Reggeon-Reggeon-Particle Vertex and Feynman Graphs

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The analytic properties of Feynman graphs are used to suggest the singularity structure of the Reggeon-Reggeon-particle vertex.

The proof by Brower and Weis[1] that the vanishing[2] of the Pomeranchukon-Reggeon-particle vertex at $t_P=0$ and arbitrary $t_R<0$ implies the asymptotic vanishing of the total cross section dealt a severe blow to the hope that the dominant Regge singularity might be a factorizable pole with intercept at 1.

Fundamental to the result of Brower and Weis is the expression they take for the Pomeranchukon-Reggeon-particle vertex. With the invariants defined as in Fig. 1, their expression for the vertex is

$$R=(\tau_P+e^{-i\pi\alpha_P})(\tau_R+e^{-i\pi\alpha_R})V(t_P,t_R,K+i\epsilon)+\tau_P\tau_R V(t_P,t_R,K-i\epsilon)\ , \qquad (1)$$

where $K=s_{12}s_{23}/s$. Enforcing the Steinmann relation allows them to write

$$V=(-K)^{-\alpha_P}V_P(t_P,t_R,K)+(-K)^{-\alpha_R}V_R(t_P,t_R,K)\ , \qquad (2)$$

with V_i free of cuts in K. Finally, requiring the poles at $\alpha_P=J_P$ or $\alpha_R=J_R$ to have residues which are polynomials in overlapping invariants of degree J_P or J_R, respectively, leads Brower and Weis to the expression

$$V_P(t_P,t_R,K)=\sum_{i=0}^{\infty}\Gamma(-\alpha_P+i)\Gamma(\alpha_P-\alpha_R-i)\times v(\alpha_P-i,t_P,t_R)(K^i/i!) \qquad (3)$$

and similarly for V_R.

A crucial feature of the expression (3) for V_P is the absence of K^{-n} $(n>0)$ terms. It is the generality of the absence of K^{-n} terms which we will discuss in this work.

First, let us note that a K^{-n} term in V_P and V_R does not violate the Steinmann relation, because it does not produce a nonvanishing double discontinuity in the overlapping invariants s_{12} and s_{23} in the physical region for the 2-to-3 process, which is what the Steinmann relation forbids.

Second, notice that a K^{-n} term cannot be ruled out directly by some asymptotic upper bound on the 2-to-3 amplitude, because in the kinematical region which we are now discussing (the double-Regge limit) K is not an asymptotic variable. In fact it is clearly bounded for fixed t_P and t_R, as can be seen from

$$K=\frac{\lambda(t_P,t_R,m^2)}{(t_Pt_R)^{1/2}\cos\omega-t_P-t_R+m^2}+O(s_{ij}{}^{-1})\ . \qquad (4)$$

Third, one may think of ruling out a K^{-n} term in (3) by continuing V to one of its t_P or t_R poles and requiring that the residue of the pole be a polynomial in overlapping invariants, hence in K. This type of reasoning does not prevent K^{-n} terms from appearing in V_P or V_R; it only shows that if they do, then they cannot appear multiplied by functions which lead to poles of V in t_P or t_R. There is no *a priori* motive to bar the possibility of terms in V which do not contain poles in t_P or t_R.

Thus, to the author's knowledge, there is no hard "axiomatic" evidence against K^{-n} terms in V_P and V_R.

As usual in such cases, one goes back to see what certain models have to say. Computations have been done with Feynman diagrams,[3] using Reggeon calculus,[4] and with the dual model.[5] All computations give results in which no K^{-n} terms are present.[6]

We will bring further evidence in support of the absence of K^{-n} terms by analyzing the analytic properties of Feynman diagrams. The argument goes as follows: A K^{-n} term in V, while not implying really that the 2-to-3 amplitude has poles at $s_{12}=0$ and $s_{23}=0$, must certainly be the asymptotic representation of some singularity in the amplitude involving both s_{12} and s_{23}. Moreover, a K^{-n} term would indicate that there must exist some point in the $\mathrm{Re}s_{12}$-$\mathrm{Re}s_{23}$ plane where the 2-to-3 amplitude is singular for any sign $\mathrm{Im}s_{12}$ or $\mathrm{Im}s_{23}$ might take. However, we show that Feynman diagrams are likely to have no point on the $(\mathrm{Im}s_{12})(\mathrm{Im}s_{23})>0$ side of the $\mathrm{Re}s_{12}$-$\mathrm{Re}s_{23}$ plane where there is a singularity produced by both s_{12} and s_{23}. Thus we conclude that no K^{-n} term can appear in V_P or V_R.

We stress that the argument does not give any preferential treatment to diagrams which are expected to contribute in the double-Regge limit.

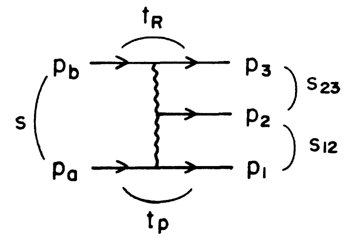

FIG. 1. The double-Regge limit of the $a+b \to 1+2+3$ process.

We show explicitly that the box diagram (Fig. 3) has the property stated above. Then we comment on the generality of that result for an arbitrary diagram, hence also for those diagrams with nonvanishing asymptotic contribution.

Consider the s_{12}-s_{23} plane. The physical region for the 2-3 process,

$$a+b \to 1+2+3 \,, \tag{5}$$

is a subregion of the physical region for the 1-to-3 process

$$c \to 1+2+3 \,, \tag{6}$$

where c is a particle of mass $\sqrt{s}$. We assume s is asymptotic and show in Fig. 2 the physical region for the process (6). For the process (5), depending upon t_P and t_R, the physical regions for single-Regge (1R), double-Regge (2R), and fixed angle (FA) limits are also shown in Fig. 2.

Let us determine the double discontinuity in s_{12} and s_{23} of the Feynman diagram shown in Fig. 3. We note immediately that the problem is to find the double spectral region for a box diagram having three external legs of fixed mass m and the fourth leg of a very large mass $\sqrt{s}$.

An easy way to proceed is to start with all four legs of the same mass m and gradually increase

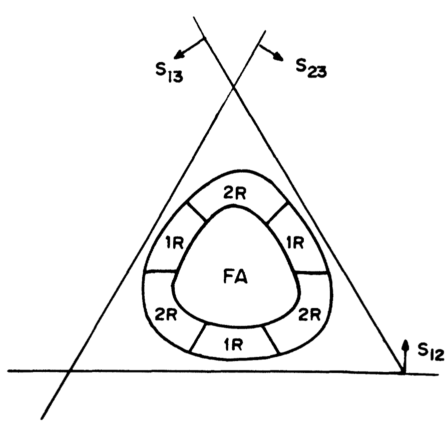

FIG. 2. The physical region for the process $c \to +1+2+3$, where $m_c = \sqrt{s} \gg m$.

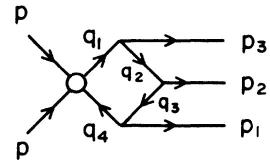

FIG. 3. Box diagram.

the mass of one leg up to the desired value $\sqrt{s}$ (our analysis is inspired by the classic book by Eden, Landshoff, Olive, and Polkinghorne[7]).

Introducing the variables

$$y_{ij} = -\frac{q_i \cdot q_j}{m_i m_j} \,, \tag{7}$$

we see that the leading Landau surface for the box diagram is given by (see Eq. 2.4.10 of Ref. 7)

$$\begin{vmatrix} 1 & -y_{12} & -y_{13} & -y_{14} \\ -y_{12} & 1 & -y_{23} & -y_{24} \\ -y_{13} & -y_{23} & 1 & -y_{34} \\ -y_{14} & -y_{24} & -y_{34} & 1 \end{vmatrix} = 0 \,. \tag{8}$$

Notice that in the present case we have

$$\begin{aligned} &y_{12} = y_{23} = y_{34} = -\tfrac{1}{2} \,, \\ &y_{13} = \frac{s_{23} - 2m^2}{2m^2} \,, \\ &y_{24} = \frac{s_{12} - 2m^2}{2m^2} \,, \\ &y_{14} = \frac{s - 2m^2}{2m^2} \,. \end{aligned} \tag{9}$$

Substituting these values into (8), we obtain the equation giving the leading Landau surface for the box diagram:

$$s_{12}{}^2 s_{23}{}^2 - 4m^2 s_{12} s_{23}(s_{12} + s_{23}) + 2(s + 5m^2) s_{12} s_{23} m^2 - 3m^4 (s - m^2)^2 = 0 \,. \tag{10}$$

Let Γ be the intersection of the leading Landau surface with the $\mathrm{Re}\,s_{12}$-$\mathrm{Re}\,s_{23}$ plane. Then Γ is tangent to the lines $\Sigma_1^\pm$, $\Sigma_2^\pm$, $\Sigma_3^\pm$, and $\Sigma_4^\pm$ representing the intersection of triangle singularity surfaces with the $\mathrm{Re}\,s_{12}$-$\mathrm{Re}\,s_{23}$ plane. Also, Γ has asymptotes, the lines $\Sigma_{13}^\pm$ and $\Sigma_{24}^\pm$, representing the normal and pseudonormal thresholds in this plane.

Since singular and nonsingular arcs of Γ are separated from each other by lower-order singularities, we shall first say a few words about the latter. The normal and pseudonormal singularities

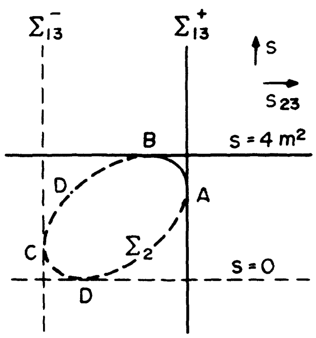

FIG. 4. Triangle singularity in the Res-Res_{23} plane.

are given by

$$\Sigma_{13}^{\pm}:\ s_{23} = (m \pm m)^2$$

and (11)

$$\Sigma_{24}^{\pm}:\ s_{12} = (m \pm m)^2\ .$$

Only Σ_{13}^+ and Σ_{24}^+ are singular. We represent singular arcs of Landau curves by full lines, and nonsingular ones by dashed lines.

The triangle singularities Σ_1, Σ_2, Σ_3, and Σ_4 correspond to $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, and $\alpha_4 = 0$, respectively. Thus Σ_1 and Σ_2 are independent of $\sqrt{s}$, while the position and nature of Σ_2 and Σ_4 vary with it. For example Σ_2 is given by

$$y_{14}{}^2 + y_{13}{}^2 + y_{34}{}^2 - 2y_{14}\,y_{13}\,y_{34} - 1 = 0\ . \tag{12}$$

In the Res-Res_{23} plane the corresponding Landau curve is shown in Fig. 4. It is well known[7] that for Re$s < 3m^2$, Σ_2 is not singular. For $3m^2 \leqslant$ Res $< 4m^2$, the arc AB is singular when approached from any direction, while the arc BC is nonsingular. We take $s =$ Re$s + i\epsilon$ with $\epsilon > 0$. Increasing Res past $4m^2$ we go on to the complex parts of Σ_2. The movement of $\Sigma_2^{\pm}$ in the s_{23} complex plane as a function of s is shown in Fig. 5. One can note at a glance that similar things happen to Σ_3 as a function of s_{12}.

Now we are ready to investigate the leading singularity of the box diagram. We skip $s = m^2$ and start with $s = 2m^2$. The situation in the Res_{12}-Res_{23} plane is shown in Fig. 6. The singularity curve Γ was determined with (10).

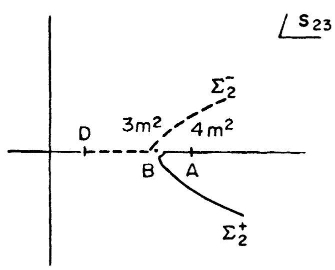

FIG. 5. Movement of the triangle singularity in the s_{23}-plane function of $s =$ Re$s + i\epsilon$, $\epsilon > 0$.

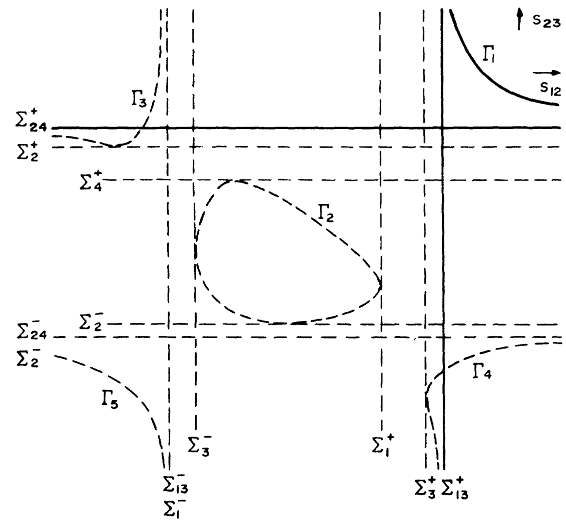

FIG. 6. Box-diagram Landau curve for $s = 2m^2$.

We will not repeat the argument of Ref. 7 showing that only the arc Γ_1 is singular and only when approached from (Ims_{12})(Ims_{23})>0, but remind the reader of the two crucial facts needed to obtain that:

(a) For $s < 4m^2$ the region where Re$s_{12} \to -\infty$ and Re$s_{23} \to -\infty$ is "safe" (Eq. 2.4.16 of Ref. 7).

(b) None of the triangle singularities have come up on the physical sheet for $s < 3m^2$.

As we increase Res past $3m^2$, triangle singularities Σ_2^+ and Σ_4^+ come up to the physical sheet. In Fig. 7 we show the situation for Re$s = 4m^2 - \epsilon$ with

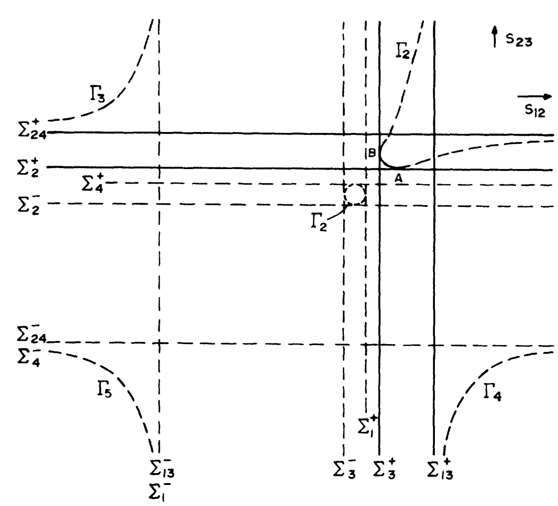

FIG. 7. Box-diagram Landau curve for $s = 4m^2 - \epsilon$, $\epsilon > 0$.

$\epsilon > 0$. There still exists the safe region, from which we can continue without problems until we cross both Σ_2^+ and Σ_3^+, which are now singular on the physical sheet. Thus again only Γ_1 could be singular. In fact only the arc AB is singular, and only when approached from $(\mathrm{Im}s_{12})(\mathrm{Im}s_{23}) > 0$, as can be seen if we recall that being in a safe region means convergence of the Feynman integral with an undistorted α hypercontour. Thus only that piece of Γ_1 corresponding to α_1, α_2, α_3, and α_4 positive will be singular when approached from $(\mathrm{Im}s_{12})(\mathrm{Im}s_{23})>0$. Remembering that Σ_2^+, Σ_3^+ and $\Sigma_{13}^+, \Sigma_{24}^+$ correspond to one and two contractions, respectively, the above conclusion follows.

As $\mathrm{Re}s$ is increased, at $\mathrm{Re}s = 4m^2$, Γ_1 forms a crunode exactly at the point of tangency to the triangle singularities Σ_2 and Σ_3 (see Fig. 8). This signals the appearance of complex singularities. Indeed, for $\mathrm{Re}s = 4m^2 + \epsilon$ with $\epsilon > 0$, the situation is as in Fig. 9. There does not exist a safe region. However, Γ_3, Γ_4, and Γ_5 and the complex surfaces attached to them are not singular since they have not gone through any cuts. So are the arcs $A\infty$ and ∞B of Γ_1. The complex surface attached to AB is singular, and so AB is singular when approached from $(\mathrm{Im}s_{12})(\mathrm{Im}s_{23}) < 0$, but not when approached from $(\mathrm{Im}s_{12})(\mathrm{Im}s_{23}) > 0$.

Increasing $\mathrm{Re}s$ to its final asymptotic value, the singularity property of the different arcs of Γ does not change. In Fig. 10 we show the $\mathrm{Re}s_{12}$-$\mathrm{Re}s_{23}$ plane for some value of $s \gg m^2$. We also show the physical region for the process (6). Notice that although Γ_1 touches the physical region for the process (6), this does not mean that the Steinmann relation is violated. Indeed this relation asserts only that the double discontinuity in overlapping invariants vanishes in the physical region, but does not preclude the existence of points in the physical region where the amplitude has a singularity produced by two overlapping invariants. That the latter might be the case without violating the Steinmann relation was emphasized by Polkinghorne[8] and Stapp[9] in connection with the interpretation of the inclusive cross section as an elementary discontinuity formula.

Thus we have proved our claim that the box diagram does not possess any point on the $(\mathrm{Im}s_{12})$ $(\mathrm{Im}s_{23}) > 0$ side of the $\mathrm{Re}s_{12}$-$\mathrm{Re}s_{23}$ plane where there is a singularity produced by both s_{12} and s_{23}. The question is whether this property belongs to

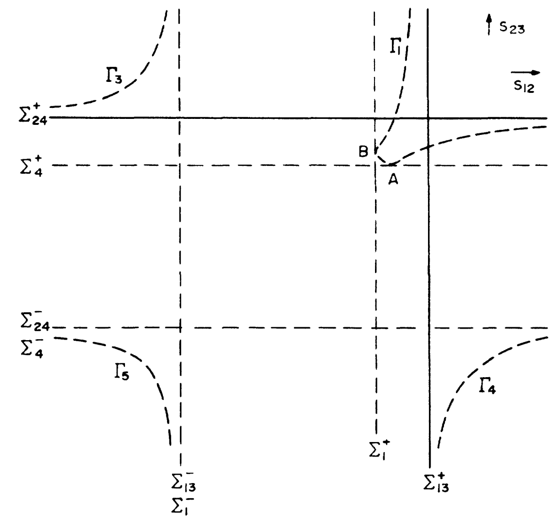

FIG. 9. Box-diagram Landau curve for $s = 4m^2 + \epsilon$, $\epsilon > 0$.

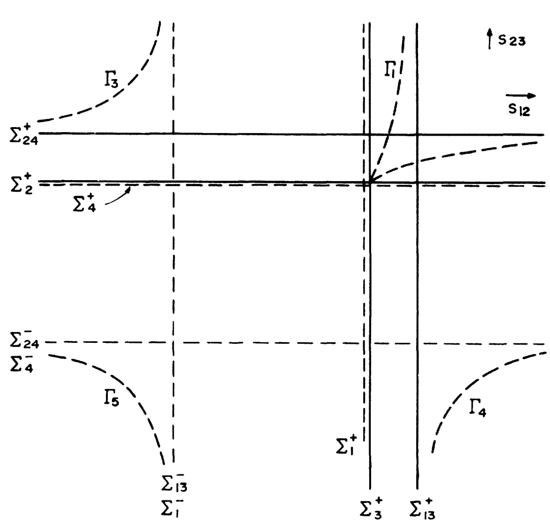

FIG. 8. Box-diagram Landau curve for $s = 4m^2$.

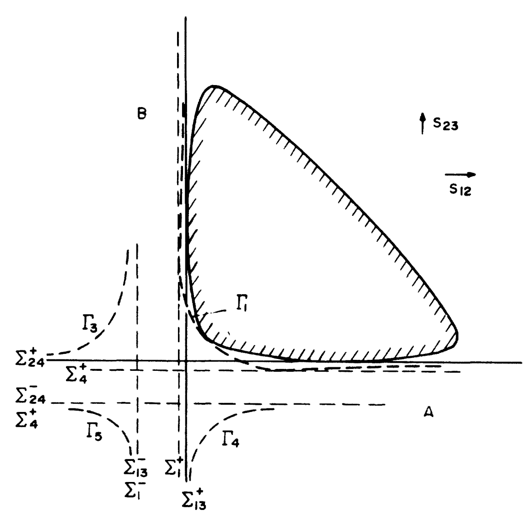

FIG. 10. Box-diagram Landau curve for $s \gg m^2$. The physical region for $c \to 1+2+3$ is also shown.

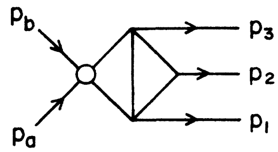

FIG. 11. Acnode graph.

any Feynman diagram or not.

We believe that the answer is yes and will sketch a rough argument indicating so. Our argumentation is based on induction, and it has the same flaw as the proof given by Eden, Landshoff, Polkinghorne, and Taylor[10] had for the absence of complex singularities.

For a given Feynman diagram, suppose that it has already been shown that all its lower-order singularities are nonsingular in s_{12} and s_{23} on the $(\mathrm{Im}s_{12})(\mathrm{Im}s_{23})>0$ side of the $\mathrm{Re}s_{12}$-$\mathrm{Re}s_{23}$ plane. Then the leading Landau curve must also be nonsingular on the $(\mathrm{Im}s_{12})(\mathrm{Im}s_{23})>0$ side of this plane. Indeed, usually singular parts are divided from nonsingular parts only by cuts attached to the lower singularities. If the latter had been shown to be harmless, then the entire leading Landau curve would have the same singularity property. But at the point of tangency with the lower-order singularity surface we have already shown that there is no (lower-order) singularity.

This completes the proof. The flaw is that there exists another mechanism for division of a surface into singular and nonsingular parts (see Eq. 2.1.13 of Ref. 7). The acnode graph shown in Fig. 11 has this property. It is this fact which prevented Eden, Landshoff, Polkinghorne, and Taylor from indeed proving the Mandelstam representation; however, we still believe the Mandelstam representation to be correct. This motivates our hope that although the argument given above is not entirely correct, the conclusion is true—no K^{-n} terms exist and hence no new singularities in the complex helicity plane are needed.[11]

In closing we will comment on some questions raised by our discussion of the analytic properties of Feynman graphs. We have seen that the box diagram (Fig. 3) has an anomalous singularity in the 2R region of the 2-to-3 process. Indeed, for $s\to\infty$, from (10) we have

$$(K-m^2)(K+3m^2)=0\ . \qquad (13)$$

But even though the singularity occurs for values of s_{12} and s_{23} in the asymptotic region, does it occur in a piece of the amplitude which contributes to the asymptotic expression or in a nonleading term which can be neglected altogether? And if it occurs in a term which survives in the asymptotic region, can the Regge expression for the amplitude accommodate such singularities?

These questions are unrelated to the main topic of this work, and so we will refrain from analyzing them here. It suffices to say that by looking at the sum of all box diagrams with an n-rung ladder insertion (Fig. 12), we conclude that it is possible to have asymptotic anomalous thresholds and that they can be accounted for in Regge expressions by the use of Regge cuts. (Being planar, the diagram in Fig. 12 gives a vanishing asymptotic contribution. However, if one separates the function into terms containing the different normal and anomalous singularities of this diagram, some of them can be asymptotically nonvanishing. Indeed satisfying no positivity condition, the sum of such asymptotically nonvanishing terms can be, and for planar diagrams is, asymptotically vanishing.)

That there may be some connection between anomalous singularities and Regge cuts is not surprising, since traditionally both have been obtained from the unitarity equations by iteration. The following observations might be new and relevant to future developments:

(1) Anomalous thresholds can be essentially different from normal thresholds in that they are moving singularities. As shown by (13), we can go into the double-Regge asymptotic region "riding" on top of an anomalous branch point. Thus, whereas normal thresholds in s_{12} and s_{23} are smeared out asymptotically, producing cuts such as those of $s_{12}^{\alpha(t_1)}$ and $s_{23}^{\alpha(t_2)}$, respectively, an anomalous singularity such as the one discussed above will have to be represented *per se* (if it occurs in a leading term).

(2) The commonly accepted assumption that asymptotically the behavior of amplitudes above and below the cut is the same, could be violated by the type of anomalous singularities discussed above. Indeed, in the physical region, the Landau curve will not be singular when approached from $\mathrm{Im}s_{12}$ and $\mathrm{Im}s_{23}>0$, but it can be singular when approached from $(\mathrm{Im}s_{12})(\mathrm{Im}s_{23})>0$ (as in the case

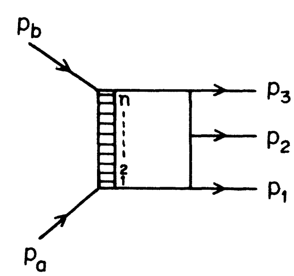

FIG. 12. Box diagram with ladder insertion.

discussed in the text).

(3) We know that there are anomalous singularities in the missing mass in the physical region for the inclusive cross section. If any of them survives in the asymptotic limit, could we detect its presence as some obvious nonanalyticity in the inclusive cross section represented as a function of the missing mass?

The possibility of having K^{-n} terms in the vertex arose in discussions with R. E. Brower and J. H. Weis regarding their powerful theorem. Helpful conversations with C. E. DeTar are gratefully acknowledged.

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Strongly Interacting Weak Intermediate Bosons*

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We reexamine the weak interaction mediated by the intermediate vector bosons which are assumed to couple to other (probably unknown) particles (or to themselves) more strongly than to the weak currents in order to see whether the theory of this type is potentially free from the divergence difficulty. Sum rules and high- and low-energy limits of the cross sections are obtained for various weak-interaction processes.

I. INTRODUCTION

Recent extensive investigations of possible synthesis of weak and electromagnetic interactions[1] associated with the Higgs mechanism[2] of gauge fields indicate a rosy future for the renormalizable theory of weak interactions.[1,3] Because of a variety of models with different symmetry schemes, these ideas do not seem to contradict any presently available experimental data. It is, however, also true that there is no evidence for any one of the propose models at present. It will probably take several years to reach energies high enough to produce the weak intermediate bosons, if any, whose masses are supposed to be around 40 GeV or even higher and to check whether this attractive idea is realized in nature. While we are waiting for some progress in increasing available energy, it is worthwhile to search for other theoretical possibilities which may have an equally good chance to be physically realized. In this paper we reexamine one of such alternative pictures of weak interactions, which is formed with the following three assumptions: (1) The weak interactions of known particles (leptons and hadrons) are mediated by weak intermediate vector bosons,[4] (2) these vector bosons couple to other (probably unkown) particles X (or to themselves) more strongly than to the weak currents of leptons and hadrons,[5-9] and (3) the above unknown coupling will suppress violent behaviors of the amplitude at high energies or at large mo-