. These involve second derivatives of the world function and first and second derivatives of the parallel propagator; they are explicitly evaluated to first order by Synge.[15] Because of the contractions on indices in (B3) these are found to depend only on the contracted Riemann tensor, i.e., the Ricci tensor, which vanishes everywhere except at the position of the mass M. Further simplifications can also be made if we wish to evaluate the scattered contributions, $A^{(1)}_\mu$, only in the large-$\omega r'$ limit.

In the second, long term on the right-hand side of (B3), there are three brackets whose evaluation must be considered. The last one involves a pure gradient with respect to x^μ – when that term is substituted into (B4) it will give rise to fields which are only spacelike and longitudinal. Therefore fields arising from that term will not contribute to the radiation, and that term can therefore be ignored (or removed by a gauge transformation). The middle term, which involves the derivative of the δ function, can be evaluated, following Synge,[1] to first order in the Riemann tensor, using

$$\Omega_{;\nu}{}^{;\nu} - 4 = \frac{1}{u_2 - u_1}\int_{u_1}^{u_2}(u-u_1)^2 R_{\alpha\beta}U^\alpha U^\beta du + O(R^2)\,, \tag{B7}$$

where u is a special parameter on the geodesic from x' to x, with end values u_1 at x' and u_2 at x. U^α is the tangent vector dx^α/du. $R_{\alpha\beta}$ is the Ricci tensor. The geometrical relation (B7) would be the only one needed if we were considering the solution to a scalar wave equation.[14] In fact since (B7) is already of first order in the Riemann tensor, all other geometrical quantities which appear as coefficients in (B3) can be evaluated as in flat space. This means that we can use the solution to the scalar wave equation[14] to write down the radiation fields arising from the middle bracket in the second term on the right-hand side of (B3):

$$A^{(1)}_{\mu,0}=\frac{2GMq}{r}\int dt'\int d\vec{r}'\delta^3(\vec{r}'-\vec{z}(t'))g_{\mu\alpha'}v^{\alpha'} \times\frac{\partial^2}{\partial t^2}\int_{r'+\vec{n}\cdot\vec{r}'}^{2r}\delta(t'-t+r-\vec{n}\cdot\vec{r}'+u)\frac{du}{u}\,. \tag{B8}$$

We now consider the first bracket of the second term on the right-hand side of (B3), which involves the undifferentiated δ function. The R^α_μ term is easy to evaluate since the Ricci tensor vanishes except at the position of the mass M. The more complicated term involves the second derivatives of the parallel propagator. Following Synge[15] we find that

$$g_{\mu\alpha';\nu}{}^{;\nu}=\frac{1}{(u_2-u_1)^2}\times\int_{u_1}^{u_2}(u-u_1)^2[R_{\beta\alpha';\mu}-R_{\beta\mu;\alpha'}]U^\beta du\,, \tag{B9}$$

where components in the integral in (B9) are referred to a standard rectangular coordinate system, the integrand already being of first order in the Riemann tensor. Also, the covariant derivatives may be replaced by ordinary derivatives for the same reason. Since we already know the solution for a term like (B7), we would like to cast (B9) in a similar form, in which there are no derivatives of the Ricci tensor. We can do this if we first define the symmetric gradient as

$$\tilde{\nabla}_k=\frac{\partial}{\partial x^k}+\frac{\partial}{\partial x^{k'}}\,,$$

giving, for example,

$$\int_{u_1}^{u_2}(u-u_1)^2R_{\beta\alpha',k}U^\beta du=-\tilde{\nabla}_k\int_{u_1}^{u_2}(u-u_1)^2R_{\beta\alpha'}U^\beta du\,.$$

Next we make use of the fact that the Ricci tensor has only equal diagonal components $R_{00}=R_{11}=R_{22}=R_{33}$, and depends only on position, not time. Further, we consider the time derivative of the perturbed potential (B4), for large r, which derivative can then be integrated by parts so that the coefficient of (B9) then involves the derivative of a δ function. In this form the solution that was used in deriving (B8) can also be applied to the terms in (B9). The net effect of these operations is to produce a rather complicated final expression. This expression, analogous to the second-retarded-time contribution in (A13), simplifies greatly in the large-$\omega r'$, nonrelativistic limit, where the dominant terms arise from derivatives acting only on the δ functions. This results in the collapse of the contributions from the second term on the right-hand side of (B4) to the simple form

$$A^{(1)}_{i,0}=-\frac{GMq}{r}\left(\frac{\dot{v}^i(r'-\vec{n}\cdot\vec{r}')+x'^i(\vec{n}\cdot\dot{\vec{v}}-\vec{r}'\cdot\dot{\vec{v}}/r')}{r'(r'+\vec{n}\cdot\vec{r}')}\right), \tag{B10}$$

with quantities to be evaluated at the retarded time $t'=t-r-r'$. It should also be noted that (B10) could also be obtained from the second-retarded-time term in (A13) by assuming large $\omega r'$, small v.

APPENDIX C: RELATION BETWEEN SOLUTIONS

We have outlined in Appendix A and Appendix B two methods of obtaining a solution to the electromagnetic field equations to first order in the potential ϕ, or equivalently, to first order in the Riemann tensor. Both solutions exhibit propagation along two null lines – a direct path which is a null geodesic from the source to observer, and a scattered path which is a null geodesic from the source to M and another null geodesic from M to the observer. The two solutions appear quite different in form. For example, the direct term of the semicovariant solution, (B6), bears little resemblance to the first-retarded-time terms of the coordinate-dependent solution (A13) and (A14). In this appendix we show that the direct contribution to (A13) and (A14) follows from (B6) if we make the approximation of small $|\omega r'\phi|$.

Starting from (B6) we explicitly write out the components of $g_{\mu\alpha'}$ from (2.15). For $g_{\mu\alpha'}v^{\alpha'}$ we obtain the components

$$\begin{aligned}g_{0\alpha'}v^{\alpha'}&=(1+\phi')+\dot{\psi}\,,\\ g_{k\alpha'}v^{\alpha'}&=-v^{k'}(1-\phi')+n^k\dot{\psi}-\psi_{,k'}(1+\vec{n}\cdot\vec{v})\,,\end{aligned} \tag{C1}$$

where $\dot{\psi}=\psi_{,k'}v^{k'}$ and $\phi'=\phi(r')$. We next add to (B6) a pure gauge term

$$A^{(g)}_{\mu,0}(\vec{r},t)=\frac{q}{r}\frac{\partial}{\partial t}\int dt'\delta(t'-t+r-\vec{n}\cdot\vec{r}'+2\psi)n_\mu\dot{\psi}, \tag{C2}$$

with $n^\mu=(1,\vec{n})$, in order to bring the choice of gauge in agreement with that made in (A13) and (A14). Substituting (C1) into (B6) and adding (C2) gives the radiation fields, to order ϕ,

$$A_{0,0}(\vec{r},t)=\frac{q}{r}\frac{\partial}{\partial t}\int dt'\delta(t'-t+r-\vec{n}\cdot\vec{r}'+2\psi)(1+\phi'+2\dot{\psi}), \tag{C3}$$

$$A_{i,0}(\vec{r},t)=\frac{-q}{r}\frac{\partial}{\partial t}\int dt'\delta(t'-t+r-\vec{n}\cdot\vec{r}'+2\psi)[v^i(1-\phi')+\psi_{,i}(1+\vec{n}\cdot\vec{v})]. \tag{C4}$$

We note that the argument of the δ functions in (C3) and (C4) involves the correct retarded time, to order ϕ, as given by (2.17). If $|\omega r'\phi|$ is small we can "expand" the δ function to first order in ϕ, giving

$$\delta(t'-t+r-\vec{n}\cdot\vec{r}'+2\psi)\approx\delta(t'-t+r-\vec{n}\cdot\vec{r}')-2\psi\frac{\partial}{\partial t}\delta(t'-t+r-\vec{n}\cdot\vec{r}'). \tag{C5}$$

Such an expansion, of course, is best done in terms of the Fourier transform of the δ functions, but the net effect is as given in (C5). We then substitute (C5) into (C3) and (C4) and keep terms up to first order in ϕ. This gives

$$A_{0,0}(\vec{r},t)=\frac{q}{r}\frac{\partial}{\partial t}\int dt'\delta(t'-t+r-\vec{n}\cdot\vec{r}')(1+\phi'+2\dot{\psi})+\frac{2GMq}{r}\frac{\partial^2}{\partial t^2}\int dt'\delta(t'-t+r-\vec{n}\cdot\vec{r}')\ln(r'+\vec{n}\cdot\vec{r}'), \tag{C6}$$

$$A_{i,0}(r't)=-\frac{q}{r}\frac{\partial}{\partial t}\int dt'\delta(t'-t+r-\vec{n}\cdot\vec{r}')[v^i(1-\phi')+\psi_{,i}(1+\vec{n}\cdot\vec{v})]$$
$$-\frac{2GMq}{r}\frac{\partial^2}{\partial t^2}\int dt'\delta(t'-t+r-\vec{n}\cdot\vec{r}')v^i\ln(r'+\vec{n}\cdot\vec{r}'), \tag{C7}$$

where we have used the definition of ψ, (2.16). Comparing (C6) and (C7) to (A12) and (A13), respectively, shows that we have reproduced the direct geometrical terms on the right-hand sides of (A12) and (A13) which involve second time derivatives.

The next step in the reduction of (C6) and (C7) involves the ψ dependence in the first terms on the right-hand side of (C6) and (C7). Explicitly, in (C6) we write

$$2\dot{\psi}=2\psi_{,k}v^k=-2GMv^k\tilde{\nabla}_k\ln(r'+\vec{n}\cdot\vec{r}), \tag{C8}$$

where $\tilde{\nabla}_k$ is defined below (A11). In (C8) the time dependence is still assumed to be in the $\vec{r}'(t')$ terms. However, we can put the time dependence into $\vec{z}(t')$ as in (A12), and commute the $\tilde{\nabla}_k$ operator with the δ function. This then reproduces the second direct geometrical term on the right-hand side of (A12); i.e., (C6) reproduces the direct terms of (A12) up to the last term, which involves the $\tilde{\nabla}^2$ operator. It can be seen that this operator does not contribute to the direct propagation signal, since $\tilde{\nabla}^2$ gives 0 everywhere except along the line $r'+\vec{n}\cdot\vec{r}'=0$. Such a term would contribute only if the source were directly behind the mass M, as seen by the observer, and in that case the second-retarded-time δ function would have the same argument as the first one and there would be no separation of the direct and scattered signals. Thus we see that (C6) reproduces the direct terms of (A12).

Similarly in (C7) we can write

$$\psi_{,i}(1+\vec{n}\cdot\vec{v})=2\psi_{,i}-\psi_{,i}(1-\vec{n}\cdot\vec{v}). \tag{C9}$$

The first term on the right-hand side of (C9) can be written as

$$2\psi_{,i}=-2GM\tilde{\nabla}_i\ln(r'+\vec{n}\cdot\vec{r}') \tag{C10}$$

and treated as above to reproduce the second operator in the direct geometrical contribution of (A13). The third operator can be reproduced if we combine the derivative in (C7) together with the δ function and the last term in (C9) to give

$$\frac{\partial}{\partial t}\delta(t'-t+r-\vec{n}\cdot\vec{r}')\psi_{,i}(1-\vec{n}\cdot\vec{v})$$
$$=\left[\frac{\partial}{\partial t'}\delta(t'-t+r-\vec{n}\cdot\vec{r}')\right]GM\tilde{\nabla}_i\ln(r'+\vec{n}\cdot\vec{r}'). \tag{C11}$$

Integrating by parts with respect to t' in (C7) then gives $\partial/\partial t'$ acting only on a function of $\vec{r}(t')$, which implies that $\partial/\partial t'$ can be written as $v^k\tilde{\nabla}_k$. As before, this gives rise to the third operator in the direct geometrical contribution of (A13). The same argument applies to the $\tilde{\nabla}^2$ operator in (A13) as applied in (A12). Thus we see that (B6), with a change of gauge and an expansion of the δ function, reproduces the direct contribution to the radiation fields (A12) and (A13).

*Work supported in part by National Science Foundation under Grant No. GP15267.

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